Intro

Welcome to 110AH! (or even to UCLA!)

I'm Jas. ("Jus")

<u>Admin stuff</u>

- see my email
  - poll for OH timings
  - generic survey thing
- my website
  math.ucla.edu/~jas
  - basic info
  - notes (like the one you're reading right now)

- HW
  - due Th 11:59PM and assigned previous Monday
    - I'll be pretty lenient on the deadline
      - If on time I'll surely get to it
      - If a few days late, almost surely
      - If significantly late, I'll try to get to it, but no guarantee.

- * problems for correctness, rest for completion.
- You're allowed (and encouraged!) to try, get it wrong, get feedback, and resubmit.

About me

- office - MS 3919
- email - jas @math.ucla.edu
- website -
- second year grad student in math (algebraic geometry / algebraic number theory)
- I took this exact class in my undergrad here

Advice

- Try to read the book before lecture
- Accept being confused a lot of the time
- start HW early, do multiple passes, and ask your classmates, myself, and Prof. Elman lots of questions

Sets and functions

Sets

For now, we suffice with:

"Definition" A set is a "collection into a whole".

That is, you collect a bunch of "objects" into a single "object" containing them all. The set is the bag itself.

Examples. - $\{1,2,3\}$

- $\{1\}$
- $\{\{1\}, \{1,2,3\}\}$
- $\emptyset = \{\}$
- $\mathbb{N}$ = natural numbers, $\{0,1,2,3,\ldots\}$
- $\mathbb{Z}$ = integers, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Q}$ = rational numbers, $\{1, \frac{1}{2}, \frac{3}{4}, -1, -\frac{1}{2}, -\frac{7}{17}, 0, \ldots\}$
- $\mathbb{R}$ = real numbers, $\{\pi, \sqrt{17}, e, 1.23411111577, 11, -5, 0, \ldots\}$
- $\mathbb{C}$ = Complex numbers, $\{a+bi \mid a,b \in \mathbb{R}\}$ where $i^2 = -1$.
- $\{x \in \mathbb{R} \mid \sin(x) = \frac{1}{3}\}$

Notation. $a \in A$ means "a is an element of A"

- $1 \in \{1, 2, 3\}$
- $\{1\} \notin \{1, 2, 3\}$

$A \subseteq B$ means "A is a subset of B",

i.e. $\underbrace{\text{for all}}_{(\forall)}$ $a \in A$, $a \in B$.

$$\forall a \ (a \in A \Rightarrow a \in B)$$

- $\{1\} \subseteq \{1, 2, 3\}$

Operations. 1. Given two sets A, B we form their cartesian product

$$A \times B = \{ (a, b) \mid a \in A, \ b \in B \}$$

e.g. $\{1, 2\} \times \{3, 4\} = \{ (1, 3), (1, 4), (2, 3), (2, 4) \}$

$\mathbb{R} \times \mathbb{R} \ (= \mathbb{R}^2)$ is the xy-plane

2. Given a set A, we form the power set

$$P(A) = \{ \text{sets } B \mid B \subseteq A \}$$

e.g. $P(\{1, 2\}) = \{ \emptyset, \{1\}, \{2\}, \{1, 2\} \}$

Functions

"How sets talk to each other"

Intuitively, given a <u>domain</u> set A and a <u>codomain</u> (or <u>target</u>) set B, a function f from A to B ($f: A \longrightarrow B$) is a way to associate an element of B to every element of A.

e.g. $f: \mathbb{R} \longrightarrow \mathbb{R}$ via $f(x) = x^3$

$f: M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R})$ via $f(x) = x^2$

$f: \mathbb{R}^2 \longrightarrow \mathbb{C}$ via $(a,b) \longmapsto a+bi$

$f: \{1, 2\} \longrightarrow \mathbb{R} \times \{\text{red}, \text{green}\}$

$1 \longmapsto (\sqrt{7}, \text{red})$

$2 \longmapsto (e, \text{green})$

$id_A : A \longrightarrow A$ via $a \longmapsto a$

Formally,

Definition. Let A, B be sets. A function from A to B is a triple (f, A, B) where $f \subseteq A \times B$ such that

