Properness of $\overline{M}_{g,n}$ (and more...)

So, Stable curves

Def. Let $C$ be a curve over $k$, a closed point $p \in C$ is a node if $\mathcal{O}_{\tilde{C}, \tilde{p}} \cong k[[x, y]]/(xy)$

Rem. Etale locally, nodes look like $\text{Spec } k[x, y]/(xy)$ for $k'/k$ finite separable.

Prop. Let $C$ be connected, nodal, and projection $\pi: C \to \tilde{C}$.

Let $p_1, \ldots, p_d$ be the nodes.

Let $C_1, \ldots, C_d$ be the irreducible components.

Let $g_i = g(C_i)$ be the geometric genus.

Then $g(C) = \sum_{i=1}^d g_i + \delta - 1$.

For $g = 4 + 6 - 4 + 1 = 7$.

Remark: $\delta - 1$ counts boundary regions!
Def. $C; P_i, P_n/n_i$ is stable if $C$ is geometrically connected, normal, projective, and $P_i, P_n \in C$ are distinct smooth points such that all smooth rational curves have marked points and all genus 2 curves have a marked point.

Prop. $C; P_i, P_n$ stably $\leq$ $Aut(C; P_i, P_n)$ finite $\leq$ $\omega_C(\sum P_i)$ ample.

Proof idea. For $C$ integral, $Aut(C)$ infinite for $|P| > 0$ and $g = 1$.

If we fix $3$ points on $|P|$, it becomes finite.

If we fix an origin on $g = 1$, it becomes finite.

Also, $\omega_{C}(3) \leq O(1)$ ample.

Also, $\omega_{E} \leq O(E)$ for $g(E) = 1$, ample $\omega_{E}$ ample.

Fact. $\omega_{C}(\sum P_i)^{\otimes 3}$ very ample.
§ 4. \( \overline{M}_{g_{10}} \)

Def. For a scheme \( S \), we define

\[
\overline{M}_{g_{10}}(S) = \left\{ (\mathcal{C}, \sigma) \mid \begin{array}{c}
\mathcal{C} \text{ an algebraic space flat over } \text{Spec}(S) \\
\mathcal{C} \to S \text{ is proper, flat, projective, finitely presented} \\
(C_5, \sigma_5) \text{ stable and geometrically stable over } S
\end{array} \right\}
\]

Rem. \( \begin{array}{c}
\phi^t : \mathbb{P}^1 \to \mathbb{P}^3 \text{ in } \mathbb{C}^5 : \langle x^2, y^2, xz, \tilde{y}, \tilde{z} \rangle \\
\end{array} \)

As \( t \to 0 \), this degenerates to \( \square \) which may in turn degenerate to \( \circ \). Thus this relaxation of curve is needed.

- There are families whose total spaces are smooth schemes so to be on stacks (i.e., to have étale descent), we must allow alg. spaces as flat spaces. This ceases to be an issue in the pro-étale setting, where stacks formulating

an issue in the pro-étale setting, where stacks formulating

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Fact: \( \circ \) is very good.
First stack theoretic properties of $\tilde{M}_{g,1}$

Def. $M_{g,1}^{all}(S) = \{(\gamma : S, \sigma_1, \ldots, \sigma_n)\}$

Facts. - $M_{g,1}^{all}$ has representable diagonal
- $M_{g,1}$ is an alg. stack locally of finite type / space
- $\tilde{M}_{g,1}$ is the universal family, so by
  \[ \text{forget } \gamma \text{ only for } h \gamma \]
- $M_{g,1}$ is a stack as quotient of alg. space
- For any automorphism of $M_{g,1}$, there is a strictly
canonical Hilbert scheme for each element of $M_{g,1}^{all}$

As for $\tilde{M}_{g,1}$, we see it is always very simple, so
we can $H = \text{Hilb}^p(\mathbb{P}^{2n}) \times (\mathbb{P}^{2n})^\gamma$ for some $p$

\[ \left\{ (\gamma, \sigma_1, \ldots, \sigma_n) | \gamma \in \pi_1(H) \right\} \]

so that $H \to M_{g,1}^{all}$ has image containing $\tilde{M}_{g,1}$.

