UNREACHABILITY OF INDUCTIVE-LIKE POINTCLASSES IN $L(\mathbb{R})$

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Abstract. In [Hjo96], Hjorth proved from $ZF + AD + DC$ that there is no sequence of distinct $\boldsymbol{\Sigma}^1_2$ sets of length $\boldsymbol{\delta}^1_2$. [Sar22] extends Hjorth's technique to show there is no sequence of distinct $\boldsymbol{\Sigma}^1_{2n}$ sets of length $\boldsymbol{\delta}^1_{2n}$. Sargsyan conjectured an analogous property is true for any regular Suslin pointclass in $L(\mathbb{R})$ — i.e. if κ is a regular Suslin cardinal in $L(\mathbb{R})$, then there is no sequence of distinct κ-Suslin sets of length κ^+ in $L(\mathbb{R})$. We prove this in the case that the pointclass $S(\kappa)$ is inductive-like.

§1. Introduction.

Definition 1.1. *For a boldface pointclass $\boldsymbol{\Gamma}$, we say λ is $\boldsymbol{\Gamma}$-reachable if there is a sequence of distinct $\boldsymbol{\Gamma}$ sets of length λ and λ is $\boldsymbol{\Gamma}$-unreachable if λ is not $\boldsymbol{\Gamma}$-reachable.*

The problem of unreachability is to determine the minimal λ which is $\boldsymbol{\Gamma}$-unreachable for each pointclass $\boldsymbol{\Gamma}$. As this problem is trivial assuming the axiom of choice, unreachability is exclusively studied under determinacy assumptions. Under AD, unreachability yields an interesting measure of the complexity of a pointclass. An early result in this area is Harrington's theorem that there is no injection of ω_1 into any pointclass strictly below the pointclass of Borel sets in the Wadge hierarchy (see [Har78]).

Theorem 1.2 (Harrington). *If $\beta < \omega_1$, then ω_1 is $\boldsymbol{\Pi}^0_{\boldsymbol{\beta}}$-unreachable.*

A recent application of Harrington's theorem was the resolution of the decomposability conjecture by Marks and Day (see [DM21]).

Prior work on unreachability has focused on levels of the projective hierarchy. Kechris gave a lower bound on the complexity of the pointclass needed to reach $\boldsymbol{\delta}^1_{2n+2}$ (see [Kec78]).

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Theorem 1.3 (Kechris). *Assume $ZF + AD + DC$. $\boldsymbol{\delta}^1_{2n+2}$ is $\boldsymbol{\Delta}^1_{2n+1}$-unreachable.*

In [Kec78], Kechris conjectured his own result could be strengthened to $\boldsymbol{\delta}^1_{2n+2}$ is $\boldsymbol{\Delta}^1_{2n+2}$-unreachable. He also made a second, stronger conjecture that $\boldsymbol{\delta}^1_{2n+2}$ is $\boldsymbol{\Sigma}^1_{2n+2}$-unreachable. Jackson proved the former in [Jac90].

Theorem 1.4 (Jackson). *Assume $ZF + AD + DC$. $\boldsymbol{\delta}^1_{2n+2}$ is $\boldsymbol{\Delta}^1_{2n+2}$-unreachable.*

[Jac90] also made progress on Kechris's second conjecture by showing there is no strictly increasing sequence of $\boldsymbol{\Sigma}^1_{2n+2}$ sets of length $\boldsymbol{\delta}^1_{2n+2}$. In fact, Jackson and Martin proved the following more general theorem.

Theorem 1.5 (Jackson). *Assume $ZF+AD+DC$. Suppose κ is a Suslin cardinal, and κ is either a successor cardinal or a regular limit cardinal. Then there is no strictly increasing (or strictly decreasing) sequence $\langle A_\alpha : \alpha < \kappa^+\rangle$ contained in $S(\kappa)$.*

But the resolution of Kechris's second conjecture eluded the traditional techniques of descriptive set theory. Hjorth pioneered the use of inner model theory in this area to resolve one case of Kechris's second conjecture (see [Hjo96]).

Theorem 1.6 (Hjorth). *Assume $ZF+AD+DC$. $\boldsymbol{\delta}^1_2$ is $\boldsymbol{\Sigma}^1_2$-unreachable.*

Kechris also pointed out the following corollary of Hjorth's result.

Corollary 1.7. *Assume $ZF + AD + DC$. A $\boldsymbol{\Pi}^1_2$ equivalence relation has either $2^{\aleph_0}$ or $\leq \aleph_1$ equivalence classes.*

Hjorth's proof of Theorem 1.6 involved an application of the Kechris-Martin Theorem, which precluded an easy generalization of his technique to other projective pointclasses. The rest of Kechris's second conjecture survived another two decades, until Sargsyan found a modification of Hjorth's proof which generalized to the rest of the projective hierarchy (see [Sar22]).

Theorem 1.8 (Sargsyan). *Assume $ZF + AD + DC$. $\boldsymbol{\delta}^1_{2n+2}$ is $\boldsymbol{\Sigma}^1_{2n+2}$-unreachable.*

The following result of Kechris shows Sargsyan's theorem is optimal.

Theorem 1.9 (Kechris). *Assume $ZF+AD+DC$. Suppose κ is a Suslin cardinal. Then there is a strictly increasing sequence $\langle A_\alpha : \alpha < \kappa\rangle$ contained in $S(\kappa)$.*

Sargsyan's theorem resolves the problem of unreachability for every level of the projective hierarchy. He conjectured an analogous result holds for every regular Suslin pointclass.

CONJECTURE 1.10 (Sargsyan). *Assume AD^+. Suppose κ is a regular Suslin cardinal. Then κ^+ is $S(\kappa)$-unreachable.*

Below, we prove part of Conjecture 1.10.

THEOREM 1.11. *Assume $ZF + AD + DC + V = L(\mathbb{R})$. Suppose κ is a regular Suslin cardinal and $S(\kappa)$ is inductive-like. Then κ^+ is $S(\kappa)$-unreachable.*

$ZF + AD + DC + V = L(\mathbb{R})$ implies AD^+, so Theorem 1.11 is a special case of Conjecture 1.10. Theorem 1.9 demonstrates this is the optimal result for inductive-like pointclasses.

Let $\mathbf{\Gamma} = S(\kappa)$ for κ as in Theorem 1.11. Then $\kappa = \boldsymbol{\delta}_{\mathbf{\Gamma}}$. $ZF + AD + DC + V = L(\mathbb{R})$ also implies any inductive-like pointclass $\mathbf{\Gamma}$ is of the form $S(\kappa)$ for some regular Suslin cardinal κ. So an equivalent formulation of Theorem 1.11 is the following:

THEOREM 1.12. *Assume $ZF + AD + DC + V = L(\mathbb{R})$. Suppose $\mathbf{\Gamma}$ is an inductive-like pointclass. Then $\boldsymbol{\delta}^+_{\mathbf{\Gamma}}$ is $\mathbf{\Gamma}$-unreachable.*

Our proof of Theorem 1.12 extends the inner model theory approach pioneered in [Hjo96]. Our technique also gives an alternative proof of Theorem 1.8.

§2. Background. We will assume the reader is familiar with the basics of descriptive set theory espoused in [Mos09] and the theory of iteration strategies for premice covered in [Ste09]. The rest of the necessary background is covered below. In Section 2.1, we summarize Steel's classification of the scaled pointclasses and Suslin pointclasses in $L(\mathbb{R})$. Section 2.2 reviews the relationship between Woodin cardinals and iteration trees. Two inner model constructions are covered in Sections 2.3 and 2.4. In Section 2.5, we review results from the core model induction demonstrating the existence of mice corresponding to inductive-like pointclasses in $L(\mathbb{R})$.

2.1. The Pointclasses of $L(\mathbb{R})$. We will assume for this section $ZF + DC + AD + V = L(\mathbb{R})$. All of the results in this section are due to Steel and are proven outright or else implicit in [Ste83].

The boldface pointclasses we are interested in all appear in a hierarchy we will now define. If $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ are non-selfdual pointclasses, say $\{\mathbf{\Gamma}, \mathbf{\Gamma^c}\} <_w \{\mathbf{\Lambda}, \mathbf{\Lambda^c}\}$ if $\mathbf{\Gamma} \subset \mathbf{\Lambda} \cap \mathbf{\Lambda^c}$. This is a wellordering by Wadge's Lemma. For $\alpha < \Theta$, consider the αth pair $\{\mathbf{\Gamma}, \mathbf{\Gamma^c}\}$ in this wellordering such that $\mathbf{\Gamma}$ or $\mathbf{\Gamma^c}$ is closed under projection. Let $\mathbf{\Sigma}^1_{\boldsymbol{\alpha}}$ denote whichever of the two is closed under projection — if both are, $\mathbf{\Sigma}^1_{\boldsymbol{\alpha}}$ denotes whichever has the separation property. Let $\mathbf{\Pi}^1_{\boldsymbol{\alpha}} = (\mathbf{\Sigma}^1_{\boldsymbol{\alpha}})^c$.

For any pointclass $\mathbf{\Gamma}$, we define

$$\mathbf{\Delta_\Gamma} = \mathbf{\Gamma} \cap \mathbf{\Gamma^c} \text{ and}$$
$$\boldsymbol{\delta}_\mathbf{\Gamma} = sup\{|\leq^*| : \leq^* \text{ is a prewellordering in } \mathbf{\Delta_\Gamma}\}.$$

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Let $\boldsymbol{\delta}^\mathbf{1}_{\boldsymbol{\alpha}} = \boldsymbol{\delta}_{\mathbf{\Sigma}^\mathbf{1}_{\boldsymbol{\alpha}}}$. The pointclasses $\{\mathbf{\Sigma}^\mathbf{1}_{\boldsymbol{n}} : n \in \omega\}$ and $\{\mathbf{\Pi}^\mathbf{1}_{\boldsymbol{n}} : n \in \omega\}$ are the usual levels of the projective hierarchy. We will refer to the collection of pointclasses $\{\mathbf{\Sigma}^\mathbf{1}_{\boldsymbol{\alpha}} : \alpha \in ON\} \cup \{\mathbf{\Pi}^\mathbf{1}_{\boldsymbol{\alpha}} : \alpha \in ON\}$ as the extended projective hierarchy.

We now define a hierarchy slightly coarser than the one above. If $n \in \omega$ and $\alpha \in ON$, we say a pointset A is in the pointclass $\mathbf{\Sigma}_{\boldsymbol{n}}(J_\alpha(\mathbb{R}))$ if there is a Σ_n formula ϕ with real parameters such that $A = \{x : J_\alpha(\mathbb{R}) \models \phi[x]\}$. $\mathbf{\Pi}_{\boldsymbol{n}}(J_\alpha(\mathbb{R}))$ is defined analogously with Π_n-formulas.[1] The Levy hierarchy consists of all pointclasses of the form $\mathbf{\Sigma}_{\boldsymbol{n}}(J_\alpha(\mathbb{R}))$ or $\mathbf{\Pi}_{\boldsymbol{n}}(J_\alpha(\mathbb{R}))$ for some n and α. It is clear any pointclass in the Levy hierarchy equals $\mathbf{\Sigma}^\mathbf{1}_{\boldsymbol{\alpha}}$ or $\mathbf{\Pi}^\mathbf{1}_{\boldsymbol{\alpha}}$ for some α, but the converse is false.

In this section, we will classify the scaled pointclasses within the Levy hierarchy, relate the Levy hierarchy to the extended projective hierarchy, and classify the regular Suslin pointclasses.

2.1.1. *Classification of Scaled Pointclasses.* A Σ_1-gap is a maximal interval $[\alpha, \beta]$ such that for any real x, the Σ_1-theory of x is the same in $J_\alpha(\mathbb{R})$ and $J_\beta(\mathbb{R})$.

We say the gap $[\alpha, \beta]$ is admissible if $J_\alpha(\mathbb{R}) \models KP$, equivalently, if the pointclass $\mathbf{\Sigma_1}(J_\alpha(\mathbb{R}))$ is closed under coprojection. Suppose $[\alpha, \beta]$ is an admissible gap. Let $n_\beta \in \mathbb{N}$ be least such that the pointclass $\mathbf{\Sigma}_{\boldsymbol{n_\beta}}(J_\beta(\mathbb{R}))$ is not contained in $J_\beta(\mathbb{R})$. We say $[\alpha, \beta]$ is a strong gap if for any $b \in J_\beta(\mathbb{R})$, there is $\beta' < \beta$ and $b' \in J_{\beta'}(\mathbb{R})$ such that the Σ_{n_β} and Π_{n_β} theories of b' in $J_{\beta'}(\mathbb{R})$ are the same as the Σ_{n_β} and Π_{n_β} theories of b in $J_\beta(\mathbb{R})$. Otherwise, we say $[\alpha, \beta]$ is weak.

THEOREM 2.1. *Suppose $\mathbf{\Gamma}$ is a pointclass in the Levy hierarchy. If $\mathbf{\Gamma}$ is scaled, then one of the following holds.*

1. $\mathbf{\Gamma} = \mathbf{\Sigma}_{2k+1}(J_\alpha(\mathbb{R}))$ *for some $k \in \omega$ and some α beginning an inadmissible gap.*
2. $\mathbf{\Gamma} = \mathbf{\Pi}_{2k+2}(J_\alpha(\mathbb{R}))$ *for some $k \in \omega$ and some α beginning an inadmissible gap.*
3. $\mathbf{\Gamma} = \mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$ *for some α beginning an admissible gap.*
4. $\mathbf{\Gamma} = \mathbf{\Sigma}_{n_\beta+2k}(J_\beta(\mathbb{R}))$ *for some $k \in \omega$ and some β ending a weak gap.*
5. $\mathbf{\Gamma} = \mathbf{\Pi}_{n_\beta+2k+1}(J_\beta(\mathbb{R}))$ *for some $k \in \omega$ and some β ending a weak gap.*

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[1]See [Ste83] for the definition of $J_\alpha(\mathbb{R})$. Alternatively, the reader will not lose too much of importance by pretending $J_\alpha(\mathbb{R}) = L_\alpha(\mathbb{R})$.

Definition 2.2. *A self-justifying system (sjs) is a countable set $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$ which is closed under complements and has the property that every $B \in \mathcal{B}$ admits a scale $\vec{\psi}$ such that $\leq_{\psi_n} \in \mathcal{B}$ for all n.*

Definition 2.3. *Let $z \in \mathbb{R}$ and $\gamma \in ON$. $OD^{<\gamma}(z)$ is the set of pointsets which are ordinal definable from the parameter z in $J_\xi(\mathbb{R})$ for some $\xi < \gamma$. $OD^{<\gamma}$ denotes $OD^{<\gamma}(0)$.*

The proof of Theorem 2.1 also gives:

Theorem 2.4. *Suppose $[\alpha, \beta]$ is an admissible gap. Let β' be the least ordinal such that there is a scale for a universal $\mathbf{\Pi_1}(J_\alpha(\mathbb{R}))$-set definable over $J_{\beta'}(\mathbb{R})$. Then there is $z \in \mathbb{R}$ and a sjs $\mathcal{B} \subset OD^{<\beta'}(z)$ such that a universal $\mathbf{\Pi_1}(J_\alpha(\mathbb{R}))$-set is in $\mathcal{B}$ and either*
1. *$[\alpha, \beta]$ is weak and $\beta' = \beta$ or*
2. *$[\alpha, \beta]$ is strong and $\beta' = \beta + 1$.*

Remark 2.5. *Suppose $\mathbf{\Gamma}$ is a boldface inductive-like pointclass in $L(\mathbb{R})$. Then*
1. *$\mathbf{\Gamma} = \mathbf{\Sigma_1}(J_\alpha(\mathbb{R}))$ for some α beginning an admissible gap,*
2. *there is $x \in \mathbb{R}$ such that letting Γ be the class of pointsets which are Σ_1-definable over $J_\alpha(\mathbb{R})$ from the parameter x, $\mathbf{\Gamma}$ is the closure of Γ under preimages by continuous functions, and*
3. *$\mathbf{\Gamma} = (\mathbf{\Sigma^2_1})^{\mathbf{\Delta_\Gamma}}$.*[2]

2.1.2. *Relationship between the Levy Hierarchy and the Extended Projective Hierarchy.*

Definition 2.6. *Suppose $\lambda < \Theta$ is a limit ordinal. We say*
- *λ is type I if $\mathbf{\Sigma^1_\lambda}$ is closed under finite intersection but not countable intersection,*
- *λ is type II if $\mathbf{\Sigma^1_\lambda}$ is not closed under finite intersection,*
- *λ is type III if $\mathbf{\Sigma^1_\lambda}$ is closed under countable intersection but not coprojection, and*
- *λ is type IV if $\mathbf{\Sigma^1_\lambda}$ is closed under coprojection.*

Let $\langle \delta_\alpha : \alpha < \Theta \rangle$ enumerate the ordinals δ such that there exist sets of reals in $J_{\delta+1}(\mathbb{R}) \backslash J_\delta(\mathbb{R})$. Let n_α be minimal such that $\mathbf{\Sigma_{n_\alpha}}(J_{\delta_\alpha}(\mathbb{R})) \not\subset J_{\delta_\alpha}(\mathbb{R})$.

Theorem 2.7. *Suppose $\alpha < \Theta$.*
1. *If $\omega\alpha$ is type I, then $\mathbf{\Sigma^1_{\omega\alpha+k}} = \mathbf{\Sigma_{n_\alpha+k}}(J_{\delta_\alpha}(\mathbb{R}))$ for all $k \in \omega$.*
2. *If $\omega\alpha$ is type II or III, then $\mathbf{\Sigma^1_{\omega\alpha+k+1}} = \mathbf{\Sigma_{n_\alpha+k}}(J_{\delta_\alpha}(\mathbb{R}))$ for all $k \in \omega$.*

[2] We say $A \subseteq \mathbb{R}$ is in $(\mathbf{\Sigma^2_1})^{\mathbf{\Delta_\Gamma}}$ if there is $z \in \mathbb{R}$ and a formula ϕ such that for all $x \in \mathbb{R}$, $x \in A \iff (\exists B \in \mathbf{\Delta_\Gamma})(\mathbb{R}, B) \models \phi(x, z)$.