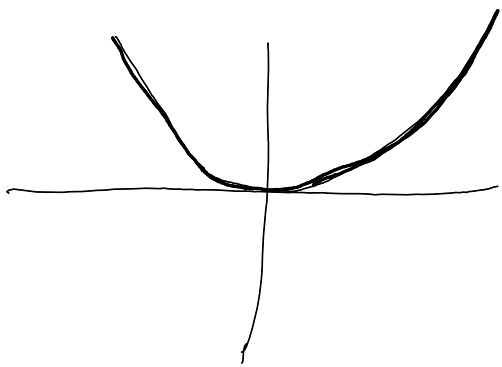
$(\forall)$ [for all $a \in A$ $\underbrace{\text{there exists}}_{(\exists)}$ a $\underbrace{\text{unique}}_{(!)}$ $b \in B$

$(\forall a \in A\ \exists!\, b \in B)$

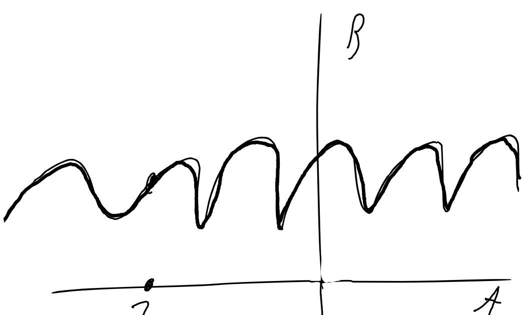
s.t. $(a, b) \in f$.

we notate this as $f: A \longrightarrow B$. If $(a,b) \in f$, we write $b = f(a)$. A is the domain of f. B is the codomain of f. (or target) e.g. "$f: \mathbb{R} \longrightarrow \mathbb{R}$ via $x \longmapsto x^2$" is formally

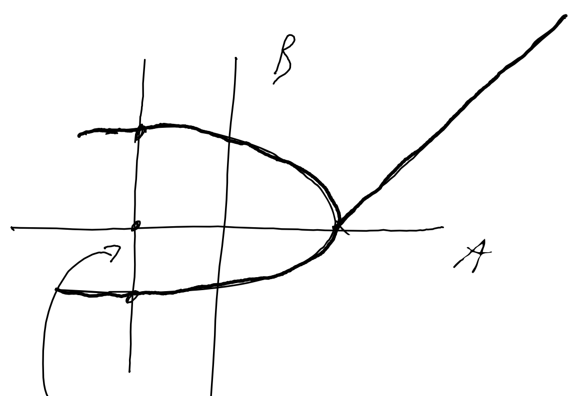
$(\{(x,x^2) \mid x \in \mathbb{R}\}, \mathbb{R}, \mathbb{R})$



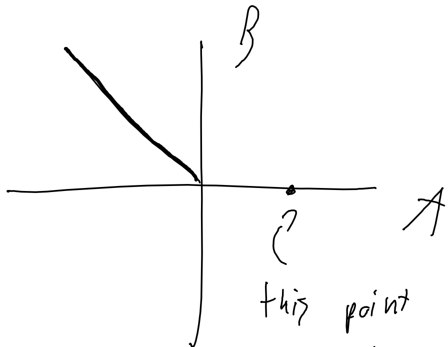
"for all $a \in A$ there is a unique $b \in B$ s.t. $(a,b) \in f$" = "vertical line test"



every point of A has something above or below it



this point has 2 things above or below it, so the "unique" part fails



this point has nothing above or below it.

Don't fret the formalism too much, but rely on it as stable footing when stuck.

