Spectral Analysis of the Kohn Laplacian on Lens Spaces

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Joint work with Elena Kim (MIT) and Yunus E. Zeytuncu (Dearborn).
Funded by the National Science Foundation (DMS-1950102 and DMS-1659203).

Weyl's Law: Spectrum (Eigenvalues) and Geometry

- $\Omega \subseteq \mathbb{R}^d$ is a bounded domain.
- $N(\lambda)$ is the number of eigenvalues less than λ (counting their multiplicities) of the standard Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}.$$

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- Weyl's law:

Theorem (Weyl-1911)

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{\mathrm{vol}(\Omega)}{2^d \pi^{d/2} \Gamma\left(\frac{d}{2}+1\right)}.$$

- One can generalize Weyl's law to Riemannian manifolds.

CR Manifold

- CR stands for either Cauchy-Riemann or complex-real.
- Smooth manifolds but with some complex structure:

$$T_p(M) = H_p(M) \oplus X_p(M),$$

where $H_p(M)$ is the complex tangent space and $X_p(M)$ is the real tangent space.

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Definition

Let M be a smooth manifold. $M \subseteq \mathbb{C}^n$ is a CR manifold if and only if $\dim H_p(M)$ is independent of p.

- **Example**: any hypersurface in $\mathbb{C}^n$, like $S^{2n-1} \subseteq \mathbb{C}^n \simeq \mathbb{R}^{2n}$.
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- **Example**: any complex manifold.
- Every CR manifold comes with a Kohn Laplacian, $\Box_b$ (CR version of standard Laplacian).

- **Goal**: Analog of Weyl's law for the Kohn Laplacian on spheres S^{2n-1},

$$\square_b : L^2\left(S^{2n-1}\right) \to L^2\left(S^{2n-1}\right).$$

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$$\Box_b : L^2\left(S^{2n-1}\right) \to L^2\left(S^{2n-1}\right).$$

- $L^2\left(S^{2n-1}\right)$ has spectral decomposition,

$$L^2\left(S^{2n-1}\right) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}_{p,q}\left(S^{2n-1}\right).$$

- Folland: Eigenvalue associated with $\mathcal{H}_{p,q}\left(S^{2n-1}\right)$ is $2q\left(p+n-1\right)$.
- $\dim \mathcal{H}_{p,q}\left(S^{2n-1}\right) = \binom{n+p-1}{p}\binom{n+q-1}{q} - \binom{n+p-2}{p-1}\binom{n+q-2}{q-1}$.

Dearborn REU 2020 (cont.)

Theorem (BGS$^+$21)

Let $N(\lambda)$ be the eigenvalue counting function (with multiplicity) for $\Box_b$ on $L^2\left(S^{2n-1}\right)$. Then,

$$\lim_{n\to\infty}\frac{N\left(\lambda\right)}{\lambda^n}=\operatorname{vol}\left(S^{2n-1}\right)\frac{n-1}{n\left(2\pi\right)^n\Gamma\left(n+1\right)}\int_{-\infty}^{\infty}\left(\frac{x}{\sinh x}\right)^n e^{-(n-2)x}\,dx.$$

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In comparison to the standard Laplacian:

Theorem (Weyl-1911)

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{n-1/2}} = \frac{\operatorname{vol}\left(S^{2n-1}\right)}{2^{2n-1}\pi^{n-1/2}\Gamma\left(n+\frac{1}{2}\right)}.$$

Lens Spaces

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- $k \in \mathbb{N}$, $\zeta = e^{2\pi i/k}$.
- $l_1, \ldots, l_n$ relatively prime to k.
- $g \in U(n)$, $gz_j = \zeta^{l_j} z_j$

$$g = \begin{bmatrix} \zeta^{l_1} & 0 & \cdots & 0 \\ 0 & \zeta^{l_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta^{l_n} \end{bmatrix}.$$

- The lens space denoted by $L(k; l_1, \ldots, l_n) = L(k; \vec{l})$ is the **quotient** of S^{2n-1} by $G = \langle g \rangle$.

