Valuative Criteria, quasicoherent sheaves and the local structure of DM stacks

§1. Valuative Criteria

Recall from schemes the following notions for $X \xrightarrow{f} Y$

i) universal closedness: $\forall T \to Y$, $f_T : X_T \to T$ is closed

ii) separatedness: $\Delta : X \to X \times_Y X$ closed embedding

iii) properness: (i) + (ii) + finite type = (iii)

These are well characterized by the valuative criteria, based on the following setup

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

Let $\mathrm{Lift} = \left\{ \alpha : \mathrm{Spec}\,A \to X \;\middle|\; \begin{tikzcd} \mathrm{Spec}\,K \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\,A \arrow[r] \arrow[ur, "\alpha"] & Y \end{tikzcd} \text{ commutes} \right\}$

Thm (valuative criteria). Let f be q.c.

i) f is universally closed $\iff |Lift| \geq 1$

ii) f is separated $\iff |Lift| \leq 1$

iii) f is proper $\iff |Lift| = 1$ and f f.t.

Idea. Recall $X(K) \cong \{(x, \iota) \mid x \in X,\ \iota : k(x) \hookrightarrow K\}$

For a valuation ring A w/ $\mathrm{Frac}(A) = K$,

$$X(A) \cong \left\{ (x_0, x_1, \iota) \;\middle|\; \begin{array}{l} x_1 \rightsquigarrow x_0 \\ (x_0 \in \overline{\{x_1\}}) \end{array},\ \iota : k(x_1) \hookrightarrow K,\ \text{s.t. } A \text{ dominates } \mathcal{O}_{\overline{\{x_1\}}, x_0} \right\}$$

Recall that primes in a valuation ring are totally ordered

$f[S] \subseteq Y$ closed $\iff$ closed under specialization

For f qc,

i) A qc map is closed $\iff$ it lifts specializations

so a qc map is univ. closed $\iff$ lift specializations along all base changes

$$\iff |Lift| \geq 1$$

ii) Δ is an immersion, so Δ closed $\iff \Delta$ univ. closed

$$\iff \begin{tikzcd} \operatorname{Spec} K \arrow[r, "x"] \arrow[d] & X \arrow[d, "\Delta"] \\ \operatorname{Spec} A \arrow[r, "(x_0, x_1)"'] \arrow[ur, dashed, "\exists ?"] & X \times_Y X \end{tikzcd}$$

i.e. $k(x,x) \rightsquigarrow (x_0, x_1)$, $x_0 = x_1$,

i.e. $\Delta(X)$ closed under specialization

i.e. lifts of specializations of f are unique

iii) ✓

Now, we define analogous notions and valuative criteria for stacks.

Defs, Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks ($\exists$ smooth presentation)

Let $f: \mathcal{X} \longrightarrow \mathcal{Y}$

i) f is universally closed if $\forall \mathcal{Y}' \to \mathcal{Y}$, $\mathcal{Y}'$ an alg stack, $f_{\mathcal{Y}'}: \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \longrightarrow \mathcal{Y}'$ induces a closed map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \longrightarrow |\mathcal{Y}'|$

ii) If f is representable (f_T $\underbrace{\text{alg space}}_{\text{étale pres.}}$) then f is proper if it's universally closed, separated, finite type

iii) f is separated if $\underbrace{\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}}_{\text{representable as } \mathcal{X}, \mathcal{Y} \text{ alg stacks}}$ is proper

iv) f is proper if it's univ. closed, sep, and f.t.