3. *If $\omega\alpha$ is type IV, then* $\mathbf{\Pi^1_{\omega\alpha}} = \mathbf{\Sigma_{n_\alpha}}(J_{\delta_\alpha}(\mathbb{R}))$ *and* $\mathbf{\Sigma^1_{\omega\alpha+k+1}} = \mathbf{\Sigma_{n_\alpha+k}}(J_{\delta_\alpha}(\mathbb{R}))$ *for all* $k \in \omega\backslash\{0\}$.

2.1.3. *Classification of Suslin Pointclasses.* There is a related classification of the Suslin pointclasses. For $\alpha < \Theta$, let κ_α be the αth Suslin cardinal. Let ν_α be the αth ordinal ν such that $\mathbf{\Sigma^1_\nu}$ or $\mathbf{\Pi^1_\nu}$ is scaled.

THEOREM 2.8. *Let* $\lambda < \boldsymbol{\delta^2_1}$ *be a limit cardinal and* $\nu = sup\{\nu_\alpha : \alpha < \lambda\}$.

1. *If* ν *is type I, then for all* $k \in \omega$
   - $\mathbf{\Sigma^1_{\nu+2k}}$ *and* $\mathbf{\Pi^1_{\nu+2k+1}}$ *are scaled,*
   - $S(\kappa_{\lambda+k}) = \mathbf{\Sigma^1_{\nu+k+1}}$,
   - $\kappa_{\lambda+2k+1} = \boldsymbol{\delta^1_{\nu+2k+1}} = (\kappa_{\lambda+2k})^+$, *and*
   - $cof(\kappa_{\lambda+2k}) = \omega$.
2. *If* ν *is type II or III, then for all* $k \in \omega$
   - $\mathbf{\Sigma^1_{\nu+2k+1}}$ *and* $\mathbf{\Pi^1_{\nu+2k}}$ *are scaled,*
   - $S(\kappa_{\lambda+k}) = \mathbf{\Sigma^1_{\nu+k+1}}$,
   - $\kappa_{\lambda+2k+2} = \boldsymbol{\delta^1_{\nu+2k+2}} = (\kappa_{\lambda+2k+1})^+$, *and*
   - $cof(\kappa_{\lambda+2k+1}) = \omega$.
3. *If* ν *is type IV, then* $\mathbf{\Pi^1_\nu}$ *is scaled,* $S(\kappa_\lambda) = \mathbf{\Pi^1_\nu}$, *and for all* $k \in \omega$, *letting* $\mu = \nu_{\lambda+1}$,
   - $\mathbf{\Sigma^1_{\mu+2k}}$ *and* $\mathbf{\Pi^1_{\mu+2k+1}}$ *are scaled,*
   - $S(\kappa_{\lambda+k+1}) = \mathbf{\Sigma^1_{\mu+k+1}}$,
   - $\kappa_{\lambda+2k+2} = \boldsymbol{\delta^1_{\mu+2k+1}} = (\kappa_{\lambda+2k+1})^+$, *and*
   - $cof(\kappa_{\lambda+2k+1}) = \omega$.

COROLLARY 2.9. *Suppose* $\mathbf{\Gamma} = S(\kappa)$ *for a regular Suslin cardinal* $\kappa \leq \boldsymbol{\delta^2_1}$. *Then one of the following holds.*

1. $\mathbf{\Gamma} = \mathbf{\Sigma}_{2k+1}(J_\alpha(\mathbb{R}))$ *for some* $k \in \omega$ *and some* α *beginning an inadmissible gap.*
2. $\mathbf{\Gamma} = \mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$ *for some* α *beginning an admissible gap.*
3. $\mathbf{\Gamma} = \mathbf{\Sigma}_{n_\beta+2k}(J_\beta(\mathbb{R}))$ *for some* $k \in \omega$ *and some* β *ending a weak gap.*

2.2. Woodin Cardinals and Iterations. We borrow most of the notation of premice and iteration trees from [Ste09]. In addition to the lightface premice defined in [Ste09], we will also consider premice built over some $a \in HC$. We write an a-premouse as $M = (J^{\vec{E}}_\alpha, \in, \vec{E} \restriction \alpha, E_\alpha, a)$, for a fine extender sequence $\vec{E} = \langle E_\eta : \eta \leq \alpha\rangle$. If $\beta \leq \alpha$, $M|\beta$ represents the premouse $(J^{\vec{E}}_\beta, \in, \vec{E} \restriction \beta, E_\beta, a)$. Unless otherwise specified, an iteration strategy will refer to an (ω_1, ω_1)-iteration strategy and a mouse will refer to a premouse with such a strategy. Under $ZF + AD$, ω_1 is measurable, so an (ω_1, ω_1)-iteration strategy induces an $(\omega_1, \omega_1 + 1)$-iteration strategy.

In particular, the theorems in this section requiring an $\omega_1 + 1$-iteration strategy will all apply to the mice we use in Section 3.

Additionally, if $\mathcal{T}$ is an iteration tree of limit length and b is a cofinal, non-dropping branch through $\mathcal{T}$, we let $M_b^{\mathcal{T}}$ be the direct limit of the models on b and let $i_b^{\mathcal{T}} : M_0^{\mathcal{T}} \to M_b^{\mathcal{T}}$ be the associated direct limit embedding.

For a model M, let δ_M denote the least Woodin cardinal of M (if one exists) and Ea_M denote Woodin's extender algebra in M at δ_M. Let κ_M be the least cardinal of M which is $< \delta_M$-strong in M. ea will refer to the generic over Ea_M. When considering the product extender algebra $Ea_M \times Ea_M$, we will write $ea_l \times ea_r$ for the generic. ea_r will typically code a pair which we shall write (ea_r^1, ea_r^2). For posets of the form $Col(\omega, X)$, $\dot{g}$ denotes a name for the generic.

Suppose M is a premouse with iteration strategy Σ. We say N is a complete iterate of M if N is the last model of an iteration tree $\mathcal{T}$ on M such that $\mathcal{T}$ is according to Σ and the branch through $\mathcal{T}$ from M to N is non-dropping.[3]

Theorem 2.10. *Let M be a countable premouse with an ω_1+1-iteration strategy such that $M \models$ "There is a Woodin cardinal." Then Ea_M is a δ_M-c.c. Boolean algebra and for any $x \in \mathbb{R}$, there is a countable, complete iterate N of M such that x is Ea_N-generic over N.*

Corollary 2.11. *Let M be a countable premouse with an ω_1+1-iteration strategy such that $M \models$ "There is a Woodin cardinal." Then for any $x \in \mathbb{R}$, there is a countable, complete iterate N of M and g which is $Col(\omega, \delta_N)$-generic over N such that $x \in N[g]$.*

See Section 7.2 of [Ste09] for a proof of Theorem 2.10 and its corollary.

Definition 2.12. *For $\kappa < \delta$ and $A \subseteq \delta$, we say κ is A-reflecting in δ if for every $\nu < \delta$, there is an extender E with critical point κ such that $i_E(\kappa) > \nu$ and $i_E(A) \cap \nu = A \cap \nu$.*

Theorem 2.13. *Suppose b and c are distinct wellfounded branches of a normal iteration tree $\mathcal{T}$ and $A \subseteq \delta(\mathcal{T})$ is in $M_b^{\mathcal{T}} \cap M_c^{\mathcal{T}}$. Then there is $\kappa < \delta(\mathcal{T})$ such that $M_b^{\mathcal{T}} \models$ "κ is A-reflecting in $\delta(\mathcal{T})$," and this is witnessed by a sequence of extenders on the extender sequence of $\mathcal{M}(\mathcal{T})$.*

See 6.9 and 6.10 of [Ste09] for definitions of $\delta(\mathcal{T})$ and $\mathcal{M}(\mathcal{T})$ and a proof of Theorem 2.13. The theorem justifies the following definitions.

Definition 2.14. *Suppose b is a wellfounded branch through a normal iteration tree $\mathcal{T}$. Let $\mathcal{Q}(b, \mathcal{T})$ be the least initial segment of $M_b^{\mathcal{T}}$ extending*

[3]This is a slight abuse of notation, since being "a complete iterate of M" is dependent on Σ as well as M. This will not cause any ambiguity, since the mice we are interested in have unique iteration strategies.

$\mathcal{M}(\mathcal{T})$ such that there is $A \subset \delta(\mathcal{T})$ which is definable over $\mathcal{Q}(b,\mathcal{T})$ and realizes $\delta(\mathcal{T})$ is not Woodin via extenders in $\mathcal{M}(\mathcal{T})$, if such an initial segment exists.

DEFINITION 2.15. *Suppose M is a premouse and $\eta \in M$. We say η is a cutpoint of M if there is no extender on the fine extender sequence of M with critical point less than η and length greater than η. η is a strong cutpoint if there is no extender on the fine extender sequence of M with critical point less than or equal to η and length greater than η.*

DEFINITION 2.16. *Suppose $\mathcal{T}$ is a normal iteration tree. Let $\mathcal{Q}(\mathcal{T})$ be the least $\delta(\mathcal{T})$-sound, ω_1+1-iterable premouse extending $\mathcal{M}(\mathcal{T})$ and projecting to $\delta(\mathcal{T})$ such that $\delta(\mathcal{T})$ is a strong cutpoint of $\mathcal{Q}(\mathcal{T})$ and there is $A \subset \delta(\mathcal{T})$ which is definable over $\mathcal{Q}(\mathcal{T})$ and realizes $\delta(\mathcal{T})$ is not Woodin via extenders in $\mathcal{M}(\mathcal{T})$, if one exists.*

It follows from Theorem 2.13 that there is at most one wellfounded branch b through $\mathcal{T}$ such that $\mathcal{Q}(\mathcal{T}) \trianglelefteq M_b^{\mathcal{T}}$. In many cases, we will be able to locate the branch a strategy Σ chooses as the unique branch which absorbs $\mathcal{Q}(\mathcal{T})$ in this sense.

Note an ω_1-iteration strategy on a countable premouse can be coded by a set of reals. For $a \in HC$ and a pointclass Γ, this allows us to define

$$Lp^{\Gamma}(a) = \bigcup\{N : N \text{ is an } \omega\text{-sound } a\text{-premouse projecting to } a \text{ with an } \omega_1\text{-iteration strategy in } \mathbf{\Delta_\Gamma}\}.$$

$Lp^{\Gamma}(a)$ can be reorganized as an a-premouse, which is what we will typically use $Lp^{\Gamma}(a)$ to refer to.

Closely related to Lp^{Γ} is the operator C_Γ. For $x \in \mathbb{R}$,

$$C_\Gamma(x) = \{z \in \mathbb{R} : z \text{ is } \Delta_\Gamma(x) \text{ in some countable ordinal}\}.$$

And for $a \in HC$,

$$C_\Gamma(a) = \{b \subseteq a : \text{for all reals } x \text{ coding } a,\ b_x \in C_\Gamma(x)\}.$$

Here b_x codes b relative to x. See [Ste16] for more details.

THEOREM 2.17. *Assume $AD^{L(\mathbb{R})}$. Suppose Γ is a (lightface) inductive-like pointclass in $L(\mathbb{R})$ and $a \in HC$. Then $C_\Gamma(a) = Lp^{\Gamma}(a) \cap P(a)$.*

REMARK 2.18. *Suppose a and b are countable, transitive sets and $a \in b$. It is easy to see from the definition of C_Γ that $C_\Gamma(a) \subseteq C_\Gamma(b)$. This, and the theorem above, implies $Lp^{\Gamma}(a) \subseteq Lp^{\Gamma}(b)$.*

2.3. The Mitchell-Steel Construction. We shall require a method of building an a-premouse inside a premouse M which contains a. Our main tool for this purpose is the fully backgrounded Mitchell-Steel construction developed in [MS17]. This section reviews the construction and its properties.

We say a premouse M is reliable if $\mathcal{C}_\omega(M)$ exists and is universal and solid. As we shall see in a moment, we will end the Mitchell-Steel construction if we reach a premouse which is not reliable. [MS17] defines reliable to include the stronger property that $\mathcal{C}_\omega(M)$ is iterable. But the weaker properties of universality and solidity are enough to propagate the construction, and our weaker requirement ensures the construction does not end prematurely when performed inside a mouse. The definitions of universality and solidity can be found in [Ste09]. In all of the cases relevant to us, universality and solidity are guaranteed and the reader will lose little by taking on faith that the construction does not end.

For the moment we will work in V and assume ZFC. Fix $z \in \mathbb{R}$. Define a sequence of z-premice $\langle \mathcal{M}_\xi : \xi \in On \rangle$ inductively as follows.

1. $\mathcal{M}_0 = (V_\omega, \in, \emptyset, \emptyset, z)$
2. Suppose we have constructed $\mathcal{M}_\xi = (J_\alpha^{\vec{E}}, \in, \vec{E}, \emptyset, z)$. Note $\mathcal{M}_\xi$ is a passive premouse. Suppose also there is an extender F^* over V, an extender F over $\mathcal{M}_\xi$, and $\nu < \alpha$ such that
   (a) $V_{\nu+\omega} \subset Ult(V, F^*)$,
   (b) ν is the support of F,
   (c) $F \restriction \nu = F^* \cap ([\nu]^{<\omega} \times \mathcal{M}_\xi)$, and
   (d) $\mathcal{N}_{\xi+1} = (J_\alpha^{\vec{E}}, \in, \vec{E}, F, z)$ is a premouse.
   If $\mathcal{N}_{\xi+1}$ is reliable, let $\mathcal{M}_{\xi+1} = \mathcal{C}_\omega(\mathcal{N}_{\xi+1})$. Otherwise, the construction ends. If there are multiple such F^*, we pick one which minimizes the support of F. We say F^* is the extender used as a background at step $\xi + 1$.
3. Suppose we have constructed $\mathcal{M}_\xi = (J_\alpha^{\vec{E}}, \in, \vec{E}, E_\alpha, z)$ and either $\mathcal{M}_\xi$ is active or $\mathcal{M}_\xi$ is passive and there is no extender F^* as above. Let $\mathcal{N}_{\xi+1} = (J_{\alpha+1}^{\vec{E}^\frown E_\alpha}, \in, \vec{E}^\frown E_\alpha, \emptyset, z)$. If $\mathcal{N}_{\xi+1}$ is reliable, let $\mathcal{M}_{\xi+1} = \mathcal{C}_\omega(\mathcal{N}_{\xi+1})$. Otherwise, the construction ends.
4. Suppose we have constructed $\langle \mathcal{M}_\xi : \xi < \lambda \rangle$ for λ a limit ordinal. Let $\eta = lim\,inf_{\xi<\lambda}(\rho_\omega(\mathcal{M}_\xi)^+)^{\mathcal{M}_\xi}$. Let $\mathcal{N}_\lambda$ be the passive premouse of height η such that $\mathcal{N}_\lambda|\beta = lim_{\xi<\lambda}\mathcal{M}_\xi|\beta$ for all $\beta < \eta$. If $\mathcal{N}_\lambda$ is reliable, let $\mathcal{M}_\lambda = \mathcal{C}_\omega(\mathcal{N}_\lambda)$. Otherwise, the construction ends.

Suppose the construction never breaks down. That is, $\mathcal{M}_\xi$ is defined for all $\xi \in On$.

THEOREM 2.19. *Suppose ζ_0 and ξ are ordinals such that $\zeta_0 < \xi$ and $\kappa = \rho_\omega(\mathcal{M}_\xi) \leq \rho_\omega(\mathcal{M}_\zeta)$ for all $\zeta \geq \zeta_0$. Then $\mathcal{M}_\xi \trianglelefteq \mathcal{M}_\eta$ for all $\eta \geq \xi$. Moreover, $\mathcal{M}_{\xi+1} \models$ "every set has cardinality at most κ."*

Let $\mathcal{M}$ be the class-sized model such that whenever $\xi \in On$ satisfies $\mathcal{M}_\xi \trianglelefteq \mathcal{M}_\eta$ for all $\eta \geq \xi$, $\mathcal{M}_\xi$ is an initial segment of $\mathcal{M}$. We call $\mathcal{M}$ the output of the Mitchell-Steel construction over z. For $\delta \in On$, we call $\mathcal{M}_\delta$ the output of the Mitchell-Steel construction of length δ over z.

THEOREM 2.20. *Assume ZFC. Suppose δ is the least ordinal such that δ is Woodin in $L(V_\delta)$. Suppose the Mitchell-Steel construction in V_δ does not break down, and let $\mathcal{M}$ be the output of the construction. Then δ in Woodin in $L(\mathcal{M})$.*

See the proof of Theorem 11.3 of [MS17].

THEOREM 2.21 (Universality). *Assume ZFC. Let δ be Woodin and $z \in \mathbb{R}$. Assume the Mitchell-Steel construction of length δ over z does not break down. Let N be the output of the construction. Suppose no initial segment of N satisfies "there is a superstrong cardinal." Let W be a premouse over x of height $\leq \delta$, and suppose P and Q are the final models above W and N, respectively, in a successful coiteration. Then $P \trianglelefteq Q$.*

See Theorem 11.1 of [Ste08].

THEOREM 2.22. *Suppose M is an $\omega_1 + 1$-iterable mouse with Woodin cardinal δ satisfying enough of ZFC and $z \in M \cap \mathbb{R}$. Then the Mitchell-Steel construction of length δ over z done inside M does not break down. Let N be the output of the construction. Then N is a z-mouse of height δ.*

The proof of Theorem 2.22 is well known. To show the construction does not break down, by [MS17] it suffices to show universality and solidity of the models $\langle \mathcal{M}_\xi : \xi < \delta \rangle$ built during the construction. [MS17] further reduces this to showing iterability for each $\mathcal{M}_\xi$. An iteration strategy for M_ξ can be defined by lifting iteration trees on $\mathcal{M}_\xi$ to trees on (an initial segment of) M and selecting the branch picked by the strategy for M. $\omega_1 + 1$-iterability of M suffices to obtain the required iterability for each $\mathcal{M}_\xi$. Similarly, N being a z-mouse follows from iterability of M.

For a premouse M satisfying enough of ZFC and $z \in M \cap \mathbb{R}$, we write $Le[M, z]$ for the output of the Mitchell-Steel construction in M over z (assuming the construction does not break down). $Le[M]$ will refer to $Le[M, \emptyset]$. $Le[M, z]$ is a z-premouse. If M is iterable, so is $Le[M, z]$.

We are most interested in cases in which M is a mouse with a Woodin cardinal δ, no largest cardinal, and no total extenders above δ. Then $Le[M|\delta, z]$ is equal to the Mitchell-Steel construction of length δ over z, done inside M, and $Le[M, z]$ is an initial segment of $L(Le[M|\delta.z])$.