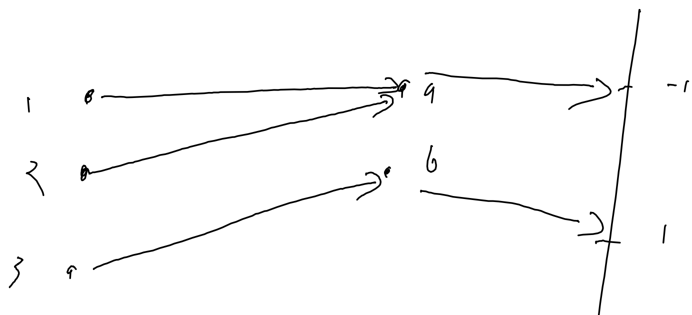
Def. Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$. We write $g\circ f: A \longrightarrow C$ for the composition, defined as $(g\circ f)(a) = \underbrace{g(\underbrace{f(\underbrace{a}_{A})}_{B})}_{C}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}
\qquad
\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

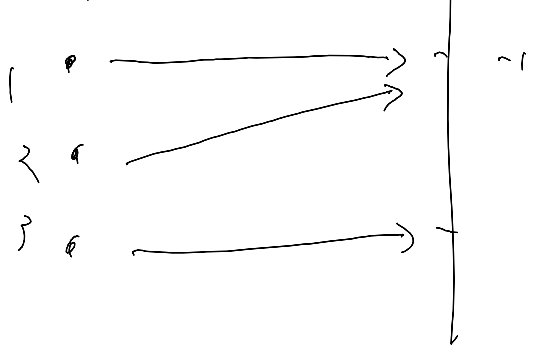
e.g. $f: \{1,2,3\} \longrightarrow \{a,b\}$ via
$$1 \longmapsto a, \quad 2 \longmapsto a, \quad 3 \longmapsto b$$

$g: \{a,b\} \longrightarrow \mathbb{R}$ via
$$a \longmapsto -1, \quad b \longmapsto 1$$

$g \circ f: \{1,2,3\} \longrightarrow \mathbb{R}$ via
$$1 \longmapsto -1, \quad 2 \longmapsto -1, \quad 3 \longmapsto 1$$



$$1 \longmapsto a \longmapsto -1$$
$$2 \longmapsto a \longmapsto -1$$
$$3 \longmapsto b \longmapsto 1$$



$$1 \longmapsto -1$$
$$2 \longmapsto -1$$
$$3 \longmapsto 1$$

e.g.

$$\begin{aligned} \mathbb{C} &\xrightarrow{\Delta} \mathbb{C}^2 \\ z &\longmapsto (z, z) \end{aligned}$$

$$\begin{aligned} \mathbb{C}^2 &\xrightarrow{\mu} \mathbb{C} \\ (z, w) &\longmapsto zw \end{aligned}$$

$$\mu \circ \Delta : \mathbb{C} \longrightarrow \mathbb{C}$$

$$z \longmapsto (z, z) \longmapsto z \cdot z$$

i.e. $z \longmapsto z^2$

Properties.

Def. Let $f : A \longrightarrow B$. We say

i. f is <u>injective</u> (or <u>one-to-one</u>) if

$f(a) = f(a') \Rightarrow a = a'$

Equivalently, $a \neq a' \Rightarrow f(a) \neq f(a')$

"f doesn't lose information"

ii. f is <u>surjective</u> (or <u>onto</u>) if

for all $b \in B$ there is $a \in A$ s.t. $f(a) = b$

"f reaches all of B"

ii. f is <u>bijective</u> if it's injective and surjective.

e.g. - $\mathbb{C}^2 \xrightarrow{\mu} \mathbb{C}$ is onto, as $z = \mu(z,1)$

$(z,w) \longmapsto zw$

is not 1-1 as $\mu(1,1) = \mu(-1,-1)$.

- $\mathbb{C} \longrightarrow \mathbb{C}^2$ is not onto, as $(1, 5)$ is never reached

$z \longmapsto (z, z^2)$ is 1-1 as $(z, z^2) = (w, w^2)$

$$\Downarrow$$

$$z = w$$

by looking at the first component

- $\mathbb{R}^2 \longrightarrow \mathbb{C}$ is 1-1 and onto.

$(a,b) \longmapsto a+bi$

- $\{a,b\} \longrightarrow \{1,2\}$ is neither 1-1 nor onto.

