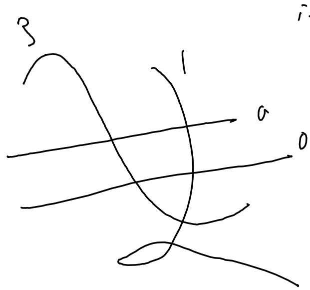
Properness of $\overline{\mathcal{M}}_{g,n}$ (and more...)

§0. Stable Curves

Def. Let C be a curve$/k$ a field. A closed point $p \in C$ is a node if $\exists\, \bar{p} \in C_{\bar{k}}$ over p so that
$$\widehat{\mathcal{O}}_{C_{\bar{k}}, \bar{p}} \cong \bar{k}[[x,y]]/(xy)$$

Rmk. Étale locally, nodes look like $\operatorname{Spec} k'[[x,y]]/(xy)$ for k'/k finite separable.

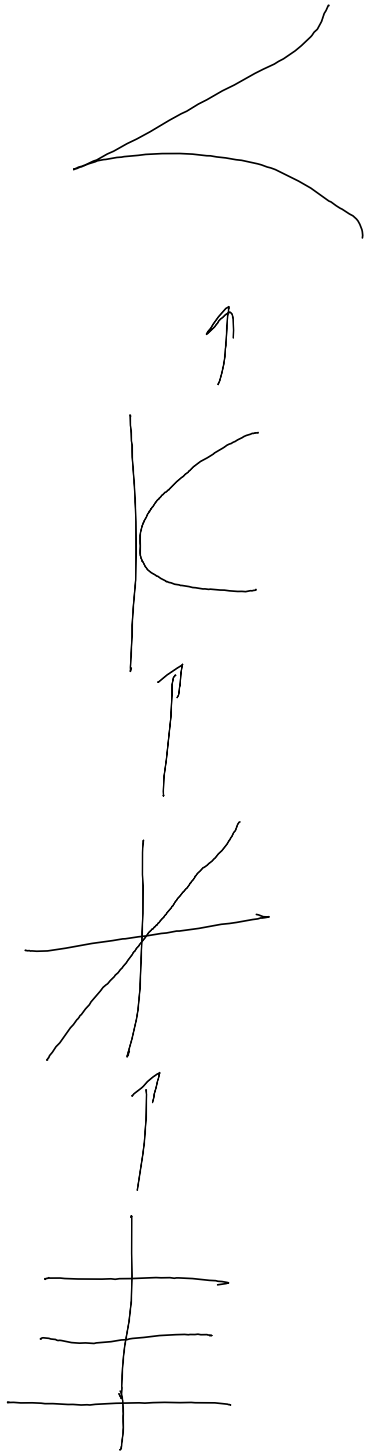
Prop. Let C be connected, nodal, and projective $/k = \bar{k}$.
Let $p_1, \dots, p_\delta$ be the nodes
Let $C_1, \dots, C_\nu$ be the irreducible components
Let $g_i = g(\widetilde{C_i})$ be the geometric genera.
Then $$g(C) = \sum_{i=1}^{\nu} g_i + \delta - \nu + 1$$



$g = 4 + 6 - 4 + 1$
$= 7$

Rmk. $\delta - \nu + 1 = \#$ bounded regions!

e.g.,



Def. $(C, p_1, \dots, p_n)_{/k}$ is stable if C is geometrically connected, nodal, projective, and $p_1, \dots, p_n \in C$ are distinct smooth points such that all smooth rational curves have 3 marked points and all genus 1 curves have a marked point.

Prop. $(C, p_1, \dots, p_n)$ stable $\iff$ $\mathrm{Aut}(C, p_1, \dots, p_n)$ finite

$\iff$ $\omega_C(\sum p_i)$ ample

Pf idea. For C integral, $\mathrm{Aut}(C)$ infinite for $\mathbb{P}^1$ and $g=1$.

If we fix 3 points on $\mathbb{P}^1$, it becomes finite.

If we fix an origin on $g=1$, it becomes finite.

