Math 115A Worksheet
Thursday, Nov 9 (Week 6)

1. Let U, V and W be vector spaces over F. Let $T: V \to W$ and $S: U \to V$ be linear transformations. Prove the following statements:

   (a) If $T \circ S$ is one-to-one, then S is one-to-one.

   Suppose $S(u) = 0_V$ for some $u \in U$. We want to show that $u = 0$. Apply T to both sides of the equation, so

   $$T(S(u)) = T(0_V)$$

   $T(S(u)) = (T \circ S)(u)$ by definition of $T \circ S$

   $T(0_V) = 0_W$ as T is linear

   so $(T \circ S)(u) = 0_W$. As $T \circ S$ is injective, $u = 0$. Thus, $S(u) = 0 \Rightarrow u = 0$, so S is injective.

   (b) Find an example where $T \circ S$ is one-to-one but T is not.

   Let $U = \mathbb{R}$, $V = \mathbb{R}^2$, $W = \mathbb{R}$.

   Let $S(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} x \\ y \end{pmatrix} = x$.

   Then $T(S(x)) = T\begin{pmatrix} x \\ 0 \end{pmatrix} = x$, so $T \circ S : \mathbb{R} \to \mathbb{R}$ is the identity function, which is injective. But $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ so $\text{null}(T) \neq 0$, so T is not injective.

   (c) If $T \circ S$ is onto, then T is onto.

   Let $w \in W$. Then as $T \circ S$ is onto, there exists some $u \in U$ so that $(T \circ S)(u) = w$.

   By definition, $(T \circ S)(u) = T(S(u))$.

   Thus, $T(S(u)) = w$, so $w \in R(T)$.

   As w was arbitrary, this shows $R(T) = W$ so T is onto.

(d) Find an example where $T \circ S$ is onto but S is not.

The same example from (b) holds

Indeed, S is not onto as $\begin{pmatrix}0\\1\end{pmatrix} \notin R(S)$. Indeed, any element of $R(S)$ is of the form $\begin{pmatrix}x\\0\end{pmatrix}$ for $x \in \mathbb{R}$, and $\begin{pmatrix}0\\1\end{pmatrix}$ is not of this form.

(e) True or False: If $T \circ S$ is an isomorphism, then both T and S are isomorphisms.

False, In the example from (b), we had $T \circ S : \mathbb{R} \longrightarrow \mathbb{R}$ via $x \longmapsto x$, which is an isomorphism.

However, $S : \mathbb{R} \longrightarrow \mathbb{R}^2$ via $x \longmapsto \begin{pmatrix}x\\0\end{pmatrix}$ is not onto as $\begin{pmatrix}0\\1\end{pmatrix} \notin R(S)$. So S is not an isomorphism,

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The above suffices to conclude the problem, but in fact $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given via $\begin{pmatrix}x\\y\end{pmatrix} \longmapsto x$ is not an isomorphism as it's not 1-1. Indeed, $T\begin{pmatrix}0\\1\end{pmatrix} = 0$, so $null(T) > 0$.

2. Let V and W be finite-dimensional vector spaces with ordered bases β and γ respectively. let $T\colon V \to W$ be an isomorphism.

   (a) Show that $[T]_\beta^\gamma$ is an invertible matrix.

Let $T^{-1}: W \to V$ be the inverse of T.

Then $id_V = T^{-1} \circ T$ and $id_W = T \circ T^{-1}$.

Hence, $[id_V]_\beta^\beta = [T^{-1} \circ T]_\beta^\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$

$$\| \\ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

and $[id_W]_\gamma^\gamma = [T \circ T^{-1}]_\gamma^\gamma = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta$

$$\| \\ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

so these matrices are inverse to one another, hence invertible.

   (b) Sow that $[T^{-1}]_\beta^\gamma = ([T]_\beta^\gamma)^{-1}$.

should be $[T^{-1}]_\gamma^\beta$

This was shown in (a)