The Goal

Theorem (2021)

Given the lens space $L(k; l_1, \dots, l_n)$, we denote the eigenvalue counting function for $\Box_b$ on the lens space by $N_L(\lambda)$, and the eigenvalue counting function for S^{2n-1} by $N(\lambda)$. We have

$$\lim_{\lambda \to \infty} \frac{N_L(\lambda)}{N(\lambda)} = \frac{1}{k}.$$

$\mathcal{H}_{p,q}^G$

G acts naturally on $L^2(S^{2n-1})$ by precomposition

$$g * f = f \circ g.$$

- $g * \mathcal{H}_{p,q}(S^{2n-1}) \subseteq \mathcal{H}_{p,q}(S^{2n-1})$.
- Denote the set of elements of $\mathcal{H}_{p,q}(S^{2n-1})$ that are fixed under the action of G by $\mathcal{H}_{p,q}^G$.

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- We have

$$L^2(L(k; l_1, \ldots, l_n)) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}_{p,q}^G.$$

- The eigenvalue for $\mathcal{H}_{p,q}^G$ for $\Box_b$ on the lens space is the same as the eigenvalue on the sphere, $2q(p+n-1)$.
- Want to compute $\dim \mathcal{H}_{p,q}^G$.

Basis for $\mathcal{H}_{p,q}(S^{2n-1})$

- For $\alpha \in \mathbb{N}^n$, define $|\alpha| = \sum_{j=1}^{n} \alpha_j$.
- Denote, for $\alpha, \beta \in \mathbb{N}^n$

$$\overline{D}^{\alpha} = \frac{\partial^{|\alpha|}}{\partial \overline{z}_1^{\alpha_1} \cdots \partial \overline{z}_n^{\alpha_n}}, \; D^{\beta} = \frac{\partial^{|\beta|}}{\partial z_1^{\beta_1} \cdots \partial z_n^{\beta_n}}.$$

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- For $p, q \in \mathbb{N}$

$$\left\{ \overline{D}^{\alpha} D^{\beta} |z|^{2-2n} : \; |\alpha| = p, \; |\beta| = q, \; \alpha_1 = 0 \text{ or } \beta_1 = 0 \right\}$$

is a basis for $\mathcal{H}_{p,q}(S^{2n-1})$.

Invariant Basis Elements of $\mathcal{H}_{p,q}(S^{2n-1})$

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$$g * f_{\alpha,\beta} = \zeta^{\sum_{j=1}^{n} l_j(\alpha_j - \beta_j)} f_{\alpha,\beta}.$$
- So the dimension of $\mathcal{H}_{p,q}^{G}$ is equal to the number of solutions to the following system:
$$|\alpha| = p,\ |\beta| = q;$$
$$\alpha_1 = 0 \text{ or } \beta_1 = 0;$$
$$\sum_{j=1}^{n} l_j(\alpha_j - \beta_j) \equiv 0 \text{ mod } k.$$

The Problem

- Eigenvalue of $\mathcal{H}_{p,q}(S^{2n-1})$ for $\Box_b$ is $2q(p+n-1)$.
- $N(\lambda),\ N_L(\lambda)$ number of positive eigenvalues of $\Box_b$ on S^{2n-1}, $L(k;\vec{l})$ (counting multiplicities) less than λ

$$N_L(\lambda) = \sum_{\substack{p\geq 0, q>0, \\ 0<2q(p+n-1)\leq\lambda}} \dim \mathcal{H}^G_{p,q}.$$

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- So $N_L(\lambda)$ is equal to the number of solutions to the following system

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$$0 < 2|\alpha|(|\beta| + n - 1) \leq \lambda$$
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- Our calculations yield: $\displaystyle\lim_{\lambda \to \infty} \frac{N_L(\lambda)}{N(\lambda)} = \frac{1}{k}.$

Can You Hear the Shape of a Lens Space?