Also, $\omega_{\mathbb{P}^1}(3) = \mathcal{O}(1)$ ample

Also, $\omega_E \cong \mathcal{O}_E$ for $g(E)=1$, and $\mathcal{O}_E(1)$ ample.

Fact. $\omega_C(\sum p_i)^{\otimes 3}$ very ample

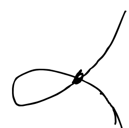
§1. $\overline{M}_{g,n}$

f.t./residue field of dimension 1

Def. For a scheme S, we let

$$\overline{M}_{g,n}(S) = \left\{ (C \to S, \sigma_1, \dots, \sigma_n) \;\middle|\; \begin{array}{l} - C \text{ an algebraic space s.t. all fibers are curves} \\ \text{and s.t. } C \to S \text{ flat, proper, finitely presented} \\ - (C_s, \sigma_1(s), \dots, \sigma_n(s)) \text{ stable } \forall \text{ geometric pts } s \in S \end{array} \right\}$$

Rmk. - $\mathbb{P}^1 \xrightarrow{\varphi_t} \mathbb{P}^3$ via $[x^3 : x^2y : xy^2 : ty^3]$

as $t \to 0$, this degenerates to  nonreduced

which may in turn degenerate to 

so this relaxation of curve is needed

- there are families whose total spaces aren't schemes, so to be a stack (i.e. to have étale descent), we must allow alg. spaces as total spaces. This ceases to be an issue in the projective setting (always true up to étale base change on S), where stable families live.

Fact. $\omega_{C/S}^{\otimes 3}$ very ample

First stack theoretic properties of $\overline{\mathcal{M}}_{g,n}$

Def. $\mathcal{M}_{g,n}^{all}(S) = \{(C \to S, \sigma_1, \dots, \sigma_n)\}$

Facts. - $\mathcal{M}_{g,n}^{all}$ has representable diagonal

- $\mathcal{M}_{g,n}^{all}$ is an alg. stack locally of finite type / Spec $\mathbb{Z}$

$\mathcal{M}_{g,n+1}^{all} \to \mathcal{M}_{g,n}$ is the universal family, so by induction we only show $n=0$.

- $\mathcal{M}_g^{all}$ is a stack as quotients of alg. spaces by étale equivalence relations are alg. spaces

- smooth presentation exist locally via a specially chosen Hilbert scheme for each element of $\mathcal{M}_g^{all}$

As for $\overline{\mathcal{M}}_{g,n}$, $\omega_{C/S}^{\otimes 3}$ is always very ample, so we can $H \subseteq \mathrm{Hilb}^P(\mathbb{P}^N) \times (\mathbb{P}^N)^n$ for some N

$\{(C, p_1, \dots, p_n) \mid p_i \in C \ \forall i\}$

so that $H \longrightarrow \mathcal{M}_{g,n}^{all}$ has image containing $\overline{\mathcal{M}}_{g,n}$.

Cor. $\overline{\mathcal{M}}_{g,n}$ an alg. stack $f.t.$ / Spec $\mathbb{Z}$

Also, by def'n of stability, the stabilizer groups are finite so $\overline{\mathcal{M}}_{g,n}$ is DM. Instead $\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(\Sigma p_i), \mathcal{O}_C) = 0$, and this is the Lie algebra of $\mathrm{Aut}(C, p_1, \dots, p_n)$

Furthermore, $\mathrm{Ext}^2_{\mathcal{O}_C}(\omega_C(\sum p_i), \mathcal{O}_C) = 0$, so all obstructions to infinitessimal lifting vanish. As such, we get smoothness.

Finally, $\mathrm{Ext}^1_{\mathcal{O}_C}(\omega_C(\sum p_i), \mathcal{O}_C)$ has dimension $3g-3+n$

Cor. Let $2g-2+n > 0$. Then $\overline{M}_{g,n}$ is qc, DM, smooth/$\mathbb{Z}$, and has relative dimension $3g-3+n$.

§2. Properness

Recall proper = f.t. + universally closed + separated

We have f.t. from before, so we are left to show two valuative criteria.

