Admin

HW4 due today (/ Monday, or really Tuesday)

Grading:

HW1 resubmissions ✓

HW2: partially done

HW3: this weekend hopefully

Midterm next Friday in class

review session

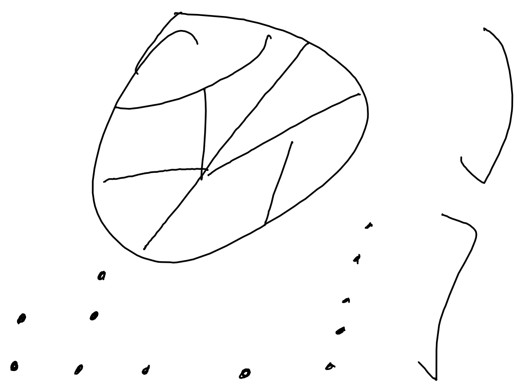
Quotients

Recall for $H \leq G$,

$$\begin{aligned} G &\longrightarrow \Sigma(G/H) \\ g &\longmapsto \lambda g \\ &\quad \lambda g(xH) = gxH \end{aligned}$$

i.e. G "acts" on G/H

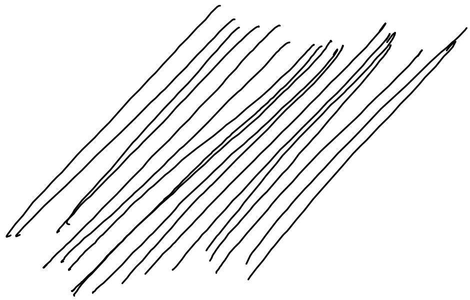
$G \twoheadrightarrow G/H$ surjective, so its fibers (cosets) yield a partition of G



partitions of same sets

But this action shows that these fibers are "homogeneous", we can translate $x \longmapsto gx$ to send one coset to another, so they have a very unique symmetry/uniformity.

e.g., $G = \mathbb{R}^2$, $H = \operatorname{span}\begin{pmatrix}1\\1\end{pmatrix}$.



is our partition of $\mathbb{R}^2$

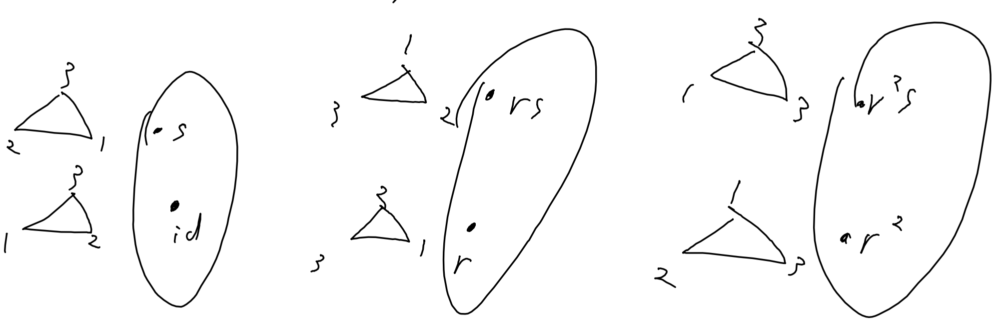
so G, as a set (or a space...) looks like taking the subgroup H and letting it translate around.

e.g., $G = \mathbb{Z}$, $H = 3\mathbb{Z}$

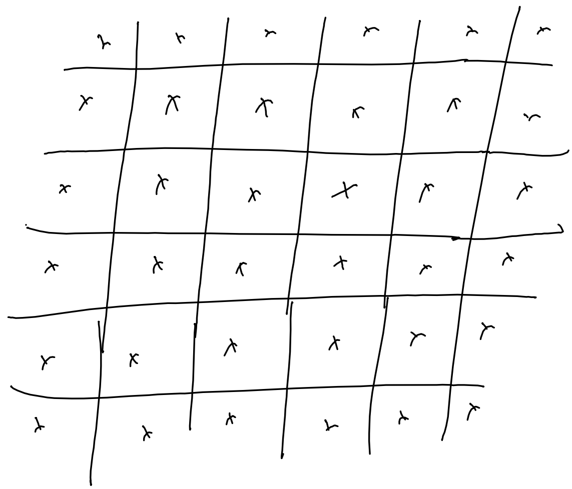
e.g., $G = D_3$, $H = \langle s \rangle$

