Admin

Midterm tomorrow

Hw today (or Monday (or later...))

be careful: if missing @ the end, I might not get to grading it

also, take-home will be stricter

getting closer on grading Hw3

No OH tomorrow

Today: Review

Next week: semidirect products

Problems from Lang Algebra Ch. 1

"Goursat's lemma" c.f. Wikipedia for a proof w/ coset bashing

5. Let $H \leq G_1 \times G_2$.

Consider $\pi_i : G_1 \times G_2 \longrightarrow G_i$, suppose $\pi_i|_H$ onto.

Let $N_1 = \ker(\pi_2|_H)$, $N_2 = \ker(\pi_1|_H)$.

i. View $N_i \trianglelefteq G_i$

ii. $G_1/N_1 \cong G_2/N_2$

iii. The image of H in $\frac{G_1}{N_1} \times \frac{G_2}{N_2}$ is the graph of the isomorphism from (ii).

Pf. i. $\ker(\pi_1) = e \times G_2$, $\ker(\pi_2) = G_1 \times e$

$\ker(\pi_1|_H) = N_2$ $\quad$ $\ker(\pi_2|_H) = N_1$

$N_i \trianglelefteq H$, so their "intersections" w/ the G_i are normal in G_i.

ii. How to find a map???

No clue.

$$\begin{tikzcd}
\frac{G_1}{N_1} & & \frac{G_2}{N_2} \\
G_1 \arrow[u] & & G_2 \arrow[u] \\
& G_1 \times G_2 \arrow[ul] \arrow[ur] & \\
& H \arrow[u] &
\end{tikzcd}$$

By first iso thm, π_2 induces $\frac{H}{N_2} \xrightarrow{\sim} G_1$

π_1 induces $\frac{H}{N_1} \xrightarrow{\sim} G_2$

so we expect $\frac{G_1}{N_1} \cong \frac{H}{N_1 N_2} \cong \frac{G_2}{N_2}$

Indeed, consider $\frac{H}{N_2} \xrightarrow[\sim]{\overline{\pi_1}} G_1 \longrightarrow G_1/N_1$

what's the kernel?

$$\begin{tikzcd}
H \arrow[r, "\pi_1"] \arrow[d] & G_1 \\
H/N_2 \arrow[ur, "\overline{\pi_1}"'] &
\end{tikzcd}
\qquad \frac{\pi_1^{-1}[N_1]}{N_2}$$

what's $\pi_1^{-1}[N_1]$?,

Lemma. $f : A \twoheadrightarrow B$, $C \le B$.

suppose $D \le A$ s.t. $f[D] = C$

Then $f^{-1}[C] = D\ker(f)$

Pf: "$\supseteq$" clear

"$\subseteq$" $f^{-1}[C] \twoheadrightarrow C$ has kernel $\ker(f)$, as $\ker(f) \subseteq C$

$$\frac{f^{-1}[C]}{\ker(f)} \cong C \text{ via } f$$

OTOH, $$\frac{D\ker(f)}{\ker(f)} \cong \frac{D}{D \cap \ker(f)}$$

$D \twoheadrightarrow C$ has kernel $D \cap \ker(f)$

so this is iso to C

Thus by correspondence (check details!), $D\ker f = f^{-1}[C]$ $\square$

$$\begin{tikzcd}
N_1 \arrow[rr, hook] \arrow[dr, hook] & & G_1 \\
& H \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"] & \\
N_2 \arrow[ur] \arrow[rr, hook] & & G_2
\end{tikzcd}$$

Apply lemma to $\pi_1 : H \twoheadrightarrow G_1$

$C = N_1$ here

$D = N_1$

to get $\pi_1^{-1}[C] = N_1 N_2$

so $\ker(H \longrightarrow G_1/N_1)$ is $\dfrac{N_1 N_2}{N_1}$

Same for $H \longrightarrow G_2/N_2$

$$\begin{tikzcd}
\dfrac{H/N_1}{N_1N_2/N_1} \arrow[d, "\sim"', "\text{``}\bar{\pi}_1\text{''}"] & \dfrac{H}{N_1N_2} \arrow[l, "\sim"', "\text{``id''}"] \arrow[r, "\sim", "\text{``id''}"'] & \dfrac{H/N_2}{N_1N_2/N_2} \arrow[d, "\sim", "\text{``}\bar{\pi}_2\text{''}"'] \\
\dfrac{G_1}{N_1} & & \dfrac{G_2}{N_2}
\end{tikzcd}$$

iii.

$$\begin{tikzcd}
& & \dfrac{G_1}{N_1} \arrow[dd, dashed, bend left=60, "\sim"] \\
\dfrac{H}{N_1N_2} \arrow[urr, bend left, "\text{``}\bar{\pi}_1\text{''}", "\sim"'] \arrow[drr, bend right, "\text{``}\bar{\pi}_2\text{''}"', "\sim"] & H \arrow[l] \arrow[r] \arrow[ur] \arrow[dr] & \dfrac{G_1}{N_1} \times \dfrac{G_2}{N_2} \\
& & \dfrac{G_2}{N_2}
\end{tikzcd}$$

image is $(h_1N_1, h_2N_2) \mid (h_1, h_2) \in H_1 \times H_2$

graph is $(g_1N_1, g_2N_2) \mid g_1N_1 \longmapsto g_2N_2$ via the iso. Then, both come from H/N_1N_2 $\square$