Thm (Stable reduction). Let R be a DVR, $K = \mathrm{Frac}(R)$,

$\Delta = \mathrm{Spec}\, R$, $\Delta^* = \mathrm{Spec}\, K$.

Let $(C^* \longrightarrow \Delta^*, \sigma_1^*, \ldots, \sigma_n^*) \in \overline{\mathcal{M}}_{g,n}(\Delta^*)$.

Then $\exists$ $\Delta' \longrightarrow \Delta$ a finite cover of DVRs and a family $(C' \longrightarrow \Delta', \sigma_1', \ldots, \sigma_n')$ extending $C^*_{\Delta'^*} \to \Delta'^*$.

To prove this, we need the birational geometry of surfaces. For one, we only do this in residue characteristic 0. Further, as the valuative criterion of universal closedness allows for any base changes of finite tr deg, we assume the residue field is algebraically closed.

Birational geometry of surfaces

Def. A surface X is an integral scheme of pure dimension 2 which is f.t. / an acf or a DVR w/ algebraically closed residue field.

Facts. - projective birational maps between smooth surfaces are compositions of blowups at smooth points

Idea. Contracting a curve factors through a blowup, by UP or explicit coordinate computations

- Embedded resolutions of curves.

X a surface, $X_0 \subseteq X$ a curve (in the central fiber if X is over a DVR)

There is a projective birational $\tilde{X} \longrightarrow X$

s.t. - $\tilde{X}$ is smooth

- the preimage $\tilde{X}_0$ of X_0 has <u>set-theoretic</u> normal crossings

- $\tilde{X}_0$ nodal

Idea. Blow up in the <u>reduced</u> preimage divisor to keep dropping the arithmetic genus

- Castelnuovo contraction criterion
  smooth rational (-1) curve can be contracted

- Existence of min'l resolutions
  $\exists!\ \tilde{X} \to X$ projective birational s.t.
  - $\tilde{X}$ smooth
  - $Y \to X$ a resolution, then
$$\begin{tikzcd} Y \arrow[r] \arrow[dr, dotted, "\exists"'] & X \\ & \tilde{X} \arrow[u] \end{tikzcd}$$

"Pf." of stable reduction,

We prove mostly by example,

As above, let the residue field be char 0 and a.c.f.

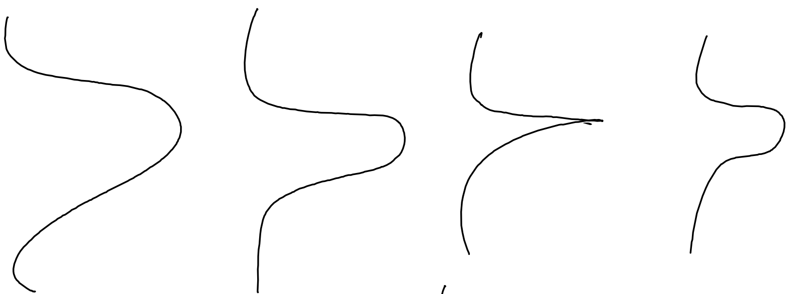
Further, reduce to $C^* \to \Delta^*$ smooth.

In the singular case, take a finite extension of K
so that each node is a K-point.
By induction on genus, do stable reduction on
the components of the normalization, then re-glue.
The smooth case handles $g=0$, as $g=0 \Rightarrow$ smooth.
The smooth case handles $C^* \to \Delta^*$ exists

Step 1. Some $C \to \Delta$ extending
$\omega_C^{\otimes 3}(\,3D^*)$ is very ample, so $C^* \hookrightarrow \mathbb{P}^{5g-6}_{\Delta^*} \leq \mathbb{P}^{5g-6}_{\Delta}$.
Take the closure, which will be flat $/\Delta$.

Now, we illustrate the remaining steps.

Consider $y^2 = x^3 + t$ for $t \in R$ a uniformizer.