Theorem (Ikeda-Yamamoto)

Two lens spaces $L(k; l_1, \ldots, l_n)$ and $L(k'; l'_1, \ldots, l'_n)$ are isometric as Riemannian manifolds if and only if

- $k = k'$, *and*
- *there exists an integer a and a permutation σ such that* $(l'_1, l'_2, \ldots, l'_n) \equiv (\pm a l_{\sigma(1)}, \pm a l_{\sigma(2)}, \ldots, \pm a l_{\sigma(n)}) \pmod{k}$.

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Theorem (Ikeda-Yamamoto, Main Theorem)

Two 3-dimensional lens spaces have the same spectrum for the real Laplacian if and only if they are isometric as Riemannian manifolds.

- Example: $L(3; 1, 1)$ and $L(3; 1, 2)$ have the same real spectrum.

Can You Hear the Shape of a CR Lens Space?

- CR isometry is a stronger condition than Riemannian isometry.

Theorem (2021)

Let $L(k; l_1, \ldots, l_n)$ *and* $L(k', l'_1, \ldots, l'_n)$*. If*

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then the spaces are CR isometric.

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- What happens with $L(3; 1, 1)$ and $L(3; 1, 2)$?

$$\text{Spectrum of } \Box_b \text{ on } L(3;1,1) = \{0, 4, 6, 10, 12, 16, 18, \ldots\}$$
$$\text{Spectrum of } \Box_b \text{ on } L(3;1,2) = \{0, 4, 6, 8, 10, 12, 14, 16, \ldots\}$$

- **Goal**: Characterize 3-dimensional lens spaces up to CR isometries via spectra.

Can You Hear the Shape of a CR Lens Space?

Conjecture

Two 3-dimensional lens spaces have the same Kohn spectrum if and only if they are CR isometric.

- This would mean the Kohn Laplacian is "more sensitive" than the standard Laplacian.

Some Partial Results

Proposition (2021)

If $L(k; l_1, l_2, \ldots, l_n)$ is isospectral to $L(k'; l'_1, l'_2, \ldots, l'_n)$ with respect to $\Box_b$, then $k = k'$.

- Analog of Weyl's law

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- Analog of Weyl's law

Theorem (2021)

Let k be a prime. If $L(k; l_1, l_2)$ and $L(k; l'_1, l'_2)$ are isospectral with respect to $\Box_b$, then they are CR isometric.

- Generating function approach

Generating Function Approach

- Given a lens space $L(k; l_1, l_2, \ldots, l_n)$, we define a generating function

$$F(z,w) = \sum_{p,q \geq 0} (\dim \mathcal{H}^G_{p,q}) z^p w^q.$$

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- Using tools from group representation theory, we obtain the closed form

$$F(z,w) = \frac{1}{k} \sum_{m=0}^{k-1} \frac{1-zw}{\prod_{i=1}^{n}(1-\zeta^{m\ell_i}z)(1-\zeta^{-m\ell_i}w)}$$

  where $\zeta = e^{2\pi i/k}$.

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  where $\zeta = e^{2\pi i/k}$.

- This equivalence relates spectral information $(\dim \mathcal{H}^G_{p,q})$ to information about the geometry of the lens space (k and $\vec{l}$).

Outline of Argument

CR isospectral

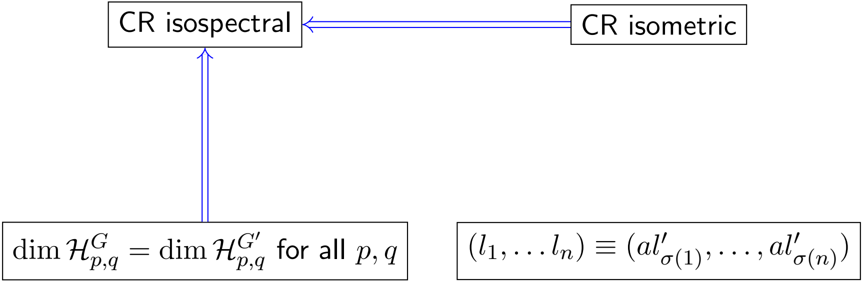
CR isometric

$$\dim \mathcal{H}_{p,q}^{G} = \dim \mathcal{H}_{p,q}^{G'} \text{ for all } p, q$$

