Permutations, Representations, and Partition Algebras

A Random Walk Through Algebraic Statistics

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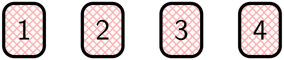
Random Walks on Groups

Consider shuffling a deck of cards in the following way.



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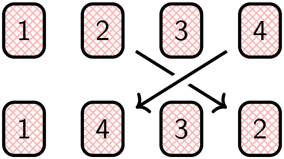

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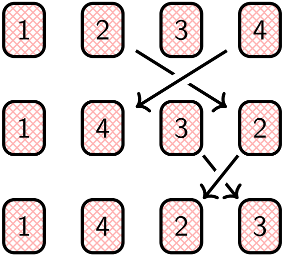

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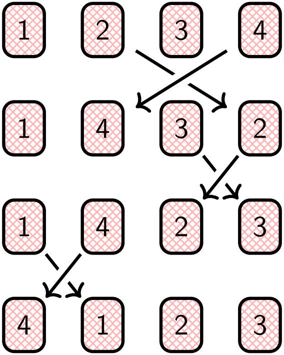

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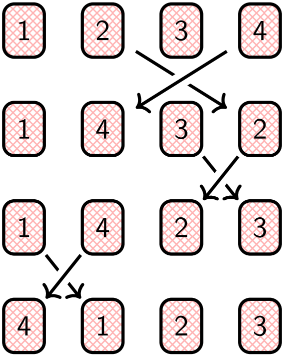

This procedure is called a *random walk* on the symmetric group S_n. I'd like to characterize the resulting permutation using *representation theory*.

Permutations

A *permutation* is a bijective function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. For example

$$\sigma : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix}$$

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One line notation:

$$\sigma = 324165.$$

Cycle notation:

$$\sigma = (1\,3\,4)(2)(5\,6).$$

The *symmetric group* S_n is the group of permutations of $\{1, 2, \ldots, n\}$ under function composition.

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For example $\sigma = 4132$ has 4 inversions:

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A *permutation statistic* is a function defined on S_n encoding information about permutations. We can define a statistic $\textsc{Inv} : S_n \to \mathbb{Z}$ by

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Characterize distributions of permutation statistics (like $\textsc{Inv}$) sampled via random walks.

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One way to characterize a statistic is through its *moments*.

$$d^{\text{th}} \text{ moment of } X = \mathbb{E}(X^d)$$

For example,

- 1^{st} moment $\rightarrow$ expected value
- 2^{nd} moment $\rightarrow$ information about variance
- 3^{rd} moment $\rightarrow$ information about skewness

Class functions on S_n

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$\sigma \in S_3$	(1)(2)(3)	(12)(3)	(13)(2)	(23)(1)	(123)	(132)
$\mathrm{INV}(\sigma)$	0	1	3	1	2	2
$\overline{\mathrm{INV}}(\sigma)$	0	5/3	5/3	5/3	2	2

The *mean statistic* $\overline{\mathrm{INV}}$ of a statistic is obtained by averaging INV over permutations with the same cycle type.

Past Results

Theorem (Rodrigues, 1839)

Let $\sigma \in S_n$ be a permutation, and let a_k be the number of k-cycles in σ. Then

$$\overline{\mathrm{INV}}(\sigma) = \frac{3n^2 - n - a_1^2 - 2a_1 n + a_1 + 2a_2}{12}$$

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Theorem (Gaetz and Ryba, 2021)

For any $d \in \mathbb{N}$, $\overline{\mathrm{INV}^d}$ is a polynomial of degree at most $2d$ in the variables $n, a_1, \ldots a_{2d}$.

This proof is based on the representation theory of the *partition algebra*, and is non-constructive.

A Polynomial

Implementing Gaetz and Ryba's argument computationally, I found the polynomial for $\overline{\textsc{Inv}^2}$.

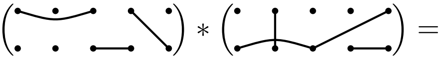
Proposition (S.)

$$\overline{\textsc{Inv}^2}(\sigma) = \frac{1}{720}(5a_1^4 + 20a_1^3 n - 14a_1^3 - 12a_1^2 a_2 + 50a_1^2 n^2 - 90a_1^2 n - 25a_1^2 - 24a_1 a_2 n + 12a_1 a_2 - 24a_1 a_3 + 60a_1 n^3 - 126a_1 n^2 + 94a_1 n + 98a_1 + 60a_2^2 - 20a_2 n^2 + 108a_2 n - 124a_2 - 24a_3 n - 48a_3 - 24a_4 + 45n^4 - 130n^3 + 111n^2 - 98n).$$

Where did this polynomial come from?