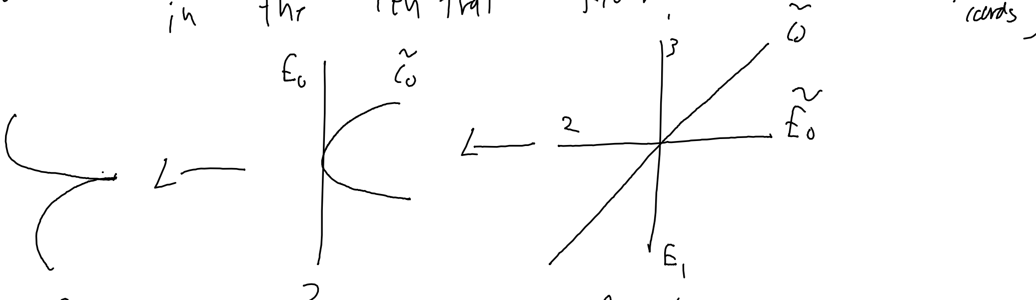
$t = 0$

We also take $n = 0$ for simplicity.

The same procedure will work for $y^2 = x^{2k+1} + t$ (A_{2k+1}).

Step 2. Perform embedded resolution of singularities in the central fiber.

(convention: (x, y) always old coords, $(\tilde{x}, \tilde{y})$ always new coords)



$y^2 = x^3$

C_0

$\tilde{x} = x$

$\tilde{y} = y/x$

$\tilde{x}^2(\tilde{y}^2 - \tilde{x}) = 0$

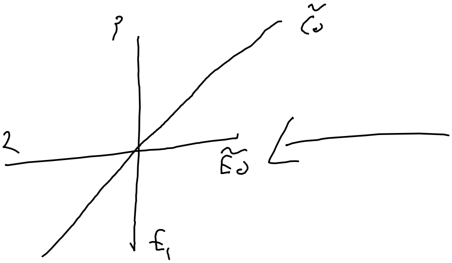
$2E_0 + \tilde{C}_0$

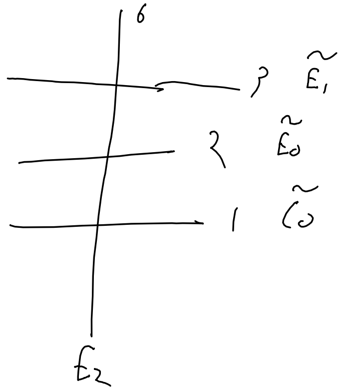
$\tilde{x} = x/y$

$\tilde{y} = y$

$\tilde{x}^2 \tilde{y}^3 (\tilde{y} - \tilde{x}) = 0$

$\tilde{C}_0 + 2\tilde{E}_0 + 3E_1$





$x^2y^3(y-x)=0$

$\tilde{C}_0+2\tilde{E}_0+3E_1$

$\tilde{x}=x$

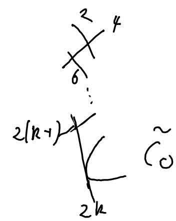
$\tilde{y}=y/x$

$\underbrace{\tilde{x}^6}_{E_2}\underbrace{\tilde{y}^3}_{E_1}\underbrace{(\tilde{y}-1)}_{C_0}$

$\tilde{C}_0+2\tilde{C}_0+3\tilde{E}_1+6E_2$

So we have set-theoretic normal crossings

(Rmk. for $y^2=x^{2k+1}+t$, we get



so we proceed just as above)

Step 3. Base change so that the normalization of the central fiber is reduced and nodal

Rmk. In the proof, let $m=\mathrm{lcm}$(multiplicities of components of C_0) and base change via $\Delta'\longrightarrow\Delta$ $t\mapsto t^m$, Then let $C'=C_{\Delta'}$ and C^ν the normalization, then exploit the étale/formal local structure of nodes/smooth points. If $\mathrm{char}(k)\,|\,m$ then $t\longmapsto t^m$ has wild ramification!

We first base change along $t \mapsto t^2$.

In other words, take a double cover $C' \to C$ branched along the central fiber $C_0 = \{t=0\}$.

