Semidirect products

Direct products

(Exc 13.7.3 in Elman)

Def. Let G be a group, $H, K \trianglelefteq G$.

We say G is an <u>internal direct product</u> of H and K if $HK = G$ and $H \cap K = 1$.

Prop. If G is an internal direct product of H and K then $G \cong H \times K$

Pf. We define a function
$$f: H \times K \longrightarrow G$$
$$(h, k) \longmapsto hk$$

and claim it's an isomorphism.

<u>homomorphic:</u> wts $f(h,k)\, f(h', k') = f(hh', kk')$

LHS $= hkh'k'$, RHS $= hh'kk'$,

so it sts that H and K commute, i.e.

$hk = kh \quad \forall h \in H, k \in K$

Indeed, consider $hkh^{-1}k^{-1}$ $(=[h,k])$

$hkh^{-1} \in K$ by normality, so $hkh^{-1}k^{-1} \in K$

$kh^{-1}k^{-1} \in H$ by normality, so $hkh^{-1}k^{-1} \in H$

$\therefore$ $hkh^{-1}k^{-1} \in H \cap K = 1$, so $hk = kh$

<u>Surjectivity:</u> $\text{im}(f) = HK = G$ by hypothesis

<u>Injectivity:</u> Let $hk = h'k'$. Then $h^{-1}h'k' \in K$, so $h^{-1}h' \in H \cap K = 1$, so $h = h'$. Thus, $k = k'$. $\square$

Rmk. $H \times K$ is the internal direct product of $H \times 1$ and $1 \times K$.

Semidirect products

Prior setup was quite special w/ both normal, which we saw forced H and K to commute.

(Exc. 13.7.10)

Def. Let $N, H \leq G$. If

- $N \trianglelefteq G$
- $NH = G$
- $N \cap H = 1$

we say G is an internal semidirect product of H and N and write $G = N \rtimes H$

In this case, the bijection $K \times H \longrightarrow G$ from before still holds, but the multiplication is off.

$$(kh)(k'h') = k\underbrace{hk'}h'$$

how to move h and k' past each other?
normality! $\theta_h(k') = hk'h^{-1} \in K$

$$\underline{hk' = \theta_h(k')h}$$

this rule lets us move k's and h's past each other

$$kh k' h' = (k\ \theta_h(k'))(hh')$$

So if we define multiplication on $K \times H$ via the rule $(k,h)(k',h') = (k\theta_h(k'), hh')$ then $K \times H$, w/ this operation, is iso to G via $(k,h) \longmapsto kh$.

The previous internal direct product arises from this in the scenario where $\theta_h(k') = k'$, i.e. H and K commute, so the "move h's past k's rule" is trivial.

In other words, the conjugation $\theta: H \longrightarrow Aut(K)$ determines the multiplication on $K \times H$. Its triviality yields a direct product.

Generalize: Let H and K be groups, so not a priori contained as subgroups in a bigger group. Can we put them in one? Sure, $K \times H$. But can we do something subtler?

If H and K are in a bigger group, we can write elements like $h_1 k_1 h_2 k_2 \cdots$ so again, how do we move h's past k's?

We need H and K to talk to each other.

So suppose we have an action of H on K. Formally,

Def. Let $\varphi: H \longrightarrow \mathrm{Aut}(K)$, $h \longmapsto \varphi_h$. We define a group called the (external) semidirect product of K and H (wrt φ), denoted $K \rtimes_\varphi H$.

- Its underlying set is $K \times H$
- Its multiplication is $(k, h)(k', h') := (k\,\varphi_h(k'), hh')$

Exer. $K \rtimes_\varphi H$ is a group w/ this operation

Hint. (e_K, e_H) is the identity

$$(k,h)^{-1} = (\varphi_{h^{-1}}(k^{-1}), h^{-1})$$

Facts. i. $K \times 1 \trianglelefteq K \rtimes_\varphi H$, $1 \times H \leq K \rtimes_\varphi H$

hence the $\rtimes$ symbol, the group on the left is the normal one

$$\begin{aligned}(1,h)(k,1)(1,h)^{-1} &= (1,h)(k,1)(1,h^{-1}) \\ &= (\varphi_h(k), h)(1,h^{-1}) \\ &= (\varphi_h(k), 1) \in K \times 1\end{aligned}$$

so the conjugation action of $1 \times H$ on $K \times 1$ is the same as the given action φ of H on K.

ii. $\dfrac{K \rtimes_\varphi H}{K} \cong H$

iii. $K \rtimes_\varphi H$ is an internal semidirect product of K and H

iv. If G is an internal semidirect product of K and H, so $G = K \rtimes H$, then

$$\begin{aligned} K \rtimes_\theta H &\longrightarrow G \\ (k,h) &\longmapsto kh \end{aligned}$$

is an iso.