$\parallel$

$\langle r, s \mid r^3 = s^2 = id,\ srs^{-1} = r^{-1} \rangle$

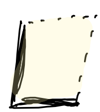


e.g. $G=\mathbb{R}^2$, $H=\mathbb{Z}^2$

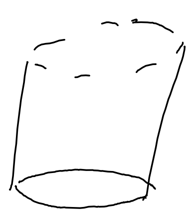


the set of all marked points is a coset

Rmk. here, $\mathbb{R}^2/\mathbb{Z}^2$ looks like

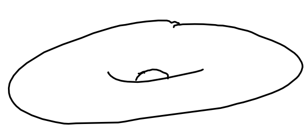


upon gluing the vertical lines



(cf. the arcade game asteroids or the QPU grid from SM64 A-press runs)

then the horizontal ones



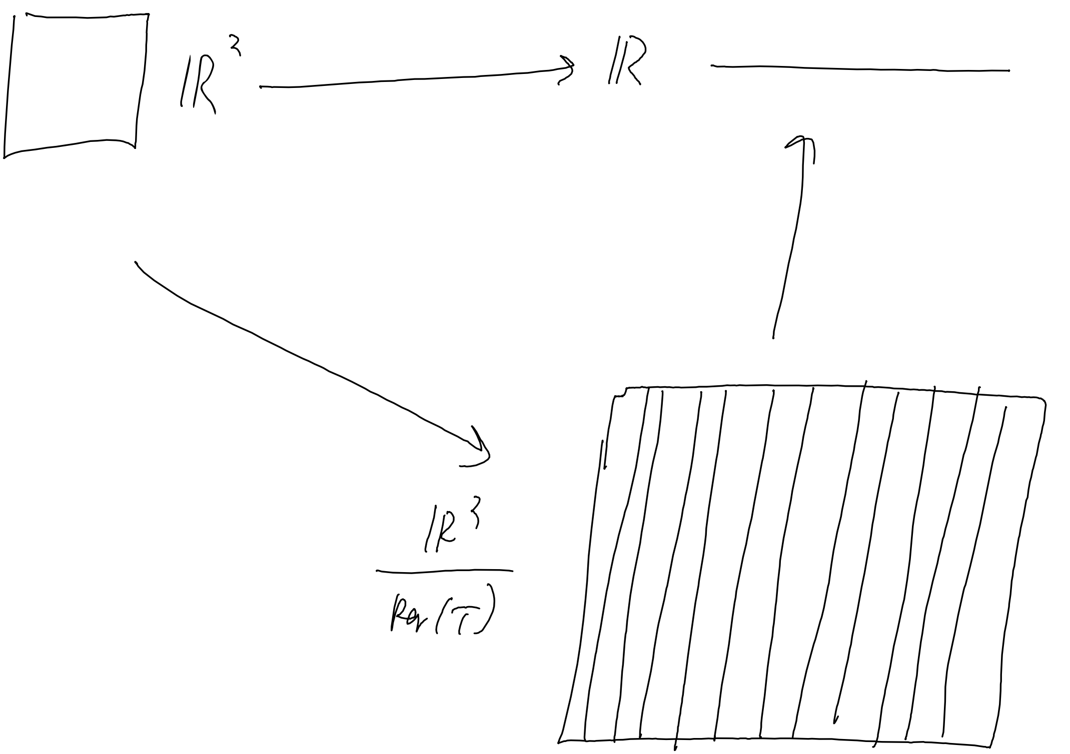
First iso thm

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x$$

$\mathrm{im}(T) = \mathbb{R}$

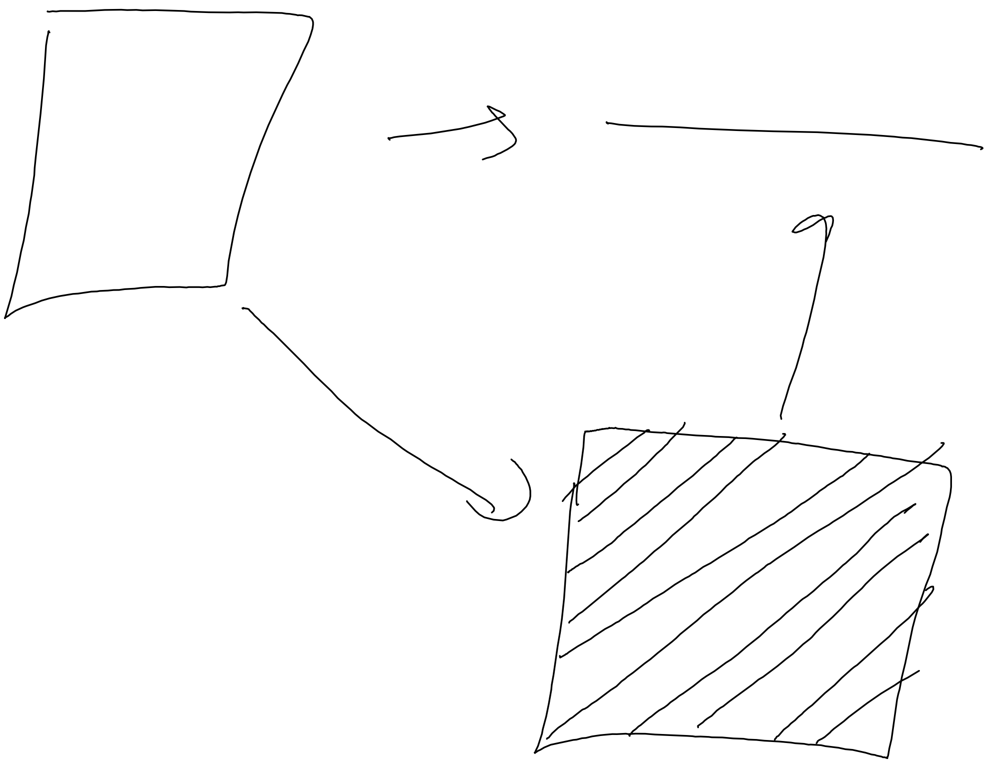
$\ker(T) = \mathrm{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