So locally, C' is $\{u^2 = t\}$.

A multiplicity m component is locally $\{t = z^m\}$,

so in C' it becomes $\{u^2 = z^m\}$.

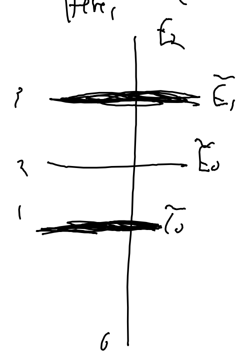
This is singular, but its normalization can be given by $v = u / z^{\lfloor m/2 \rfloor}$ and replacing u by v.

So $v^2 = \begin{cases} z & m \text{ odd} \\ 1 & m \text{ even} \end{cases}$

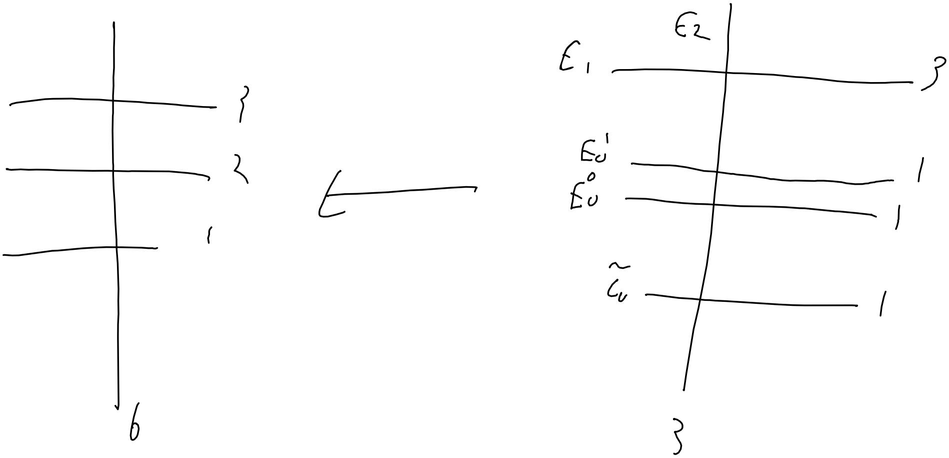
So $\tilde{C}' \longrightarrow C$ is a degree 2 cover branched along the central fiber $C_0 = \{t_0 = 0\}$ reduced mod 2, i.e. replacing all multiplicities w/ mod 2 representatives 0, 1.

This works for 2 replaced by any prime.

Here, $\tilde{C}' \longrightarrow C$ is thus branched over $\tilde{C}_0 + \tilde{E}_1$.



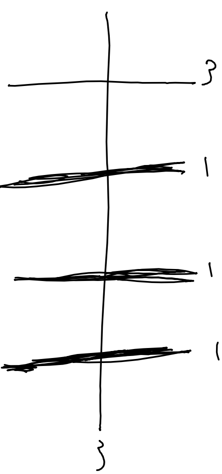
- inverse image of $E_2 \longrightarrow E_2$ branched at 2 pts, & it's $\mathbb{P}^1$, call it E_2
- inv. image of $\tilde{E}_0 \longrightarrow \tilde{E}_0$ is unramified and $\tilde{E}_0$ is rational, so $\tilde{E}_0$ is 2 disjoint $\mathbb{P}^1$'s, E_0^1 E_0^2
- $\{u=0\} = \frac{1}{2}\{t=0\}$, so multiplicities are cut in half

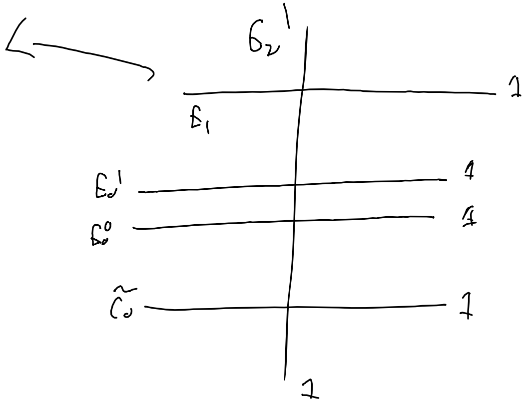


Now let $C' := \tilde{C'}$ and $\tilde{C'}$ be the normalization of the base change $t \mapsto t^3$.