Rmk, Why do we insist Δ is proper and not a closed embedding? Recall

stack	Δ
alg space	mono
DM stack	unramified
alg stack	*

and proper + mono = closed embedding

Furth. univ. closedness is smooth local on target, so it can be tested on a smooth presentation of the target

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\text{univ. closed}"] \\ U \arrow[r, two heads] & \mathcal{Y} \end{tikzcd} \iff \begin{tikzcd} \mathcal{X} \times_{\mathcal{Y}} U \arrow[d, "\text{univ. closed}"] \\ U \end{tikzcd}$$

e.g. Suppose $\mathcal{X} \longrightarrow S$ is separated for S a scheme and $\mathcal{X}$ an alg space, so $\mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X}$ is proper

Recall
$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

If $\mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X}$ is affine as well, then $G_x \longrightarrow \operatorname{Spec} k$ is proper and affine, hence finite.

e.g. - $\mathcal{M}_g = [H' / PGL_{5g-5}]$ has affine diagonal as H' does, and indeed its stabilizers are finite

- $\mathrm{Bun}_{r,d}(C)$ has affine diagonal, as it's locally of the form $[Q' / PGL_N]$, Q' a locally closed subscheme of a Quot scheme. But stabilizers here are generally infinite, so this is not separated.

Setup for valuative criteria for $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ a morphism of algebraic stacks.

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "x"] \arrow[d, "i"'] \arrow[dr, phantom, "\Nearrow"] & \mathcal{X} \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r, "y"'] & \mathcal{Y}
\end{tikzcd}$$

2-commutative via γ

Def. A lift is a triple

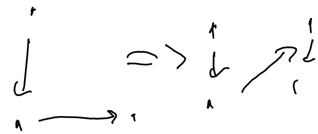
$$\left(a : \operatorname{Spec} A \to \mathcal{X},\ \alpha : ai \Rightarrow x,\ \beta : y \Rightarrow fa \right)$$

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] \arrow[r, phantom, shift right=3ex, "\alpha \Nearrow"] & \mathcal{X} \arrow[d] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, "a"] \arrow[r, phantom, shift left=3ex, "\beta \Nearrow"] & \mathcal{Y}
\end{tikzcd}$$

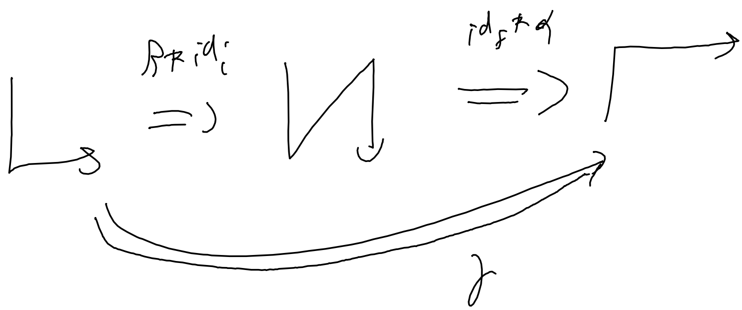
such that $\gamma = (\mathrm{id}_f * \alpha) \circ (\beta * \mathrm{id}_i)$

\begin{tikzcd}
\underline{\operatorname{Spec} K} \arrow[r, bend left, "i"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\Downarrow id_i"] & \underline{\operatorname{Spec} A} \arrow[r, bend left, "y"] \arrow[r, bend right, "fa"'] \arrow[r, phantom, "\Downarrow \beta"] & Y
\end{tikzcd}

\begin{tikzcd}
\underline{\operatorname{Spec} K} \arrow[r, bend left, "yi"] \arrow[r, bend right, "fai"'] \arrow[r, phantom, "\Downarrow \beta * id_i"] & Y
\end{tikzcd}



Similarly, $id_f * \alpha : fai \Rightarrow fx$

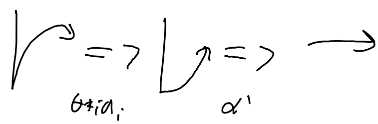


A map of lifts is $\theta : a \Rightarrow a'$

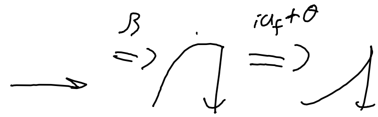
$(a, \alpha, \beta) \to (a', \alpha', \beta')$