If we write $(k,1)$ as k and $(1,h)$ as h, then we again reckon with $khk'h'$

$$= (k,h)(k',h')$$
$$= (k\varphi_h(k'), hh')$$
$$= (k\varphi_h(k'))(hh')$$

and $hkh^{-1} = \varphi_h(k)$ as before, so $\underbrace{hk = \varphi_h(k)h}_{}$

our rule to move h's and k's past each other

so elts of $K \rtimes_\varphi H$ are words in K and H, reduced as usual via the rules of K and H, as well as the extra relation $hk = \varphi_h(k)h$.

Rmk. If φ is trivial, $K \rtimes_\varphi H$ is the external direct product

Examples

1. $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}/n\mathbb{Z}$ via $x \longmapsto -x$

i.e. $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$

$\bar{1} \longmapsto (x \mapsto -x)$

in other words, $\mathbb{Z}/2\mathbb{Z} \longrightarrow \underline{(\mathbb{Z}/n\mathbb{Z})^\times}$

$\bar{1} \longmapsto \overline{-1}$

For notation's sake, write $\mathbb{Z}/2\mathbb{Z} = \langle a \rangle$ multiplicatively

and $\mathbb{Z}/n\mathbb{Z}$ as $\langle b \rangle$ multiplicatively,

$\mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$ consists of words in a and

b s.t. $a^2 = e$, $b^n = e$, and $aba^{-1} = b^{-1}$.

That is, $\mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z} \cong D_n$

$a \longleftrightarrow f$

$b \longleftrightarrow r$

2. $\underset{\langle b \rangle}{\mathbb{Z}} \rtimes_\varphi \underset{\langle a \rangle}{\mathbb{Z}/2\mathbb{Z}}$ via $x \longmapsto -x$ (Rmk. $\mathbb{Z}/2\mathbb{Z} = \mathrm{Aut}(\mathbb{Z})$)

This is $\langle a, b \mid a^2 = e,\ aba^{-1} = b^{-1} \rangle$

called the infinite dihedral group

3. A an abelian group, $\mathbb{Z}/2\mathbb{Z} \circlearrowright A$ via $a \longmapsto a^{-1}$

4. $\mathbb{Z}/2\mathbb{Z} \circlearrowright SO(n)$ as follows;

Pick some reflection matrix R, so $R^2 = I$.

Then for $A \in SO(n)$, the action of $\bar{1}$ on A is $\underbrace{RAR^{-1}}_{\substack{\text{change basis}\\ \text{to opposite}\\ \text{orientation}}}$

Then $\underbrace{O(n)}_{\text{symmetries}} \cong \underbrace{SO(n)}_{\text{rotations}} \rtimes_\varphi \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\text{flips}}$

This is a continuous analog of $D_n \cong \mathbb{Z}/2\mathbb{Z} \rtimes_\varphi \mathbb{Z}/n\mathbb{Z}$

5. $O(2) \circlearrowright \mathbb{R}^2$ by matrix multiplication.

View elements of $\mathbb{R}^2$ as translation maps $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. A fact of classical geometry is that all isometries $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are uniquely the composition of a translation and a linear isometry (an elt of $O(2)$).

Then one can check $\mathbb{R}^2 \rtimes_\varphi O(2)$ is the group of plane isometries

6. Let G be the 2×2 upper triangular matrices with determinant 1. (what I thought $ST_2\mathbb{R}$ was...)

all of the form $\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}$ $\quad a \in \mathbb{R}^\times$, $b \in \mathbb{R}$

$$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & b' \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & b+b' \\ & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\begin{pmatrix} a' & \\ & (a')^{-1} \end{pmatrix} = \begin{pmatrix} aa' & \\ & (aa')^{-1} \end{pmatrix}$$

so G contains a copy of $\mathbb{R}$ and of $\mathbb{R}^\times$.