By the same token $\tilde{C'} \to C$ is a degree 3 cover branched at the max'l divisor $\tilde{C} + E_0^0 + E_0'$

- E_1 is disjoint from the branch locus, so it splits into 3 $\mathbb{P}^1$'s $E_1^0 + E_1^1 + E_1^2$
- E_2 is branched at 3 points, so its preimage is an elliptic curve E_2'

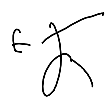




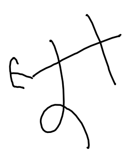
So the central fiber is now reduced and nodal, upon $\Delta' \longrightarrow \Delta$ finite

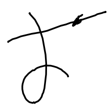
Step 4. Take a minimal resolution and contract negative curves

Def. A rational tail $E \subseteq C$ is a rational curve s.t. $E \cdot \overline{(C-E)} = 1$ and E has no marked points



- A rational bridge is one s.t. $E \cdot \overline{(C-E)} = 2$ or $E \cdot \overline{(C-E)} = 1$ and E has a marked pt





rat'l tails + bridges must be contracted for stability

In our case, E_1, E_0^0, E_0^1 must be contracted

and we reach the (a, a priori) stable limit.

Thus, $\overline{M}_{g,n}$ is <u>universally closed</u> □

Thm. $\overline{M}_{g,n}$ is separated.

Explicitly, let R a DVR, $K = \mathrm{Frac}(R)$

$\Delta = \mathrm{Spec}\, R$ $\quad \Delta^* = \mathrm{Spec}\, K$

Let $C, D \in \overline{M}_{g,n}(\Delta)$ and $\alpha^* : $

$$\begin{tikzcd} C^* \arrow[rr, "\sim"] \arrow[dr] & & D^* \arrow[dl] \\ & \Delta^* & \end{tikzcd}$$

Then $\exists !\ \alpha :$

$$\begin{tikzcd} C \arrow[rr, "\sim"] \arrow[dr] & & D \arrow[dl] \\ & \Delta & \end{tikzcd}$$

Pf. Let $\tilde{C} \to C$, $\tilde{D} \to D$ be minimal resolutions,

Let Γ be the graph of α^* and $\tilde{\Gamma} \to \Gamma$ the minimal resolution.

$$\begin{tikzcd} & \tilde{\Gamma} \arrow[dl] \arrow[dr] & \\ \tilde{C} \arrow[d] & & \tilde{D} \arrow[d] \\ C \arrow[dr] & & D \arrow[dl] \\ & \Delta & \end{tikzcd}$$

$$H^0(\omega^{\otimes m}_{\tilde{C}/\Delta}) \xrightarrow{\sim} H^0(\omega^{\otimes m}_{\tilde{\Gamma}/\Delta}) \xleftarrow{\sim} H^0(\omega^{\otimes m}_{\tilde{D}/\Delta})$$

so $\tilde{C}$ and $\tilde{D}$ have the same canonical models,

As C, D are stable families w/out marked pts, $\tilde{C} \to C$, $\tilde{D} \to D$ arise by contracting rational tails + bridges, so C and D are the canonical models of $\tilde{C}$ and $\tilde{D}$, hence are iso $\square$

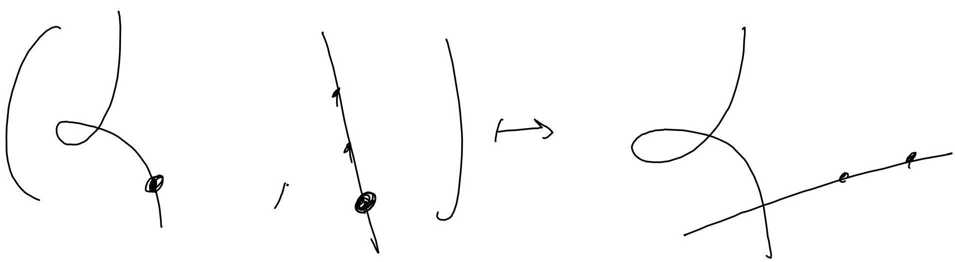
Cor. $\overline{M}_{g,n} \longrightarrow \operatorname{Spec} \mathbb{Z}$ is proper