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[ur, bend left, "a"] \arrow[ur, bend right, "a'"'] \arrow[ur, phantom, "\Downarrow \theta"] & \cdot \end{tikzcd}$$

s.t. $\alpha = \alpha' \circ (\theta * \mathrm{id}_i)$



$\beta' = (\mathrm{id}_f * \theta) \circ \beta$



This defines a groupoid Lift

e.g.,

$$\begin{tikzcd} \mathrm{Spec}\, \mathbb{C}((t)) \arrow[r, "{(y^2z = (x-2)(x+2)(x-(t+1)),\ [t+1:1:0])}"] \arrow[d, "i"'] & \mathcal{M}_{1,1} \arrow[d, "u"] \\ \mathrm{Spec}\, \mathbb{C}[[t]] \arrow[r, "{y^2z = (x-2)(x+2)(x-t)}"'] \arrow[ur, phantom, "\gamma"] & \mathcal{M}_1 \end{tikzcd}$$

$\exists!\ \mathcal{M}_1(\mathbb{C}((t))) \longrightarrow \mathcal{M}_1(\mathbb{C}((t)))$ via base change along $t \mapsto t+1$

we have lifts a, a' via

$$\begin{tikzcd}
& & \mathcal{M}_{1,1} \\
\operatorname{Spec} \mathbb{C}[[t]] \arrow[urr, bend left, "{\{y^2z = (x-2)(x+2)(x-(t+1)),\ [t+1:1:\mathrm{id}]\}\ a}"] \arrow[urr, bend right, "{a'\ \{y^2z = (x-2)(x+2)(x-t),\ [t:1:\mathrm{id}]\}}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{C}((t)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha = \mathrm{id}"] & \mathcal{M}_{1,1} \arrow[d] \\
\operatorname{Spec} \mathbb{C}[[t]] \arrow[r] \arrow[ur, "a"] \arrow[ur, phantom, "\Nearrow \beta = t+1 \text{ base change}"'] & \mathcal{M}_1
\end{tikzcd}$$

$\beta : \mathcal{M}_1(\mathbb{C}[[t]]) \longrightarrow \mathcal{M}_1(\mathbb{C}[[t]])$

pull back along $t \longmapsto t+1$

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha' = (t \mapsto t+1)"] & {} \arrow[d] \\
{} \arrow[r] \arrow[ur, "a'"] \arrow[ur, phantom, "\Nearrow \beta' = \mathrm{id}"'] & {}
\end{tikzcd}$$

Def. $X \xrightarrow{f} Y$ satisfies the uniqueness part of the valuative criterion if *Lift* is a discrete connected groupoid (potentially empty!)

That is, $\exists!$ lift up to unique iso

Def. $X \xrightarrow{f} Y$ satisfies the existence part of the valuative criterion if given any

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists ?"] & Y \end{tikzcd}$$

There is an extension K'/K and $A' \subseteq A$ dominating A such that the outer rectangle

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A' \arrow[r] & \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

has a nonempty *Lift*

Rmk. why the extension in uniqueness?

e.g. $Y = \mathbb{A}^1_{\mathbb{R}}$, $U = \mathbb{A}^1_{\mathbb{C}}$.

Then $U \times_Y U = \Delta(U) \amalg \bar{\Delta}(U)$, where

$\bar{\Delta} = (\mathrm{id}, \text{conjugation})$

Then $R = \Delta(U) \amalg \bar{\Delta}(U - \{0\})$ is an étale equivalence relation on U/Y.

Let $X = U/R$, so $X \to Y$.

Then $X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is $\mathbb{A}^1_{\mathbb{C}}$ w/ double origin $\to \mathbb{A}^1_{\mathbb{C}}$ is fco.

