Math 115A Worksheet
Thursday, Nov 30 (Week 9)

1. Let A, B be two matrices in $M_{n\times n}(F)$ such that B is invertible:

   (a) Show that A and $B^{-1}AB$ have the same eigenvalues.

   We show that A and $B^{-1}AB$ have the same characteristic polynomial, i.e. $\chi_A(t) = \chi_{B^{-1}AB}(t)$.

   Indeed, $\chi_{B^{-1}AB}(t) = \det(B^{-1}AB - tI)$
   $= \det(B^{-1}(A-tI)B)$
   $= \det(B)^{-1}\det(A-tI)\det(B)$
   $= \chi_A(t)$

   Eigenvalues are precisely roots of the characteristic polynomial, so A and $B^{-1}AB$ do indeed have the same eigenvalues.

   (b) What is the relationship between the eigenvectors of A and $B^{-1}AB$?

   Let v be an eigenvector of A of eigenvalue λ. That is, $v \neq 0$ and $Av = \lambda v$. Then let $w = B^{-1}v$. We have $B^{-1}ABw = B^{-1}ABB^{-1}v$
   $= B^{-1}Av$
   $= B^{-1}\lambda v$
   $= \lambda w$

   so $B^{-1}ABw = \lambda w$. Furthermore, $v \neq 0$ and B^{-1} is invertible, hence injective, so $w \neq 0$. Thus, if v is an eigenvector of A of eigenvalue λ then $B^{-1}v$ is an eigenvector of $B^{-1}AB$ of eigenvalue λ.
   The same method shows that if v is an eigenvector of $B^{-1}AB$ of eigenvalue λ that Bv is an eigenvector of A of eigenvalue λ.

   In short,
   $B^{-1}(E_\lambda(A))$
   $\parallel$
   $E_\lambda(B^{-1}AB)$

   (c) Using the above, show that given a linear operator T on a finite-dimensional vector space V, the eigenvalues of $[T]_\beta$ do not depend on the choice of an ordered basis β for V.

   Let β, γ be two ordered bases of V. Then we have the equation

   $$[T]_\gamma^\gamma = [id]_\beta^\gamma [T]_\beta^\beta [id]_\gamma^\beta$$

   Let $A = [T]_\beta^\beta$, $B = [id]_\gamma^\beta$. Then this equation says

   $$[T]_\gamma^\gamma = B^{-1}AB$$

   So by part (a), $A = [T]_\beta^\beta$ and $B^{-1}AB = [T]_\gamma^\gamma$ have the same eigenvalues.

   (Aside: This explains (b) as well, let $Tv = \lambda v$ for $v \in V$ nonzero. Then we have $[T]_\beta^\beta [v]_\beta = [T(v)]_\beta = [\lambda v]_\beta = \lambda[v]_\beta$, so $[v]_\beta$ is an eigenvector of $[T]_\beta^\beta$ of eigenvalue λ. Similarly, $[T]_\gamma^\gamma [v]_\gamma = \lambda [v]_\gamma$. That is, $A[v]_\beta = \lambda[v]_\beta$ and $B^{-1}AB[v]_\gamma = \lambda[v]_\gamma$. And indeed, $[v]_\beta$ and $[v]_\gamma$ are related by $[v]_\gamma = [id]_\beta^\gamma [v]_\beta = B^{-1}[v]_\beta$ as in (b).)

(Keep these 2 examples in mind! A fails to be diagonalizable over $\mathbb{R}$, but it's fixed by going to the bigger field $\mathbb{C}$, B fails to be diagonalizable more substantially, as it fails to be so for any field)

2. In this problem, we investigate the diagonalizability of matrices over $\mathbb{C}$:

(a) Recall that $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$, whose associated left multiplication transformation L_A is counter-clockwise rotation by 90 degrees on $\mathbb{R}^2$, does not have real eigenvalues or eigenvectors.

Now consider the same matrix A as a matrix in the $\mathbb{C}$-vector space $M_{2\times 2}(\mathbb{C})$. Find the complex eigenvalues and eigenvectors of A, and diagonalize the matrix.

$\chi_A(t) = \det\begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix} = t^2+1 = (t-i)(t+i)$.

Hence, the eigenvalues of A are $\pm i$.

$E_i(A) = \ker(A-iI) = \ker\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$

$E_{-i}(A) = \ker(A+iI) = \ker\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$

Thus, $A = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{-1}$

(Indeed, if $\beta = \{e_1, e_2\}$ and $\gamma = \left\{\begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$, this equation says $[L_A]_\beta^\beta = [id]_\gamma^\beta [L_A]_\gamma^\gamma [id]_\beta^\gamma$, and $[id]_\gamma^\beta$ has columns given by the vectors in γ written in the basis β, exactly as we have.)

(b) Now consider the matrix $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is B diagonalizable over $\mathbb{R}$, i.e. is B considered as a matrix in $M_{2\times 2}(\mathbb{R})$ diagonalizable? Is B diagonalizable over $\mathbb{C}$?