$$(l_1, \dots l_n) \equiv (al'_{\sigma(1)}, \dots, al'_{\sigma(n)})$$

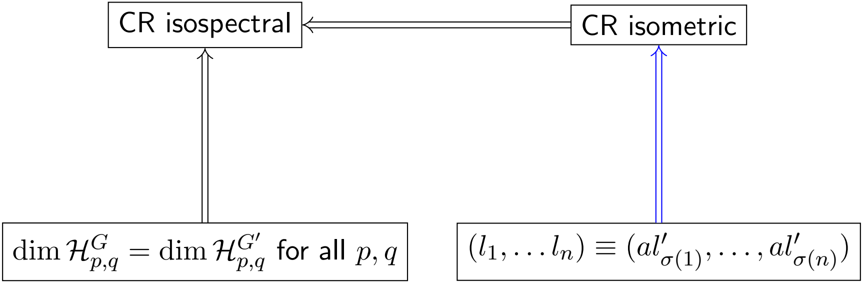
- Fix two lens spaces $L(k, \ell_1, \dots, \ell_n)$ and $L(k, \ell'_1, \dots, \ell'_n)$.
- We would like to show that if two lens spaces are CR isospectral, they are CR isometric.

Outline of Argument



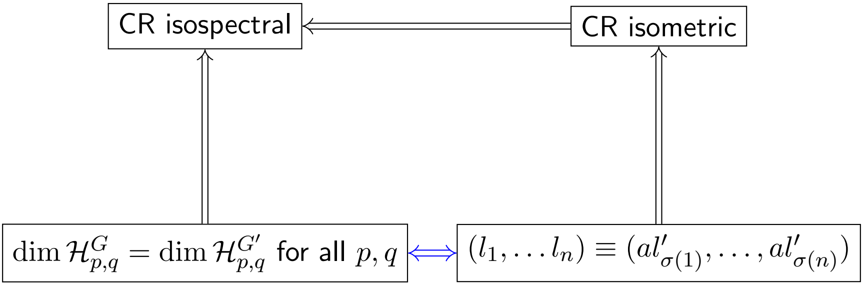
- These directions quickly follow from the definitions.

Outline of Argument



- We can show this direction by extending a result from Ikeda and Yamamoto.

Outline of Argument



- This equivalence comes from the generating function

$$\sum_{p,q\geq 0} (\dim \mathcal{H}_{p,q}^{G}) z^p w^q = \frac{1}{k} \sum_{m=0}^{k-1} \frac{1 - zw}{\prod_{i=1}^{n}(1 - \zeta^{m\ell_i} z)(1 - \zeta^{-m\ell_i} w)}.$$

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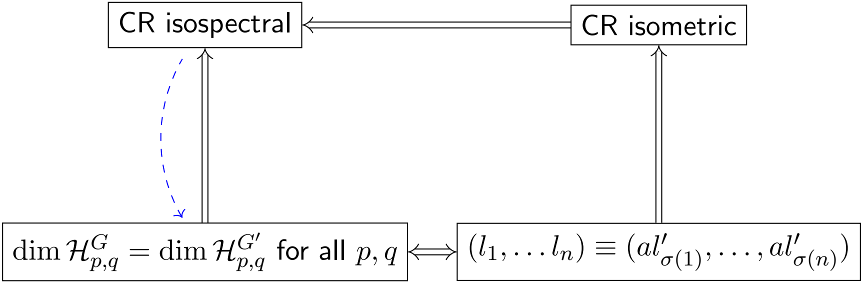

- We have shown the dotted arrow in the case when $n = 2$ and k is prime.
- We have conjectured that it is true for all k when $n = 2$.

Acknowledgements

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- and the rest of the Dearborn REU!