Math 115A Worksheet
Thursday, Oct 26 (Week 4)

1. Recall the Replacement Theorem from Week 3 lectures (§1.6 Theorem 1.10):

   Theorem. *Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.*

   Prove the following results using the Replacement Theorem (These are Corollaries (c) and (d) from lecture):

   (a) Let V be a vector space of dimension n. Then every linearly independent subset of V can be extended to a basis for V, i.e. if L is a linearly independent subset of V, then there is a basis β for V such that $L \subset \beta$.

   (*Hint:* You may also use Corollary (a), which says that any finite generating set for V has at least n vectors, and a generating set for V with exactly n vectors is a basis for V.)

Let $|\beta| = n$ and $|L| = m$.

As β is a basis of V, β spans V. Hence, take $G = \beta$ in the replacement theorem. Then by the replacement theorem, there is some $H \subseteq \beta$ with $|H| = n-m$ so that $L \cup H$ spans V.

Then $|L \cup H| \leq |L| + |H| = m + (n-m) = n = |\beta|$.
As $L \cup H$ spans V, corollary (a) says $|L \cup H| \geq n$. Thus, $n \leq |L \cup H| \leq n$ so $|L \cup H| = n$.
So $L \cup H$ is a basis for V by corollary (a), and $L \subseteq L \cup H$.

   (b) Let V be a vector space of dimension n, and let W be a subspace of V. Then $\dim W \leq \dim V$, and if $\dim W = \dim V$, then $W = V$.

Let β be a basis for W. Then $\beta \subseteq W \subseteq V$ and β is linearly independent, so by (a) there is some basis γ of V so that $\beta \subseteq \gamma$. Then $|\beta| \leq |\gamma|$ so $\dim W \leq \dim V$.

Suppose $\dim W = \dim V$. Then for a basis β of W, $|\beta| = \dim V$. Let γ be a basis of V, so $|\gamma| = |\beta| = n$. By the replacement theorem, there is a subset $H \subseteq \gamma$ with $|H| = n-n = 0$ and $\beta \cup H$ spans V. But $|H| = 0$ means $H = \emptyset$, so $\beta \cup H = \beta$ spans V. Thus, $W = \operatorname{span}(\beta) = V$.

2. Consider the real vector space $\mathbb{R}^\infty$ consisting of all infinite sequences of real numbers. Define the "shift map", $T\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$, to be the map which maps

$$(x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$$

(a) Show that this map T is a linear transformation.

Let $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in \mathbb{R}^\infty$.

$$T(x+y) = T((x_1+y_1, x_2+y_2, \ldots))$$
$$= (x_2+y_2, x_3+y_3, \ldots)$$
$$= (x_2, x_3, \ldots) + (y_2, y_3, \ldots)$$
$$= T(x) + T(y)$$

Let $c \in \mathbb{R}$. $T(cx) = T((cx_1, cx_2, \ldots))$
$$= (cx_2, cx_3, \ldots)$$
$$= c(x_2, x_3, \ldots)$$
$$= c\,T(x)$$

(b) If this map injective? Is this map surjective?

T is not injective. For example, $T((1,0,0,\ldots)) = (0,0,\ldots)$ so $\mathrm{null}(T) \neq 0$. Thus, T is not injective.

T is onto. Indeed, let $y \in \mathbb{R}^\infty$. Write $y = (y_1, y_2, \ldots)$

Then let $x = (0, y_1, y_2, \ldots)$.

We compute $T(x) = (y_1, y_2, \ldots) = y$

Hence, $y \in R(T)$. As y was arbitrary, this shows $R(T) = \mathbb{R}^\infty$, so T is onto.

3. Find a map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(0) = 0$ but f is not a linear transformation.

Let $f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x^3 \\ y^2 \end{pmatrix}$

Then $f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0^3 \\ 0^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

However, $f\left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$,

but $2f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = 2\begin{pmatrix} 1^3 \\ 1^2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

and $\begin{pmatrix} 3 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ so $f\left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \neq 2f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$.

Thus, f is not linear.