This is universally closed, so $X \to Y$ is

But
$$\begin{array}{ccc} \operatorname{Spec} \mathbb{R}((t)) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{R}[[t]] & \longrightarrow & Y \end{array}$$
cannot be lifted. The fiber X_0 is $\operatorname{Spec}(\mathbb{C})$, so $\operatorname{Spec} \mathbb{R}[[t]] \to X$ cannot exist, lest the special fiber induces an $\mathbb{R}$-point of X_0

Rmk. X, Y are alg. spaces, A map between separated alg spaces needs involve this K'/K for existence in the valuative criterion.

e.g. Consider $\mathbb{C}((x)) \longrightarrow \mathbb{C}((y))$

$$x \longmapsto y^2$$

$\rightsquigarrow$ $\operatorname{Spec} \mathbb{C}((y)) \longrightarrow \operatorname{Spec} \mathbb{C}((x))$, a $\mathbb{Z}/2$-torsor, corresponding to a point $\operatorname{Spec} \mathbb{C}((x)) \longrightarrow B_{\mathbb{C}}(\mathbb{Z}/2)$

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{C}((x)) \arrow[r] \arrow[d, hook] & B_{\mathbb{C}} \mathbb{Z}/2 \arrow[d] \\
\operatorname{Spec} \mathbb{C}[[x]] \arrow[r] \arrow[ur, dashed, "\exists ?"] & \operatorname{Spec} \mathbb{C}
\end{tikzcd}$$

Any $\mathbb{Z}/2$ torsor over $\operatorname{Spec} \mathbb{C}[[x]]$ is trivial as it's a local strictly Henselian ring, but $\operatorname{Spec} \mathbb{C}((x)) \longrightarrow B_{\mathbb{C}} \mathbb{Z}/2$ is <u>not</u> trivial

Base changing to $\mathbb{C}((y))$ trivializes the torsor, whereas a lift exists.

Indeed, $B_{\mathbb{Z}} G \longrightarrow \operatorname{Spec} \mathbb{Z}$ is proper for any finite group G.

Thm (valuative criteria for alg. stacks).

Let $f: \mathcal{X} \longrightarrow \mathcal{Y}$ be a qs morphism of alg stacks.

i) f is universally closed $\iff$ existence in valuative criteria

ii) f is separated $\iff$ uniqueness in valuative criteria

iii) f proper $\iff$ (i) + (ii) + f.t.

Pf idea. As before, reduce to only showing (i). Use again the lifting specializations property for qc morphisms to get a lift over an extension K'/K.

§2. Sheaves

Let $\mathcal{X}$ be a DM stack

Def. The little étale site of $\mathcal{X}$ is

$$\mathcal{X}_{\acute{e}t} = \begin{cases} \text{obj } & U \longrightarrow \mathcal{X} \text{ étale, } U \text{ a sch} \\ \text{mor } & /\mathcal{X} \\ \text{covering } & \{U_i \to U\} \text{ étale, jointly surjective} \end{cases}$$

We thus get a notion of sheaves on $\mathcal{X}_{\acute{e}t}$.

e.g. - $\mathcal{O}_{\mathcal{X}}(U) = \Gamma(U, \mathcal{O}_U)$, a sheaf of rings

- $\Omega_{\mathcal{X}/S}(U) = \Gamma(U, \Omega_{U/S})$

- $\mathcal{H}$ on $\mathcal{M}_g$ the Hodge bundle as such.

Let $U \longrightarrow \mathcal{M}_g$ étale, corresponding to $C \longrightarrow U$,

Let $\mathcal{H}(U) = \Gamma(C, \Omega_{C/U})$.

