Math 115A Worksheet
Thursday, Dec 7 (Week 10)

1. Let V be an inner product space, over a field F ($F = \mathbb{R}$ or $F = \mathbb{C}$).

   (a) Let $\vec{x}, \vec{y} \in V$. Prove that if $\langle \vec{x}, \vec{z} \rangle = \langle \vec{y}, \vec{z} \rangle$ for all $\vec{z} \in V$, then $\vec{x} = \vec{y}$.

   If $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in V$ then $\langle x, z \rangle - \langle y, z \rangle = 0$ for all $z \in V$. By axiom 1, $\langle x, z \rangle - \langle y, z \rangle = \langle x-y, z \rangle$. Hence, $\langle x-y, z \rangle = 0$ for all $z \in V$. Hence, the equation is true for $z = x-y$. That is, $\langle x-y, x-y \rangle = 0$. By axiom 4 as 0 is not positive, $x - y = 0$. Hence, $x = y$.

   of an inner product

   (the contrapositive of)

   Now let β be a basis of V. (Note: You should be able to do this problem without assuming that V is finite-dimensional.)

   (b) Let $x \in V$. Prove that if $\langle \vec{x}, \vec{z} \rangle = 0$ for all $\vec{z} \in \beta$, then $\vec{x} = \vec{0}$.

   Let $y \in V$. Write $y = \sum_{i=1}^{k} a_i z_i$ for $a_i \in F$, $z_i \in \beta$. Then

   $\langle x, y \rangle = \langle x, \sum_{i=1}^{k} a_i z_i \rangle = \sum_{i=1}^{k} a_i \langle x, z_i \rangle$. By hypothesis, $\langle x, z_i \rangle = 0$ as $z_i \in \beta$. Thus, $\langle x, y \rangle = 0$ for all $y \in V$. Thus, $\langle x, y \rangle = \langle 0, y \rangle$ for all $y \in V$, so by part (a) we conclude $x = 0$.

   (c) Prove that for any $\vec{x}, \vec{y} \in V$, if $\langle \vec{x}, \vec{z} \rangle = \langle \vec{y}, \vec{z} \rangle$ for all $\vec{z} \in \beta$, then $\vec{x} = \vec{y}$.

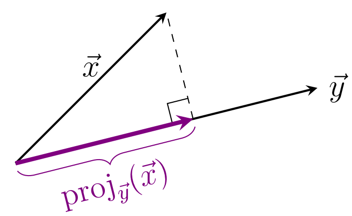
   $\langle x, z \rangle - \langle y, z \rangle = \langle x-y, z \rangle$.

   Thus, we are supposing $\langle x-y, z \rangle = 0$ for all $z \in V$. As such, by part (b) $x - y = 0$. Hence, $x = y$.

2. Recall the following from Math 32A or Math 33A, or some other course in which you learned basic vector operations:

   For any two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, we define the *orthogonal projection of $\vec{x}$ onto $\vec{y}$* as follows:

$$\operatorname{proj}_{\vec{y}}(\vec{x}) = \begin{cases} \left(\dfrac{\vec{x} \bullet \vec{y}}{\vec{y} \bullet \vec{y}}\right) \vec{y} & \text{if } \vec{y} \neq \vec{0} \\ \vec{0} & \text{if } \vec{y} = \vec{0} \end{cases}$$

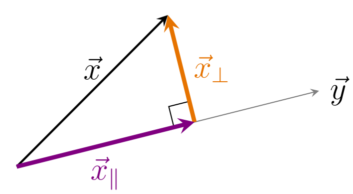


   Then we can define $\vec{x}_\parallel$ and $\vec{x}_\perp$ by

$$\vec{x}_\parallel = \operatorname{proj}_{\vec{y}}(\vec{x}) \qquad \text{and} \qquad \vec{x}_\perp = \vec{x} - \vec{x}_\parallel. \tag{1}$$

   Then (as you can see in the diagram below, and can prove easily using the properties of dot products) these satisfy the following three properties:

   (i) $\vec{x} = \vec{x}_\parallel + \vec{x}_\perp$
   (ii) $\vec{x}_\parallel$ is parallel to (a scalar multiple of) $\vec{y}$.
   (iii) $\vec{x}_\perp$ is orthogonal to $\vec{y}$ (i.e., $\vec{x}_\perp \bullet \vec{y} = 0$).



   We may generalize the above construction to any inner product space V, as follows: Let $\vec{x}, \vec{y} \in V$. Define

$$\operatorname{proj}_{\vec{y}}(\vec{x}) = \begin{cases} \left(\dfrac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle}\right) \vec{y} & \text{if } \vec{y} \neq \vec{0} \\ \vec{0} & \text{if } \vec{y} = \vec{0} \end{cases}$$

   Then we can define $\vec{x}_\parallel$ and $\vec{x}_\perp$ as in equation (1) above, and it's easy to prove that properties (i), (ii), and (iii) are still true in this setting.