$a \longmapsto 1$

$b \longmapsto 1$

Alternative characterization

Consider $f: A \longrightarrow B$ an injection.

Let $a, a' \in A$.

Consider maps $i_a : \{0\} \longrightarrow A$ and
$$0 \longmapsto a$$

$$i_{a'} : \{0\} \longrightarrow A$$
$$0 \longmapsto a'$$

Then if $f \circ i_a = f \circ i_{a'}$, we have
$$\begin{matrix} f(i_a(0)) & = & f(i_{a'}(0)) \\ \| & & \| \\ f(a) & & f(a') \end{matrix}$$

So $a = a'$ by injectivity.

Thus, $f \circ i_a = f \circ i_{a'} \implies a = a' \implies i_a = i_{a'}$

More generally,

Proposition. $f: A \longrightarrow B$ is injective if and only if

$f \circ g = f \circ h \implies g = h$ for all functions

$g, h: Z \longrightarrow A$.

$$Z \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B$$

Proof. ($\Rightarrow$) Let f be an injection and $f \circ g = f \circ h$ for $g, h: Z \longrightarrow A$.

Let $z \in Z$. We want to show $g(z) = h(z)$.

We have $f \circ g = f \circ h$, so $f(g(z)) = f(h(z))$.

By injectivity, $g(z) = h(z)$.

Thus, $z \in Z \Rightarrow g(z) = h(z)$ so $g = h$.

($\Leftarrow$). Say $f \circ g = f \circ h$ for all $g, h: Z \to A$.

Then we can in particular use $g = i_a$ and $h = i_{a'}$

$f \circ i_a = f \circ i_{a'} \Leftrightarrow f(a) = f(a')$

$i_a = i_{a'} \Leftrightarrow a = a'$ $\square$

What about surjections?

Proposition. $f: A \longrightarrow B$ is onto if and only if $g \circ f = h \circ f \Rightarrow g = h$ for all $g, h: B \longrightarrow Z$.

Observe that this is "opposite" or "dual" to injectivity!

Idea. All inputs to g, h are outputs of f.

Pf. ($\Rightarrow$) Let $f: A \longrightarrow B$ be onto and $g, h: B \longrightarrow Z$ s.t. $g \circ f = h \circ f$.

Let $b \in B$. We want to show $g(b) = h(b)$.

By surjectivity, there is $a \in A$ s.t. $f(a) = b$.

By hypothesis, $g \circ f = h \circ f$ so $\underbrace{g(f(a))}_{g(b)} = \underbrace{h(f(a))}_{h(b)}$

as desired.

($\Leftarrow$) Proving this is equivalent to showing that f is not onto

$\Downarrow$

there exist $g, h: B \longrightarrow Z$ s.t. $g \circ f = h \circ f$ but $g \neq h$.

Indeed, if f is not onto then is a $b' \in B$ s.t. for all $a \in A$, $f(a) \neq b'$.

Let $Z = \{0, 1\}$ and define

$g: B \longrightarrow Z$ via

$$g(b) = \begin{cases} 1 & b = b' \\ 0 & b \neq b' \end{cases}$$

and $h: B \longrightarrow Z$ via $h(b) = 0$.

Then $h(f(a)) = 0$ for all $a \in A$ and

$g(f(a)) = 0$ for all $a \in A$, as $f(a) \neq b'$!

Hence, $h \circ f = g \circ f$ but $g \neq h$. $\square$

Bijectivity?

Proposition. $f: A \longrightarrow B$ is bijective if and only if there is a function $g: B \to A$ s.t. $f \circ g = id_B$ and $g \circ f = id_A$

Pf. ($\Leftarrow$) If such g exists, then both prior "cancellative properties" hold, so f is 1-1 and onto.

($\Rightarrow$) Let f be bijective.

For $b \in B$, there exists $a \in A$ s.t. $f(a) = b$ as f is onto.

That a is unique as f is 1-1.

So we let $g(b)$ be that corresponding a.

