Math 115A Worksheet
Thursday, Oct 12 (Week 2)

Recall: For a field F, a *vector space over* F is a set V together with two operations, *addition* ($V \times V \to V$) and *scalar multiplication* ($F \times V \to V$), which satisfy all of the following properties (axioms):

(VS 0) For all $\vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} \in V$. (V is *closed under addition.*)

and

For all $a \in F$ and all $\vec{x} \in V$, $a\vec{x} \in V$. (V is *closed under scalar multiplication.*)

(VS 1) For all $\vec{x}, \vec{y} \in V$,

$$\vec{x} + \vec{y} = \vec{y} + \vec{x}.$$

(Addition in V is *commutative.*)

(VS 2) For all $\vec{x}, \vec{y}, \vec{z} \in V$,

$$(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z}).$$

(Addition in V is *associative.*)

(VS 3) There exists an element $\vec{0} \in V$ such that, for all $\vec{x} \in V$,

$$\vec{0} + \vec{x} = \vec{x}.$$

(There is an *additive identity* element. Furthermore, we proved it is unique.)

(VS 4) For all $\vec{x} \in V$, there exists $\vec{y} \in V$ such that

$$\vec{x} + \vec{y} = \vec{0}.$$

(Every element has an *additive inverse.*)

(VS 5) For all $\vec{x} \in V$,

$$1\vec{x} = \vec{x}.$$

(VS 6) For all $a, b \in F$ and all $\vec{x} \in V$,

$$a(b\vec{x}) = (ab)\vec{x}.$$

(Scalar multiplication is *associative.*)

(VS 7) For all $a \in F$ and all $\vec{x}, \vec{y} \in V$,

$$a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}.$$

(Scalar multiplication is *distributive* on the left.)

(VS 8) For all $a, b \in F$ and all $\vec{x} \in V$,

$$(a + b)\vec{x} = a\vec{x} + b\vec{x}.$$

(Scalar multiplication is *distributive* on the right.)

You will need the above definition for all of the questions on this worksheet.

1. Last week, we proved the uniqueness of the additive inverse and the multiplicative inverse for any element of a field. Here you will do the same for additive inverses in a vector space.

   First, complete the statement of a theorem below about the uniqueness of additive inverses for elements of a vector space V. (Be careful about the order of quantifiers!) Then prove your theorem as a corollary of the cancellation law for vector addition (Theorem 1.1 in §1.2).

   Theorem. *Let F be a field, and let V be a vector space over F.* Let $\vec{v} \in V$. If $\vec{w}_1 + \vec{v} = \vec{0}$ and $\vec{w}_2 + \vec{v} = \vec{0}$ then $\vec{w}_1 = \vec{w}_2$.

   Proof. Suppose $\vec{w}_1 + \vec{v} = \vec{w}_2 + \vec{v}$. By the cancellation law, i.e. Thm 1.1 in §1.2, $\vec{w}_1 = \vec{w}_2$. □

2. Last week, we proved cancellation laws for addition and multiplication in a field. There are *two* cancellation laws for scalar multiplication in a vector space. In this problem, you will prove the first of them. Fill in the two blanks in the following statement of a theorem, and then prove the theorem. In your proof, try to be explicit about every field axiom (**F** [**0-5**]) and every vector space axiom (**VS** [**0-8**]) that you use.

   Theorem. *Let V be a vector space over a field F. For all $a \in F$ and all $\vec{x}, \vec{y} \in V$, if $a\vec{x} = a\vec{y}$ and* $a \neq 0$, *then* $\vec{x} = \vec{y}$.

   Proof. Let $a\vec{x} = a\vec{y}$ with $a \neq 0$. As $a \neq 0$, F4 says there exists a $b \in F$ s.t. $ab = 1$. By F1, $ba = ab$ so $ba = 1$.
   multiply both sides of $a\vec{x} = a\vec{y}$ by b, so $b(a\vec{x}) = b(a\vec{y})$.
   By VS 6, $(ba)\vec{x} = (ba)\vec{y}$. Then $1 \cdot \vec{x} = 1 \cdot \vec{y}$.
   By VS 5, $1 \cdot \vec{x} = \vec{x}$ and $1 \cdot \vec{y} = \vec{y}$. Hence, $\vec{x} = \vec{y}$. □

For the next two problems, you will need the following theorem. (This is Theorem 1.2 in your textbook.)

Theorem. *Let V be a vector space over a field F.*

(a) For each $\vec{x} \in V$, $0\vec{x} = \vec{0}$.
(b) For each $a \in F$ and each $\vec{x} \in V$, $(-a)\vec{x} = -(a\vec{x}) = a(-\vec{x})$.
(c) For each $a \in F$, $a\vec{0} = \vec{0}$.