Remark 2.23. *Suppose M is an ω_1+1-iterable premouse, $z \in M \cap \mathbb{R}$, and κ is inaccessible in M. Let $\langle \mathcal{M}_\xi : \xi < \kappa\rangle$ be the models of the Mitchell-Steel construction in M of length κ over z. Suppose an extender is added at step $\xi+1$ in the construction. Let F^*, F, and ν be as in Case 2 of the construction. Then there is $F' \in M|\kappa$ such that $M \models V_{\nu+\omega} \subset Ult(M, F')$ and $F' \cap ([\nu]^{<\omega} \times \mathcal{M}_\xi) = F \restriction \nu$. So we may assume if F^* is used as a background in the construction of length κ, then $F^* \in M|\kappa$.*

In particular, if M is an ω_1+1-iterable premouse, $z \in M \cap \mathbb{R}$, and κ is inaccessible in M, then $Le[M|\kappa, z]$ equals the Mitchell-Steel construction of length κ over z, done in M.

2.4. S-constructions. Below we outline the S-construction (this was introduced as the P-construction in [SS09]).

Suppose $M = (J_\gamma^{\vec{E}}, \in, \vec{E} \restriction \gamma, E_\gamma, a)$ is a countable a-premouse and $\delta \in M$ is a cardinal and cutpoint of M. Suppose $ON \cap \bar{S} = \delta + \omega$, δ is a Woodin cardinal of $\bar{S}$, $\bar{S}$ is definable over M, and there is a generic G (for the version of Woodin's extender algebra with δ propositional letters) such that $\bar{S}[G] = M|\delta+1$. Inductively define a sequence $\langle S_\alpha : \delta+1 \le \alpha \le \gamma\rangle$ as follows. $S_{\delta+1}$ is set to be $\bar{S}$. At a limit λ, $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$. If $M|\lambda$ is active, add a predicate for $E_\lambda \cap S_\lambda$ to S_λ. For the successor step, we define $S_{\alpha+1}$ by constructing one more level over S_α. The construction proceeds until we construct S_γ, or we reach some S_α such that δ is not Woodin in S_α. We refer to S_γ as the maximal S-construction in M over $\bar{S}$ if the construction reaches γ. We are primarily interested in cases where δ is Woodin in M, in which case the construction is guaranteed to reach γ.

Lemma 2.24. *Suppose $M, \bar{S}, \delta, \gamma$, and G are as above. Assume also M is (ω_1, ω_1+1)-iterable, ω-sound, and $\rho_\omega(M) \ge \delta$. If the construction reaches γ, then for each α such that $\delta+1 \le \alpha \le \gamma$, S_α is an $\bar{S}$-mouse and $S_\alpha[G] = M|\alpha$. If also $\alpha < \gamma$, or $\alpha = \gamma$ and δ is definably Woodin over S_α, then $\rho_n(S_\alpha) = \rho_n(M|\alpha)$ for all n and S_α is ω-sound.*

Lemma 1.5 of [SS09] gives everything in Lemma 2.24 except the iterability of S_γ. The iteration strategy for S_γ in Lemma 2.24 comes from lifting an iteration tree on S_γ to iteration trees on M above δ. In particular, we have:

Fact 2.25. *Suppose $M, \bar{S}, \delta, \gamma$, and G are as in Lemma 2.24. Then the iteration strategy for S_γ (as an $\bar{S}$-premouse) is projective in the iteration strategy for M restricted to iteration trees above δ.*

The S-construction serves two purposes in what follows. It allows us to "undo" generic extensions from Woodin's extender algebra. And combined with the fully-backgrounded Mitchell-Steel construction, it provides an inner model of a premouse with convenient properties.

Definition 2.26. *Let M be an ω_1+1-iterable premouse with a Woodin cardinal and $z \in M \cap \mathbb{R}$. Let $\bar{S}$ be the result of constructing one level of the $\mathcal{J}$-hierarchy over $Le[M|\delta_M, z]$. Let $StrLe[M,z]$ denote the maximal S-construction in M over $\bar{S}$.*

2.5. Suitable Mice. We now review some results from the core model induction. Most of the concepts below are from [SS], with some minor additions. We need to work with mice with an inaccessible cardinal above a Woodin, so in Definition 2.28 we introduce a modification of the standard notion of a suitable premouse. [SS] proves the existence of terms in suitable mice capturing certain sets of reals. We will need analogous lemmas for our modified definition. In fact we require more than is stated in [SS] — it is essential for our purposes that there is a canonical term capturing each set. Fortunately, this stronger claim is already implicit in the proofs of [SS].

For the remainder of this section, we will asssume $ZF+AD+DC+V=L(\mathbb{R})$ and fix a boldface inductive-like pointclass $\mathbf{\Gamma}$ such that $\mathbf{\Gamma} \neq \mathbf{\Sigma^2_1}$. We then have $\mathbf{\Gamma} = \mathbf{\Sigma_1}(J_{\alpha_0}(\mathbb{R}))$ for some α_0 beginning an admissible Σ_1-gap $[\alpha_0, \beta_0]$. Fix a lightface pointclass Γ as in Remark 2.5 such that $\mathbf{\Gamma}$ is the closure of Γ under preimages by continuous functions.

Definition 2.27. *Suppose $x \in HC$. Say an x-premouse N is Γ-suitable if N is countable and*

1. *$N \models$ there is exactly one Woodin cardinal δ_N.*
2. *Letting $N_0 = Lp^\Gamma(N|\delta_N)$ and $N_{i+1} = Lp^\Gamma(N_i)$, we have that $N = \bigcup_{i<\omega} N_i$.*
3. *If $\xi < \delta_N$, then $Lp^\Gamma(N|\xi) \models \xi$ is not Woodin.*

Definition 2.28. *Suppose $x \in HC$. Say an x-premouse N is Γ-super-suitable (Γ-ss) if N is countable and*

1. *$N \models$ There is exactly one Woodin cardinal δ_N.*
2. *$N \models$ There is exactly one inaccessible cardinal above δ_N. We denote this inaccessible by ν_N.*
3. *Letting $N_0 = Lp^\Gamma(N|\nu_N)$ and $N_{i+1} = Lp^\Gamma(N_i)$, we have that $N = \bigcup_{i<\omega} N_i$.*
4. *For each $\xi \geq \delta_N$, $N|(\xi^+)^N = Lp^\Gamma(N|\xi)$.*
5. *If $\xi < \delta_N$, then $Lp^\Gamma(N|\xi) \models \xi$ is not Woodin.*

Definition 2.29. *Let N be a mouse and $\delta \in N$. We say δ is a Γ-Woodin of N if δ is Woodin in $Lp^\Gamma(N|\delta)$.*

A Γ-suitable premouse is a minimal premouse with a Γ-Woodin cardinal which is closed under Lp^Γ, in that none of its initial segments have this property. Similarly, a Γ-ss premouse can be considered a minimal premouse with a Γ-Woodin which is closed under Lp^Γ and has an inaccessible cardinal above its Γ-Woodin. The existence of a Γ-suitable (or Γ-ss) premouse is not any stronger than the existence of a premouse with a Γ-Woodin cardinal. For suppose $N = Lp^\Gamma(N|\delta)$, δ is Woodin in N, and no $\xi < \delta$ is a Γ-Woodin of N. If $Q \triangleright N$, $\rho(Q) \leq \delta$, Q is δ-sound, and a set definable over Q witnesses that δ is not Woodin, then Q must have a Γ-Woodin cardinal above δ. Otherwise, Q would be iterable by Q-structures in $\mathbf{\Delta_\Gamma}$ and hence in $Lp^\Gamma(N|\delta)$. Then we may build a Γ-suitable (Γ-ss) premouse by closing under Lp^Γ as many times as is necessary, since this will never construct a Γ-Woodin.

Definition 2.30. *Let $A \subseteq \mathbb{R}$, N a countable premouse, η an uncountable cardinal of N, and $\tau \in N^{Col(\omega,\eta)}$. We say that τ weakly captures A over N if whenever g is $Col(\omega,\eta)$-generic over N, $\tau[g] = A \cap N[g]$.*

Lemma 2.31. *Suppose $\mathcal{B}$ is a self-justifying system and N and M are transitive models of enough of ZFC such that $N \in M$. Let $\mathcal{C}$ be a comeager set of $Col(\omega, N)$ generics over M and suppose for each $B \in \mathcal{B}$ there is a term $\tau_B \in M$ such that if $g \in \mathcal{C}$, then $\tau_B[g] = B \cap M[g]$. Let $\pi : \bar{M} \to M$ be elementary with $\{N\} \cup \{\tau_B : B \in \mathcal{B}\} \subset ran(\pi)$. Let $(N, \tau_B) = \pi(\bar{N}, \bar{\tau}_B)$. Then whenever g is $Col(\omega, \bar{N})$-generic over $\bar{M}$, $\bar{\tau}_B[g] = B \cap \bar{M}[g]$.*

See Lemma 3.7.2 of [SS].

Let β' be the least ordinal greater than α_0 such that there is a scale for a universal $\mathbf{\Pi_1}(J_{\alpha_0}(\mathbb{R}))$ set definable over $J_{\beta'}(\mathbb{R})$. By Theorem 2.4, $\beta' = \beta_0$ or $\beta' = \beta_0 + 1$ and there is a self-justifying system $\mathcal{G} = \{G_n : n \in \omega\}$ such that

$$\begin{aligned} G_0 = \{(x, y) : \ & x \text{ codes some transitive set } a \text{ and } y \text{ codes an } \omega\text{-sound} \\ & a\text{-premouse } R \text{ such that } R \text{ projects to } a \text{ and } R \text{ has an} \\ & \omega_1\text{-iteration strategy in } \mathbf{\Delta}\}, \end{aligned}$$

and $\mathcal{G}$ is contained in $OD^{<\beta'}(z)$ for some $z \in \mathbb{R}$. Note $G_0 \in \mathbf{\Gamma}$, by part 3 of Remark 2.5. In fact, G_0 is a universal $\mathbf{\Gamma}$-set (we will not need this property specifically for G_0, but we will use that $\mathcal{G}$ contains some universal $\mathbf{\Gamma}$-set). For ease of notation, assume $\mathcal{G} \subset OD^{<\beta'}$.

Definition 2.32. *Suppose $B \subset \mathbb{R}$, N is a premouse, and η is a cardinal of N. Let $\tau^N_{B,\eta}$ be the set of pairs $(\sigma, p) \in N$ such that*

1. *σ is a $Col(\omega,\eta)$-standard term for a real,*
2. *$p \in Col(\omega, \eta)$, and*
3. *for comeager many $g \subset Col(\omega, \eta)$ which are $Col(\omega, \eta)$-generic over N such that $p \in g$, $\sigma[g] \in B$.*

For $n \in \omega$, let $\tau^N_{n,\eta} = \tau^N_{G_n,\eta}$ and if N has a Woodin cardinal let $\tau^N_n = \tau^N_{n,\delta_N}$.

Lemma 2.33. *Suppose N is a Γ-suitable or Γ-ss premouse, $z \in N$, $B \in OD^{<\beta'}(z)$, and η is a cardinal of N. Then $\tau^N_{B,\eta}$ is in N.*

See the proof of Lemma 3.7.5 of [SS]. In Lemma 5.4.3 of [SS], Lemma 2.31 is used to show:

LEMMA 2.34 (Woodin). *Suppose $z \in \mathbb{R}$, N is a Γ-suitable (or Γ-ss) z-premouse, and $\mathcal{B}$ is a sjs containing a universal $\mathbf{\Sigma_1}(J_{\alpha_0}(\mathbb{R}))$-set such that each $B \in \mathcal{B}$ is $OD^{<\beta'}(z)$. Suppose $\pi : M \to N$ is Σ_1-elementary and for every $B \in \mathcal{B}$ and $\eta \geq \delta_N$, $\tau^N_{B,\eta} \in range(\pi)$. Then*

1. *M is Γ-suitable (Γ-ss) and*
2. *$\pi(\tau^M_{B,\bar{\eta}}) = \tau^N_{B,\eta}$, where $\bar{\eta}$ is such that $\pi(\bar{\eta}) = \eta$.*

As a result of Lemmas 2.31 and 2.33 we have:

COROLLARY 2.35. *If N is Γ-suitable or Γ-ss and η is an uncountable cardinal of N, then $\tau^N_{n,\eta}$ weakly captures G_n.*

DEFINITION 2.36. *Let $\mathcal{T}$ be a normal iteration tree on a Γ-suitable (or Γ-ss) premouse N. Suppose also $\mathcal{T}$ is below δ_N. Say $\mathcal{T}$ is Γ-short if for all limit $\xi \leq lh(\mathcal{T})$, $Lp^\Gamma(\mathcal{M}(\mathcal{T} \restriction \xi)) \models \delta(\mathcal{T} \restriction \xi)$ is not Woodin. Otherwise, say $\mathcal{T}$ is Γ-maximal.*

DEFINITION 2.37. *Let N be a Γ-suitable (Γ-ss) premouse with an (ω_1, ω_1)-iteration strategy Σ. Say Σ is fullness-preserving if whenever P is an iterate of N by Σ via an iteration below δ_N, then*

1. *if the branch to P does not drop, then P is Γ-suitable (Γ-ss), and*
2. *if the branch to P does drop, then P has an ω_1-iteration strategy in $J_{\alpha_0}(\mathbb{R})$.*

REMARK 2.38. *Let N be a Γ-suitable (or Γ-ss) mouse with a fullness-preserving iteration strategy Σ. Suppose $P \lhd N|\delta_N$, and Σ' is the iteration strategy for P given by restricting the domain of Σ to trees on P. Suppose $\mathcal{T}$ is an iteration tree on P according to Σ'. Then the branch b through $\mathcal{T}$ chosen by Σ' can be determined from $\mathcal{Q}(\mathcal{T})$. And $\mathcal{Q}(\mathcal{T})$ is the unique $\mathcal{M}(\mathcal{T})$-mouse projecting to ω with an iteration strategy in $\mathbf{\Delta}$. It follows from Remark 2.5 and the uniqueness of $\mathcal{Q}(\mathcal{T})$ that Σ' is coded by a set in $\mathbf{\Delta}$.*

DEFINITION 2.39. *Let $\mathcal{T}$ be a Γ-maximal iteration tree on a Γ-suitable (or Γ-ss) premouse N and let b be a cofinal branch through $\mathcal{T}$. Say b respects $\vec{G}_n$ if $i^{\mathcal{T}}_b(\tau^N_{k,\eta}) = \tau^{M^{\mathcal{T}}_b}_{k,i_b(\eta)}$ for all $k < n$ and every cardinal η of N above δ_N.*

DEFINITION 2.40. *Let N be a Γ-suitable (or Γ-ss) mouse with a fullness-preserving iteration strategy Σ. Say Σ is guided by $\mathcal{G}$ if whenever $\mathcal{T}$ is an iteration tree according to Σ of limit length and $b = \Sigma(\mathcal{T})$, then*

1. *if $\mathcal{T}$ is Γ-short, then $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{Q}(b, \mathcal{T}) \in Lp^\Gamma(\mathcal{M}(\mathcal{T}))$, and*
2. *if T is Γ-maximal, then $\Sigma(b)$ respects $\vec{G}_n$ for all $n \in \omega$.*

LEMMA 2.41. *If N is Γ-suitable (or Γ-ss) and Σ is an iteration strategy for N which is guided by $\mathcal{G}$, then Σ is not in $\mathbf{\Gamma}$.*

PROOF. There is $n \in \omega$ such that G_n is a universal $\mathbf{\Gamma^c}$-set. Then $y \in G_n$ if and only if there exists a countable, complete iterate N^* of N according to Σ and $g \in \mathbb{R}$ which is $Col(\omega, \mathbb{R})$-generic over N^* such that $y \in \tau_n^{N^*}[g]$. Since $\mathbf{\Gamma}$ is closed under projection, if Σ were in $\mathbf{\Gamma}$, G_n would also be in $\mathbf{\Gamma}$. ⊣

THEOREM 2.42 (Woodin). *For any $x \in HC$, there is a (unique) ω-sound, Γ-suitable x-mouse W_x projecting to x with a (unique) iteration strategy that is fullness-preserving, condenses well,[4] and is guided by $\mathcal{G}$. Similarly, there is a (unique) ω-sound, Γ-ss x-mouse M_x projecting to x with a (unique) iteration strategy that is fullness-preserving, condenses well, and is guided by $\mathcal{G}$.*

Chapter 5 of [SS] demonstrates the existence of such a Γ-suitable mouse. It is not difficult to see this gives the existence of the required Γ-ss mouse as well.

For any Γ-suitable (or Γ-ss) premouse N and any $n \in \omega$, let

$$\gamma_n^N = Hull^N(\{\tau_i^N : i < n\}) \cap \delta_N.$$

The regularity of δ_N in N implies each γ_n^N is an ordinal. Lemma 2.34 can be used to show:

FACT 2.43. *$\langle \gamma_n^N : n \in \omega \rangle$ is cofinal in δ_N.*

LEMMA 2.44. *Let $\mathcal{T}$ be a normal iteration tree on a Γ-suitable (or Γ-ss) premouse N and let b and c be branches through $\mathcal{T}$ which respect $\vec{G}_n$. Then $i_b^{\mathcal{T}} \restriction \gamma_n^N = i_c^{\mathcal{T}} \restriction \gamma_n^N$. Moreover, if b and c both respect $\vec{G}_n$ for all n, then $b = c$.*

See Lemma 6.25 of [SW16].

Lemma 2.44 implies if b is the branch through $\mathcal{T}$ chosen by the nice iteration strategy for a Γ-suitable premouse given by Theorem 2.42 and c is any branch respecting $\vec{G}_n$, then $i_b^{\mathcal{T}}$ and $i_c^{\mathcal{T}}$ agree up to γ_n^N. In particular, to track the iteration of a Γ-suitable mouse up to some point below its least Woodin, it is sufficient to know finitely many of the sets in $\mathcal{G}$.

Suppose M is a countable premouse with an $(\omega_1, \omega_1 + 1)$-iteration strategy.[5] Together, the Comparison Lemma and the Dodd-Jensen Lemma imply the collection of countable, complete iterates of M, together with the iteration maps between them, forms a directed system.

[4] In the sense of Definition 5.3.7 of [SS].