By construction, $f \circ g = id_B$ and $g \circ f = id_A$. $\square$

Remarks. - g is unique and written f^{-1}

- f is 1-1 $\Leftrightarrow$ there is a "left inverse" g s.t. $g \circ f = id$

- f is onto $\Leftrightarrow$ there is a "right inverse" g s.t. $f \circ g = id$

Further operations

Def. Let $f: A \longrightarrow B$.

i. Let $S \subseteq A$. Define

$$f(S) = \{f(a) \mid a \in S\}$$

"the image of S under f"

(*) This is the smallest subset of B so that the notation

$$f: S \longrightarrow f(S)$$

makes sense

ii. Let $T \subseteq B$. Define

$$f^{-1}(T) = \{a \in A \mid f(a) \in T\}$$

"the preimage of T under f"

If $T = \{b\}$ we often write $f^{-1}[b]$. This is called the "fiber of f over b".

(*) This is the largest subset of A so that the notation

$$f: f^{-1}(T) \longrightarrow T$$

makes sense

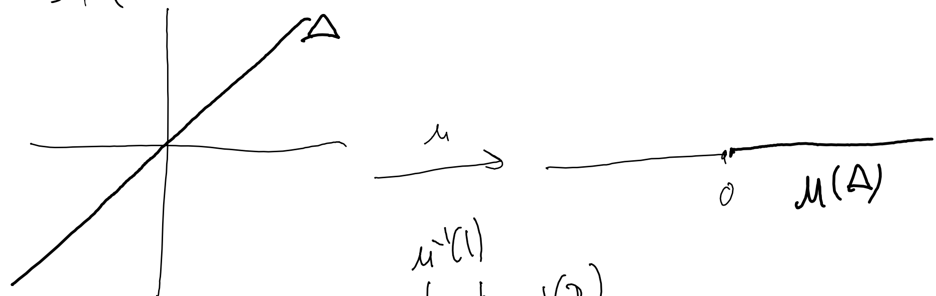
(*) This intuition was taken from a student in 110AH!
If you restrict the domain you can restrict the codomain via the image.
If you want to restrict the codomain, you can restrict the domain via the preimage

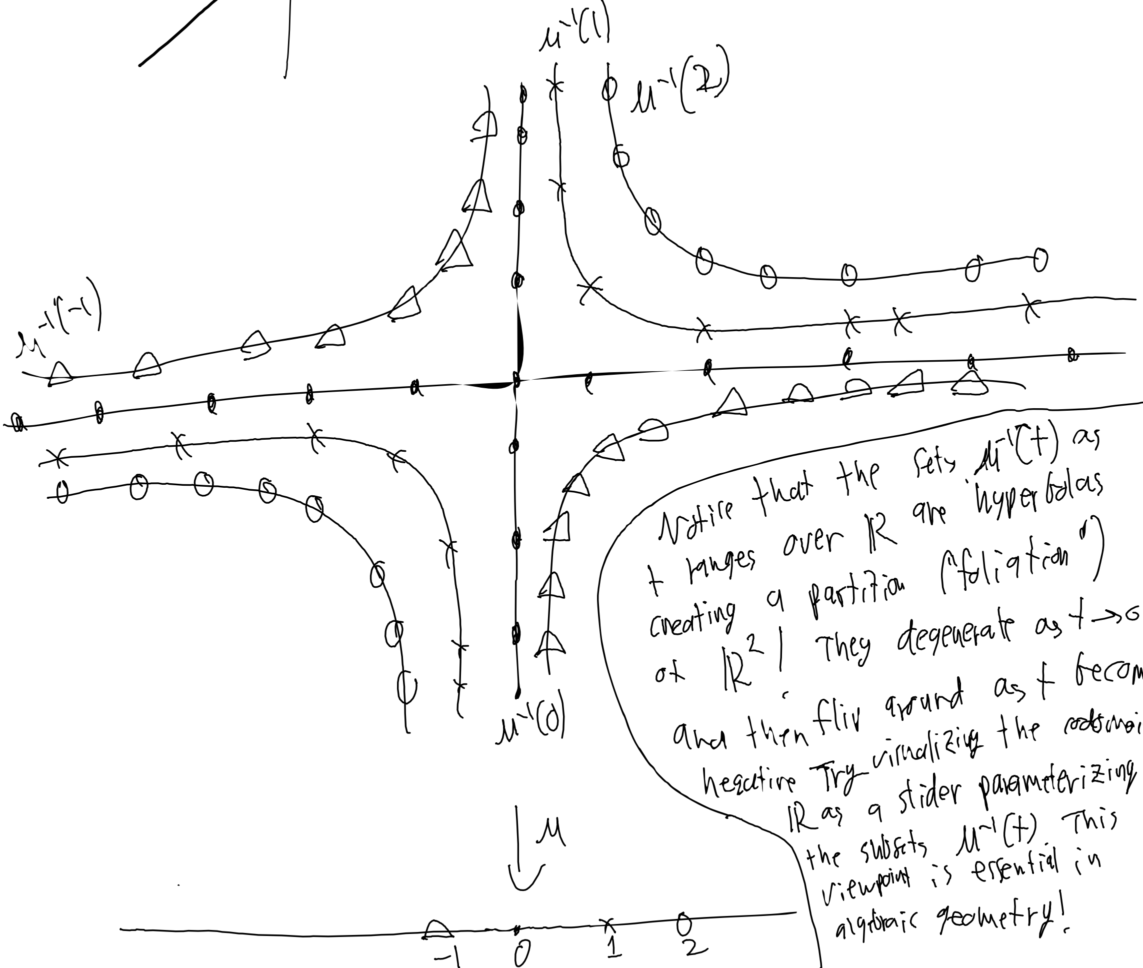
e.g. $\mu : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(a,b) \longmapsto ab$