T is constant along these lines

$$\mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x-y$$



$x-y$ is constant along these lines

General feature:

$$f: G \longrightarrow H$$

If $f(g_0) = h_0$ then $f^{-1}[h_0] = h_0 \ker(f)$.

e.g. in $\mathbb{R}^3 \xrightarrow{x-y} \mathbb{R}$ the set of solutions to $x-y=1$ is $\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\text{particular solution}} + \underbrace{\operatorname{span}\begin{pmatrix}1\\1\end{pmatrix}}_{\text{kernel}}$

e.g. Consider $C^\infty(\mathbb{R}) = \{f: \mathbb{R} \to \mathbb{R} \text{ smooth}\}$.

Let $D: C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ via $f \longmapsto f'$.

$\operatorname{im}(D) = C^\infty(\mathbb{R})$.
$\ker(D) = \mathbb{R}$, viewed as constant functions.

$$\frac{C^\infty(\mathbb{R})}{\mathbb{R}} \xrightarrow{\sim} C^\infty(\mathbb{R})$$

What is the inverse? Integration! This isn't a single function, rather a coset. e.g. $I(x^2) = \frac{1}{3}x^3 + \mathbb{R}$

Universal property of quotients

Thm. Let $f: G \to H$ and $N \trianglelefteq G$ s.t. $f|_N$ is trivial (i.e. $N \leq \ker(G)$). Then there is a unique map

$\bar{f}: G/N \to H$ s.t. $\pi \circ \bar{f} = f$, where

$\pi: G \to G/N$, i.e.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & H \\ G/N \arrow[ur, dashed, "\exists! \bar{f}"'] & \end{tikzcd}$$

we say f "factors through" G/N.

Proof. There is no choice but to set

$$\bar{f}(gN) = f(g)$$

- is this well defined? Moral: f can't "see" N.

  Pf. Let $gN = g'N$, so $g^{-1}g' \in N$. Then $f(g^{-1}g') = e_H$
  $$\overset{\shortparallel}{f(g')f(g)^{-1}}$$

  so $f(g) = f(g')$.

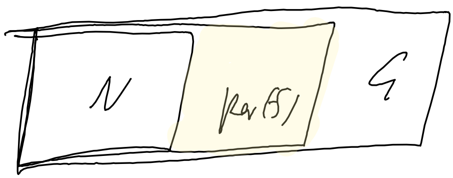
- is it a group homomorphism? ✓

– does this commute? ✓

– is it unique? ✓

$\square$

what is the kernel of $\bar{f}$?



everything in $\ker(f) - N$ is still sent to e_H but hasn't been killed

Proposition. $\ker(\bar{f}) = \dfrac{\ker(f)}{N}$

Pf. Let $\bar{f}(gN) = e_H$. $\bar{f}(gN) = f(g)$, so $f(g) = e_H$, i.e. $g \in \ker(f)$.

$\square$

what is the image of $\bar{f}$?

Prop. $\operatorname{im}(\bar{f}) = \operatorname{im}(f)$

Pf. π is onto.

$\square$

Moral. f can't "see" N so you may as well kill it to move to a smaller/more favorable domain.

Examples. Let G be a finite group of order N and fix $g \in G$.

Let $f: \mathbb{Z} \to G$

$1 \mapsto g$

(thus, $n \mapsto g^n$ is forced)

f vanishes on $N\mathbb{Z}$ by Lagrange's theorem, so this factors through a map

$\mathbb{Z}/N\mathbb{Z} \to G$

$1 + N\mathbb{Z} \mapsto g$

Its kernel is $\dfrac{|g|\mathbb{Z}}{N\mathbb{Z}}$. Apply the first iso thm to get

$$\frac{\mathbb{Z}/N\mathbb{Z}}{|g|\mathbb{Z}/N\mathbb{Z}} \to G$$

$|g|\mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}/|g|\mathbb{Z}$ by 3rd iso thm

Application: Dual spaces

Recall for a vector space V over a field k, $V^* = \{f : V \to k \text{ linear}\}$.

Given $T : V \to W$, we get a map $T^t : W^* \to V^*$ via $f \mapsto f \circ T$

For instance, consider $\iota : W \hookrightarrow V$ and
$\pi : V \to V/W$

(we write $0 \to W \to V \to V/W \to 0$ a "short exact sequence")

Then we have $\pi^t : (V/W)^* \to V^*$
$\iota^t : V^* \to W^*$

what is $\text{im}(\pi^t)$? the functionals vanishing at W,
essentially by UP of quotients

$\ker(\pi^t) = 0$

$\mathrm{im}(\tau^t)$? all, as can extend bases

$\ker(\tau^t)$? precisely $\mathrm{im}(\pi^t)$

(so $0 \to (V/W)^* \longrightarrow V^* \longrightarrow W^* \longrightarrow 0$ exact)