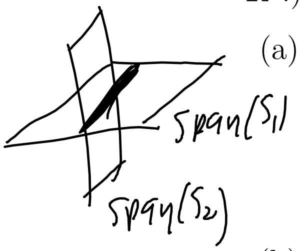
Math 115A Worksheet
Thursday, October 19 (Week 3)

1. Consider the vector space $V = \mathbb{R}^3$, and let $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, and $\vec{e}_3 = (0,0,1)$. (You might recall from Math 33A that these vectors are called the *standard basis* of $\mathbb{R}^3$.)



   (a) Let $S_1 = \{\vec{e}_1, \vec{e}_2\}$, and let $S_2 = \{\vec{e}_2, \vec{e}_3\}$. What is $\text{span}(S_1)$? What is $\text{span}(S_2)$?

   $\text{Span}(S_1) = \{a_1\vec{e}_1 + a_2\vec{e}_2 \mid a_1, a_2 \in \mathbb{R}\} = \{a_1\begin{pmatrix}1\\0\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\end{pmatrix} \mid a_1, a_2 \in \mathbb{R}\} = \{\begin{pmatrix}a_1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\a_2\\0\end{pmatrix} \mid a_1, a_2 \in \mathbb{R}\} = \{\begin{pmatrix}a_1\\a_2\\0\end{pmatrix} \mid a_1, a_2 \in \mathbb{R}\}$, which is the xy-plane!

   $\text{Span}(S_2) = \{a_1\vec{e}_2 + a_2\vec{e}_3 \mid a_1, a_2 \in \mathbb{R}\} = \{a_1\begin{pmatrix}0\\1\\0\end{pmatrix} + a_2\begin{pmatrix}0\\0\\1\end{pmatrix} \mid a_1, a_2 \in \mathbb{R}\} = \{\begin{pmatrix}0\\a_1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\a_2\end{pmatrix} \mid a_1, a_2 \in \mathbb{R}\} = \{\begin{pmatrix}0\\a_1\\a_2\end{pmatrix} \mid a_1, a_2 \in \mathbb{R}\}$, which is the yz-plane!

   (b) Using the same sets S_1 and S_2 as in part (a), what is $S_1 \cap S_2$? What is $\text{span}(S_1 \cap S_2)$? Is this the same as $\text{span}(S_1) \cap \text{span}(S_2)$?

   $S_1 \cap S_2 = \{\vec{e}_2\}$, $\text{span}(S_1 \cap S_2) = \text{span}(\{\vec{e}_2\}) = \{a\begin{pmatrix}0\\1\\0\end{pmatrix} \mid a \in \mathbb{R}\} = \{\begin{pmatrix}0\\a\\0\end{pmatrix} \mid a \in \mathbb{R}\}$, i.e. the y-axis

   $\text{span}(S_1) \cap \text{span}(S_2) = \{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid x = 0 \text{ and } z = 0\}$, i.e. the y-axis. Hence, these are indeed equal.

   (c) Come up with a completely different set of vectors T such that $\text{span}(T)$ is the same subspace as $\text{span}(S_1)$, but so that neither $\vec{e}_1$ nor $\vec{e}_2$ are in T. In that case, what is $\text{span}(S_1 \cap T)$? And what is $\text{span}(S_1) \cap \text{span}(T)$?

   Let $T = \{2\vec{e}_1, 2\vec{e}_2\}$. Then $\text{span}(T) = \text{span}(S_1)$, so

   $\text{span}(T) \cap \text{span}(S_1) = \text{span}(S_1) = \text{span}(T)$ = the xy-plane,

   $\text{span}(S_1 \cap T) = \text{span}(\emptyset) = \{\vec{0}\}$.

2. Let V be a vector space over a field F. Suppose $\vec{x}, \vec{y} \in V$ with $\vec{x} \neq \vec{y}$. If the set $\{\vec{x}, \vec{y}\}$ is linearly dependent, what does this tell you about $\vec{x}$ and $\vec{y}$? Come up with a necessary and sufficient condition, and prove it:

Theorem. *The set $\{\vec{x}, \vec{y}\}$ is linearly dependent if and only if* $\vec{y}$ is a multiple of $\vec{x}$ or $\vec{x}$ is a multiple of $\vec{y}$.

Proof.

Let $\{\vec{x}, \vec{y}\}$ be linearly dependent. Then $a\vec{x} + b\vec{y} = \vec{0}$ for some $a, b \in F$ with at least one nonzero.

