Moduli Stack of Curves Learning Seminar
Spring 2023

Abstract

The goal of this learning seminar is to understand the construction and properties of the moduli stack of stable genus g curves with n marked points $\overline{\mathcal{M}}_{g,n}$. We will follow the outline of Jarod Alper's course [Alper] and refer to Harris and Morrison's text on *Moduli of Curves* [HM98]. We also refer to [Vistoli] for further information about descent. We are particularly interested in reaching stable reduction of curves and properness of $\overline{\mathcal{M}}_{g,n}$ over $\operatorname{Spec}\mathbb{Z}$.

Week 1. *Introduction to moduli problems and examples*
We introduce the idea of moduli functors and give specific examples such as $\mathcal{M}_g$, the Hilbert scheme, and potentially others. Furthermore, we discuss the issue of representability and universal families of moduli functors, especially as it relates to $\mathcal{M}_g$. We furthermore briefly discuss classic computations (going back as far as Riemann) of the dimension of $\mathcal{M}_g$ and its relation to other moduli spaces.

Week 2. *Sites, sheaves, and prestacks*
We quickly recall the notion of a site, and focus especially on the Zariski and étale sites on Sch. With time, we discuss the fppf site as well. We then define prestacks and give first properties and examples, especially quotients.

Week 3. *Stacks and representability*
We define the notion of a stack and give examples. Furthermore, we discuss representability of stacks and define algebraic spaces and Deligne-Mumford stacks. We then discuss the construction of $\mathcal{M}_g$ as a stack quotient and its algebraicity.

Week 4. *Geometry of stacks I - local, topological, and separation properties*
We begin exploring how to talk about stacks in a geometric fashion by discussing étale/smooth local and topological properties of DM/algebraic stacks. Additionally, we discuss the diagonal and separation properties of stacks.

Week 5. *Geometry of stacks II - dimension, tangent spaces, and residual gerbes*
We continue to phrase geometric properties of stacks by defining dimension, tangent spaces, and residual gerbes.

Week 6. *Geometry of stacks III - infinitesimal lifting criteria and characterization of DM stacks*
We discuss the lifting criteria for smooth, unramified, and étale morphisms of stacks. We furthermore discuss when an algebraic stack is DM and conclude that $\mathcal{M}_g$ is a smooth DM stack over $\operatorname{Spec}\mathbb{Z}$ with relative dimension $3g-3$.

Week 7. *Geometry of stacks IV - properness, sheaves, and local structure of DM stacks*
We define properness and state the valuative criterion. Furthermore, we define quasi-coherent sheaves on DM stacks and their cohomology. Finally, we state the local structure of DM stacks.

Week 8. *Stable curves*
We define stable pointed curves and define the compactification $\overline{\mathcal{M}}_{g,n}$. We determine stack-theoretic properties of $\overline{\mathcal{M}}_{g,n}$, namely that it is a quasi-compact DM stack smooth over $\operatorname{Spec}\mathbb{Z}$ with relative dimension $3g-3+n$.

Week 9. *Stable reduction*
We discuss the stable reduction theorem and aspects of its proof.

Week 10. *Irreducibility and projectivity*
We conclude by discussing irreduciblity of the moduli stack of curves as well as the existence of a projective coarse moduli space.

References

[HM98] Joe Harris and Ian Morrison. *Moduli of curves*. Graduate tests in mathematics 187. New York: Springer, 1998. ISBN: 9780387984384 9780387984292.

[Vistoli] Angelo Vistoli. *Notes on Grothendieck topologies, fibered categories and descent theory*. 2007. arXiv: math/0412512 [math.AG].

[Alper] Jarod Alper. *Math 582C: Introduction to stacks and moduli*. URL: https://sites.math.washington.edu/~jarod/math582C-winter21.html.