Rmk. F can be extended to $\mathcal{U} \xrightarrow{\acute{e}t} \mathcal{X}$ w/ $\mathcal{U}$ DM by

picking a presentation $U \longrightarrow \mathcal{U}$ via $R \longrightarrow U \times_{\mathcal{U}} U$ and

letting $F(\mathcal{U}) = \text{equalizer}\left(F(U) \rightrightarrows F(R)\right)$

Hence, there exists a global sections functor $\Gamma = \mathrm{Hom}(\mathbb{Z}, -)$

Rmk. Have f^{-1} + f_* exactly as expected

$\operatorname{colim}_{V \to Y,\, U \to V \times_Y X} G(V)$ $\swarrow$ $\qquad \rightarrow f_* F(U) = F(U \times_Y X)$

$\mathcal{O}_X$ - Mod

Now, $\mathcal{O}_X$ is a ring object in $Ab(X_{et})$, so there is the usual notion of an $\mathcal{O}_X$ - Module,

$\rightsquigarrow$ f^*, f_* as usual

$$f^* G = f^{-1}G \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

$\mathcal{O}_X$ - Mod is an abelian category

qcoh

Def. An $\mathcal{O}_X$ - Module F is quasicoherent if there for all $U \xrightarrow{et} X$ $\exists \{U_i \to U\}$ cover and a presentation $\mathcal{O}_{U_i}^{\oplus J} \longrightarrow \mathcal{O}_{U_i}^{\oplus I} \longrightarrow F|_{U_i} \longrightarrow 0$

Rmks. - F qcoh $\iff$ $F|_{U_{zar}}$ qcoh $\forall$ $U \xrightarrow{et} X$

and

$\forall$ $U \xrightarrow{f} V$ étale, $f^*(F|_{V_{zar}}) \xrightarrow{\sim} F|_{U_{zar}}$
in X

- $U \to X$ étale pres, $R = U \times_X U$, Let $G \in QCoh(U)$ and let $\alpha: p_1^* G \xrightarrow{\sim} p_2^* G$ an iso in $QCoh(R)$ which satisfies the cocycle condition $p_{23}^* \alpha \circ p_{12}^* \alpha = p_{13}^* \alpha$. Then G descends to a unique QCoh Sheaf on X

e.g. G a finite group $/k$, A an affine k-Alg.

A quasicoherent sheaf on $[\operatorname{Spec} A/G]$ is an A module with an A-linear G-action

The other usual stuff, like flat base change, relative Spec, reduction, normalization all work

Cohomology

Prop. $Ab(\mathcal{X}_{ét})$, $\mathcal{O}_{\mathcal{X}}$-Mod have enough injectives. As does $QCoh(\mathcal{X})$, if $\mathcal{X}$ is qs.

Pf. Exactly as the usual sheaf are, $f^* \dashv f_*$ and f^* is exact. This is purely formal.

We hence define $H^i(\mathcal{X}_{ét}, F)$ and $R^i f_* F$

Prop. $H^i((\operatorname{Spec} A)_{ét}, F) = 0$ $\forall i \geq 1$, $F \in QCoh((\operatorname{Spec} A)_{ét})$

As usual, we can use Čech cohomology on "good covers"

Prop. $F \in Ab(\mathcal{X}_{ét})$, $\{U_i \to \mathcal{X}\}_{i \in I}$ an étale cover s.t. $H^i(U_{j_0} \times_{\mathcal{X}} \cdots U_{j_n}, F) = 0$ for all $i \geq 1$, $j_0, \ldots, j_n \in I$. Then $\check{H}^i(\mathcal{U}, F) = H^i(\mathcal{X}_{ét}, F)$

Combining these, let $\mathcal{X}$ be a DM stack w/ affine diagonal. Let $\mathcal{U} = \{U_i \to \mathcal{X}\}$ an étale covering by affines. Then $\check{H}^i(\mathcal{U}, \mathcal{F}) = H^i(\mathcal{X}_{ét}, \mathcal{F})$.