If $a \neq 0$, $\vec{y} = \frac{-b}{a}\vec{x}$, so $\vec{y}$ is a multiple of $\vec{x}$,

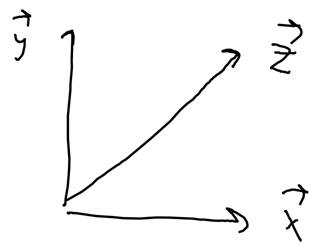
If $b \neq 0$, $\vec{x} = \frac{-a}{b}\vec{y}$, so $\vec{x}$ is a multiple of $\vec{y}$,

Thus, $\vec{y}$ is a multiple of $\vec{x}$ or $\vec{x}$ is a multiple of $\vec{y}$.

Conversely, suppose $\vec{y}$ is a multiple of $\vec{x}$ or $\vec{x}$ is a multiple of $\vec{y}$. In the first case, there is some $a \in F$ s.t. $\vec{y} = a\vec{x}$ so $a\vec{x} - \vec{y} = \vec{0}$ and $-1 \neq 0$, so $\{\vec{x}, \vec{y}\}$ is linearly dependent. In the second case, there is some $a \in F$ s.t. $\vec{x} = a\vec{y}$, so $a\vec{y} - \vec{x} = \vec{0}$ and $-1 \neq 0$, so $\{\vec{x}, \vec{y}\}$ is linearly dependent. So $\{\vec{x}, \vec{y}\}$ is linearly dependent in either case.

3. Come up with three vectors $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ such that the set $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly dependent, but none of the three vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ is a scalar multiple of any of the other ones.

Let $\vec{x} = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{y} = \begin{pmatrix}0\\1\end{pmatrix}$, $\vec{z} = \begin{pmatrix}1\\1\end{pmatrix}$. Certainly no two are multiples of each other, but $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent as $1 \cdot \vec{x} + 1 \cdot \vec{y} - 1 \cdot \vec{z} = \vec{0}$.



4. Recall that for a nonnegative integer n, $P_n(F)$ denotes the subspace of $P(F)$ consisting of polynomials of *degree at most* n:

$$P_n(F) = \{\, p \in P(F) \mid \deg(p) \leq n \,\}$$

(a) Let n be any nonnegative integer. Find a basis of $P_n(F)$.
(Note: You don't have to write up a proof that your answer is correct, but you should think through the details.)

$\{1, x, \ldots, x^n\}$

(b) What is the dimension of $P_n(F)$?

$n+1$

5. Let n be a positive integer, and fix some $a \in \mathbb{R}$. $P_n(\mathbb{R})$ denotes the vector space consisting of polynomials (with real coefficients this time) of degree at most n. Consider the subspace

$$W = \{ f \in P_n(\mathbb{R}) \mid f(a) = 0 \}$$

(a) Find a polynomial in $P_n(\mathbb{R}) \setminus W$ (that is, a polynomial that is in $P_n(\mathbb{R})$ that is *not* in W). Conclude that W is a *proper* subspace of $P_n(\mathbb{R})$. What does this tell you about $\dim(W)$?

Consider $f(x) = 1$. $f \in P_n(\mathbb{R})$ but $f(a) = 1 \neq 0$, so $f \notin W$. Hence, W is a proper subspace of $P_n(\mathbb{R})$. Thus, $\dim(W) < n+1 = \dim P_n(\mathbb{R})$

(b) Come up with a conjecture about the dimension of W. Discuss with the other members of your group to see if you all agree.

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(c) Prove your conjecture from part (b).
Hint: Come up with an actual basis for W. Of course, you must then prove it's a basis. If you use the result from part (a), together with results recently covered in class, you might only need to prove one of the two conditions needed for a basis (that it generates W, or that it's linearly independent... which one?)

Consider $S = \{ x-a, x(x-a), \ldots, x^{n-1}(x-a) \}$, we claim S is a basis for W.

Span: Let $f \in W$. Then $f(a) = 0$, so $f = (x-a)g(x)$ for some polynomial $g(x)$. As $f \in P_n(\mathbb{R})$, $g \in P_{n-1}(\mathbb{R})$ so we write $g(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$. Hence,

$$f(x) = (a_0 + a_1 x + \cdots + a_{n-1}x^{n-1})(x-a)$$
$$= a_0(x-a) + a_1 x(x-a) + \cdots + a_{n-1}x^{n-1}(x-a)$$

which is in $\mathrm{Span}(S)$.

Linear independence: Let $a_0(x-a) + a_1 x(x-a) + \cdots + a_{n-1}x^{n-1}(x-a) = 0$. Then $(x-a)(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}) = 0$. The right hand side here is the 0 polynomial, so as $x-a$ is not the zero polynomial, $a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} = 0$. Thus, all $a_i = 0$ so this set is linearly independent.

Thus, S is a basis of W. As $|S| = n$, $\dim W = n$.