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & \\ & a \end{pmatrix}$$

$$= \begin{pmatrix} a & ab \\ & a^{-1} \end{pmatrix}\begin{pmatrix} a^{-1} & \\ & a \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a^2 b \\ & 1 \end{pmatrix}$$

Let $\mathbb{R}^\times \xrightarrow{\varphi} \mathrm{Aut}(\mathbb{R})$ via $a \longmapsto (b \longmapsto a^2 b)$. Then $G \cong \mathbb{R} \rtimes_\varphi \mathbb{R}^\times$.

7. Non-example.

$\mathbb{Z}/4\mathbb{Z}$ contains $2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$

the quotient is $\dfrac{\mathbb{Z}/4\mathbb{Z}}{2\mathbb{Z}/4\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$

but $\mathbb{Z}/4\mathbb{Z}$ is <u>not</u> a semidirect product of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$

Indeed, $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\times} = 1$, so the only semi direct products $\mathbb{Z}/2\mathbb{Z} \rtimes_{\varphi} H$ are direct products. But $\mathbb{Z}/4\mathbb{Z} \not\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,

8. Quaternions Q_8 aren't a semidirect product of smaller groups via a takehome problem 2c.

Fancy Stuff

(aka homological algebra)

We say a sequence

$$\cdots \to G_{i+1} \xrightarrow{d_{i+1}} G_i \xrightarrow{d_i} G_{i-1} \to \cdots$$

is <u>exact</u> if $\ker(d_i) = \operatorname{im}(d_{i+1})$ $\forall i$

e.g. $1 \longrightarrow G \xrightarrow{f} H$ is exact iff f is mono

$G \xrightarrow{f} H \longrightarrow 1$ is exact iff f is epic

$1 \longrightarrow G \to H \to 1$ is exact iff f is an iso

If $1 \longrightarrow K \xrightarrow{f} G \xrightarrow{g} H \longrightarrow 1$ is exact, we call it a short exact sequence.

f realizes K as a (normal) subgroup of G and H as the quotient of G by K.

e.g. $1 \longrightarrow N \hookrightarrow G \longrightarrow G/N \longrightarrow 1$ for $N \trianglelefteq G$ is a short exact sequence

Intuitively, G is built out of K and H.

Note:

	quotient of G	subgroup of G
K	no	yes
H	yes	no

We say $1 \to K \xrightarrow{f} G \xrightarrow{g} H \to 1$ "splits" if there is a group homomorphism $H \xrightarrow{s} G$ s.t. $g \circ s = id_H$ (called a "section" of g). s is automatically injective. This allows H to be viewed as a subgroup of G.

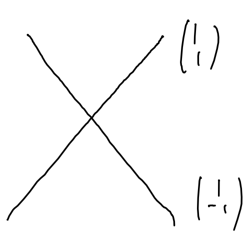
e.g. i. $0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R} \to 0$

$x \mapsto \begin{pmatrix} x \\ x \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x - y$

splits via $\mathbb{R}^2 \leftarrow \mathbb{R}$

$\begin{pmatrix} x \\ -x \end{pmatrix} \leftarrow x$



More generally, $0 \to W \to V \to V/W \to 0$

splits via
$$\begin{tikzcd} V \arrow[r] \arrow[d] & W^{\perp} \arrow[d, hook] \\ V/W \arrow[r] & V \end{tikzcd}$$
orthogonal projection (for fdvs/$\mathbb{R}$)

or really via writing $V = W \oplus U$ for some U

and taking $V \to U$ projection along W

(needn't be orthogonal) (any field/dimension)

i.e. $1 \to K \to K \rtimes_\varphi H \to H \to 1$ has a section

$$K \rtimes_\varphi H \leftarrow H$$
$$(1,h) \mapsfrom h$$

Given a section, we have
$$\begin{tikzcd} H \arrow[r, "s"] \arrow[rr, bend right, "\varphi"'] & G \arrow[r, "\theta"] & \mathrm{Aut}(K) \end{tikzcd}$$

where $G \xrightarrow{\theta} \mathrm{Aut}(K)$ is conjugation of G on $f[K]$,

pulled back to K via $K \xrightarrow{\sim} f[K]$.

In this case, $G \cong K \rtimes_\varphi H$.

Thm. A short exact sequence $1 \to K \to G \to H \to 1$ is split if and only if G is a semidirect product of K and H.