Intro to the moduli stack of curves

§ Intro

Algebro-geometric objects are often parameterized by other algebro-geometric objects.

e.g. $\mathbb{P}^5$ parameterizes conic sections

we call these moduli spaces.

Goal. Construct and analyze the moduli space of smooth genus g $\underbrace{\text{curves}}_{\text{integral, projective}}$ M_g

So M_g must be a "space".

e.g. $\mathbb{P}^5$ has meaningful geometry, informing us about the space of plane conics.

$\operatorname{Spec} k \longrightarrow \mathbb{P}^5 \quad \longleftrightarrow$ conic section $/k$

$\mathbb{P}^1 \longrightarrow \mathbb{P}^5 \quad \longleftrightarrow$ pencil of plane conics

So similarly, we ought to say things like

$\operatorname{Spec} k \longrightarrow M_g \quad \longleftrightarrow$ curve of genus g/k

$\mathbb{P}^1 \longrightarrow M_g \quad \longleftrightarrow$ pencil of curves of genus g

§ Moduli functors

We proceed purely formally.

Def. $\mathcal{M}_g : \mathrm{Sch}^{op} \longrightarrow \mathrm{Set}$

$S \longmapsto \{ \mathcal{C} \to S$ smooth, proper w/ geometric fibers genus g smth curves $\} / \cong$

so $\mathcal{M}_g(k) = \{$ genus g smth curves $/k \} / \cong$

$\mathcal{M}_g(\mathbb{P}^1) = \{ \mathbb{P}^1$ - families of genus g curves $\} / \cong$

e.g. - X a scheme, $\underline{X} : \mathrm{Sch}^{op} \longrightarrow \mathrm{Set}$

$S \longmapsto \mathrm{Sch}(S, X)$

- $VB_{C,r,d}(S) = \{$ vector bundles on $C \times S$ w/ rank r and degree $d \} / \cong$
where C is a smooth curve

- $Gr_{k,n}(S) = \{ \mathcal{O}_S^{n+1} \twoheadrightarrow F \mid F$ locally free of rank $k \} / \cong$

- $BG(S) = \{$ princ. G-bundles on $S \} / \cong$
G an alg. group

Yoneda lemma

Let $\mathcal{C}$ be a category, $X \in \mathrm{Obj}(\mathcal{C})$, $F : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$

Then $\mathrm{Nat}(\underline{X}, F) \xleftrightarrow{\sim} F(X)$

$\eta \longmapsto \eta_X(\mathrm{id}_X)$

$(f \mapsto F(f)(c)) \longleftarrow\!\!\shortmid c$

Thus, we can formally write e.g.

$$\underline{\mathbb{P}^1} \Longrightarrow \mathcal{M}_g$$

as a pencil of curves or

$$\underline{\operatorname{spec} k} \Longrightarrow \mathcal{M}_g$$

as a curve $/k$

Universal families

Yoneda says that representable functors are dominated by id of the representing object.

e.g. $F : Sch^{op} \longrightarrow Set$

$$S \longmapsto \{(L, s_0, \dots, s_n) \mid L \text{ inv. sheaf on } S \text{ globally generated by } s_0, \dots, s_n\}/\cong$$

we know a point in $F(\mathbb{P}^n) \ni (\mathcal{O}(1), x_0, \dots, x_n)$

Thus, we get a map

$$\mathbb{P}^n \Longrightarrow F$$
$$id_{\mathbb{P}^n} \longmapsto (\mathcal{O}(1), x_0, \dots, x_n)$$
$$S \xrightarrow{f} \mathbb{P}^n \longmapsto (f^*\mathcal{O}(1), f^*x_0, \dots, f^*x_n)$$

Fact. This is a nat. iso.

We call $(\mathcal{O}(1), x_0, \dots, x_n)$ the universal family of F.

e.g. In topology we have BG classifying principal G-bundles of a space X. There's a universal family $EG \longrightarrow BG$. For $G = GL_n$, BGL_n classifies rank n vector bundles.

Recall that Chern classes are a map $K_0 \Rightarrow H^*(-;\mathbb{Z})$. K_0 in degree n is repr. by BGL_n, so to get this map it suffices to compute $H^*(BGL_n;\mathbb{Z})$. This is known to be $\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ w/ σ_i the elementary symmetric polynomials. Hence, we've defined Chern classes!

Prop. $\mathcal{M}_g$ is not representable by a scheme.

