Math 115A Worksheet
Thursday, Nov 2 (Week 5)

1. Consider the real vector space $\mathbb{R}^\infty$ consisting of all infinite sequences of real numbers. Define the "shift map", $T\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$, to be the map which maps

$$(x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$$

   (a) Show that this map T is a linear transformation.

   See wk 4

   (b) Is this map injective? Is this map surjective?

   See wk 4

2. Find a map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(0) = 0$ but f is not a linear transformation.

See wk 4

The following problems are similar to Problem 5 on Homework 4:

Definition. Let V be a vector space, and let W_1 and W_2 be subspaces such that $V = W_1 \oplus W_2$. The *projection of* V *onto* W_1 *along* W_2 is the linear function $T\colon V \to V$ defined as follows: for any $\vec{x} \in V$, let $\vec{w}_1$ and $\vec{w}_2$ be the ***unique*** vectors in W_1 and W_2, respectively, such that $\vec{x} = \vec{w}_1 + \vec{w}_2$. Then

$$T(\vec{x}) := \vec{w}_1$$

3. Let $V = \mathbb{R}^3$, and suppose

$$W_1 = \{\, (x, 0, z) \mid x, z \in \mathbb{R} \,\} \qquad \text{(the } xz\text{-plane)}$$
$$W_2 = \{\, (0, y, 0) \mid y \in \mathbb{R} \,\} \qquad \text{(the } y\text{-axis)}$$

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the projection of $\mathbb{R}^3$ onto W_1 along W_2.

(a) How do you write any vector $(x, y, z) \in \mathbb{R}^3$ as a sum of a vector in W_1 and a vector in W_2? (This is really easy.)

$$(x,y,z) = \underbrace{(x,0,z)}_{\in W_1} + \underbrace{(0,y,0)}_{\in W_2}$$

(b) Based on your answer to part (a), and the definition of *projection* above, write down a formula for $T(x, y, z)$.

$$T(x,y,z) = (x,0,z)$$

(Note: The above is an example of an *orthogonal projection*, which you saw in 33A.)

4. Let $V = \mathbb{R}^3$ again, and this time, let

$$W_1 = \{\, (0, a, a) \mid a \in \mathbb{R} \,\} \qquad \text{(the line } x = 0, y = z\text{)}$$
$$W_2 = \{\, (x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0 \,\} \qquad \text{(the plane } z = -x - y\text{)}$$

Once again let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the projection of $\mathbb{R}^3$ onto W_1 along W_2.
(This example is the more general type of projection, that need *not* be orthogonal.)

(a) Now how do you write any vector $(x, y, z) \in \mathbb{R}^3$ as a sum of a vector in W_1 and a vector in W_2? (This is a little harder.)

We wish to solve $(x,y,z) = (0,a,a) + (b,c,d)$ with $a,b,c,d \in \mathbb{R}$ and $b+c+d = 0$

Hence, $x = 0 + b$, $y = a + c$, $z = a + d$, and $b + c + d = 0$

One way to solve this system is to row reduce the matrix

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & x \\ 1 & 0 & 1 & 0 & y \\ 1 & 0 & 0 & 1 & z \\ 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

yielding $a = \frac{x+y+z}{2}$, $b = x$, $c = \frac{-x+y-z}{2}$, $d = \frac{-x-y+z}{2}$

Thus, $(x,y,z) = \underbrace{\left(0, \frac{x+y+z}{2}, \frac{x+y+z}{2}\right)}_{\in W_1} + \underbrace{\left(x, \frac{-x+y-z}{2}, \frac{-x-y+z}{2}\right)}_{\in W_2}$

(b) Based on your answer to part (a), and the definition of *projection* above, write down a formula for $T(x, y, z)$ in this case.

$$T(x,y,z) = \left(0, \frac{x+y+z}{2}, \frac{x+y+z}{2}\right)$$