Car. $\tilde{M}_{g,1}$ as alg. stack $\tilde{M}_{g,1}/\text{Spec } \mathbb{Z}$

All. By def'n of stability, the stabilizer groups are finite
so $\tilde{M}_{g,1}$ is DM. Then $\text{End}_{\tilde{M}_{g,1}}(\psi_c(\Theta_1), \Theta_c) = 0$, and
this is the Lie algebra of $\text{Aut} (\gamma, \psi_1, \ldots, \psi_n)$
Furthermore, $\operatorname{Ext}^1_{\mathcal{O}_c}(\omega_c(\mathcal{E} v_1), \mathcal{O}_c) = 0$, so all obstruction to infinitesimal lifting vanish. As such, we get smoothness.

Finally, $\operatorname{Ext}^1_{\mathcal{O}_c}(\omega_c(\mathcal{E} v_1), \mathcal{O}_c)$ has dimension $3g-3+n$ (or, let $2g-2+n > 0$. Then $\widetilde{M}_{g,1}$ is qc, DM, smooth, and has relative dimension $3g-3+n$.}
§2. Properties

Recall proper = $\text{Sep}$, + universally closed + separated
we have $\text{Sep}$ from before, so we are left to show two
uniqueness criteria.

Thm (Stable reduction). Let $R$ be a DVR, $K = \text{Frac}(R)$,

$\Delta = \text{Spec } R$, $\Delta^* = \text{Spec } K$.

Let $\left( C^* \rightarrow \Delta^*, \sigma_1^*, \ldots, \sigma_n^* \right) \in \overline{M}_{g,n}(\Delta^*)$.

Then $\exists \Delta' \rightarrow \Delta$ a finite cover of DVRs, and

a family $\left( C' \rightarrow \Delta', \sigma_1', \ldots, \sigma_n' \right)$ extending $C^* \rightarrow \Delta^*$.

To prove this, we need the birational geometry of surfaces.

For one, we only do this in residue characteristic $0$.

Further, as the valuative criteria at universal closedness
allows for any base change of finite type, we assume the
residue field is algebraically closed.
Birational geometry of surfaces

Def. A surface $X$ is an integral scheme of pure dimension 2, which is flat over a DVR, and algebraically closed residue field.

Facts. - Projective birational maps between smooth surfaces are compositions of blowups at smooth points. (Ideas: Contracting a curve factor through a blowup, by using explicit coordinate computations)

- Embedded resolution of curves,

$\tilde{X}$ a surface, $\pi: \tilde{X} \to X$ a curve (in the central fiber if $X$ is over a DVR)

There is a projective birational $\tilde{X} \to X$

So $\tilde{X}$ smooth.

- The primes $\mathfrak{p}$ at $\mathfrak{p}_0$ have $\mathfrak{p}_0$-th power.

- $\tilde{X}$ normal.

Idea. Blow up in the reduced principal divisor to keep dropping the ambient genus.
- Castelnuovo contraction criterion
  smooth rational (-1)-curve can be contracted

- Existence of mild resolution
  1. \( \tilde{X} \to X \) projective birational $\mathbb{Q}$-div.
  2. \( \tilde{X} \to X \) smooth
  3. \( Y \to X \) is resolution, then \( Y \to \tilde{X} \)

\[ \text{Pf. of stable reduction,} \]

We prove mostly by example.

As above, fix the residue field by char 0 and act.

Further, reduce to \( \mathbb{C}^k \to \Delta^k \) smooth.

In the singular case, take a finite extension of \( K \)

so that each node is a \( k \)-point.

By induction on genus, do stable reduction on

By induction on genus, do stable reduction on

The smooth case handles \( g = 0 \), as \( g = 0 \Rightarrow \text{smooth} \).

Take the closure, which will be \( \text{fl}_{/\Delta} \).

Step 1. Some \( C \to \Delta \) extending \( C^k \to \Delta^k \) exists

\( \omega^g \) is very ample, \( \text{ie} C^g \to 10^{g-6} \Delta^g \leq 10^{6} \Delta \)
Now, we illustrate the remaining steps.

Consider \( y^2 = x^3 + t \) for \( t \in \mathbb{Q} \), a uniformizer.

We also take \( h = 0 \) for simplicity.

The next procedure will work for \( y^2 = x^{2r+1} + t \) (where \( 2r+1 \) odd).