§3. Irreducibility and the boundary

Prop. $\overline{\mathcal{M}}_{i,n} \times \overline{\mathcal{M}}_{g-i,m} \longrightarrow \overline{\mathcal{M}}_{g,n+m-2}$

$$((C,p_1,\dots,p_n),(C',p_1',\dots,p_m')) \longmapsto (C\cup C', p_1,\dots,p_{n-1},p_1',\dots,p_{m-1}')$$

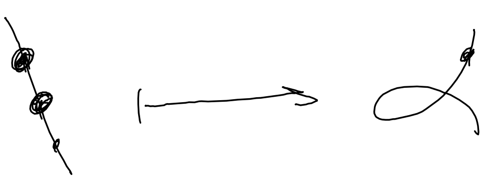
glue nodally along $p_n \sim p_m'$



and

$$\overline{\mathcal{M}}_{g-1,n} \longrightarrow \overline{\mathcal{M}}_{g,n-2}$$

$$(C,p_1,\dots,p_n) \longmapsto (C/p_{n-1}\sim p_n, p_1,\dots,p_{n-2})$$



are morphisms of stacks.

Rmk. These maps are proper by 2/3 and equidimensional, so they're finite! They're also representable by alg. spaces.

Def. Let $\delta_0 = \mathrm{im}(\overline{\mathcal{M}}_{g-1,2} \longrightarrow \overline{\mathcal{M}}_g)$

$\delta_i = \mathrm{im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \longrightarrow \overline{\mathcal{M}}_g)$

for $i = 1, \dots, \lfloor g/2 \rfloor$

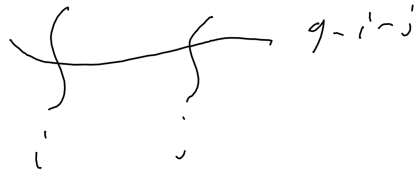
closed substacks

Facts. - $\delta = \overline{\mathcal{M}}_g - \mathcal{M}_g = \bigcup_{i=0}^{\lfloor g/2 \rfloor} \delta_i$ is a normal crossings divisor

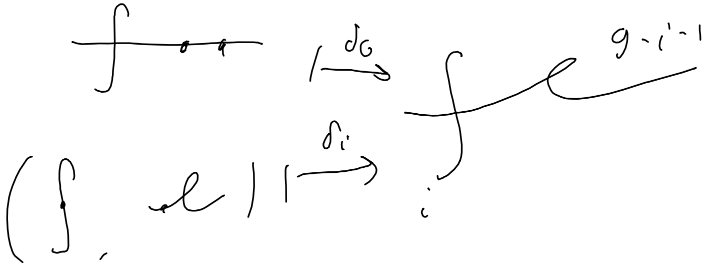
- If $\overline{\mathcal{M}}_{g',n'}$ is irred for all $g' < g$ then δ_i is connected