[5] The suitable mice from Theorem 2.42 satisfy this. We only explicitly required these to have (ω_1, ω_1)-iteration strategies, but since $ZF + AD$ implies ω_1 is measurable, an (ω_1, ω_1)-iteration strategy induces an $(\omega_1, \omega_1 + 1)$-iteration strategy.

[Ste95b] presents work of Steel and Woodin analyzing the direct limit of all countable, complete iterates of $M_\omega^\#$. This direct limit cut to its least Woodin is $(HOD||\Theta)^{L(\mathbb{R})}$. [SW16] goes further in showing that the entire class $HOD^{L(\mathbb{R})}$ is a strategy mouse. The iteration maps through trees on $M_\omega^\#$ are approximated using indiscernibles, analogously to the use of terms in Lemma 2.44. These approximations are merged to give an ordinal definable definition of the direct limit in $L(\mathbb{R})$. In particular, initial segments of the direct limit maps are definable from finitely many indiscernibles.

In place of $M_\omega^\#$, we shall analyze the direct limit of a Γ-suitable mouse and prove that portions of the direct limit maps are definable within a Γ-ss mouse.

Our task is simpler in that we only need to reach up to $\boldsymbol{\delta}_{\boldsymbol{\Gamma}}^{+}$, which we show in Section 3.1 is below the least Woodin of our direct limit. So a single approximation using only finitely many sets from $\mathcal{G}$ will suffice. Another advantage we have is that there is no harm in working over a real parameter, so we can work in a Γ-ss mouse over a real which codes W_0. On the other hand, we will have some extra work to do in Section 3.2 ensuring enough information about Γ and $\mathcal{G}$ is definable in a Γ-ss mouse before we internalize the directed system in Section 3.3.

[SW16] also makes use of the fact that the derived model of $M_\omega^\#$ is essentially $L(\mathbb{R})$. So for $x \in M_\omega^\# \cap \mathbb{R}$, a Σ_1^2 statement about x is true if and only if it holds in the derived model of $M_\omega^\#$. In particular, there is a natural way to ask about Σ_1^2 truth inside of $M_\omega^\#$. A second, though minor, inconvenience of having to use a Γ-suitable mouse is we cannot talk about its derived model, since it only has one Woodin. Instead we will use the fine-structural witness condition of [SS].

REMARK 2.45. *We can associate to any Σ_1-formula ϕ a sequence of formulas $\langle \phi^k : k < \omega \rangle$ such that for any ordinal γ and any real z, $J_{\gamma+1}(\mathbb{R}) \models \phi[z] \iff (\exists k) J_\gamma(\mathbb{R}) \models \phi^k[z]$. Moreover, the map $\phi \to \langle \phi^k : k < \omega \rangle$ is recursive.*

DEFINITION 2.46. *Suppose $\phi(v)$ is a Σ_1-formula and $z \in \mathbb{R}$. A $\langle \phi, z \rangle$-witness is an ω-sound z-mouse N in which there are $\delta_0 < ... < \delta_9$, $\mathcal{S}$, and $\mathcal{T}$ such that N satisfies the formulae expressing*

1. *ZFC,*
2. *$\delta_0 < ... < \delta_9$ are Woodin,*
3. *$\mathcal{S}$ and $\mathcal{T}$ are trees on some $\omega \times \eta$ which are absolutely complementing in $V^{Col(\omega,\delta_9)}$, and*
4. *For some $k < \omega$, $\rho[T]$ is the Σ_{k+3}-theory (in the language with names for each real) of $J_\gamma(\mathbb{R})$, where γ is least such that $J_\gamma(\mathbb{R}) \models \phi^k[z]$.*

Other than iterability, the rest of the properties of being a $\langle\phi, z\rangle$-witness are first order. The following two lemmas illustrate the usefulness of this definition.

LEMMA 2.47. *If there is a $\langle\phi, z\rangle$-witness, then $L(\mathbb{R}) \models \phi[z]$.*

LEMMA 2.48. *Suppose ϕ is a Σ_1-formula, $z \in \mathbb{R}$, γ is a limit ordinal, and $J_\gamma(\mathbb{R}) \models \phi[z]$. Then there is a $\langle\phi, z\rangle$-witness N such that the iteration strategy for N restricted to countable trees is in $J_\gamma(\mathbb{R})$. By taking a Skolem hull, we can also ensure $\rho_\omega(N) = \omega$.*

§3. The Inductive-Like Case. In this section we will prove Theorem 1.12. We now assume $ZF + AD + DC + V = L(\mathbb{R})$ and fix a boldface inductive-like pointclass $\boldsymbol{\Gamma}$. By a reflection argument, we may assume $\boldsymbol{\Gamma} \neq \boldsymbol{\Sigma}^2_1$.[6]

Let $\boldsymbol{\Delta} = \boldsymbol{\Delta}_{\boldsymbol{\Gamma}}$ and let $[\alpha_0, \beta_0], \beta'$, Γ, and $\mathcal{G}$ be as in Section 2.5. We will also refer to the mouse operators $x \to W_x$ and $x \to M_x$ from Theorem 2.42 and use the notation for standard terms from Definition 2.32.

In Sections 3.1 through 3.3 we analyze the directed system of iterates of a suitable mouse and show the directed system can be approximated inside a larger suitable mouse. Section 3.4 covers some lemmas about the StrLe construction inside a suitable mouse. Section 3.5 contains a lemma we will use to obtain witnesses for Σ_1 statements inside an initial segment of a suitable mouse. Finally, Theorem 1.12 is proven in Section 3.6.

One of the key ideas to our proof of Theorem 1.12 is a different coding than the one used in [Hjo96] and [Sar22]. In [Sar22], $\boldsymbol{\Sigma}^1_{2n+2}$ sets are coded by conditions in the extender algebra at the least Woodin of some complete iterate N of $M^{\#}_{2n+1}$. The reflection argument from [Hjo96] ensures a code for each $\boldsymbol{\Sigma}^1_{2n+2}$ set appears below the least $< \delta_N$-strong cardinal κ_N of some iterate N (in fact it gives a uniform bound below κ_N). But this reflection argument depends upon the pointclass $\boldsymbol{\Sigma}^1_{2n+2}$ not being closed under coprojection.

Our proof of Theorem 1.12 instead codes $\boldsymbol{\Gamma}$-sets by sets of conditions in the extender algebra of some Γ-suitable mouse N. A weaker reflection argument than the one in [Hjo96] is used to contain each code in $N|\kappa_N$.

[6]Suppose the theorem fails for $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}^2_1$. Then an initial segment of $L(\mathbb{R})$ satisfying a large fragment of $ZF + AD + DC$ satisfies this. Reflecting this gives an inital segment N of $L(\mathbb{R})$ below $\boldsymbol{\delta}^2_1$ such that $\boldsymbol{\Gamma}' = (\boldsymbol{\Sigma}^2_1)^N$ is an inductive-like pointclass in $L(\mathbb{R})$ and N satisfies that there exists a sequence of distinct $\boldsymbol{\Gamma}'$ sets of length $\boldsymbol{\delta}_{\boldsymbol{\Gamma}}{}^+$. Since N satisfies enough of $ZF + AD + DC + V = L(\mathbb{R})$, the proof that follows will give a contradiction in N. In this case, the iteration strategy for the Γ'-suitable mouse used in the proof will not be in N. But this does not effect the argument — it is enough that the strategy is in $L(\mathbb{R})$.

This weaker reflection is sufficient for the proof.

3.1. The Direct Limit. Let $W = W_0$ and let $\mathcal{I}$ be the directed system of countable, complete iterates of W according to its (ω_1, ω_1)-iteration strategy. Let M_∞ be the direct limit of $\mathcal{I}$. For $M, N \in \mathcal{I}$ and N an iterate of M, let $\pi_{M,N} : M \to N$ be the iteration map and $\pi_{M,\infty} : M \to M_\infty$ the direct limit map. Here we demonstrate a few properties of M_∞. The proofs of this section are generalizations of arguments in [Ste09] and [SW16] giving analogous properties of the direct limit of all countable, complete iterates of $M_\omega^\#$.

Lemma 3.1. $\kappa_{M_\infty} \leq \boldsymbol{\delta_\Gamma}$ [7]

Proof. Suppose $\xi < \kappa_{M_\infty}$. Let $M \in \mathcal{I}$ and $\bar{\xi} \in M$ be such that $\pi_{M,\infty}(\bar{\xi}) = \xi$. Let P be an initial segment of M such that $\bar{\xi} \in P$ and the largest cardinal of P is both a cutpoint and a cardinal of M. The iteration strategy Σ for P is in $\boldsymbol{\Delta}$ by Remark 2.38. Let $\mathcal{I}_P$ be the directed system of countable, complete iterates of P by Σ. Then $\bar{\xi}$ is sent to ξ by the direct limit map of this system, since the largest cardinal of P is a cutpoint and a cardinal of M. So a prewellordering of height ξ is projective in Σ and therefore $\boldsymbol{\delta_\Gamma} > \xi$. ⊣

Lemma 3.2. $\delta_{M_\infty} > (\boldsymbol{\delta_\Gamma})^+$

Proof. Let Σ be the (ω_1, ω_1)-iteration strategy for W. Recall Σ is not in $\boldsymbol{\Gamma}$. We will show Σ is in $S(\delta_{M_\infty}) \backslash S(\boldsymbol{\delta_\Gamma}^+)$.

Claim 3.3. *Σ is δ_{M_∞}-Suslin.*

Proof. Let $\mathcal{T}$ be a tree on $(\omega \times \omega) \times \delta_{M_\infty}$ such that $(x, y, f) \in [\mathcal{T}]$ if and only if x codes a countable iteration tree $\mathcal{S}$ on W of limit length, y codes a cofinal, wellfounded branch b through $\mathcal{S}$, and f codes an embedding $\pi : M_b^{\mathcal{S}} \to M_\infty$ such that $\pi \circ i_b^{\mathcal{S}} = \pi_{W,\infty}$. Let $\Sigma' = \rho[\mathcal{T}]$.

If $(x, y) \in \Sigma$, then x codes an iteration tree $\mathcal{S}$ on W according to Σ and y codes the cofinal, wellfounded branch b through $\mathcal{S}$ chosen by Σ. And $\pi_{M_b^{\mathcal{S}},\infty} \circ i_b^{\mathcal{S}} = \pi_{W,\infty}$. So if $f : \omega \to \delta_{M_\infty}$ codes the embedding $\pi_{M_b^{\mathcal{S}},\infty}$, then $(x, y, f) \in [\mathcal{T}]$. Thus $(x, y) \in \Sigma'$.

On the other hand, suppose $(x, y) \in \Sigma'$ and x codes an iteration tree $\mathcal{S}$ according to Σ. Fix $f : \omega \to \delta_{M_\infty}$ such that $(x, y, f) \in [\mathcal{T}]$. Let b be the branch coded by y and π the embedding coded by f.

Subclaim 3.4. *For all n, $\pi_b^{\mathcal{S}}(\tau_n^W) = \tau_n^{M_b^{\mathcal{S}}}$.*

Proof. Let $Q \in \mathcal{I}$ be such that $range(\pi) \subseteq range(\pi_{Q,\infty})$. Let $\pi' = \pi_{Q,\infty}^{-1} \circ \pi$. Then $\pi' : M_b^{\mathcal{S}} \to Q$ and $\pi'(i_b^{\mathcal{S}}(\tau_n^W)) = \tau_n^Q$. Then by Lemma 2.34, $i_b^{\mathcal{S}}(\tau_n^W) = \tau_n^{M_b^{\mathcal{S}}}$. ⊣

[7] In fact $\kappa_{M_\infty} = \boldsymbol{\delta_\Gamma}$, but we don't need this.

From the subclaim and the last part of Lemma 2.44, we have that $(x, y) \in \Sigma$.

We can now characterize Σ as the set of $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that

1. x codes an iteration tree $\mathcal{S}$ on W of limit length,
2. y codes a cofinal, wellfounded branch through $\mathcal{S}$,
3. $(x, y) \in \Sigma'$, and
4. for any $(x_0, y_0) \leq_T x$ such that x_0 codes a proper initial segment $\mathcal{S}_0$ of $\mathcal{S}$ of limit length and y_0 codes the branch through $\mathcal{S}_0$ determined by $\mathcal{S}$, $(x_0, y_0) \in \Sigma'$.

Condition 4 is just to guarantee $\mathcal{S}$ is in the domain of Σ. It does so because any proper initial segment $\mathcal{S}_0$ of $\mathcal{S}$ is coded by some real computable from x. From this, and the preceding paragraphs, it is clear these conditions characterize Σ. Since Σ' is δ_{M_∞}-Suslin, this characterization of Σ makes plain that Σ is also δ_{M_∞}-Suslin. $\dashv$

Claim 3.5. *$\boldsymbol{\Gamma} = S(\boldsymbol{\delta_\Gamma})$.*

Proof. First, let's establish $\boldsymbol{\Gamma}$ is Suslin (we say a pointclass is Suslin if it equals $S(\lambda)$ for some cardinal λ). Let

$$\Omega = \{\boldsymbol{\Sigma_1}(J_\gamma(\mathbb{R})) : \gamma < \alpha_0 \text{ and } \gamma \text{ begins a } \Sigma_1\text{-gap}\}.$$

It follows from Theorem 2.7 that $\boldsymbol{\Gamma}$ is the minimal non-selfdual pointclass closed under projection which contains every pointclass in Ω. Let

$$\Psi = \{\boldsymbol{\Sigma_1}(J_\gamma(\mathbb{R})) \in \Omega : \boldsymbol{\Sigma_1}(J_\gamma(\mathbb{R})) \text{ is Suslin}\}.$$

By Theorem 2.8, Ψ is cofinal in Ω. But the minimal Suslin pointclass larger than any element of Ψ is just the minimal non-selfdual pointclass closed under projection which contains every pointclass in Ω (by part 3 of Theorem 2.8). Since Ψ is cofinal in Ω, this is $\boldsymbol{\Gamma}$.

So $\boldsymbol{\Gamma} = S(\lambda)$ for some cardinal λ. By the Kunen-Martin Theorem, there is a prewellordering of length λ in $\boldsymbol{\Gamma}$ but no prewellordering of length λ^+. The latter implies that $\lambda \geq \boldsymbol{\delta_\Gamma}$, since $\boldsymbol{\delta_\Gamma}$ is a limit cardinal,[8] and since there are prewellorderings of length α in $\boldsymbol{\Gamma}$ for all $\alpha < \boldsymbol{\delta_\Gamma}$. The former implies that $\lambda \leq \boldsymbol{\delta_\Gamma}$, since there is no prewellordering of length $\boldsymbol{\delta_\Gamma^+}$ in $\boldsymbol{\Gamma}$ (Otherwise a proper initial segment of this prewellordering would be of length $\boldsymbol{\delta_\Gamma}$, giving a prewellordering of length $\boldsymbol{\delta_\Gamma}$ in $\boldsymbol{\Delta}$). So $\boldsymbol{\delta_\Gamma} = \lambda$. $\dashv$

By the previous two claims, $\Sigma \in S(\delta_{M_\infty}) \backslash S(\boldsymbol{\delta_\Gamma})$. In particular, $\delta_{M_\infty} \geq \lambda'$ where λ' is the next Suslin cardinal after $\boldsymbol{\delta_\Gamma}$.[9] But $cof(\lambda') = \omega$ by part 3 of Theorem 2.8, so $\delta_{M_\infty} \geq \lambda' > \boldsymbol{\delta_\Gamma^+}$. $\dashv$

Lemma 3.6. *Suppose $\mu < \delta_{M_\infty}$ is a regular cardinal of M_∞. Then μ is not measurable in M_∞ if and only if μ has cofinality ω in $L(\mathbb{R})$.*

[8]See Theorem 7D.8 of [Mos09].

[9]In fact $\delta_{M_\infty} = \lambda'$, but we don't need this.

PROOF. Suppose μ is not measurable in M_∞. Fix $M \in \mathcal{I}$ and $\bar{\mu}$ such that $\pi_{M,\infty}(\bar{\mu}) = \mu$. Then $\bar{\mu}$ is regular but not measurable in M. Since M is countable, there is a sequence of ordinals $\langle \bar{\xi}_n : n < \omega \rangle$ cofinal in $\bar{\mu}$. Let $\xi_n = \pi_{M,\infty}(\bar{\xi}_n)$. Since $\bar{\mu}$ is regular and not measurable in M, $\pi_{M,\infty}$ is continuous at $\bar{\mu}$ (This is because $\pi_{M,\infty}$ is essentially an iteration embedding — in fact it is an iteration embedding in $V^{Col(\omega,\mathbb{R})}$. And any iteration embedding is continuous at a cardinal which is regular but not measurable, since ultrapower embeddings are continuous at such cardinals). So $\langle \xi_n : n < \omega \rangle$ is cofinal in μ.

Now suppose μ has cofinality ω in $L(\mathbb{R})$. Let $\langle \xi_n : n < \omega \rangle$ be cofinal in μ. Fix $M \in \mathcal{I}$ such that there is $\bar{\mu} \in M$ and $\langle \bar{\xi}_n : n < \omega \rangle \subset M$ with $\pi_{M,\infty}(\bar{\mu}) = \mu$ and $\pi_{M,\infty}(\bar{\xi}_n) = \xi_n$. If μ is measurable in M_∞, then there is a total extender F on the fine extender sequence of M with critical point $\bar{\mu}$. Let M' be the ultrapower of M by F and $j : M \to M'$ the embedding induced by F. Then for any $n < \omega$,

$$\begin{aligned}\xi_n &= \pi_{M,\infty}(\bar{\xi}_n) \\ &= \pi_{M',\infty} \circ j(\bar{\xi}_n) \\ &= \pi_{M',\infty}(\bar{\xi}_n) \\ &< \pi_{M',\infty}(\bar{\mu}) \\ &< \pi_{M',\infty} \circ j(\bar{\mu}) \\ &= \mu.\end{aligned}$$

So $\pi_{M',\infty}(\bar{\mu})$ is an upper bound for $\bar{\xi}_n$ below μ, a contradiction. ⊣

3.2. Definability in Suitable Mice.

LEMMA 3.7. *Suppose N is a premouse satisfying enough of ZFC, ν is a cardinal of N, $Lp^\Gamma(a) \subset N$ for each $a \in N|\nu$, and $\tau \in N^{Col(\omega,\nu)}$ weakly captures G_0. Then the map with domain $N|\nu$ defined by $a \mapsto Lp^\Gamma(a)$ is definable in N from τ.*

PROOF. Recall

$$\begin{aligned}G_0 = \{(x,y) :\ & x \text{ codes some transitive set } a \text{ and } y \text{ codes an } \omega\text{-sound} \\ & a\text{-premouse } R \text{ such that } R \text{ projects to } a \text{ and } R \text{ has an} \\ & \omega_1\text{-iteration strategy in } \mathbf{\Delta}\},\end{aligned}$$

Fix $a \in N|\nu$. If R is any set in $N|\nu$ and g is any $Col(\omega,\nu)$-generic over N, then there are reals x and y in $N[g]$ coding a and R, respectively. It is easy to see from this that $Lp^\Gamma(a)$ is

$$\bigcup\{R \in N : \emptyset \Vdash^N_{Col(\omega,\nu)} (\exists x,y)[(x,y) \in \tau \wedge x \text{ codes } a \wedge y \text{ codes } R]\}.$$

⊣

Corollary 3.8. *If P is Γ-ss, then the map with domain $P|\nu_P$ defined by $a \mapsto Lp^{\Gamma}(a)$ is definable in P from τ^P_{0,ν_P}.*

Proof. It is clear from Remark 2.18 and Corollary 2.35 that P and τ^P_{0,ν_P} satisfy the conditions of Lemma 3.7. ⊣

Lemma 3.9. *Suppose P is Γ-ss and $N \in P|\nu_P$ is Γ-suitable. Then $\{\tau^N_{n,\mu} : \mu$ is an uncountable cardinal of $N\}$ is definable in P from N and τ^P_{n,ν_P} (uniformly in P and N).*

Proof. Let μ be an uncountable cardinal of N.