$\chi_B(t) = \det(B-tI) = \det\begin{pmatrix} -t & 1 \\ 0 & -t \end{pmatrix} = t^2$.

Hence, the only eigenvalue of B is 0.

$E_0(B) = \ker(B) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

Hence, any set of eigenvectors of B is contained in $E_0(B)$, a 1-dimensional space, so no 2 eigenvectors can be linearly independent. Thus, no eigenbasis exists and B is not diagonalizable.

B is not diagonalizable over $\mathbb{R}$ or over $\mathbb{C}$.

This proof shows this. In fact, it works for any field F.

(c) Let V be a finite-dimensional vector space over F with some ordered basis $\beta = (v_1, \ldots, v_n)$, and let T be a linear operator on V with matrix $[T]_\beta$. Suppose that the first column of the matrix $[T]_\beta$ is a vector of the form $(a, 0, \ldots, 0)$ for some $a \in F$.

i. Show that a is an eigenvalue of T with corresponding eigenvector v_1.

$v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$ so $[v_1]_\beta = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Then $[T(v_1)]_\beta = [T]_\beta^\beta [v_1]_\beta = [T]_\beta^\beta \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a[v_1]_\beta$

Then $[T(v_1)]_\beta = a[v_1]_\beta = [av_1]_\beta$. As the linear transformation $V \longrightarrow F^n$ sending $v \mapsto [v]_\beta$ is an isomorphism, it's injective so $[T(v_1)]_\beta = [av_1]_\beta$ implies $T(v_1) = av_1$. As $v_1 \in \beta$ a linearly independent set, we have $v_1 \neq 0$. Thus, v_1 is an eigenvector of T with eigenvalue a.

ii. Explain why this example shows that if V is a vector space over $\mathbb{R}$, then there are linear transformations T for which there is no basis β such that $[T]_\beta$ is upper triangular.

Does the same argument work if V is a vector space over $\mathbb{C}$?

Consider $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $90°$ counterclockwise rotation. Suppose there was a basis β of $\mathbb{R}^2$ so that $[T]_\beta^\beta$ was upper triangular. Write

$$[T]_\beta^\beta = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$$

Then by (c), T has an eigenvalue $a \in \mathbb{R}$. But T has no real eigenvalues, so this is a contradiction. So no such β exists.

Over $\mathbb{C}$ this argument fails, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \mathbb{C}^2 \to \mathbb{C}^2$ has eigenvalues. In fact, let $T : V \to V$ be any linear transformation for V a fdvs/$\mathbb{C}$. Then $\chi_T(t)$ is a complex polynomial and thus (by the Fundamental Theorem of Algebra) has some complex root. That is, T has an eigenvalue.

(Hint: induction)

Exr. Use this fact about $\mathbb{C}$ to show that for all linear transformations $V \to V$ for V a fdvs/$\mathbb{C}$ there is a basis β of V so that $[T]_\beta^\beta$ is upper triangular.

3. Find all the eigenvalues for the transformation $D: P(\mathbb{R}) \to P(\mathbb{R})$ that maps a polynomial $p(x)$ to $p'(x)$. Your answer should include a list of all the eigenvalue(s), and an argument for why you have found all of them.

Let $Dp = \lambda p$.

Then $p' = \lambda p$

Suppose $\deg(p) \geq 1$. Then $\deg(p') = \deg(p) - 1$. Furthermore,

$$\deg(\lambda p) = \begin{cases} -\infty & \lambda = 0 \\ \deg(p) & \lambda \neq 0 \end{cases}$$

(Note. $\deg(0) = -\infty$ in my convention)

and in either case, $\deg(\lambda p) \neq \deg(p) - 1$. Hence, we must have $\deg(p) < 1$ for $p' = \lambda p$ to be possible.

Thus, p must be a constant. In that case,

$$p' = 0 = 0 \cdot p$$

Thus, the only eigenvectors of D are the nonzero constants, which have eigenvalue 0.

Rmk 1. The "matrix" for D in the infinite basis $\{1, x, x^2, \ldots\}$ is

$$\begin{pmatrix} 0 & 2 & 0 & 0 & \ddots & \\ 0 & 0 & 3 & 0 & & \ddots \\ \vdots & \vdots & \vdots & 4 & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

which looks the kind of matrix whose only eigenvalue is 0 with the eigenvector given by the first basis vector

Rmk 2. if you dislike this argument, solve $p' = \sum_{i=1}^{n} i a_i x^{i-1} = \lambda \sum_{i=0}^{n} a_i x^i = \lambda p$ to show p must be constant

we say f is smooth or infinitely differentiable at all $x \in \mathbb{R}$

Rmk 3. If we replace $P(\mathbb{R})$ with $C^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f'(x), f''(x), f'''(x), \ldots$ all exist for all $x \in \mathbb{R}\}$

then $D(e^{\lambda x}) = \lambda e^{\lambda x}$, so all real numbers λ are eigenvalues of D (and the eigenspaces are all 1 dimensional)