Step 2. Perform embedded resolution at \( \text{Sing}(\mathcal{O}_{\tilde{Y}, \tilde{y}}) \), i.e.,

replace \((x, y, t)\) old curves \((\tilde{x}, \tilde{y}, \tilde{t})\) new curves.

\( y^2 = x^3 \)

\( \tilde{y} = y/x \)

\( \tilde{x}^2 (\tilde{y}^2 - \tilde{x}) = 0 \)

\( 2 \tilde{E}_0 + \tilde{C}_0 \)

\( \tilde{x}_0 \tilde{y}_0^3 (\tilde{y}_0 - \tilde{x}_0) = 0 \)

\( \tilde{C}_0 + 2 \tilde{E}_0 + 3 \tilde{E}_1 \)
$x^2 y^3 (y-x) = 0$

$c_0 + 2 \sim \mathbb{E}_0 + \mathbb{E}_1$

$\tilde{c}_0 + 2 \sim \mathbb{E}_0 + \mathbb{E}_1 + \mathbb{E}_2$

$\tilde{x} = x$

$\tilde{y} = y(y - 1)$

$\tilde{E}_0 \sim \mathbb{E}_0 + \mathbb{E}_1 + \mathbb{E}_2$

So we have $S \sim 1$. Therefore $v_{an}$ is crossing.

(Rem. for $y \sim x^{2k+1}$, we get $x^{2k-1} = 2 \times \tilde{c}_0$ so we proceed just as above.)

Step 1: 

Basic change so that the normalization of the central fiber is reduced and homal.

(Rem. In the proof, let $m = lcm(multiplities of \gamma)$

and basic change via $\Delta \to \tilde{\Delta} \to \tilde{\Delta}' \to \tilde{\Delta}'' \to \tilde{\Delta}'''$.

The lift $C' \subseteq \tilde{\Delta}'$ and $C''$ the normalization, then reduct the formal local structure at nodes/ramified points.

If $chur(k)$ in then $\tilde{\Delta}' \to \tilde{\Delta}'''$ has wild ramification!
In other words, take a double cover \( C \rightarrow C \) branched along the central fiber \( C_0 = \{ t = u^2 \} \) locally, \( C \) is \( \{ x^2 = t^2 \} \).

A multiplicity \( m \) on \( C \) at \( C_0 = \{ t = u^2 \} \) locally \( f^2 = z^{m^2} \), so in \( \tilde{C} \) it becomes \( f^2 = z^{m^2} \).

This is singular. But the normalization \( \text{can be given} \) by \( v = u / z \), and replacing \( u \) by \( v \).

\[ \tilde{C} \cong \tilde{C} \] is a degree 2 cover branched along the central fiber \( C_0 = \{ tu = u^2 \} \) reduced mod 2, i.e., reducing the multiplicity \( w \) mod 2 representatives \( 0, 1 \).

This works for \( 2 \) replaced by any prime.

Here, \( \tilde{C} \rightarrow C \) is thus branched over \( C_0 + \tilde{E}_1 \).

\[ \text{invariant mass of } E \rightarrow E_2 \rightarrow E_2 \] is \( \text{trivial, call it } E_2 \).

\[ \text{pull image at } \tilde{E}_0 \rightarrow \tilde{E}_0 \text{ is unramified and } \tilde{E}_0 \text{ is } \in \text{L} \text{, so } \tilde{E}_0 \text{ is 2 disjoint } \text{L} \text{, } \tilde{E}_0/\text{E}_0 \text{,} \]

\( f(u) = u^2 + 4 \), so multiplicity are rational.
Now let $C^\prime := \tilde{C}^\prime$ and $\tilde{C}^\prime$ be the normalization of the base change $\tilde{t}_! \to \tilde{t}^\prime$.

As the same takes $C^\prime \to C$, is a divisor.

Cover branched at the union $E_1 \cup E_0 \cup E_0^\prime$.

- $E_1$ is disjoint from the branch locus, so it splits into 3 111's, $E_1 \cup E_1^\prime \cup E_0^2$.
- $E_2$ is branched at 3 points, so its preimage is an elliptic curve $E_2^\prime$. 

\[ \begin{array}{c}
E_1 \\
E_1^\prime \\
E_0 \\
\tilde{E}_0 \\
1 \\
3 \\
1 \\
1 \\
4 \\
4 \\
1 \\
1 \\
\end{array} \]
So the central fiber is now reduced and nodal, given \( \Delta' \rightarrow \Delta \) for it.