3. Fill in the blank in the following theorem, to get the "zero product property" for vector spaces. (Recall that we proved the analogous fact for fields in class on Jan 10.) Then prove the theorem.

   Theorem. *Let V be a vector space over a field F. For any $a \in F$ and any $\vec{x} \in V$, if $a\vec{x} = \vec{0}$, then* $a = 0$ or $\vec{x} = \vec{0}$.

   Proof.
   (Hint: Remember how to prove an either-or statement: "P or Q" is logically equivalent to "If P is false, then Q is true", and also equivalent to "If Q is false, then P is true".)

   Let $a\vec{x} = \vec{0}$. By (c) above, $a\vec{0} = \vec{0}$.
   Suppose $a \neq 0$. Then $a\vec{x} = \vec{0} = a\vec{0}$, so by the cancellation law on the previous page, $\vec{x} = \vec{0}$.

4. In this problem, you will prove the *second* of the cancellation laws for scalar multiplication in a vector space. Fill in the two blanks in the following statement of a theorem, and then prove the theorem.

   Theorem. *Let V be a vector space over a field F. For all $a, b \in F$ and all $\vec{x} \in V$, if $a\vec{x} = b\vec{x}$ and* $\vec{x} \neq \vec{0}$, *then* $a = b$.

   Proof.
   (Hint: This one is harder than the other one. But the previous problem will be helpful!)

   We denote by $-(b\vec{x})$ the additive inverse of $b\vec{x}$, as in VS4.

   Let $a\vec{x} = b\vec{x}$ with $\vec{x} \neq \vec{0}$. Then $a\vec{x} + -(b\vec{x}) = b\vec{x} + -(b\vec{x}) = \vec{0}$
   $\vec{0} = a\vec{x} + -(b\vec{x}) = a\vec{x} + (-b)\vec{x}$ by (b) in the above theorem.
   $= (a + (-b))\vec{x}$ by VS8

   Then by the zero product property, $a + (-b) = 0$ or $\vec{x} = \vec{0}$. We assumed $\vec{x} \neq \vec{0}$ so $a + (-b) = 0$. Then $(a + (-b)) + b = 0 + b = b$ by F3. Additionally we have
   $(a + (-b)) + b = a + (-b + b) = a + 0 = 0 + a = a$
   (F2) (F1) (F2).

   Hence, we have shown $a = b$. □

In the next homework assignment (Homework 3), you will be using the following two important definitions in several problems. Here is the first of those definitions:

Definition. Let V be a vector space over a field F, and let X and Y be nonempty subsets of V. Then the *sum* of X and Y, denoted $X + Y$, is the set

$$\{\, x + y \mid x \in X \text{ and } y \in Y \,\}.$$

5. In the vector space $\mathbb{R}^2$, consider the subsets $X = \{(1,1),(2,2),(3,3)\}$ and $Y = \{(1,0),(0,1)\}$. Compute the set $X + Y$.

   (Note that neither of these sets are *subspaces* of $\mathbb{R}^2$. This problem is merely an example to get you used to the idea of what the subset $X + Y$ means.)

$$X+Y = \left\{ \begin{matrix} (1,1)+(0,1), & (2,2)+(0,1), & (3,3)+(0,1), \\ (1,1)+(1,0), & (2,2)+(1,0), & (3,3)+(1,0) \end{matrix} \right\} = \left\{ \begin{matrix} (1,2), & (2,3), & (3,4), \\ (2,1), & (3,2), & (4,3) \end{matrix} \right\}$$

6. Recall that two sets are equal if (and only if) they contain exactly the same elements. Therefore, given two sets A and B, to show that $A = B$, the standard way is to show that (1) for all $x \in A$, it is also true that $x \in B$, and (2) for all $x \in B$, $x \in A$ as well. Note that (1) is the same as showing $A \subseteq B$ and (2) is the same as showing that $B \subseteq A$.

   Let $V = \mathbb{R}^3$, and let

$$X = \operatorname{span}\big(\{(1,1,0)\}\big) = \{\,(a,a,0) \mid a \in \mathbb{R}\,\} \qquad \text{and}$$
$$Y = \operatorname{span}\big(\{(1,-1,0)\}\big) = \{\,(a,-a,0) \mid a \in \mathbb{R}\,\}.$$

   Let $W = \{\,(a_1,a_2,a_3) \in \mathbb{R}^3 \mid a_3 = 0\,\}$. Prove that $X + Y = W$.

i) we show $X+Y \subseteq W$.
Let $\vec{v} \in X+X$. Then $\vec{v} = \vec{x}+\vec{y}$ for some $\vec{x} \in X$, $\vec{y} \in Y$.
As $\vec{x} \in X$, $\vec{x} = (a,a,0)$ for some $a \in \mathbb{R}$.
As $\vec{y} \in X$, $\vec{y} = (b,-b,0)$ for some $b \in \mathbb{R}$.
Then $\vec{v} = \vec{x}+\vec{y} = (a,a,0)+(b,-b,0) = (a+b, a-b, 0) \in W$, so $X+Y \subseteq W$.