The *partition algebra* is an associative algebra whose elements are diagrams like the ones shown. It's representations are closely related to those of the symmetric group, so many questions about the symmetric group can be rephrased as questions about the partition algebra and vice versa.

Multiplication in $\mathrm{Par}_k(n)$:



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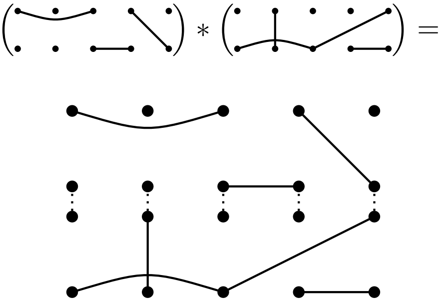

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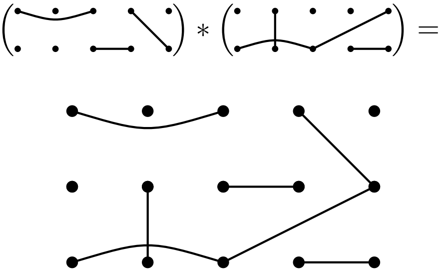

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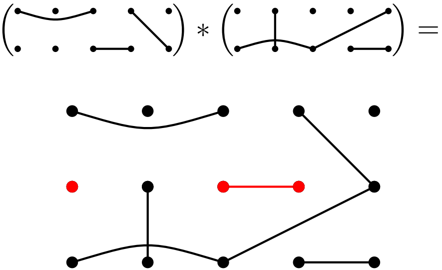

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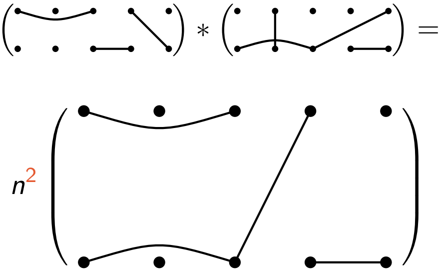

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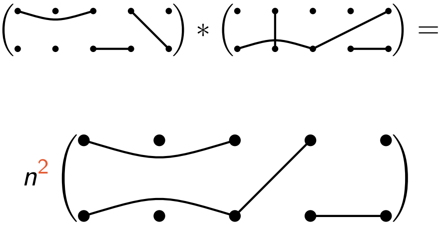

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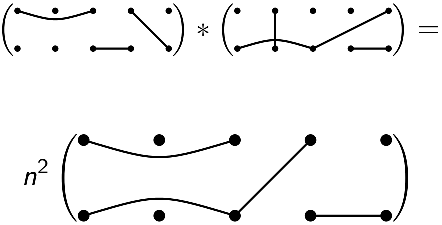

Results

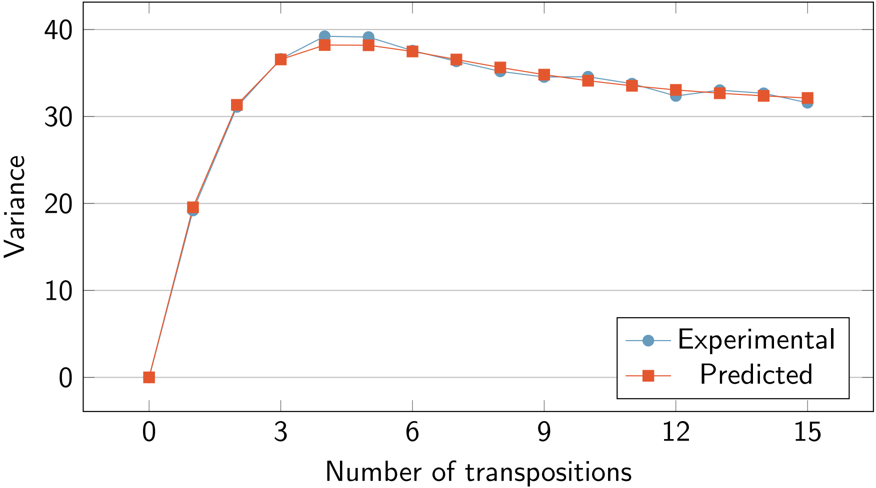

Figure: Variance in the number of inversions in the product of t random transpositions from S_{10}.

Acknowledgements

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