Math 197: Senior Thesis

Permutations, Representations, and Partition Algebras: A Random Walk through Algebraic Statistics

Ian Shors

Introduction

Consider shuffling a deck of cards pairwise; pick any two cards, swap them, and repeat. How can we characterize the resulting distribution?

Using representation theory, we can compute expected values of permutation statistics after t steps of a random walk like this one. I set out to find analogous formulas for higher order information like variance. We can characterize variance by understanding the behavior of *squares* of permutation statistics.

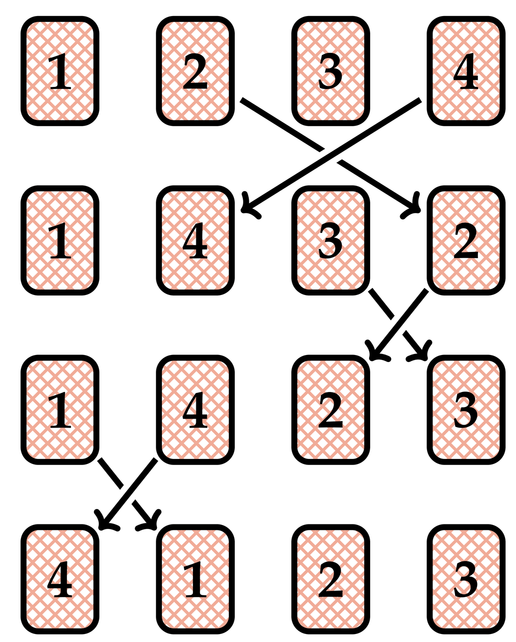


Figure 1: An example application of the pairwise shuffle on to a deck of 4 cards.

Background

An *inversion* in a permutation is a pair $i < j$ such that $\sigma(i) > \sigma(j)$. For example, $342156 \in S_6$ has 5 inversions, which are

$$342156,\ 342156,\ 342156,\ 342156,\ \text{and } 342156.$$

Let

$$\text{INV}(\sigma) = \text{number of inversions in } \sigma.$$

INV measures how much 'reversing' the permutation does.

Theorem [2]

Let $\sigma \in S_n$ be a permutation, and let m_i be the number of i-cycles in σ. Then

$$\overline{\text{INV}}(\sigma) = \frac{3n^2 - n - m_1^2 - 2m_1n + m_1 + 2m_2}{12}.$$

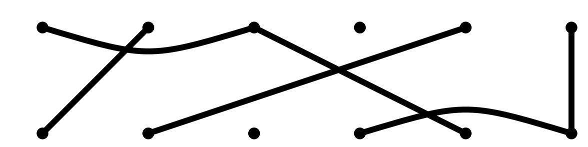
Here $\overline{\text{INV}}$ denotes the average of INV over conjugacy classes of S_n. Gaetz and Ryba [1] proved that analogous polynomials exist for higher moments of INV.

Corollary of [1]

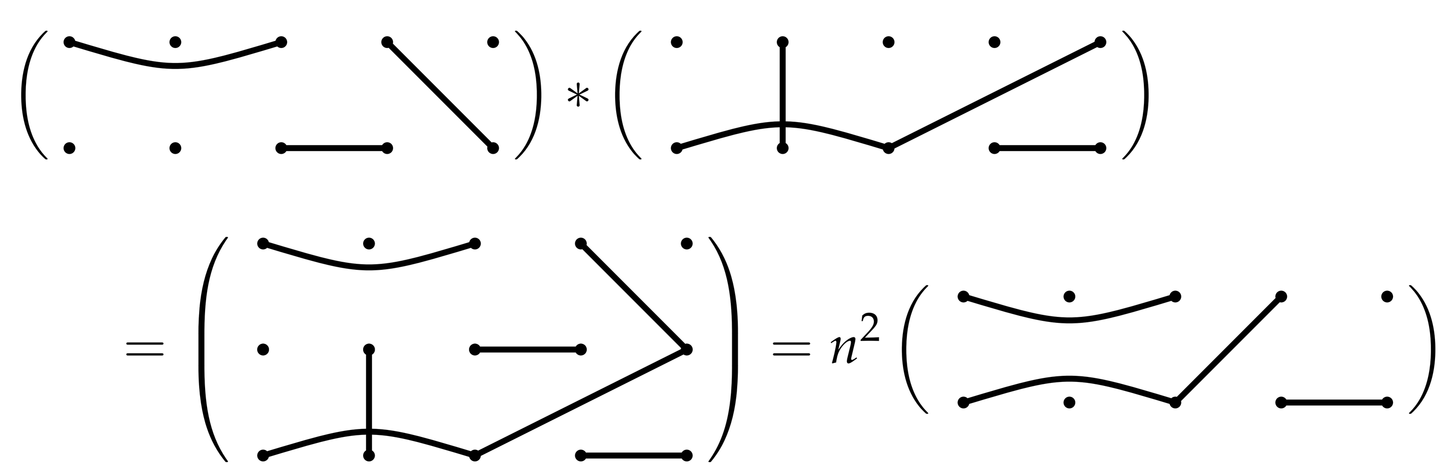
For any positive integer d, $\overline{\text{INV}^d}$ is a polynomial of degree at most $2d$ in the variables $n, m_1, \ldots, m_{2d}$, where n has degree 1 and m_i has degree i.