Step 4. Take a minimal resolution and contract negative curves.

Def. A rational tail \( E \subset C \) is a rational curve.

\[ E \in \text{rk} (C - E) = 1 \] and \( E \) has no minimal point.

\[ E \]

A rational bridge \( ii \) on \( C \), \( E \in (C - E) = 2 \)

A rational bridge \( ii \) on \( C \), \( E \in (C - E) = 2 \)

In our case, \( E_1, E_0, E_0' \) must be contracted for stability.

In our case, \( E_1, E_0, E_0' \) must be contracted.

\[ \text{end} \]

\[ \text{end} \]

Thus, \( M_{x,n} \) is universally (closed).

\( \square \)
Thm. \( \overline{M}_{g_{1g}} \) is separated.

Explicitly, let \( R \) a DVR, \( K = \text{Frac}(R) \), \( \Delta = \text{Spec} K \).

Let \( C, D \in \overline{M}_{g_{1g}}(\Delta) \) and \( \phi : C \to D \).

Then \exists ! \delta : (\quad \to \quad) \quad \Delta \quad \Delta \)

pf. Let \( \overline{C} \to C \to D \to D \) be multi resolution,

Let \( \pi \) be the graph of \( \phi \) and \( \overline{\pi} \) the minimal resolution.

\[ \begin{array}{ccc}
\overline{C} & \xrightarrow{\delta} & \overline{D} \\
\downarrow & & \downarrow \\
\pi & \xrightarrow{\phi} & D \\
\downarrow & & \downarrow \\
\overline{\delta} & \xrightarrow{\phi} & D \\
\end{array} \]

So \( \overline{C} \) and \( \overline{D} \) have the same canonical models.

As \( C, D \) are stable families w/ many pts, \( \overline{C} \to C, \overline{D} \to D \)

arise by cutting (rational tails + bridges), so \( C \) and \( D \) an the canonical models of \( \overline{C} \) and \( \overline{D} \), hence are isomorphic. \( \square \)
Cor. \( \mathcal{M}_{g_0} \rightarrow \text{Spec } \mathbb{Z} \rightarrow \text{proper} \)
Prop. \( \overline{\mathcal{M}}_{g, c, n} \times \overline{\mathcal{M}}_{g', c', n'} \to \overline{\mathcal{M}}_{g, n+m-2} \)

\(( (C, \pi_1, \pi_n), (C', \pi'_1, \ldots, \pi'_{n'}) ) \to ( C \cup C', \pi_1, \ldots, \pi_{n-1}, \pi'_2, \ldots, \pi'_{n'} ) \)

Glu along \( \pi_n \sim \pi'_{n'} \)

and

\( \overline{\mathcal{M}}_{g-1, n} \to \overline{\mathcal{M}}_{g, n-2} \)

\(( (C, \pi_1, \pi_n), \pi_{n-1} \sim \pi_n \) \to \( (C/\pi_{n-1}, \pi_1, \ldots, \pi_{n-2}) \)

are morphisms of stacks.
Rank. These maps are proper by 2/3 and equidimensional, so they're finite.

This is also representable by any space.

**Def.** Let $\delta_0 = \text{im} \left( \overline{M}_{g-1,2} \rightarrow \overline{M}_g \right)$

$\delta_i = \text{im} \left( \overline{M}_{i,1} \times \overline{M}_{g-1,i} \rightarrow \overline{M}_g \right)$

for $i = 1, \ldots, \left\lfloor g/2 \right\rfloor$.

closed subsets $\overline{M}_{i,1} \times \overline{M}_{g-1,i}$

**Facts.** $\delta = \overline{M}_g - M_g = \bigcup_{i=0}^{\left\lfloor g/2 \right\rfloor} \delta_i$

- $J_t \overline{M}_{i,m}$ is irreducible for all $g \geq 3$ union $\delta_i$.

**Remark**

**Idea.** The hypothesis ensures all $\delta_i$ are connected.

Also, all $\delta_i \cap \delta_j$ are normal.