Note if g is $Col(\omega, \nu_P)$-generic over P and $f \in P$ is a surjection of ν_P onto μ, then $f \circ g$ is P-generic for $Col(\omega, \mu)$. In particular, $f \circ g$ is N-generic for $Col(\omega, \mu)$. Fix such an f which is minimal in the constructibility order of P. Let

$$\begin{aligned}\tau_{n,\mu} = \{(\sigma, p) : \sigma &\text{ is a } Col(\omega,\mu)\text{-standard term for a real}, p \in Col(\omega, \mu),\\ &\text{and } \emptyset \Vdash^P_{Col(\omega,\nu_P)} (\check{p} \in \check{f} \circ \dot{g} \to \check{\sigma}[\check{f} \circ \dot{g}] \in \tau^P_{n,\nu_P})\}\end{aligned}$$

It is clear that $\tau_{n,\mu}$ is definable in P from N, μ, and τ^P_{n,ν_P}. It suffices to show $\tau_{n,\mu} = \tau^N_{n,\mu}$.

$\tau_{n,\mu} \subseteq \tau^N_{n,\mu}$ by Definition 2.32 and that comeager many $h \subset Col(\omega, \mu)$ which are generic over N are of the form $f \circ g$ for some g which is $Col(\omega, \nu_P)$-generic over P.

On the other hand, suppose $(\sigma, p) \in \tau^N_{n,\mu}$. By Corollary 2.35, $\sigma[h] \in G_n$ for any h which is $Col(\omega, \mu)$-generic over N such that $p \in h$. In particular, $\sigma[f \circ g] \in \tau^P_{n,\nu_P}[g]$ for any g which is $Col(\omega, \nu_P)$-generic over P such that $p \in f \circ g$. Thus $(\sigma, p) \in \tau_{n,\mu}$. ⊣

We will also need versions of Corollary 3.8 and Lemma 3.9 in generic extensions of Γ-ss mice.

Lemma 3.10. *Suppose $B \subseteq \mathbb{R}$, P is a premouse, δ is Woodin in P, $\mu \geq \delta$, $\tau \in P^{Col(\omega,\mu)}$ weakly captures B over P, and y is Ea_P-generic over P. Then there is $\tau' \in P[y]^{Col(\omega,\mu)}$ which weakly captures B over $P[y]$. Moreover, τ' is definable in $P[y]$ from τ and y (uniformly).*

Proof. $Col(\omega, \mu)$ is universal for pointclasses of size μ. So there is a complete embedding $\Phi : Ea_p \times Col(\omega, \mu) \to Col(\omega, \mu)$.[10] If g is $Col(\omega, \mu)$-generic over P, let (y_g, f_g) be the $Ea_P \times Col(\omega, \mu)$-generic consisting of all conditions $(p, q) \in Ea_P \times Col(\omega, \mu)$ such that $\Phi((p, q)) \in g$ (see Chapter 7,

[10]In the sense of Definition 7.1 of Chapter 7 of [Kun83].

Theorem 7.5 of [Kun83]). Let

$$\begin{aligned}\tau^* = \{(\sigma,(p,q)) : &\sigma \text{ is an } Ea_P\text{-term for a } Col(\omega,\mu)\text{-standard term}\\ &\text{for a real, } (p,q) \in Ea_P \times Col(\omega,\mu), \text{ and}\\ &\Phi((p,q)) \Vdash^P_{Col(\omega,\mu)} \sigma[y_g][f_g] \in \tau[g]\}.\end{aligned}$$

CLAIM 3.11. *For any (y,f) which is $Ea_P \times Col(\omega,\mu)$-generic over P, $\tau^*[y][f] = B \cap P[y][f]$.*

PROOF. Suppose $x \in \tau^*[y][f]$. $x = \sigma[y][f]$ for some $(\sigma,(p,q)) \in \tau^*$ such that $p \in y$ and $q \in f$. Let g be $Col(\omega,\mu)$-generic such that $y_g = y$ and $f_g = f$. In particular, $\Phi((p,q)) \in g$. Then $P[g] \models \sigma[y_g][f_g] \in \tau[g]$. Since $x = \sigma[y_g][f_g]$ and $\tau[g] = B \cap P[g]$, $x \in B \cap P[g]$.

Now suppose $x \in B \cap P[y][f]$. Let σ be an Ea_P-term for a $Col(\omega,\mu)$-standard term for a real such that $x = \sigma[y][f]$.

$\bigcup \Phi''\{(p,q) : (p,q) \in y \times g\}$ is a function $g_1 : S \to \mu$ for some $S \subseteq \omega$. Let

$$\mathbb{Q} = \{r \in Col(\omega,\mu) : domain(r) \cap S = \emptyset\}$$

($\mathbb{Q}$ is the quotient of $Col(\omega,\mu)$ by g_1). Let g_2 be $\mathbb{Q}$-generic over $P[g_1]$. Then $g = g_1 \cup g_2$ is $Col(\omega,\mu)$-generic over P.

We have $x \in \tau[g]$. Pick $s \in g$ such that $s \Vdash^P_{Col(\omega,\mu)} \sigma[y_g][f_g] \in \tau[g]$. $s = r_1 \cup r_2$ for some $r_1 \in g_1$ and $r_2 \in g_2$.

SUBCLAIM 3.12. $r_1 \Vdash^P_{Col(\omega,\mu)} \sigma[y_g][f_g] \in \tau$.

PROOF. Suppose not. Then there is g_2' which is $\mathbb{Q}$-generic over $P[g_1]$ such that, letting $g' = g_1 \cup g_2'$, $\sigma[y_{g'}][f_{g'}] \notin \tau[g']$. $\sigma[y_{g'}][f_{g'}] = x$, since y_g and f_g depend only on $g \restriction S$. But then $x \in (B \cap P[g']) \setminus \tau[g']$, contradicting that τ weakly captures B. ⊣

Pick $p \in y$ and $q \in f$ such that $\Phi((p,q))$ extends r_1. Then $(\sigma,(p,q)) \in \tau^*$. So $x \in \tau^*[y][f]$. ⊣

Let

$$\tau' = \{(\sigma[y],q) : \exists p \in y \text{ such that } (\sigma,(p,q)) \in \tau^*\}.$$

τ' is definable in $P[y]$ from τ and y. It is clear from Claim 3.11 that τ' weakly captures B over $P[y]$. ⊣

LEMMA 3.13. *Let P be Γ-ss and y be Ea_P-generic over P. Then for any $a \in P[y]$, $Lp^\Gamma(a) \subset P[y]$.*

PROOF. Let N be a Γ-suitable mouse built over P. N has a Woodin cardinal δ_N above δ_P. The iteration strategy for any proper initial segment of $N|\delta_N$ restricted to trees above δ_P is in $\mathbf{\Delta}$. And no initial segment of N above δ_N projects strictly below δ_N. It follows that any cardinal of P remains a cardinal in N. In particular, δ_P remains Woodin in N and y is also Ea_P-generic over N.

Suppose R is an ω-sound a-premouse with an ω_1-iteration strategy in $\boldsymbol{\Delta}$ such that R projects to a. It suffices to show $R \in P[y]$.

Let α be the height of R. Iterating N above R if necessary, we may assume there is a real g which is $Col(\omega, \delta_N)$-generic over N such that some real in $N[y][g]$ codes R. By Lemma 3.10, there is a $Col(\omega, \delta_N)$-term τ in $N[y]$ which weakly captures G_0. Then R is the unique premouse in $N[y][g]$ of height α such that if x_a codes a and x_R codes R, then $(x_a, x_R) \in \tau[g]$. By homogeneity of the forcing, for any g' which is $Col(\omega, \delta_N)$-generic over N, there is a premouse $R' \in N[y][g']$ of height α and reals x_a and $x_{R'}$ in $N[y][g']$ coding a and R', respectively, such that $(x_a, x_{R'}) \in \tau[g']$. The uniqueness of R implies $R \in N[y]$. Since R is coded by a subset of a, $R \in P[y]$. $\dashv$

Corollary 3.14. *If P is Γ-ss and y is Ea_P-generic over P, then the map with domain $P[y]|\nu_P$ defined by $a \mapsto Lp^\Gamma(a)$ is definable in $P[y]$ from τ^P_{0,ν_P} and y (uniformly in P and y).*

Proof. $P[y]$ is Lp^Γ-closed by Lemma 3.13. Then by Lemma 3.7, the map $a \to Lp^\Gamma(a)$ with domain $P[y]|\nu_P$ is definable from any term $\tau \in P[y]^{Col(\omega,\nu_P)}$ which weakly captures G_0 over $P[y]$.

Lemma 3.10 shows there is a term $\tau \in P[y]^{Col(\omega,\nu_P)}$ which weakly captures G_0 over $P[y]$ and is definable from τ^P_{0,ν_P} and y in $P[y]$. $\dashv$

Corollary 3.15. *Suppose P is Γ-ss, y is Ea_P-generic over P, and $N \in P[y]|\nu_P$ is Γ-suitable. Then $\{\tau^N_{n,\mu} : \mu$ is an uncountable cardinal of $N\}$ is definable in $P[y]$ from N, y, and τ^P_{n,ν_P} (uniformly in P, y, and N).*

Proof. This is by the proof of Lemma 3.9, using from Lemma 3.10 that there is a term in P which weakly captures G_n over $P[y]$ and is definable from τ^P_{n,ν_P} and y. $\dashv$

3.3. Internalizing the Direct Limit. Let $x_0 \in \mathbb{R}$ be any real which is Turing above some real coding W and consider some M which is a countable, complete iterate of M_{x_0}. For elements of $M|\nu_M$, being a Γ-suitable premouse, a Γ-short iteration tree, or a Γ-maximal iteration tree is definable over M from τ^M_{0,ν_M} (This follows easily from Corollary 3.8). Let

$$\mathcal{I}^M = \{P \in M|\nu_M : P \in \mathcal{I}\}.$$

Lemma 3.16. *Let $\mathcal{T} \in M|\nu_M$ be a Γ-short tree on some Γ-suitable $P \in M$. Then the branch b picked by the iteration strategy for P is in M and b is definable in M from $\mathcal{T}$ and τ^M_{0,ν_M} (uniformly). In particular, $M^\mathcal{T}_b$ and the iteration map $i^\mathcal{T}_b : P \to M^\mathcal{T}_b$ are definable in M from $\mathcal{T}$ and τ^M_{0,ν_M}.*

Proof. Let g be $Col(\omega, \nu_M)$-generic over M. Note b is the unique branch through $\mathcal{T}$ which absorbs $\mathcal{Q}(\mathcal{T})$. So by Shoenfield absoluteness,

$b \in M[g]$ (in $M[g]$ the existence of such a branch is a Σ^1_2 statement about reals). But b is independent of the generic g, so $b \in M$.

It then follows from Corollary 3.8 that b, and therefore also $M^{\mathcal{T}}_b$ and $i^{\mathcal{T}}_b$, are definable in M from τ^M_{0,ν_M}. ⊣

Corollary 3.17. *Suppose $P \in \mathcal{I}^M$ and Σ is the iteration strategy for P. Suppose also $\mathcal{T} \in M|\nu_M$ is an iteration tree on P below δ_P of limit length. Whether $\mathcal{T}$ is according to Σ is definable in M from parameter τ^M_{0,ν_M} by a formula independent of $\mathcal{T}$ and the choice of Γ-ss mouse M.*

Lemma 3.18. *Suppose $P, Q \in \mathcal{I}^M$. Then there is $R \in \mathcal{I}^M$ and normal iteration trees $\mathcal{T}$ and $\mathcal{U}$ on P and Q, respectively, such that*

1. *$\mathcal{T}$ realizes R is a complete iterate of P,*
2. *$\mathcal{U}$ realizes R is a complete iterate of Q,*
3. *$\mathcal{T} \restriction lh(\mathcal{T}) \in M|\nu_M$,*
4. *$\mathcal{U} \restriction lh(\mathcal{U}) \in M|\nu_M$, and*
5. *R is definable in M from P, Q, and τ^M_{0,ν_M} (uniformly).*

Proof. We perform a coiteration of P and Q inside M. Suppose so far from the coiteration we have obtained iteration trees $\mathcal{T}$ and $\mathcal{U}$ on P and Q, respectively.

Suppose $\mathcal{T}$ and $\mathcal{U}$ have successor length. Let P' and Q' be the last models of $\mathcal{T}$ and $\mathcal{U}$, respectively. First consider the case $P' \trianglelefteq Q'$ or $P' \trianglelefteq Q'$. If either is a proper initial segment of the other, or there are any drops on the branches to P' or Q', we have violated the Dodd-Jensen property. So $P' = Q'$ and P' is a common, complete iterate of P and Q. Otherwise, we continue the coiteration as usual by applying the extender at the least point of disagreement between the last models of $\mathcal{T}$ and $\mathcal{U}$, respectively.

Now suppose $\mathcal{T}$ and $\mathcal{U}$ are of limit length. In this case $\mathcal{M}(\mathcal{T}) = \mathcal{M}(\mathcal{U})$. If $\mathcal{T}$ is Γ-short, so is $\mathcal{U}$, and by Lemma 3.16, M can identify the branches the iteration strategies for P and Q pick through $\mathcal{T}$ and $\mathcal{U}$, respectively. So the coiteration can be continued inside M. Otherwise, $\mathcal{T}$ and $\mathcal{U}$ are Γ-maximal. In this case let R be the unique Γ-suitable mouse extending $\mathcal{M}(\mathcal{T})$. R is just the result of applying Lp^Γ to $\mathcal{M}(\mathcal{T})$ ω times, so M can identify R by Lemma 3.7. Then R is a complete iterate of P and Q.

The proof of the Comparison Lemma gives the coiteration terminates in fewer than ν_M steps. Then the argument above implies the trees from this coiteration, without their last branches, are in $M|\nu_M$ and definable in M. ⊣

The lemma implies $\mathcal{I}^M$ is a directed system. $\mathcal{I}^M$ is countable and contained in $\mathcal{I}$, so we may define the direct limit $\mathcal{H}^M$ of $\mathcal{I}^M$, and $\mathcal{H}^M \in \mathcal{I}$.

Let

$$\begin{aligned}\tilde{\mathcal{I}}^M = \{P \in \mathcal{I}^M : &\text{ there is a normal iteration tree } \mathcal{T} \text{ such that } \mathcal{T} \text{ realizes} \\ &P \text{ is a complete iterate of } W \text{ and } \mathcal{T} \restriction lh(\mathcal{T}) \in M|\nu_M\}.\end{aligned}$$

. $\tilde{\mathcal{I}}^M$ is definable in M by Corollary 3.17.

Lemma 3.19. *$\tilde{\mathcal{I}}^M$ is cofinal in $\mathcal{I}^M$. In particular, the direct limit of $\tilde{\mathcal{I}}^M$ is $\mathcal{H}^M$.*

Proof. Suppose $P \in \mathcal{I}^M$. By Lemma 3.18, there is $R \in \mathcal{I}^M$ which is a common, complete, normal iterate of both P and W by trees which are in M (modulo their final branches). Then R is below P in $\mathcal{I}^M$ and $R \in \tilde{\mathcal{I}}^M$. ⊣

Lemma 3.20. *Suppose $P \in \mathcal{I}^M$. Let Σ be the (unique) iteration strategy for P. Suppose $\mathcal{T} \in M|\nu_M$ is an iteration tree on P according to Σ. Let $b = \Sigma(\mathcal{T})$ and let $Q = M_b^{\mathcal{T}}$. Then Q is definable in M from $\mathcal{T}$ and τ^M_{0,ν_M}. And $\pi_{P,Q} \restriction \gamma_n^P$ is definable in M from $\mathcal{T}$ and $\langle \tau^M_{k,\nu_M} : k < n\rangle$ (uniformly).*

Proof. If $\mathcal{T}$ is Γ-short, then this is by Lemma 3.16.