Methods

The *partition algebra* $P_k(n)$ is an associative algebra whose elements are (k,k)*-set partition diagrams* like the one shown below.



These diagrams can be multiplied by stacking them vertically:



The partition algebra $P_n(k)$ exhibits *Schur-Weyl duality* with the symmetric group S_n, meaning their representation theories are very closely linked.

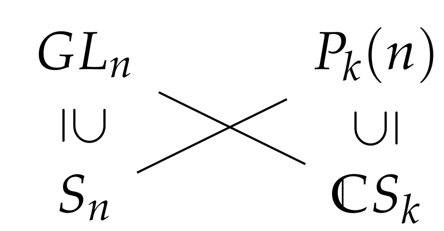


Figure 2: Schur-Weyl duality relationships S_k and GL_n, and between S_n and $P_k(n)$

Using this relationship, we can convert questions about INV on S_n into character calculations over $P_k(n)$. In particular, we may write $\overline{\text{INV}^d}$ as a linear combination of traces of the images of partition algebra elements under some representation $\Phi_{k,n}$ of $P_k(n)$.

$$\overline{\text{INV}^d}(\sigma) = \sum_{\alpha \in \Gamma(12,d)} \sum_{\beta \in \Gamma(21,d)} \sum_{P} a_P^{\alpha,\beta} \text{tr}_{V^{\otimes 2d}}(\Phi_{2d,n}(P)\sigma).$$

Acknowledgments

I'm immensely grateful to my advisor, Professor Michael Orrison, for his invaluable advice and encouragement during this project. I'm also very thankful to my reader, Professor Gizem Karaali, and my colleagues Maxwell Thum, Hannah Friedman, Tomás Aguilar-Fraga, and Kausik Das for their helpful comments and support.

Results

Extending the methods of [1] and using the previous equation, I was able to prove that for any $\sigma \in S_n$,

$$\overline{\text{INV}^2}(\sigma) = \tfrac{1}{720}(98m_1 - 25m_1^2 - 14m_1^3 + 5m_1^4 - 124m_2 + 12m_1m_2 - 12m_1^2m_2 + 60m_2^2 - 48m_3 - 24m_1m_3 - 24m_4 - 98n + 34m_1n - 30m_1^2n + 20m_1^3n - 12m_2n - 24m_1m_2n - 24m_3n - 9n^2 + 54m_1n^2 - 10m_1^2n^2 + 100m_2n^2 - 10n^3 - 60m_1n^3 + 45n^4).$$

This polynomial allows us to find exact formulas for variances in the number of inversions in permutations sampled via random walks.