Pf. Suppose $\mathcal{M}_g \cong \underline{M_g}$. Then there is a universal family $U_g \longrightarrow M_g$

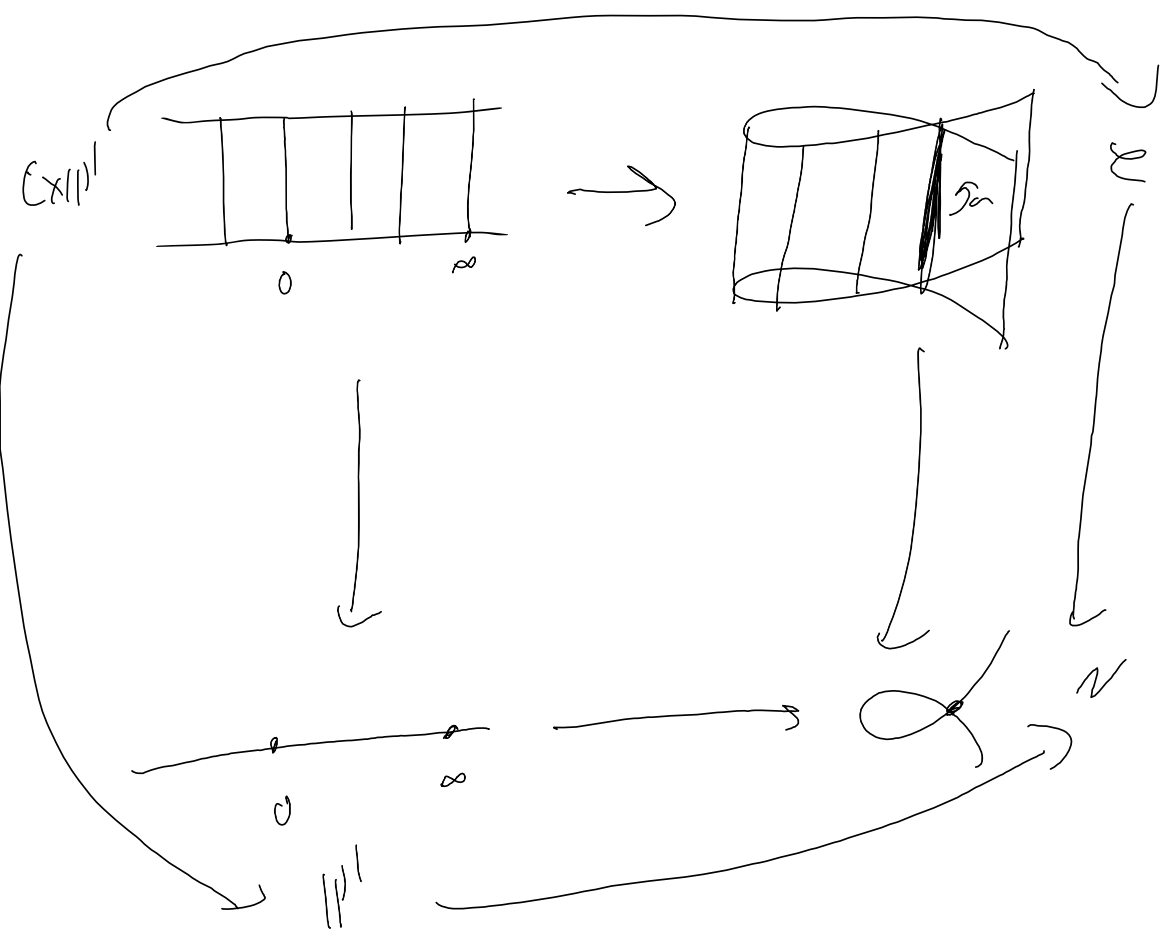
Let's base change everything to $\mathbb{C}$.

Let C be a smth genus g curve with a nontrivial automorphism σ.

$$\text{Let } \mathcal{C} = \frac{C \times \mathbb{P}^1}{(x,0) \sim (\sigma(x), \infty)}$$

$$\downarrow$$

$$N = \mathbb{P}^1 / \{0 \sim \infty\}$$



By universality,
$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U_g \arrow[d] \\ N \arrow[r] & M_g \end{tikzcd}$$
$\begin{matrix}\mathcal{C}\\ \downarrow \\ N\end{matrix}$ is isotrivial, i.e. all fibers are isomorphic to C.