Suppose $\mathcal{T}$ is Γ-maximal. Then $Q = \bigcup_{i<\omega} Q_i$, where $Q_0 = \mathcal{M}(\mathcal{T})$ and $Q_{i+1} = Lp^\Gamma(Q_i)$. So Q is definable from $\mathcal{M}(\mathcal{T})$ and τ^M_{0,ν_M} by Corollary 3.8. And $\pi_{P,Q} \restriction \gamma_n^P = \pi_c \restriction \gamma_n^P$, where c is any branch through $\mathcal{T}$ respecting $\vec{G}_n$. The argument of Lemma 3.16 shows there is a branch c in M respecting $\vec{G}_n$. Then $\pi_{P,Q} \restriction \gamma_n^P = \pi_c \restriction \gamma_n^P$ for any wellfounded branch $c \in M$ through $\mathcal{T}$ such that $\pi_c(\langle \tau_k^P : k < n\rangle) = \langle \tau_k^Q : k < n\rangle$. $\langle \tau_k^P : k < n\rangle$ and $\langle \tau_k^Q : k < n\rangle$ are definable in M from P, Q, and $\langle \tau^M_{k,\nu_M} : k < n\rangle$ by Lemma 3.9. So $\pi_{P,Q} \restriction \gamma_n^P$ is definable in M from $\mathcal{T}$ and $\langle \tau^M_{k,\nu_M} : k < n\rangle$. ⊣

It follows from the previous lemmas that for any $P \in \mathcal{I}^M$, $\pi_{P,\mathcal{H}^M} \restriction \gamma_n^P$ is definable in M from P and $\langle \tau^M_{k,\nu_M} : k < n\rangle$ (uniformly in M). The same lemmas hold in $M[y]$ for y Ea_M-generic over M. In particular, we have:

Lemma 3.21. *Suppose y is Ea_M-generic over M and $P \in \mathcal{I} \cap M[y]|\nu_M$. Let Σ be the (unique) iteration strategy for P. Suppose $\mathcal{T} \in M[y]|\nu_M$ is an iteration tree on P according to Σ. Let $b = \Sigma(\mathcal{T})$ and let $Q = M_b^{\mathcal{T}}$. Then Q is definable in $M[y]$ from $\mathcal{T}$ and τ^M_{0,ν_M}. And $\pi_{P,Q} \restriction \gamma_n^P$ is definable in $M[y]$ from $\mathcal{T}$ and $\langle \tau^M_{k,\nu_M} : k < n\rangle$ (uniformly). Moreover, the definition is independent not just of the choice of Γ-ss mouse M, but also of the generic y.*

Lemma 3.22. *Suppose $p \in Ea_M$ and $\dot{S}$ is an Ea_M-name in $M|\nu_M$ such that $p \Vdash_{Ea_M}$ "$\dot{S}$ is a complete iterate of W." Then there is $R \in \tilde{I}^M$ such that R is a complete iterate of $S[y]$ for every $y \in \mathbb{R}$ which is Ea_M-generic over M. Moreover, we can pick R such that R is (uniformly) definable in M from parameters $\dot{S}$ and p.*

Proof. Let $\mathbb{P}$ be the finite support product $\Pi_{j<\omega}\mathbb{P}_j$, where each $\mathbb{P}_j$ is a copy of the part of Ea_M below p. Let H be $\mathbb{P}$-generic over M. We can represent H as $\Pi_{j<\omega}H_j$, where H_j is $\mathbb{P}_j$-generic over M. Let $\dot{S}_j$ be a $\mathbb{P}$-name for $\dot{S}[H_j]$. Let $S_j = \dot{S}_j[H]$ for $j \in \omega$ and $S_{-1} = W$.

Lemma 3.21 tells us that $M[H]$ can perform the simultaneous coiteration of all of the S_j for $j \in [-1, \omega)$ (except possibly finding the last branches). The proof of the Comparison Lemma gives that this coiteration terminates after fewer than ν_M steps. Let R_j be the last model of the iteration tree on S_j produced by the coiteration. Since each S_j is a complete iterate of W, the Dodd-Jensen property implies there are no drops on the branches from S_j to R_j and $R_j = R_i$ for all $i, j \in [-1, \omega)$. Let $R = R_j$ for some (equivalently all) $j \in [-1, \omega)$. Then R is a complete iterate of M and R is a complete iterate of S_j for each $j \in \omega$. Let $\mathcal{U}$ be the iteration tree on W from the coiteration.

Claim 3.23. *R is independent of the choice of generic H.*

Proof. Code R by a set of ordinals X contained in ν_M. Let $\dot{X}$ be a name for X. If R is not independent of H, then there is $\alpha < \nu_M$ and $q_1, q_2 \in \mathbb{P}$ such that $q_1 \Vdash \check{\alpha} \in \dot{X}$ and $q_2 \Vdash \check{\alpha} \notin \dot{X}$.

Let $N > max(support(q_2))$. Let $\bar{q}_1$ be the condition q_1 shifted over by N — that is, $support(\bar{q}_1) = \{j \in [N, \omega) : j - N \in support(q_1)\}$ and for $j \in support(\bar{q}_1)$, $\bar{q}_1(j) = q_1(j - N)$. So $\bar{q}_1$ is compatible with q_2 and by symmetry, $\bar{q}_1 \Vdash \check{\alpha} \in \dot{X}$. But then there is $r \leq q_2, \bar{q}_1$ which forces both $\check{\alpha} \in \dot{X}$ and $\check{\alpha} \notin \dot{X}$. ⊣

Claim 3.24. *$\mathcal{U} \restriction lh(\mathcal{U})$ is independent of the choice of generic H.*

Proof. The same proof as in Claim 3.23 works. ⊣

Claim 3.23 implies $R \in M|\nu_M$ and R is a complete iterate of $S[y]$ for any y which is Ea_M-generic over M. Claim 3.24 gives that $\mathcal{U} \restriction lh(\mathcal{U}) \in M|\nu_M$ and thus $R \in \tilde{\mathcal{I}}^M$. ⊣

3.4. The StrLe Construction. Recall the mouse operator $x \to M_x$ defined in Section 2.5. In the following lemmas let $z, x \in \mathbb{R}$ be such that $z \in M_x$ and let $M = M_x$.

Lemma 3.25. *Suppose $P = StrLe[M, z]$. Then P is Γ-ss and $\delta_P = \delta_M$.*

Proof. Let $\delta = \delta_M$. By Lemma 2.24, the cardinals of P above δ are the same as the cardinals of M and ν_M is inaccessible in P. Any inaccessible of P above δ is inaccessible in M, since M is a generic extension of P by a δ-c.c. forcing. In particular, ν_M is the unique inaccessible of P above δ. Then it suffices to show the following claim.

Claim 3.26. *(a) If $\eta < \delta$, then $Lp^\Gamma(P|\eta) \triangleleft P$.*

(b) δ is a Γ-Woodin of P. That is, δ is Woodin in $Lp^\Gamma(P|\delta)$.
(c) If $\eta \in P$ and $\eta \geq \delta$, then $P|(\eta^+)^P \trianglelefteq Lp^\Gamma(P|\eta)$.
(d) $P \models \delta$ is Woodin.
(e) If $\eta < \delta$, η is not Woodin in $Lp^\Gamma(P|\eta)$.
(f) If $\eta \in P$ and $\delta \leq \eta$, then $Lp^\Gamma(P|\eta) \subseteq P$.

PROOF. To prove (a), it suffices to show if $\eta < \delta$, $R \triangleleft Lp^\Gamma(P|\eta)$, and $\rho_\omega(R) = \eta$, then $R \triangleleft P$. Coiterate R against $Le[M, z]$. Suppose $\mathcal{T}$ and $\mathcal{U}$ are the iteration trees on R and $Le[M, z]$, respectively, from the coiteration. $\mathcal{T}$ is above η because $Le[M, z]|\eta = R|\eta$ and η is a cutpoint of R. Let $\lambda < lh(\mathcal{T})$ be a limit ordinal and $Q = \mathcal{Q}(\mathcal{T}) = \mathcal{Q}(\mathcal{U})$. Since $R \in Lp^\Gamma(P|\eta)$ and $\mathcal{T}$ is above η, $Q \in Lp^\Gamma(\mathcal{M}(\mathcal{T}))$. $[0, \lambda]_T$ and $[0, \lambda]_U$ are the unique branches through $\mathcal{T}$ and $\mathcal{U}$, respectively, which absorb Q. By Corollary 3.8, these branches can be identified in M. In particular, the coiteration of R and $Le[M, z]$ can be performed in M. Theorem 2.21 gives that R cannot outiterate $Le[M, z]$. Then since R is ω-sound, R projects to η, and $Le[M, z]$ does not project to η, R is a proper initial segment of $Le[M, z]|(\eta^+)^{Le[M,z]}$. $Le[M, z]$ agrees with P up to δ, so $R \triangleleft P$.

(b) is by the proof of Theorem 11.3 of [MS17]. For (c), the iteration strategies for initial segments of $P|(\eta^+)^P$ restricted to iteration trees above δ are in $\boldsymbol{\Delta}$ by Fact 2.25. (d) is immediate from (b) and (c). See Sublemma 7.4 of [SW16] for a proof of (e).

Towards (f), let $Q = Lp^\Gamma(P|\eta)$. Let $\mathbb{P}$ be the extender algebra in P at δ with δ generators. $M|\delta$ is $\mathbb{P}$-generic over P. Note δ is Woodin in Q by (b). In particular, $\mathbb{P}$ is also δ-c.c. in Q, so any antichain of $\mathbb{P}$ in Q is also in P and $M|\delta$ is also $\mathbb{P}$-generic over Q.

Let $B \in Lp^\Gamma(P|\eta)$. B is in $M = P[M|\delta]$ since $P|\eta$ is in M and M is closed under Lp^Γ. So let $\dot{B}$ be a $\mathbb{P}$-name in P such that $\dot{B}[M|\delta] = B$.

Choose $p \in \mathbb{P}$ such that $p \Vdash^Q_{\mathbb{P}} \dot{B} = \check{B}$.

Any G which is $\mathbb{P}$-generic over P is also $\mathbb{P}$-generic over Q. So for any G which is $\mathbb{P}$-generic over P such that $p \in G$, $\dot{B}[G] = B$. But then B is in P, since $B = \{\xi < \delta : p \Vdash^P_{\mathbb{P}} \check{\xi} \in \dot{B}\}$.[11] $\dashv$

$\dashv$

LEMMA 3.27. *Suppose $P = StrLe[M, z]$. Let $\mu \geq \delta_P$ be a cardinal of P. $\tau^P_{n,\mu}$ is definable in M from $\tau^M_{n,\mu}$ and z.*

PROOF. Let

$$\tau = \{(\sigma, p) \in P : \sigma \text{ is a } Col(\omega, \mu)\text{-standard term for a real}, p \in Col(\omega, \mu), \text{ and } p \Vdash^M_{Col(\omega,\mu)} \sigma \in P[\dot{g}] \cap \tau^M_{n,\mu}\}.$$

[11] Viewing B as a subset of δ.

Since $\tau^M_{n,\mu} \in M$ and P is definable over M from z, $\tau \in M$ and is definable from $\tau^M_{n,\mu}$ and z. Then it suffices to show the following claim.

CLAIM 3.28. $\tau = \tau^P_{n,\mu}$

PROOF. Clearly $\tau \subseteq \tau^P_{n,\mu}$.

Suppose $(\sigma, p) \in \tau$. Let $\mathcal{C}$ be the set of g which are $Col(\omega, \mu)$-generic over M such that $p \in g$. For any $g \in \mathcal{C}$, $\sigma[g] \in \tau^M_{n,\mu}[g]$. In particular, $\sigma[g] \in G_n$. Since $\mathcal{C}$ is comeager in the set of $Col(\omega, \mu)$-generics over P which extend p, $\sigma \in \tau^P_{n,\mu}$. ⊣

⊣

LEMMA 3.29. *Suppose $P = StrLe[M, z]$. The iteration strategy for P is fullness-preserving and guided by $\mathcal{G}$.*

PROOF. Let Σ be the (unique) iteration strategy for P. The proof of Theorem 2.22 gives that Σ is determined by lifting an iteration on P to one on M. More precisely, if $\mathcal{T}$ is a non-dropping[12] iteration tree on $P_0 = P$ with $\langle P_\alpha \rangle$ the models of the iteration and $i_{\beta,\alpha}$ the associated iteration maps for $\beta <_T \alpha$, then we maintain an iteration tree $\mathcal{T}^*$ on $M_0 = M$ with models $\langle M_\alpha \rangle$ and associated iteration embeddings $i^*_{\beta,\alpha}$. We also maintain embeddings $\pi_\alpha : P_\alpha \to StrLe[M_\alpha, z]$ such that $\pi_\alpha \circ i_{\beta,\alpha} = i^*_{\beta,\alpha} \circ \pi_\beta$ and $\pi_0 = id$. In particular, $\pi_\alpha \circ i_{0,\alpha} = i^*_{0,\alpha}$.

Suppose μ is a cardinal of P and $\mu > \delta_P$. By Lemma 3.27, $i^*_{0,\alpha}(\tau^P_{n,\mu}) = \tau^{StrLe[M_\alpha,z]}_{n,\mu}$ for each $n < \omega$. Then $\pi_\alpha \circ i_{0,\alpha}(\tau^P_{n,\mu}) = \tau^{StrLe[M_\alpha,z]}_{n,\mu}$. Then by Lemma 2.34, P_α is Γ-ss and $\pi_\alpha(\tau^{P_\alpha}_{n,\mu}) = \tau^{StrLe[M_\alpha,z]}_{n,\mu}$. This gives Σ is fullness-preserving. A second application of Lemma 2.34 gives $i_{0,\alpha}(\tau^P_{n,\mu}) = \tau^{P_\alpha}_{n,\mu}$. So Σ is guided by $\mathcal{G}$. ⊣

COROLLARY 3.30. *Suppose $P = StrLe[M, z]$. Then the ω_1-iteration strategy for P is not in $\mathbf{\Gamma}$.*

PROOF. Immediate from Lemmas 2.41 and 3.29. ⊣

LEMMA 3.31. *Suppose $x, z \in \mathbb{R}$ and x codes a mouse N which is a complete iterate of M_z. Let $P = StrLe[M_x, z]$. Then P is a complete iterate of N below δ_N.*

PROOF. Coiterate N and P. Let $\mathcal{T}$ and $\mathcal{U}$ be the iteration trees on N and P, respectively, from the coiteration. Let N^* and P^* be the last models of $\mathcal{T}$ and $\mathcal{U}$, respectively.

Suppose P outiterates N. One possibility is that there is a drop on the branch of $\mathcal{T}$ from P to P^*. Since the iteration strategy for P is fullness-preserving by Lemma 3.29, P^* has an ω_1-iteration strategy in $\mathbf{\Delta}$. But the

[12]We leave to the reader the task of proving the case where $\mathcal{T}$ drops, as well as showing that Σ condenses well.

strategy for N is fullness-preserving and guided by $\mathcal{G}$. So N^* cannot have an iteration strategy in $\boldsymbol{\Delta}$, contradicting that $N^* \trianglelefteq P^*$.

If there is no drop between P and P^*, then $N^* \triangleleft P$. Since neither side of the coiteration drops, N^* and P^* are both Γ-ss. But no Γ-ss mouse can have a proper initial segment which is Γ-ss.

An identical argument shows N cannot outiterate P. Thus $N^* = P^*$ and $\mathcal{T}$ and $\mathcal{U}$ realize N^* and P^* are complete iterates of N and P, respectively. Since there are no total extenders on N above δ_N, $\mathcal{T}$ is below δ_N. Similarly, $\mathcal{U}$ is below δ_P. Then stationarity of the Mitchell-Steel construction[13] implies that $P^* = P$. So $\mathcal{T}$ realizes that P is a complete iterate of N. $\dashv$

3.5. A Reflection Lemma. In this section we prove a lemma that any Σ_1 statement true in M_x also holds in some $N \triangleleft M_x|\kappa_{M_x}$ with the property that $StrLe[N] \triangleleft StrLe[M]$. A thorough reader not already familiar with the fully-backgrounded Mitchell-Steel construction may wish to review Section 2.3 before proceeding. A lazy one may read the statement of Lemma 3.35 and skip to Section 3.6.

First, we need to show M_x can compute the iteration strategies of its own initial segments below its Woodin cardinal. More precisely, we have:

LEMMA 3.32. *Let $x \in \mathbb{R}$, $N \triangleleft M_x|\delta_{M_x}$ and $\mathcal{T} \in M_x$ be an iteration tree on N of limit length $< \delta_{M_x}$, according to the (unique) iteration strategy for N. The cofinal branch b through $\mathcal{T}$ determined by the iteration strategy for N is definable in M_x (uniformly in N and $\mathcal{T}$, from the parameter $\tau^{M_x}_{0,\nu_{M_x}}$).*

PROOF. Let $M = M_x$. By Corollary 3.8, the function $a \mapsto Lp^\Gamma(a)$ with domain $M|\delta_M$ is definable in M from the parameter τ^M_{0,ν_M}.

Let N and $\mathcal{T}$ be as in the statement of the lemma. Let $S = \mathcal{M}(\mathcal{T})$. Clearly S is definable from $\mathcal{T}$. Let $Q = \mathcal{Q}(\mathcal{T})$. Q is an initial segment of $Lp^\Gamma(S)$. The previous paragraph implies Q is definable in M from S and τ^M_{0,ν_M}. The branch b through $\mathcal{T}$ chosen by the iteration strategy for N is the unique branch which absorbs Q.

It remains to show b is in M. Iterate M to M' well above where $\mathcal{T}$ is constructed to make some g generic over $Ea^{M'}_{\delta_{M'}}$ so that g codes b. $M'[g]$ satisfies that b is the unique branch which absorbs Q. Since b is in fact the unique such branch in V, symmetry of the forcing gives b is in M'. But the iteration from M to M' does not add any subsets of $lh(\mathcal{T})$, so in fact b is in M. $\dashv$

We need to put down a few more properties of the Mitchell-Steel construction before proving the main lemma of this section.

LEMMA 3.33. *Suppose N is a mouse with a Woodin cardinal δ_N. Let $z \in N \cap \mathbb{R}$. There is a club C of $\tau < \delta_N$ such that $Le[N|\delta_N, z]|\tau = \mathcal{M}_\tau$,*

[13] See e.g. 3.23 of [ST16].

where $\mathcal{M}_\tau$ is the Mitchell-Steel construction of length τ in $N|\delta_N$. Moreover, we can take C to be definable in N.

Proof. Let $\langle \mathcal{M}_\xi : \xi < \delta_N \rangle$ be the models from the Mitchell-Steel construction of length δ_N over z, done inside $N|\delta_N$. Let $C' \subset \delta_N$ be the set of $\tau < \delta_N$ such that $\mathcal{M}_\tau$ has height τ and $\rho_\omega(\mathcal{M}_\xi) \geq \tau$ whenever ξ is between τ and the height of N. It is not hard to see from the material in Section 2.3 that C is a club and if $\tau \in C$, then $\mathcal{M}_\tau = Le[N|\delta_N, z]|\tau$. $\dashv$

Corollary 3.34. *Let N, z, and C be as in Lemma 3.33. Let S be the set of inaccessibles of N below δ_N. Then $Le[N|\delta_N, z] = \bigcup_{\tau \in C \cap S} Le[N|\tau, z]$.*

Proof. Since δ_N is Woodin in N, $N \models$ "S is stationary." And C is definable in N, so $C \cap S$ is cofinal in δ_N. Since $Le[N|\delta_N, z]$ has height δ_N, $Le[N|\delta_N, z] = \bigcup_{\tau \in C \cap S} Le[N|\delta_N, z]|\tau$. So it suffices to show if $\tau \in C \cap S$, then $Le[N, z]|\tau = Le[N|\tau, z]$.