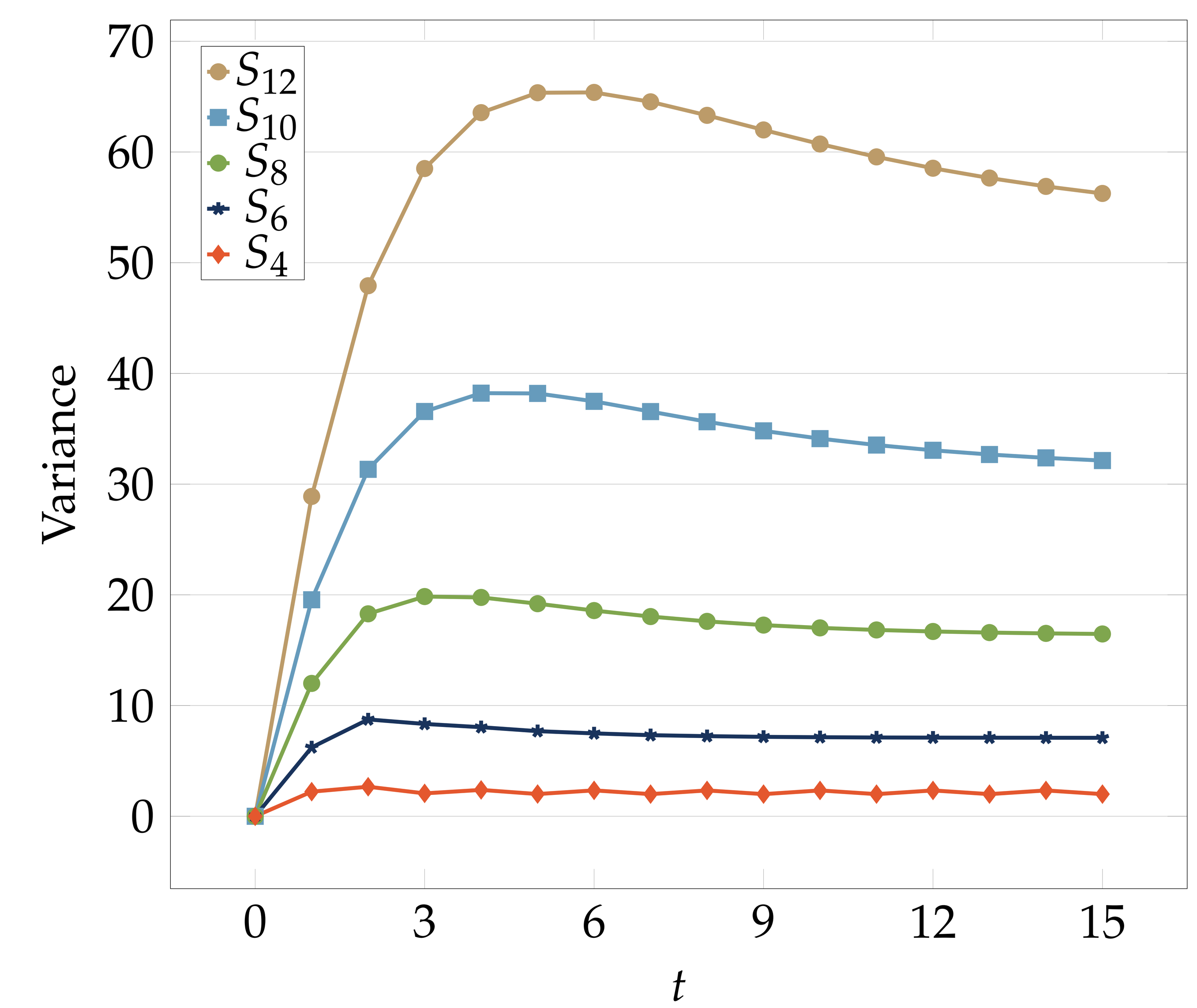


Figure 3: Variance in the number of inversions of t random transpositions from S_n, for $n = 4, 6, 8, 10$ and 12.

References

[1] C. Gaetz and C. Ryba. Stable characters from permutation patterns. *Selecta Mathematica*, 27(4):70, 2021.

[2] O. Rodrigues. Note sur les inversions, ou dérangements produits dans les permutations. *Journal de Mathématiques Pures et Appliquées*, 1e série, 4, 1839.

Advisor: Michael Orrison
Reader: Gizem Karaali

Email: ishors@hmc.edu
Website: https://sites.google.com/g.hmc.edu/ishors