Permutations, Representations, and Partition Algebras: A Random Walk through Algebraic Statistics

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Math 197: Senior Thesis

Introduction
Consider shuffling a deck of cards pairwise: pick any two cards, swap them, and repeat. How can we characterize the resulting distribution?

Using representation theory, we can compute expected values of permutation statistics after $t$ steps of a random walk like this one. I set out to find analogous formulas for higher order information like variance. We can characterize variance by understanding the behavior of squares of permutation statistics.

Background
An inversion in a permutation is a pair $i < j$ such that $i(j) > i(j)$. For example, 342156 ∈ $S_6$ has 5 inversions, which are 342156, 342156, 342156, 342156, 342156.

Let $\text{INV}(\sigma) =$ number of inversions in $\sigma$.

INV measures how much ‘reversing’ the permutation does.

Methods
The partition algebra $P_k(n)$ is an associative algebra whose elements are $(k,k)$-set partition diagrams like the one shown below.

These diagrams can be multiplied by stacking them vertically:

$$\begin{align*}
\begin{array}{c}
\cdots \\
\end{array}
\begin{array}{c}
\cdots \\
\end{array}
\end{align*}
\right)\left( \\
\right) = n^2\begin{align*}
\begin{array}{c}
\cdots \\
\end{array}
\begin{array}{c}
\cdots \\
\end{array}
\end{align*}.

The partition algebra $P_k(n)$ exhibits Schur-Weyl duality with the symmetric group $S_n$.

Extending the methods of [1] and using the previous equation, I was able to prove that for any $\sigma \in S_n$,

$$\text{INV}(\sigma) = \frac{1}{2}(98m_1 - 25m_1^2 - 14m_1 + 5m_1^4 - 124m_2 + 12m_1m_2 - 12m_1m_2 - 60m_2^2 - 48m_3 - 24m_1m_3 - 24m_2^2 - 98n - 34m_1n - 30m_1n^2 - 20m_1n - 12m_2n - 24m_1m_2 - 24m_3 - 9n^2 + 54m_1n^2 - 10m_1n^2 + 100m_2n^2 - 10m_3^2 - 60m_1n^3 + 45n^4).$$

This polynomial allows us to find exact formulas for variances in the number of inversions in permutations sampled via random walks.

Results

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References

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