ii) we show $W \subseteq X+Y$.
Let $\vec{w} \in W$. Then $\vec{w} = (a_1, a_2, 0)$. We want to find $\vec{x} \in X, \vec{y} \in Y$ so that $\vec{w} = \vec{x}+\vec{y}$. That is, we seek $a,b \in \mathbb{R}$ so that $\vec{w} = (a,a,0)+(b,-b,0)$.
So we have $a+b = a_1$, $a-b = a_2$. This has a solution $a = \frac{a_1+a_2}{2}$, $b = \frac{a_1-a_2}{2}$.
That is, $\vec{w} = \frac{a_1+a_2}{2}(1,1,0) + \frac{a_1-a_2}{2}(1,-1,0)$. The first term is in X and the second is in Y, so $\vec{w} \in X+Y$. Thus, $W \subseteq X+Y$.

So we have shown $W = X+Y$. $\square$

Here is the second of the two definitions from Homework 3:

Definition. Let V be a vector space, and let W_1 and W_2 be subspaces of V. We say that V is the *(internal) direct sum* of W_1 and W_2 if *both* of the following are true:

(i) $V = W_1 + W_2$ (see the previous definition) and

(ii) $W_1 \cap W_2 = \{\vec{0}\}$.

If V is the direct sum of W_1 and W_2, we write $V = W_1 \oplus W_2$.

7. Let V be a vector space over a field F, and suppose W_1 and W_2 are two subspaces of V such that $V = W_1 \oplus W_2$. Let $\vec{x} \in V$. Then by part (i) of the above definition, we have $\vec{x} \in W_1 + W_2$, which means that there must exist some $\vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$ such that $\vec{x} = \vec{w}_1 + \vec{w}_2$. Prove that $\vec{w}_1$ and $\vec{w}_2$ are unique.
(Hint: Remember how to prove something is unique! It will help to first carefully state more clearly what is meant by "$\vec{w}_1$ and $\vec{w}_2$ are unique".)

Suppose $\vec{x} = \vec{w}_1 + \vec{w}_2$ and $\vec{x} = \vec{v}_1 + \vec{v}_2$ for $\vec{v}_1, \vec{w}_1 \in W_1$, $\vec{v}_2, \vec{w}_2 \in W_2$.

Then $\vec{w}_1 + \vec{w}_2 = \vec{v}_1 + \vec{v}_2$, so $\vec{w}_1 - \vec{v}_1 = \vec{v}_2 - \vec{w}_2$. The left hand side is in W_1 and the right hand side is in W_2. Thus, both sides of the equation are in $W_1 \cap W_2 = \{\vec{0}\}$, so $\vec{w}_1 - \vec{v}_1 = \vec{0}$ and $\vec{v}_2 - \vec{w}_2 = \vec{0}$. Hence, $\vec{w}_1 = \vec{v}_1$ and $\vec{v}_2 = \vec{w}_2$, showing the desired uniqueness. $\square$

8. Once again let V be a vector space over a field F, and suppose W_1 and W_2 are subspaces of V. This time, suppose that $W_1 + W_2 = V$, but $W_1 \cap W_2 \neq \{\vec{0}\}$. Let $\vec{x} \in V$. Once again, since $W_1 + W_2 = V$, we have $\vec{x} \in W_1 + W_2$, so there must exist $\vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$ such that $\vec{x} = \vec{w}_1 + \vec{w}_2$. This time, however, show that $\vec{w}_1$ and $\vec{w}_2$ are *not* the unique vectors satisfying this.

Pick some $\vec{u} \in W_1 \cap W_2$ so that $\vec{u} \neq \vec{0}$.

Note that $\vec{x} = (\vec{w}_1 + \vec{u}) + (\vec{w}_2 - \vec{u})$.

As $\vec{u} \neq \vec{0}$, $\vec{w}_1 + \vec{u} \neq \vec{w}_1$ and $\vec{w}_2 - \vec{u} \neq \vec{w}_2$.

Additionally, as $\vec{u} \in W_1 \cap W_2$ and $\vec{w}_1 \in W_1$, $\vec{w}_1 + \vec{u} \in W_1$.

Similarly, as $\vec{u} \in W_1 \cap W_2$ and $\vec{w}_2 \in W_2$, $\vec{w}_2 - \vec{u} \in W_2$.

Hence, $\vec{x} = (\vec{w}_1 + \vec{u}) + (\vec{w}_2 - \vec{u})$ is an expression of $\vec{x}$ as a sum of a vector in W_1 and in W_2 respectively which is distinct from our first expression $\vec{x} = \vec{w}_1 + \vec{w}_2$. So this expression is not unique. $\square$