Let $\langle \mathcal{M}_\xi : \xi < \delta_N \rangle$ be the models from the Mitchell-Steel construction of length δ_N over z, done inside N. $\tau \in C$ guarantees $Le[N, z]|\tau = \mathcal{M}_\tau$. And by Remark 2.23, $\tau \in S$ gives $\mathcal{M}_\tau = Le[N|\tau, z]$. So $Le[N, z]|\tau = Le[N|\tau, z]$ for $\tau \in C \cap S$. $\dashv$

Lemma 3.35. *Suppose $M_x \models \phi[\vec{a}, \delta_{M_x}]$ for some Σ_1 formula ϕ, $z \in \mathbb{R} \cap M_x$, and $\vec{a} \in \mathbb{R}^{|\vec{a}|} \cap M_x$. Then there exists $N \lhd M_x|\kappa_{M_x}$ such that*

(a) N has one Woodin cardinal,
(b) δ_N is an inaccessible cardinal of M_x,
(c) $N \models \phi[\vec{a}, \delta_N]$, and
(d) $StrLe[N, z] \lhd StrLe[M_x, z]$.

Proof. Denote M_x by M. For ease of notation we will assume $z = 0$. Let μ be a cardinal of M above δ_M such that $M|\mu \models \phi[\vec{a}]$.

Claim 3.36. *There is a stationary set of $\tau < \delta_M$ such that τ is inaccessible in M and if $\tau \leq \zeta < \delta_M$, then ζ is not definable in $M|\mu$ from parameters below τ.*

Proof. Work in M. Let S be the set of inaccessible cardinals below δ_M. Since δ_M is Woodin, S is stationary. Define $f : S \to \delta_M$ by setting $f(\zeta)$ to be the least η such that there is $\zeta \leq \iota < \delta_M$ definable in $M|\mu$ from parameters in η. If the claim is false, then f is regressive on a stationary set. Then by Fodor's Lemma, there is a stationary set S_0 and $\eta < \delta_M$ such that $f''S_0 = \{\eta\}$. But $cof(\delta_M) > |\eta^{<\omega}| \times \aleph_0$, so we cannot have cofinally many elements of δ_M defined by some formula and parameters from η. $\dashv$

Fix τ as in Lemma 3.33 and Claim 3.36. Let $H = Hull^{M|\mu}(\tau)$. Let N be the transitive collapse of H and $\pi : N \to M|\mu$ the anti-collapse map.

By condensation, $N \triangleleft M|\mu$.[14] Clearly $N \triangleleft M|\delta_M$, $N \models \phi[\vec{a}, \delta_N]$, τ is the unique Woodin of N, τ is inaccessible in M, and $\rho_\omega(N) = \tau$.

Claim 3.37. $Le[N|\tau] \triangleleft Le[M|\delta_M]$

Proof. For $\zeta < \tau$, $Le[N|\zeta] \triangleleft Le[N|\tau] \iff Le[M|\zeta] \triangleleft Le[M|\delta_M]$ by elementarity. But $Le[N|\zeta] = Le[M|\zeta]$ for $\zeta < \tau$. So if $Le[N|\zeta]$ is an initial segment of $Le[N|\tau]$, then it is also an initial segment of $Le[M|\delta_M]$. But this implies $Le[N|\tau] \triangleleft Le[M|\delta_M]$, since by Corollary 3.34, $Le[N|\tau]$ is a union of mice of the form $Le[N|\zeta]$ for $\zeta < \tau$. $\dashv$

We have found $N \triangleleft M|\delta_M$ satisfying (a), (b), (c), $\rho_\omega(N) = \delta_N$, and $Le[N|\delta_N] \triangleleft Le[M|\delta_M]$ (since $\delta_N = \tau$). Our next step is to reflect this below κ_M. Let F be a total extender in M such that the strength of F is greater than $On \cap N$. In particular, we have $N \triangleleft Ult(M|\delta_M, F)$.

Claim 3.38. $Le[N|\tau] \triangleleft Le[Ult(M|\delta_M, F)]$.

Proof. τ is inaccessible in $Ult(M|\delta_M, F)$. So by Remark 2.23, $Le[N|\tau]$ equals the Mitchell-Steel construction of length τ in $Ult(M|\delta_M, F)$.

Suppose the claim fails. Then there is a mouse Q built during the Mitchell-Steel construction in $Ult(M|\delta_M, F)$ after $Le[N|\tau]$ is constructed, such that Q projects to some $\beta < \tau$. Pick such a Q which minimizes β. By Lemma 3.32, any initial segment of M below δ_M is iterable in M. Then M has iteration strategies for $Ult(P, F)$ for any $P \triangleleft M|\delta_M$. Q is a mouse built during the Mitchell-Steel construction in $Ult(P, F)$ for some $P \triangleleft M|\delta_M$, so Q is also iterable in M. Let $Q' = \mathcal{C}_\omega(Q)$. Then Q' is an ω-sound mouse over $Le[N|\tau]|\beta$ projecting to β which is iterable in M. It follows from Theorem 2.21 that $Le[M|\delta_M]$ outiterates Q'. Since both extend $Le[N|\tau]|\beta$, and Q' is ω-sound and projects to β, $Q' \triangleleft Le[M|\delta_M]$. But then since τ is inaccessible in M, $Le[M|\tau]$ has height τ, and $Le[M|\tau] \triangleleft Le[M|\delta_M]$, Q' is in $Le[M|\tau]$. This is a contradiction, since a subset of β which is not in $Le[M|\tau]$ is definable over Q'. $\dashv$

By elementarity of the ultrapower embedding induced by F, there exists $N \triangleleft M|\kappa_M$ satisfying (a), (b), (c), $\rho_\omega(N) = \delta_N$, and $Le[N|\delta_N] \triangleleft Le[M|\delta_M]$. It remains to prove the following claim.

Claim 3.39. $StrLe[N] \triangleleft StrLe[M]$.

Proof. Since N projects to δ_N, so does $StrLe[N]$ (by Lemma 2.24). And $StrLe[N]$ agrees with $StrLe[M]$ up to δ_N since $Le[N|\delta_N] \triangleleft Le[M|\delta_M]$. So it suffices to show $StrLe[M]$ outiterates $StrLe[N]$. But $StrLe[N]$ has an iteration strategy in $\mathbf{\Gamma}$, and $StrLe[M]$ cannot by Lemma 3.30. $\dashv$

$\dashv$

[14] See Theorem 5.1 of [Ste09].

3.6. Main Theorem. We are ready to prove Theorem 1.12. Suppose for contradiction $\langle A_\alpha | \alpha < \boldsymbol{\delta}^+_{\boldsymbol{\Gamma}} \rangle$ is a sequence of distinct $\boldsymbol{\Gamma}$ sets. Let $U \subset \mathbb{R} \times \mathbb{R}$ be a universal $\boldsymbol{\Gamma}$ set.

Recall in Section 3.1 we defined $\mathcal{I}$ as the direct limit of all countable, complete iterates of the Γ-suitable mouse W. Let $\mathcal{J} = \{(P, \xi) : P \in \mathcal{I} \wedge \xi < \delta_P\}$. Say $(P, \xi) \leq_* (Q, \zeta)$ if $(P, \xi), (Q, \zeta) \in \mathcal{J}$ and whenever S is a complete iterate of both P and Q, $\pi_{P,S}(\xi) \leq \pi_{Q,S}(\zeta)$. By Lemma 3.2, the relation $\leq_*$ has length $> \boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$. Fix n such that for some (equivalently any) $P \in \mathcal{I}$, $\pi_{P,\infty}(\gamma^P_n) > \boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$. Let $\leq'_*$ be $\leq_*$ restricted to pairs (P, ξ) such that $\xi < \gamma^P_n$. Then $\leq'_*$ has length $\geq \boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$ and $\leq'_*$ is in $J_{\beta'}(\mathbb{R})$.[15] Let $B_\alpha = \{y : U_y = A_\alpha\}$. By the Coding Lemma there is a set D in $J_{\beta'}(\mathbb{R})$ such that $(x, y) \in D$ implies x codes a pair in the domain of $\leq'_*$ and $y \in B_{|x|_{\leq'_*}}$, and D_x is nonempty for all x in the domain of $\leq'_*$.

Let $z_0 \in \mathbb{R}$ be such that z_0 codes W and $D \in OD^{<\beta'}(z_0)$. Let $\mathcal{I}'$ be the directed system of all countable, complete iterates of M_{z_0}. Let M'_∞ be the direct limit of $\mathcal{I}'$. For $M, N \in \mathcal{I}'$ and N an iterate of M, let $\pi_{M,N} : M \to N$ be the iteration map and $\pi_{M,\infty} : M \to M'_\infty$ the direct limit map (We also used $\pi_{M,N}$ and $\pi_{M,\infty}$ for $M, N \in \mathcal{I}$, but this should not cause any confusion).

For $M \in \mathcal{I}'$, let $\tau^M = \tau^M_{D,\delta_M}$. There is a slight issue in that our current definitions do not obviously guarantee that τ^M is moved correctly. That is, we might have a complete iterate N of M such that $\pi_{M,N}(\tau^M) \neq \tau^N$. This can happen because we defined the operator $x \mapsto M_x$ so that M_x is guided by $\mathcal{G}$, but it is possible $D \notin \mathcal{G}$. There is no real issue here, since we can expand $\mathcal{G}$ to a larger self-justifying system $\mathcal{G}'$ such that $D \in \mathcal{G}'$ and require M_x be guided by $\mathcal{G}'$. However, we should leave the operator $x \to W_x$ as is, otherwise we risk altering our construction of D. This raises another minor complication, because in Sections 3.2 and 3.3 we assumed our Γ-ss mouse M was guided by the same self-justifying system as our Γ-suitable mouse W. Fortunately, the results of those sections remain true so long as $\mathcal{G} \subseteq \mathcal{G}'$, modulo increasing the number of terms required as parameters in some of the lemmas. For simplicity, in what follows we will just assume τ^M is moved correctly.

Definition 3.40. *Say $M \in \mathcal{I}'$ is locally α-stable if there is $\xi \in M$ such that $\pi_{\mathcal{H}^M,\infty}(\xi) = \alpha$. Write α_M for this ordinal ξ.*

Definition 3.41. *Say $M \in \mathcal{I}'$ is α-stable if M is locally α-stable and whenever $N \in \mathcal{I}'$ is a complete iterate of M, $\pi_{M,N}(\alpha_M) = \alpha_N$.*

Lemma 3.42. *For any $\alpha < \boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$, there is an α-stable $M \in \mathcal{I}'$.*

[15] This is done by similar arguments to those in Section 3.3.

Proof. This is essentially the same as the proof of the analogous lemma in [Sar22]. We will show for any $P \in \mathcal{I}'$, there is an iterate of P which is α-stable.

Claim 3.43. *For any $P \in \mathcal{I}'$, there is a countable, complete iterate R of P which is locally α-stable*

Proof. Fix $S \in \mathcal{I}$ and $\zeta \in S$ such that $\pi_{S,\infty}(\zeta) = \alpha$. Let R be a countable, complete iterate of P such that S is Ea_R-generic over R.

Let $\dot{S}$ be an Ea_R-name for S such that $\emptyset \Vdash^R_{Ea_R}$ "$\dot{S}$ is a complete iterate of W." Applying Lemma 3.22 yields $S' \in \mathcal{I}^R$ which is a complete iterate of S. Then

$$\begin{aligned} \pi_{\mathcal{H}^R,\infty} \circ \pi_{S',\mathcal{H}^R} \circ \pi_{S,S'}(\zeta) &= \pi_{S,\infty}(\zeta) \\ &= \alpha. \end{aligned}$$

In particular, $\alpha \in range(\pi_{\mathcal{H}^R,\infty})$. ⊣

Now suppose no $M \in \mathcal{I}'$ is α-stable. Let $\langle R_j : j < \omega\rangle$ be a sequence in $\mathcal{I}'$ such that for all j, R_j is locally α-stable and R_{j+1} is an iterate of R_j, but $\pi_{R_j,R_{j+1}}(\alpha_{R_j}) \neq \alpha_{R_{j+1}}$.

Claim 3.44. $\pi_{R_j,R_{j+1}}(\alpha_{R_j}) \geq \alpha_{R_{j+1}}$

Proof. By elementarity, $\pi_{R_j,R_{j+1}} \restriction \mathcal{H}^{R_j}$ is an embedding of $\mathcal{H}^{R_j}$ into $\mathcal{H}^{R_{j+1}}$. Then the Dodd-Jensen property implies for any common, complete iterate Q of $\mathcal{H}^{R_j}$ and $\mathcal{H}^{R_{j+1}}$,

$$\pi_{\mathcal{H}^{R_{j+1}},Q} \circ \pi_{R_j,R_{j+1}}(\alpha_{R_j}) \geq \pi_{\mathcal{H}^{R_j},Q}(\alpha_{R_j}).$$

Then

$$\begin{aligned} \pi_{\mathcal{H}^{R_{j+1}},\infty} \circ \pi_{R_j,R_{j+1}}(\alpha_{R_j}) &\geq \pi_{\mathcal{H}^{R_j},\infty}(\alpha_{R_j}) \\ &= \alpha \\ &= \pi_{\mathcal{H}^{R_{j+1}},\infty}(\alpha_{R_{j+1}}). \end{aligned}$$

So $\pi_{R_j,R_{j+1}}(\alpha_{R_j}) \geq \alpha_{R_{j+1}}$. ⊣

Let R_ω be the direct limit of the sequence $\langle R_j : j < \omega\rangle$. Let $\alpha_j = \pi_{R_j,R_\omega}(\alpha_{R_j})$. Claim 3.44 implies $\alpha_{j+1} < \alpha_j$ for all j, contradicting the wellfoundedness of R_ω. ⊣

Let A be a maximal antichain in Ea_M such that $p \in A$ implies p forces the generic ea is a pair (ea^1, ea^2), where ea^1 codes a pair (R_{ea^1}, ξ_{ea^1}) such that there exists an iteration tree on W (according to the strategy for W) with last model R_{ea^1} and $\xi_{ea^1} < \delta_{R_{ea^1}}$.[16] Since Ea_M is δ_M-c.c., $|A|^M < \delta_M$. Then the disjunction of conditions in A is also a condition in Ea_M. Pick $p^M \in Ea_M$ to be the least condition in the constructability order of M

[16]This is first order by Corollary 3.14 and Lemma 3.21.

which is the disjunction of conditions in some A as above, to ensure p^M is definable in M. p^M is a maximal condition forcing the property above, in that any condition $p' \in Ea_M$ which forces the same property is compatible with p^M.

LEMMA 3.45. *There is $Q^M \in \mathcal{I}^M$ such that p^M forces Q^M is a complete iterate of R_{ea^1}. Moreover, Q^M is definable in M from parameter p^M (uniformly in M).*

PROOF. Apply Lemma 3.22 to the condition p^M and a name for R_{ea^1}. ⊣

DEFINITION 3.46. *For α-stable $M \in \mathcal{I}'$, say $p \in Ea_M$ is α-good if p extends p^M and p forces*
1. $\pi_{\check{Q}^M,\mathcal{H}^M} \circ \pi_{R_{ea^1},\check{Q}^M}(\xi_{ea^1}) = \alpha_M$ and
2. $(ea^1, ea^2) \in \tau^M$.

REMARK 3.47. *If $\alpha < \boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$, being α-good is definable over α-stable $M \in \mathcal{I}'$ from α_M, τ^M, and $\langle \tau^M_{k,\nu_M} : k < n \rangle$ (uniformly in M). This follows from Lemmas 3.20 and 3.21.*

Let p^M_α be the maximal α-good condition in M which is least in the construction of M. Note if M is α-stable and N is a complete iterate of M, then $\pi_{M,N}(p^M_\alpha) = p^N_\alpha$.

For $w \in \mathbb{R} \cap M$ and a Σ_1 formula $\psi(w)$, write $M \models [\psi(w)]$ to mean whenever g is $Col(\omega, \delta_M)$-generic over M, there is a proper initial segment of $M[g]$ which is a $\langle \psi', g \rangle$-witness, where $\psi'(x)$ is a formula expressing "$\psi(f(x))$" for some computable function f such that $f(g) = w$. Note "$M \models [\psi(w)]$" is Σ_1 over M if M is iterable.

For α-stable $M \in \mathcal{I}'$, let S^M_α be the set of conditions q such that there exist $N, r \in M$ satisfying

(a) $N \lhd M|\kappa_M$,
(b) N has one Woodin,
(c) δ_N is a cardinal of M,
(d) $q, r \in Ea_N$ and $(q, r) \Vdash^N_{Ea_N \times Ea_N} [U(ea_l, ea^2_r)]$,[17] and
(e) r is compatible with p^M_α.

Let $S_\alpha = \pi_{M,\infty}(S^M_\alpha)$ for some (equivalently any) α-stable $M \in \mathcal{I}'$. S_α can be viewed as an element of $P(\kappa_{M'_\infty})^{M'_\infty}$.

Let A'_α be the set of reals x such that for any α-stable $\bar{M} \in \mathcal{I}'$ there is a countable, complete iterate M of $\bar{M}$ and $q \in M$ satisfying

1. $q \in S^M_\alpha$,
2. $x \models q$, and
3. x is Ea_M-generic over M.

[17]Here by U we really mean some fixed Σ_1-formula defining U in $J_{\alpha_0}(\mathbb{R})$.