For $i, j \in [0, \left\lfloor g/2 \right\rfloor]$ and $\delta_0 \cap \delta_i$,

$$
\text{for } 1 \leq j < i \leq \left\lfloor g/2 \right\rfloor, \quad \delta_i \cap \delta_j
$$

(\text{for } i = 0, 1 \rightarrow \delta_i)$
- all smooth curves degenerate to the boundary of
\[ \tilde{\mathcal{M}}_{g,n} \rightarrow \tilde{\mathcal{M}}_{g,n-1} \cup \tilde{\mathcal{C}} \]

\[ (C, p_1, \ldots, p_n) \rightarrow (C^r, p_r, \ldots, p_{n-1}), \] the stack model, i.e. contract all rational curves and bridges

\[ \mathcal{B} \rightarrow \infty \]

is the universal family

Then \( \tilde{\mathcal{M}}_{g,n} \) is irreducible.

Ps. By smoothness, it suffices to show connectedness
\[ \tilde{\mathcal{M}}_{g,n} \rightarrow \tilde{\mathcal{M}}_{g,n-1} \] is the universal family, so it has connected fibers, by induction, it suffices to show \( \tilde{\mathcal{M}}_g \) is connected.

- Smooth curves degenerate to \( \mathcal{D} \)
- \( \mathcal{D} \) is connected

\( \square \)
(cr. Any 2 smooth curves of genus \( g \) are in a 1d smooth family.)

p.s. \( \text{My} \) is connected to \( \text{pick a curve} \) \( C \leq \text{My} \)
connecting the 2 points. Then \( C \to \text{My} \)
yields the family.

(crf. Any 2 compact connected Riemann surfaces of genus \( g \) are diffeomorphic.)

p.s. Let \( \mathcal{X} \to C \) be a smooth family
w/ \( mv 2 \) fibers as fibers. By
Ehresmann's theorem, this is a locally trivial smooth fiber bundle.

\( \square \)
§4. Biregional geometry of $\overline{M}_g$

Fact. $\overline{M}_{g,n}$ admits a coarse projective moduli space for $2g-2+n > 0$, call it $\overline{M}_{g,n}$.

Idea. $\text{DM} + \text{several}$ = the coarse moduli space by the Keel-Mori theorem.

\[(\text{in fact, } \overline{M}_{g,n} \to \overline{M}_{g,n} \text{ is a proper universal homeomorphism})\]

For projectivity, $\overline{M}_{g,n} \to \overline{M}_{g,n}$ is projective and $\chi := \deg \left( \mathcal{W}_{g,n}(\overline{M}_g) \right)$ is defined, to give an ample line bundle on $\overline{M}_g$ for $g \geq 10$.

Thm. (Harris-Mumford) $\overline{M}_g$ is general type for $g \geq 10$.

\[ (\mathcal{U}(x) = \text{tr.deg}(K(x))) \to \mathcal{U} / \overline{M}_{g,n} \geq 1 \]

Thm (Ferrand) $\overline{M}_g$ is unirational for $g \leq 10$.

Rem. Unirationality means $\exists \text{dom} \overline{M}_g$ rational, i.e. we can "write" $\overline{M}_g$ the way genus $g$ curves!
Several conjectures \(\tilde{\text{M}}_g\) are always true for
but by Harris-Mumford, this is false.

Conj. \[ U(\tilde{\text{M}}_g) = -\infty \quad \text{for} \quad g \leq 22 \]

The proof of Harris-Mumford is beyond me, but
Harris-Mumford alleges it uses intersection theory on \(\tilde{\text{M}}_g\).

(Indeed, our very own \([70+99]\) defined \(\text{Min}^1\) of \(\text{B}G\)!) (Indeed, our very own \([70+99]\) defined \(\text{Min}^1\) of \(\text{B}G\)!

Namely, for \(g \geq 3\),

\[ \text{Pic}^2(\tilde{\text{M}}_g) \otimes \mathbb{Q} = \mathbb{Z} < \lambda, \delta_0, \delta_1(2) > \text{ freely} \]

Harris, Arbarello, Cornalba.

Mumford, \[ U_{\tilde{\text{M}}_g} = \langle \lambda - 2\delta_0, -3\delta_1 - 2\delta_2, \ldots, 2\delta_{g/2} \rangle \]