We may use this to prove the **Cauchy–Schwarz inequality**:

For any inner product space V, for all $\vec{x}, \vec{y} \in V$, $\quad \left|\langle \vec{x}, \vec{y} \rangle\right| \leq \|\vec{x}\| \|\vec{y}\|$

Fill in the blanks to complete the following proof.

Proof. If $\vec{y} = \vec{0}$, then $|\langle x, y \rangle| = 0 = \|x\| \|y\|$, so indeed $|\langle x, y \rangle| \leq \|x\| \|y\|$.

Assume that $\vec{y} \neq \vec{0}$. Let $\vec{x}_\parallel$ and $\vec{x}_\perp$ be as above, so that $\vec{x} = \vec{x}_\parallel + \vec{x}_\perp$, and $\vec{x}_\perp$ is orthogonal to $\vec{x}_\parallel$. Then

$$\|\vec{x}\|^2 = \left\|\vec{x}_\parallel + \vec{x}_\perp\right\|^2 = \left\|\vec{x}_\parallel\right\|^2 + \left\|\vec{x}_\perp\right\|^2 \geq \left\|\vec{x}_\parallel\right\|^2$$

where the second equality is by the Pythagorean theorem, and the inequality at the end is because $\|x_\perp\|^2 \geq 0$. Therefore

$$\|\vec{x}\| \geq \left\|\vec{x}_\parallel\right\| = \left\| \frac{\langle x, y \rangle}{\langle y, y \rangle} y \right\|$$

$$= \left| \frac{\langle x, y \rangle}{\langle y, y \rangle} \right| \|y\|$$

$$= \frac{|\langle x, y \rangle|}{\|y\|^2} \cdot \|y\|$$

$$= \frac{|\langle x, y \rangle|}{\|y\|}$$

Thus, $\|x\| \geq \dfrac{|\langle x, y \rangle|}{\|y\|}$, so $|\langle x, y \rangle| \leq \|x\| \|y\|$.

$\square$

3. Let V be an inner product space, and let $\vec{x}, \vec{y} \in V$. Prove that the Cauchy–Schwarz inequality ($|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|$) is an *equality*, that is

$$|\langle \vec{x}, \vec{y} \rangle| = \|\vec{x}\|\|\vec{y}\|$$

if and only if one of the two vectors ($\vec{x}$ or $\vec{y}$) is a scalar multiple of the other.

(Hint: One direction is easy. For the other direction, look back through the proof of the Cauchy–Schwarz inequality on the previous page, and find where the ***inequality*** *originated. Show that this must be an* ***equality*** *in order to have equality in Cauchy–Schwarz. And from this equality, you can deduce easily that $\vec{x}$ is a scalar multiple of $\vec{y}$. You should also deal separately with the special case where $\vec{y} = \vec{0}$.)*

If $y = 0$ then, $|\langle x, y \rangle| = |\langle x, 0 \rangle| = 0$ and

$\|x\| \|y\| = \|x\| \|0\| = \|x\| 0 = 0$.

Thus, if $y = 0$ then $|\langle x, y \rangle| = \|x\| \|y\|$.

Now suppose $y \neq 0$

($\Leftarrow$) If x or y are scalar multiples of each other then there is a scalar $c \in F$ so that $x = cy$ or $y = cx$.

If $y = cx$, then as $y \neq 0$ we must have $c \neq 0$.

Hence, $\frac{1}{c} y = x$, so in either case, i.e., x is a scalar multiple of y.

Write instead then $x = cy$. Then $|\langle x, y \rangle| = |\langle cy, y \rangle|$

$= |c| \langle y, y \rangle$

$= |c| \|y\|^2$

On the other hand $\|x\|\|y\| = \|cy\| \|y\| = |c| \|y\|^2$. Hence, $|\langle x, y \rangle| = \|x\| \|y\|$ in this case.

($\Rightarrow$) Suppose conversely that $|\langle x, y \rangle| = \|x\| \|y\|$. As $y \neq 0$, the proof of Cauchy-Schwarz only introduced inequality in the step $\|x\| \geq \|x^{\parallel}\|$

so $|\langle x, y \rangle| = \|x\| \|y\| \Leftrightarrow \|x\| = \|x^{\parallel}\|$. By the Pythagorean theorem, $\|x\| = \|x^{\parallel}\| + \|x^{\perp}\|$,

so $\|x\| = \|x^{\parallel}\| \Rightarrow \|x^{\perp}\| = 0 \Leftrightarrow x^{\perp} = 0$. As $x = x^{\parallel} + x^{\perp}$, this is equivalent to $x = x^{\parallel}$. And $x^{\parallel} = \frac{\langle x, y \rangle}{\langle y, y \rangle} y$ is a scalar multiple of y, so x is a scalar multiple of y.