The iso class of C is given by a map $\operatorname{Spec}\mathbb{C} \to M_g$,

so we have a factorization
$$\begin{tikzcd} N \arrow[r] \arrow[d] & M_g \\ \operatorname{Spec}\mathbb{C} \arrow[ur] & \end{tikzcd}$$

Thus

$$\begin{tikzcd}
\mathcal{C} \arrow[rr] \arrow[dr] \arrow[dd] & & U_g \arrow[dd] \\
& \operatorname{Spec}\mathbb{C}\times C \arrow[ur] \arrow[dd] & \\
N \arrow[rr] \arrow[dr] & & M_g \\
& \operatorname{Spec}\mathbb{C} \arrow[ur] &
\end{tikzcd}$$

so $\mathcal{C} \to N$ is trivial $\ast$ $\square$

Rmk. This says that $\mathcal{M}_g$ is not a sheaf in Sch_{Zar}, but representable schemes are Zariski sheaves.

Issue: gluing schemes requires equality on triple cocycles, but sheaves only see double intersections

$$F(\cup U_i) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j) \left(\substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} F(U_i \cap U_j \cap U_k) \right)$$

idea: stacks are sheaves valued in groupoid or another $(2,1)$ category.

§ Examples

i) Let $p \in \mathbb{Q}[x]$.

Def. $\mathcal{H}ilb_n^p : Sch^{op} \longrightarrow Set$

$$S \longmapsto \left\{ \begin{matrix} S \times \mathbb{P}^n \\ \downarrow \\ S \end{matrix} \text{ flat} \;\middle|\; \begin{matrix} \text{fibers are subvarieties} \\ \text{of } \mathbb{P}^n \text{ w/ Hilbert} \\ \text{polynomial } p \end{matrix} \right\} \Big/ \cong$$

Fact. This is representable by a scheme $Hilb_n^p$ which is projective over $\operatorname{Spec} \mathbb{Z}$.

i) $\mathcal{H}ur_{d,g} : Sch^{op} \longrightarrow Set$

$$S \longmapsto \left\{ \begin{matrix} \text{branched} \\ \text{covers} \\ \mathcal{C} \longrightarrow \mathbb{P}^1_S \end{matrix} \;\middle|\; \begin{matrix} \text{w/ branch locus} \\ \text{deg } d \text{ and genus } g \end{matrix} \right\} \Big/ \cong$$

Fact. Repr. by a scheme $Hur_{d,g}$ projective over $\operatorname{Spec} \mathbb{Z}$.

§ Complex analytic cases

$$\begin{aligned} \mathrm{Hur}_{d,g}(\mathbb{C}) &\xrightarrow{\pi} \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1(\mathbb{C}) \\ f &\longmapsto \text{branch locus of } f \end{aligned}$$

This is a finite dominant map,

Restrict to $U \subseteq \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1(\mathbb{C})$ consisting of $\{p_1, \dots, p_{2d+2g-2}\}$ where all p_i are distinct

e.g. if a point has a mult. $> d$, it can't be a branch locus

$\pi^{-1}[U] \longrightarrow U$ is a finite covering, so we make $\pi^{-1}[U]$ into a complex manifold as such.

$$\begin{tikzcd} \mathrm{Hur}_{d,C} \arrow[r] & \mathrm{Hur}_{d,g}(\mathbb{C}) \arrow[r] \arrow[d, "\psi"] & \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1(\mathbb{C}) \\ & C \in \mathcal{M}_g(\mathbb{C}) & \end{tikzcd}$$

$\mathrm{Hur}_{d,C}$ fiber

$f : C \to \mathbb{P}^1 \longmapsto C$

By the above, $\dim \mathrm{Hur}_{d,g}(\mathbb{C}) = \dim \pi^{-1}[U] = \dim U = \dim \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1(\mathbb{C})$

$= 2d+2g-2$

What is $\dim \mathrm{Hur}_{d,C}$?

Given C, how to define $C \longrightarrow \mathbb{P}^1$

- Pick an effective divisor D of degree d

$\mathrm{Sym}^d C$ many

- Given D, pick a section

$h^0(C, \mathcal{O}(D))$ many
$= d - g + 1$ by RR (for $d \geq 2g$)

So $\dim \mathrm{Hur}_{d,C} = d + (d - g + 1)$

$\therefore \dim \mathcal{M}_g(\mathbb{C}) = (2d + 2g - 2) - (d + (d - g + 1))$
$= 3g - 3$

Also, a monodromy analysis of $\mathrm{Hur}_{d,g}(\mathbb{C}) \longrightarrow \mathrm{Sym}^{2d+2g-2} \mathbb{P}^1(\mathbb{C})$ shows connectedness. As it's smooth, it's irreducible.

Thus, $\mathcal{M}_g(\mathbb{C})$ is irreducible.