Idea. The hypothesis ensures all δ_i are connected

Also, all $\delta_i \cap \delta_j$ are nonempty



$\in \delta_i \cap \delta_j$



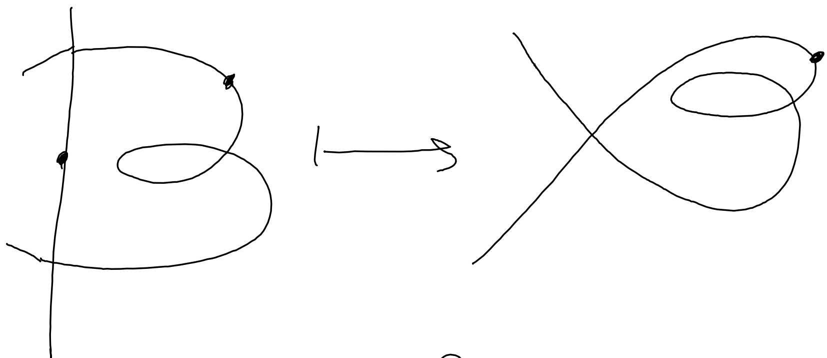
$\in \delta_0 \cap \delta_i$

- all smooth curves degenerate to the boundary δ

- $\overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n-1}$ via

$(C, p_1, \dots, p_n) \longmapsto (C^{st}, p_1, \dots, p_{n-1})$, the stable model,

i.e. contract all rational curves and bridges



is the universal family

Thm $\overline{\mathcal{M}}_{g,n}$ is irreducible.

Pf. By smoothness, it suffices to show connectedness. By the universal family, so $\overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n-1}$ is the universal family, so it has connected fibers. By induction, it suffices to show $\overline{\mathcal{M}}_g$ is connected.

- Smooth curves degenerate to δ
- δ is connected

$\square$

Cor. Any 2 smooth curves of genus g are in a 1d smooth family

Pf. M_g is connected so pick a curve $C \subseteq M_g$ connecting the 2 points. Then $\tilde{C} \to M_g$ yields the family. $\square$

Cor. Any 2 compact connected Riemann surfaces of genus g are diffeomorphic.

Pf. Let $\mathcal{X} \to C$ be a smooth family w/ our 2 friends as fibers. By Ehresmann's theorem, this is a locally trivial smooth fiber bundle. $\square$

§4. Birational geometry of $\mathcal{M}_g$

Fact. $\overline{\mathcal{M}}_{g,n}$ admits a coarse projective moduli space for $2g-2+n>0$, call it $\overline{M}_{g,n}$

Idea. DM + sep + f.t. $\Rightarrow$ the coarse moduli space by the Keel–Mori theorem.

(in fact, $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ is a proper universal homeomorphism)

For projectivity, $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_{g,n-1}$ is projective and $\lambda_k = \det\left(\pi_*\left(\omega^{\otimes k}_{\mathcal{M}_{g,1}/\overline{\mathcal{M}}_g}\right)\right)$ descends to an ample line bundle on $\overline{M}_g$ for $k \gg 0$.

Thm. (Harris–Mumford) $\overline{\mathcal{M}}_g$ is general type for $g \geq 24$ $\leftarrow$ $(\kappa(X) = \dim(X))$

$(\kappa(X) = \operatorname{trdeg}(K(X))) \to \kappa(\overline{\mathcal{M}}_{23}) \geq 1$

$\mathbb{P}^5 \dashrightarrow \overline{\mathcal{M}}_2$
$y^2 = x^6 + \sum a_i x^i$
$\mathbb{P}^{14} \dashrightarrow \overline{\mathcal{M}}_3$ plane quartics
$g=4$, c.i. of quadric + cubic in $\mathbb{P}^3$
...

Thm (Severi) $\overline{\mathcal{M}}_g$ is unirational for $g \leq 10$.

Rmk. Unirationality means $\exists\, \mathbb{P}^N \dashrightarrow \overline{\mathcal{M}}_g$ dominant rational, i.e. one can "write down" the general genus g curve!

Severi conjectured $\overline{M}_g$ is always unirational!
but by Harris-Mumford, this is false!

Conj. $\kappa(\overline{M}_g) = -\infty$ for $g \leq 22$

The proof of Harris-Mumford is beyond me, but Harris-Morrison alleges it uses intersection theory on $\overline{M}_g$.
(Indeed, our very own [Tot 99] defined Chow rings of BG!)

Namely, for $g \geq 3$ $\underbrace{\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}}_{\text{Harer; Arbarello-Cornalba}} = \mathbb{Z}\langle \lambda, \delta_0, \delta_{\lfloor g/2 \rfloor} \rangle$ freely

Mumford: $K_{\overline{M}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor g/2 \rfloor}$