Lemma 3.48. $A'_\alpha = A_\alpha$

Corollary 3.49. $\alpha \neq \beta \Longrightarrow S_\alpha \neq S_\beta$

Proof. Suppose $S_\alpha = S_\beta$. Let $x \in A_\alpha$. Let $\bar{M} \in \mathcal{I}'$ be β-stable. There is a countable, complete iterate $\hat{M}$ of $\bar{M}$ which is also α-stable. By Lemma 3.48, $x \in A'_\alpha$, so there is a countable, complete iterate M of $\hat{M}$ such that $q \in S^M_\alpha$, $x \models q$, and x is Ea_M-generic over M. M is also a complete iterate of $\bar{M}$, so we have shown $x \in A'_\beta$. Applying Lemma 3.48 again, $x \in A_\beta$. Similarly, $x \in A_\beta \Longrightarrow x \in A_\alpha$, so $A_\alpha = A_\beta$ and thus $\alpha = \beta$. ⊣

It suffices to show Lemma 3.48. By the same proof as for M_∞ given in Lemma 3.1, $\kappa_{M'_\infty} \leq \boldsymbol{\delta_\Gamma}$. Then by Corollary 3.49, we have $\boldsymbol{\delta}^+_{\boldsymbol{\Gamma}}$ distinct subsets of $\boldsymbol{\delta_\Gamma}$ in M'_∞. Then the successor of $\boldsymbol{\delta_\Gamma}$ in M'_∞ is the successor of $\boldsymbol{\delta_\Gamma}$ in $L(\mathbb{R})$, contradicting the following claim.

Claim 3.50. *Let* $\eta = \boldsymbol{\delta_\Gamma}$. *Then* $(\eta^+)^{M'_\infty} < (\eta^+)^{L(\mathbb{R})}$

Proof. Let $\lambda = (\eta^+)^{M'_\infty}$. Since λ is regular in M'_∞ but not measurable, Lemma 3.6 implies λ has cofinality ω in $L(\mathbb{R})$.

Let $f \in L(\mathbb{R})$ be a cofinal function from ω to λ. Let $\langle g_\xi : \xi < \lambda\rangle$ be a sequence of functions in M'_∞ such that $g_\xi : \eta \to \xi$ is a surjection. Such a sequence exists because M'_∞ satisfies AC. Then in $L(\mathbb{R})$ we can construct from f and $\langle g_\xi\rangle$ a surjection from η onto λ. ⊣

Proof of Lemma 3.48. First suppose $x \in A_\alpha$. Let $\bar{M} \in \mathcal{I}'$ be α-stable. Pick $y \in \mathbb{R}$ such that $y = (y^1, y^2)$, $D(y^1, y^2)$ holds, and $|y^1|_{\leq_*} = \alpha$. Let z be a real coding $\bar{M}$ and let $P = M_{\langle x,y,z\rangle}$. Let $S = StrLe[P, z_0]$.

Claim 3.51. *x and y are Ea_S-generic over S.*[18]

Claim 3.52. *S is a complete iterate of $\bar{M}$ by an iteration below $\delta_{\bar{M}}$.*

Proof. See Lemma 3.31. ⊣

Claim 3.53. *There is $r \in Ea_S$ such that r is α-good and $y \models r$.*

Proof. Note by choice of y, y^1 codes a pair (R, ξ) such that R is a complete iterate of W, $\pi_{R,\mathcal{H}^S}(\xi) = \alpha_S$, and $D(y^1, y^2)$ holds. Then there is $r \in Ea_S$ such that $y \models r$, r forces $\pi_{\check{Q}^S,\mathcal{H}^S} \circ \pi_{R_{ea^1},\check{Q}^S}(\xi_{ea^1}) = \alpha_S$, and $(ea^1, ea^2) \in \tau^S$. ⊣

Claim 3.54. *There exist conditions $q, r \in Ea_S$ such that $x \models q$, $y \models r$, and $(q, r) \Vdash^S_{Ea_S \times Ea_S} [U(ea_l, ea^2_r)]$.*

Proof. By Claim 3.53, y satisfies some α-good condition r. Let y_0 be $S[x]$-generic such that $y_0 \models r$. Then by the definition of α-good, $y_0 =$

[18] This is a standard property of the fully-backgrounded construction - see Section 1.7 of [Sar22].

(y_0^1, y_0^2) where $(y_0^1, y_0^2) \in D$ and $|y_0^1|_{\leq^*} = \alpha$. It follows that $U_{y_0^2} = A_\alpha$. So $x \in U_{y_0^2}$.

SUBCLAIM 3.55. $S[x][y_0] \models [U(x, y_0^2)]$.

PROOF. Let g be $Col(\omega, \delta_S)$-generic over $S[x][y_0]$. Note $S[x][y_0][g] = S[g]$ is a g-mouse. By the proof of Lemma 3.13, $Lp^\Gamma(g)$ is contained in $S[g]$. Let f be a computable function such that $f(g) = (x, y_0^2)$ and let $U'(v)$ be a formula expressing $U(f(v))$ holds. By Lemma 2.48, there is a $\langle U', g\rangle$-witness which is sound, projects to ω, and has an iteration strategy in $\mathbf{\Delta}$. Since $Lp^\Gamma(g) \subseteq S[g]$, this witness is an initial segment of $S[g]$. ⊣

We have shown $S[x][y_0] \models [U(x, y_0^2)]$ for any y_0 which satisfies r and is $S[x]$-generic. Thus there is $q \in Ea_S$ such that x satisfies q and $(q, r) \Vdash [U(ea_l, ea_r^2)]$. ⊣

We next would like to find some $N \lhd S|\kappa_S$ with the properties of S we obtained above. Note Claims 3.51 and 3.54 are not first order over S, since x and y are not in S. So a straightforward reflection argument inside S will not suffice. The point of introducing P and obtaining S as a construction inside P is that these claims are first order in P. The next claim demonstrates we can perform a reflection in P to obtain the desired initial segment of S.

CLAIM 3.56. *There is $N \lhd S|\kappa_S$ such that N has one Woodin, δ_N is an inaccessible cardinal of S, x and y are generic for Ea_N, and there exist $q, r \in Ea_N \times Ea_N$ such that $x \models q$, $y \models r$, and $(q, r) \Vdash [U(ea_l, ea_r^2)]$.*

PROOF. By Claims 3.51 and 3.54, P satisfies

1. x and y are $Ea_{StrLe[P,z_0]}$-generic over $StrLe[P, z_0]$ and
2. there exist conditions $q, r \in StrLe[P, z_0]$ such that $x \models q$, $y \models r$, and $(q, r) \Vdash^{StrLe[P,z_0]}_{Ea_{StrLe[P,z_0]} \times Ea_{StrLe[P,z_0]}} [U(ea_l, ea_r^2)]$.

Both properties are Σ_1 over P in parameters x, y, z_0, and δ_P. Then we may apply Lemma 3.35 to obtain $P' \lhd P|\kappa_P$ such that P' has one Woodin cardinal, $\delta_{P'}$ is an inaccessible cardinal of P, $StrLe[P', z_0] \lhd S$, and P' satisfies properties 1 and 2.

Let $N = StrLe[P', z_0]$. Note $\delta_N = \delta_{P'}$ is an inaccessible cardinal of S. Then all the properties we required of N are apparent except that $N \lhd S|\kappa_S$. Standard properties of the Mitchell-Steel construction imply that $\kappa_S \geq \kappa_P$.[19] Then N has cardinality less than κ_S in P, since N is contained in P'. Since also $N \lhd S$, we have $N \lhd S|\kappa_S$. ⊣

To get $x \in A'_\alpha$, it remains to show the following claim.

[19]Suppose $\lambda < \delta_S = \delta_P$ and E is an extender on the fine extender sequence of S witnessing κ_S is λ-strong in S. Let E^* be the background extender for E on the fine extender sequence of P. Then E^* witnesses κ_S is λ-strong in P.

Claim 3.57. *r is compatible with p^S_α.*

Proof. By Claim 3.53, y satisfies some α-good condition p. We may assume p extends r. p is α-good, so by maximality p is compatible with p^S_α. Then r is compatible with p^S_α as well. $\dashv$

Now suppose $x \in A'_\alpha$. Let M, q realize this (for whichever $\bar{M} \in \mathcal{I}'$ you please) and let N, r realize $q \in S^M_\alpha$. Let y be $M[x]$-generic for Ea_M such that $y \models r \wedge p^M_\alpha$. Since $y \models p^M_\alpha$, $y = (y^1, y^2)$ where $U_{y^2} = A_\alpha$. Since $(x, y) \models (q, r)$, $M[x][y] \models [U(x, y^2)]$. Let $g \subset Col(\omega, \delta_M)$ be $M[x][y]$-generic. Then $M[x][y][g] = M[g]$ has an initial segment R witnessing $U(x, y^2)$. By taking the least such R, we may assume R projects to ω and hence $R \in Lp^\Gamma(g)$. It follows that $x \in U_{y^2} = A_\alpha$. $\dashv$

§4. Remarks on Some Projective-Like Cases. Here we provide a few brief comments on the problem of unreachability for projective-like cases. Section 4.1 covers the projective pointclasses. In Section 4.2, we discuss what appears to be the main obstacle to proving the rest of the following conjecture.

Conjecture 4.1. *Assume $ZF+AD+DC+V = L(\mathbb{R})$. Suppose $\kappa \leq \boldsymbol{\delta^2_1}$ is a Suslin cardinal and κ is either a successor cardinal or a regular limit cardinal. Then κ^+ is $S(\kappa)$-unreachable.*

4.1. The Projective Cases. In the introduction, we discussed a theorem of Sargsysan solving the problem of unreachability for the projective pointclasses:

Theorem 4.2 (Sargsyan). *Assume $ZF + AD + DC$. Then $\boldsymbol{\delta^1_{2n+2}}$ is $\boldsymbol{\Sigma^1_{2n+2}}$-unreachable.*

Our technique for proving Theorem 1.12 gives another proof of Sargsyan's theorem, which we outline below. We will assume $ZF + AD + DC$ for the rest of this section.

Let $W = M^\#_{2n+1}$. Let $\mathcal{I}$ be the directed system of countable, complete iterates of W and let M_∞ be the direct limit of $\mathcal{I}$.

Fact 4.3. *$\kappa_{M_\infty} < \boldsymbol{\delta^1_{2n+2}}$ and $\delta_{M_\infty} > (\boldsymbol{\delta^1_{2n+2}})^+$.*

The iteration strategy Σ for W is guided by indiscernibles, analogously to how the iteration strategies for Γ-suitable mice are guided by terms for sets in a sjs.[20] [Sar13] covers this analysis of the iteration strategy for W

[20]More explicitly, for an appropriate sequence of indiscernibles $\langle v_i : i < \omega\rangle$, Σ is the unique iteration strategy witnessing that W is strongly $\langle v_0, .., v_i\rangle$-iterable (in the sense of [Sar13]) for every $i < \omega$.

in detail. Also analogously to Sections 3.2 and 3.3, inside an iterate M of $M^{\#}_{2n+1}(x_0)$ for some $x_0 \in \mathbb{R}$ coding W, we can form the direct limit $\mathcal{H}^M$ of countable iterates of W in M and approximate the iteration maps from W to $\mathcal{H}^M$. This internalization is covered in [Sar22].

The following fact gives us an analogue of the notion of a $\langle \phi, z \rangle$-witness.

FACT 4.4. *There is a computable function which sends a Σ^1_{2n+2}-formula ϕ to a formula $\phi^* = \phi^*(u_0, ..., u_{2n-1}, v)$ in the language of mice such that the following hold:*

1. *If $x \in \mathbb{R}$, M is a countable, $\omega_1 + 1$-iterable x-premouse, $M \models ZFC$, M has $2n$ Woodin cardinals $\delta_0, ..., \delta_{2n-1}$, ϕ is a Σ^1_{2n+2} formula, and $M \models \phi^*[\delta_0, ..., \delta_{2n-1}, x]$, then $\phi(x)$ holds.*
2. *If $x \in \mathbb{R}$, $\delta_0, ..., \delta_{2n-1}$ are the Woodin cardinals of $M^{\#}_{2n}(x)$, ϕ is a Σ^1_{2n+2} formula, and $\phi(x)$ holds, then a proper initial segment of M above δ_{2n-1} satisfies ZFC and $\phi^*[\delta_0, ..., \delta_{2n-1}, x]$.*

With these tools it is not difficult to adapt our proof of Theorem 1.12 into a proof of Theorem 4.2.

Here is a brief overview of the proof of Theorem 4.2 in [Sar22]. The basis of this proof is also studying the directed system $\mathcal{I}'$ of countable iterates of $M^{\#}_{2n+1}(z_0)$ for some $z_0 \in \mathbb{R}$. Suppose $\langle A_\alpha : \alpha < \boldsymbol{\delta^1_{2n+2}} \rangle$ is a sequence of distinct $\boldsymbol{\Sigma^1_{2n+2}}$ sets. Fix a $\Pi^1_{2n+3} \backslash \Sigma^1_{2n+3}$ set $A \subset \omega$. If $n \in A$, this is witnessed in a proper initial segment of any M_{2n+1}-like Π^1_{2n+2}-iterable premouse M. Then there is a Σ^1_{2n+3} set $A' \subset A$ consisting of, roughly speaking, all $n \in \omega$ which are witnessed in such an M before some $x \in A_\alpha$ is witnessed. There is $n_0 \in A' \backslash A$. This is witnessed in some proper initial segment $\bar{N}_M$ of $M|\kappa_M$ for any $M \in \mathcal{I}'$. A coding set S^M is defined analogously to our coding sets in the proof of Theorem 1.12, but with the additional requirement that the conditions appear below $\bar{N}_M$. The coding sets are used to show a $\boldsymbol{\Sigma^1_{2n+2}}$ code for A_α is small generic over M. The contradiction is obtained from this.

The technique described in the previous paragraph is a stronger argument than the one we used for Theorem 1.12, since it gives coding sets which are uniformly bounded below the least strong cardinal. It is not clear whether a similar argument could work for inductive-like pointclasses. There is no obvious analogue of the $\Pi^1_{2n+3} \backslash \Sigma^1_{2n+3}$ set A for an inductive-like pointclass Γ, since there is no universal $\Gamma \backslash \Gamma^c$ set of integers. So the proof from [Sar22] is not applicable to inductive-like pointclasses. On the other hand, the techniques of Section 3 are applicable to the projective pointclasses. And this yields a substantially simpler proof of Theorem 4.2, since it eliminates the need for a uniform bound on our coding sets.

4.2. Mouse Sets and Open Problems. In this section we discuss the relationship between the problem of unreachability and well-known conjectures on mouse sets. We will assume $ZF + AD + DC + V = L(\mathbb{R})$, although this is overkill for some of the results stated below.

DEFINITION 4.5. *$X \subset \mathbb{R}$ is a mouse set if there is an $\omega_1 + 1$-iterable premouse M such that $X = M \cap \mathbb{R}$.*

THEOREM 4.6 (Steel). *Suppose $\Gamma = \Sigma^1_{n+2}$ for some $n \in \omega$. Then C_Γ is a mouse set.*

THEOREM 4.7 (Woodin). *Suppose λ is a limit ordinal and let $\Gamma = \{A \subseteq \mathbb{R} : A$ is definable in $J_\beta(\mathbb{R})$ for some $\beta < \lambda\}$. Then C_Γ is a mouse set.*

See [Ste95a] and [Ste16] for proofs of Theorems 4.6 and 4.7, respectively. [Ste16] also gives the following conjecture.

CONJECTURE 4.8 (Steel). *Suppose Γ is a level of the (lightface) Levy hierarchy.*[21] *Then C_Γ is a mouse set.*

Conjecture 4.8 is a way of asking if there is a mouse corresponding exactly to the pointclass Γ. For each Γ in the Levy hierarchy, the core model induction constructs a mouse which contains C_Γ, but in some cases the mouse constructed is too large. For example, let J be the mouse operator $J(x) = \bigcup_{n<\omega} M_n^\#(x)$. If $\Gamma = \Sigma_{n+2}(J_2(\mathbb{R}))$, then

$$M_n^{J^\#} \cap \mathbb{R} \subsetneq C_\Gamma \subsetneq M_{n+1}^{J^\#} \cap \mathbb{R}.$$

There are many similar cases in which the mice constructed in [SS] skip the (hypothesized) mouse realizing Conjecture 4.8. Recent progress has been made towards Conjecture 4.8 in [Rud23], which resolves the case $\Gamma = \Sigma_2(J_2(\mathbb{R}))$.

The problem of unreachability is connected to a boldface version of Conjecture 4.8.

CONJECTURE 4.9. *Suppose $\alpha \in ON$ and $n \in \omega$. For $x \in \mathbb{R}$, let Γ_x consist of all pointsets A for which there is a Σ_n formula ϕ with parameter x such that $A = \{y : J_\alpha(\mathbb{R}) \models \phi[y]\}$. Then for any $y \in \mathbb{R}$, there is $x \in \mathbb{R}$ such that $y \leq_T x$ and C_{Γ_x} is a mouse set.*

Presumably a proof of Conjecture 4.8 would relativize, so a proof of Conjecture 4.8 would also resolve Conjecture 4.9.

The mouse operator $x \mapsto M_{2k}^\#(x)$ realizes Conjecture 4.9 holds for $\alpha = 1$ and $n = 2k + 2$. To prove $\boldsymbol{\delta^1_{2n+2}}$ is $\boldsymbol{\Sigma^1_{2n+2}}$-unreachable, we studied the

[21] I.e. $\Gamma = \Sigma_n(J_\alpha(\mathbb{R}))$ for some $\alpha \in On$ and $n \in \omega$.

direct limit of $M = M^{\#}_{2n+1}(x_0)$ for some $x_0 \in \mathbb{R}$. Note if g is $Col(\omega, \delta_M)$-generic over M, then $M[g] = M^{\#}_{2n}(g)$.

For α admissible, the mouse operator $x \mapsto M_x$ of Theorem 2.42 realizes Conjecture 4.9 holds in the case $n = 1$. Note if g is $Col(\omega, \delta_{M_x})$-generic over M_x, then $M_x[g] \cap \mathbb{R} = Lp^{\Gamma}(g) \cap \mathbb{R} = C_{\Gamma}(g)$. So in the inductive-like case as well we studied the direct limit of a mouse such that collapsing its least Woodin yields a mouse realizing one case of Conjecture 4.9.

Thus for each pointclass $\boldsymbol{\Sigma_n}(J_\alpha(\mathbb{R}))$ for which we have proven Conjecture 1.10 holds, we used a mouse operator realizing Conjecture 4.9 holds for α and n. It seems likely a proof of Conjecture 4.1 would involve proving Conjecture 4.9 for each α and n such that $\boldsymbol{\Sigma_n}(J_\alpha(\mathbb{R})) = S(\kappa)$ for some Suslin cardinal κ which is a successor cardinal or a regular limit cardinal.

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