Now we compare Ab, Mod, QCoh

Prop. - Let $f: \mathcal{X} \to \mathcal{Y}$ be a morphism of DM stacks. Let $\mathcal{F} \in \mathcal{O}_{\mathcal{X}}$-Mod. Then $R^i f_* \mathcal{F}$ as an abelian sheaf on $\mathcal{Y}_{ét}$ is the i^{th} right derived functor of $f_*: Ab(\mathcal{X}_{ét}) \to Ab(\mathcal{Y}_{ét})$

- If f is qc and $\mathcal{X}, \mathcal{Y}$ have affine diagonals, the same goes for $QCoh \to Mod$.

e.g. Let G a finite group/k, V a G-Rep.

$H^i(G, V)$ is the i^{th} right derived functor $Rep(G) \to k\text{-Vect}$
$V \mapsto V^G$

$Rep(G) = QCoh(B_k G)$, so by the above, $H^i(G, V) = H^i(B_k G, V)$

Have an étale cover $\mathcal{U} = \{\operatorname{Spec} k \to B_k G\}$. What is the Čech cohomology?

$$\check{C}^n(\mathcal{U}, V) = V((\operatorname{Spec} k)^{n+1}) \cong \Gamma(G, \mathcal{O}_G)^{\otimes n} \otimes V \cong \operatorname{Map}(G^n, V)$$

and we recover the usual bar resolution!

§3. Local Structure of DM Stacks

Quotients by finite groups have nice properties

Prop. Let $G \curvearrowright A$. Then $A^G \hookrightarrow A$ is integral and if A is f.t./R, then A^G is f.t./R and hence $A^G \hookrightarrow A$ is finite

$\operatorname{Spec} A^G$ is a good model for the quotient of A by G.

Def. Let $G \curvearrowright U$ an algebraic space. A G-equivariant map $U \to X$ (alg space) is a geometric quotient if

- $U(k)/G \xrightarrow{\sim} X(k)$ $\forall\, k = \bar{k}$
- If $U \to Y$ (alg space) is G-equivariant then

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Y \\ X \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

Prop. $\operatorname{Spec} A \longrightarrow \operatorname{Spec} A^G$ is a geometric quotient

In the proof, one shows $\mathrm{AlgSpace}([\operatorname{Spec} A/G], Y) = \mathrm{AlgSpace}(\operatorname{Spec} A^G, Y)$.

Thm. Let $\mathcal{X}$ be a separated DM stack

Let $x \in \mathcal{X}$ a point w/ geometric stabilizer G_x.

($\operatorname{Spec} \bar{k} \to x$)

$\exists$ affine étale $([\operatorname{Spec} A / G_x], w) \xrightarrow{f} (\mathcal{X}, x)$

inducing an iso $G_w \cong G_x$.

Construction.

Let $(U, u) \to (\mathcal{X}, x)$ an étale representable map from an affine scheme U.

Let $d := |\operatorname{Spec} k \times_{\mathcal{X}} U|$ the degree of the cover.

$U \to \mathcal{X}$ is affine as $\mathcal{X}$ is separated

Let $(U/\mathcal{X})^d := U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U$ d times

So maps $S \to (U/\mathcal{X})^d$ are choices of d sections $U_S \to S$.

Let $(U/\mathcal{X})^d_0$ be the complement of all pairwise diagonals,

So that maps $S \to (U/\mathcal{X})^d_0$ are given by disjoint sections $U_S \to S$.

$S_d \circlearrowright (U/\mathcal{X})^d$ and $(U/\mathcal{X})^d_0$ is S_d-equivariant.

k-points of $[(U/\mathcal{X})^d_0 / S_d]$ are principal S_d-bundles, i.e., finite étale covers of degree d, of $\operatorname{Spec} k$. One such point is the fiber $\operatorname{Spec} k \times_{\mathcal{X}} U$. Let this be $w \in [(U/\mathcal{X})^d_0 / S_d](k)$.

Let $W = (G/x)_0^d$. Then consider $[W/G_x] \to \mathcal{X}$

an étale representable affine map inducing isos on

stabilizers $w \longmapsto x$,