Let $\Delta = \{(a,a) \mid a \in \mathbb{R}\} \subseteq \mathbb{R}^2$

$\mu(\Delta) = \{a^2 \mid a \in \mathbb{R}\} = (0, \infty)$





Notice that the sets $\mu^{-1}(t)$ as t ranges over $\mathbb{R}$ are hyperbolas creating a partition ("foliation") of $\mathbb{R}^2$! They degenerate as $t \to 0$ and then flip around as t becomes negative. Try visualizing the codomain $\mathbb{R}$ as a slider parameterizing the subsets $\mu^{-1}(t)$. This viewpoint is essential in algebraic geometry!

Fact. - $f: A \to B$ is onto $\iff f^{-1}[b] \neq \emptyset$ for all $b \in B$

- $f: A \to B$ is 1-1 $\iff f^{-1}[b]$ has at most 1 element for all $b \in B$.

- Thus, $f: A \to B$ is bijective $\iff f^{-1}[b]$ has one element for all $b \in B$.

Def. $\mathrm{Fun}(A, B) = \{\text{fns } A \to B\}$. This is often written B^A.

Theorem. There is a bijection

$$\mathrm{Fun}(A, \{0,1\}) \xrightarrow{T} P(A)$$
$$f \longmapsto f^{-1}[1]$$

Idea: If $f(a)=1$ then a is "on", i.e. in the subset. If $f(a)=0$, a is "off", i.e. not in the subset.

Pf. We define an inverse $P(A) \xrightarrow{U} \mathrm{Fun}(A, \{0,1\})$
$$S \longmapsto \chi_S$$

where $\chi_S(a) = \begin{cases} 1 & a \in S \\ 0 & a \notin S. \end{cases}$ Then $\chi_S^{-1}[1] = S$, so $T(U(S)) = S$.

On the other hand, let $f: A \longrightarrow \{0,1\}$.

$U(T(f)) = \chi_{f^{-1}[1]}$,

Let $a \in A$. $\chi_{f^{-1}[1]}(a) = \begin{cases} 1 & a \in f^{-1}[1] \\ 0 & a \notin f^{-1}[1] \end{cases}$

$= \begin{cases} 1 & f(a) = 1 \\ 0 & f(a) \neq 1 \end{cases}$

$= \begin{cases} 1 & f(a) = 1 \\ 0 & f(a) = 0 \end{cases}$

So $U(T(f)) = f$. $\square$