

The tree property on long intervals of regular cardinals

James Cummings*

*Department of Mathematical Sciences, Carnegie Mellon University
Pittsburgh, PA 15213-3890, USA
jcumming@andrew.cmu.edu*

Yair Hayut

*Einstein Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram. Jerusalem, 9190401, Israel
yair.hayut@mail.huji.ac.il*

Menachem Magidor

*Einstein Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram. Jerusalem, 9190401, Israel
mensara@savion.huji.ac.il*

Itay Neeman

*Department of Mathematics, University of California Los Angeles
Los Angeles, CA 90095-1555, USA
ineeman@math.ucla.edu*

Dima Sinapova

*Department of Mathematics, Rutgers University
Piscataway, NJ 08854-8019, USA
dima.sinapova@rutgers.edu*

Spencer Unger

*Department of Mathematics, University of Toronto
Toronto, ON M5S 2E4, Canada
spencer.unger@utoronto.ca*

In this paper we prove that the tree property can hold on regular cardinals in an interval which overlaps a strong limit cardinal. This is a crucial milestone in the long term project, tracing back to a question raised by Foreman and Magidor in the 1980s, of obtaining the tree property at every regular cardinal above ω_1 .

Keywords: Tree property; forcing; large cardinals; compactness; Aronszajn trees.

03E35

*Carnegie Mellon University

1. Introduction

Let κ be a regular infinite cardinal. A κ -tree is a tree of height κ where every level has cardinality less than κ , and κ has the *tree property* if every κ -tree has a branch of length κ . A κ -Aronszajn tree is a counterexample to the tree property at κ , that is to say a κ -tree with no branch of length κ . A λ^+ -tree T is *special* if there is a function $f : T \rightarrow \lambda$ such that $u <_T v \implies f(u) \neq f(v)$: such a tree is a robust counterexample to the tree property, in the sense that it is a λ^+ -Aronszajn tree in any outer model where λ^+ remains a cardinal.

The tree property belongs to a class of *compactness properties*, which are of great interest in combinatorial set theory. Significant results about the tree property include:

- (König [9, 1927]) ω has the tree property.
- (Specker [22, 1949]) If $\kappa^{<\kappa} = \kappa$ then there is a special κ^+ -tree.
- If κ is strongly inaccessible then:
 - (Keisler and Tarski [8, 1963]) If κ has the tree property then κ is weakly compact.
 - (Monk and Scott [14, 1964]) If κ is weakly compact then κ has the tree property.
- (Silver [13, Theorem 5.9, 1972]) If κ is uncountable and has the tree property, then κ is weakly compact in L .
- (Magidor and Shelah [12, 1996]) If λ is a singular limit of cardinals which are λ^+ -strongly compact, then λ^+ has the tree property.
- (Combining results of Foreman, Magidor and Schindler [4, 2001], Schimmerling and Zeman [17, 2004], and Jensen and Steel [7, 2013]) If κ and κ^+ are successive regular cardinals with the tree property, then there is an inner model with a Woodin cardinal.
- (Combining results of Schimmerling and Zeman [17, 2004], and Jensen and Steel [7, 2013]) If λ is a singular cardinal such that λ^+ has the tree property, then there is an inner model with a Woodin cardinal.

It is known to be consistent that certain small regular cardinals can have the tree property. Mitchell [13] showed that if $\lambda < \kappa$ with λ regular and κ weakly compact, then there is a generic extension by $< \lambda$ -closed κ -cc forcing in which $2^\lambda = \kappa = \lambda^{++}$ and the tree property of κ is preserved. Magidor and Shelah [12] showed it to be consistent modulo a hypothesis at the level of huge cardinals that $\aleph_{\omega+1}$ is strong limit and has the tree property.

A natural question, raised by Foreman and by Magidor among others, asks whether it is consistent that all regular $\kappa > \aleph_1$ should simultaneously have the tree property. There are many obstacles to be overcome in resolving this question: in particular we need a model where GCH fails everywhere and Jensen’s “weak square” principle $\square_\lambda^*$ fails for every λ . On a closely related point, we seem to need instances of strong compactness in order to violate weak square for singular λ , but there

is tension here with Solovay's theorem that SCH holds above a strongly compact cardinal.

Our main result is:

Theorem 1.1. *Modulo a suitable large cardinal assumption, it is consistent that $\aleph_{\omega^2}$ is strong limit and the tree property holds for all regular cardinals κ such that $\aleph_2 \leq \kappa \leq \aleph_{\omega^2+3}$.*

Theorem 1.1 is the first known instance where the tree property holds on regular cardinals in an interval which overlaps a strong limit cardinal. To be more precise, observe that if κ is a singular strong limit cardinal and SCH holds at κ , then by Specker's theorem there is a special κ^{++} -tree. It follows to get a model where all regular cardinals above ω_1 have the tree property, we are required to produce a singular strong limit cardinal κ where regular cardinals between ω_1 and κ all have the tree property, $2^\kappa \geq \kappa^{++}$, and κ^+ has the tree property.

This has long been considered the next key step in the longstanding goal of obtaining the tree property everywhere. Notice that in Theorem 1.1 the strong limit cardinal which is overlapped is $\aleph_{\omega^2}$, not $\aleph_\omega$. This sidesteps another key question, which is still open, as to whether the failure of the Singular Cardinals Hypothesis at $\aleph_\omega$ is consistent with the tree property at $\aleph_{\omega+1}$.

The history behind Theorem 1.1 and the ingredients that go into its proof is a long one. We survey this history very briefly, where the price of brevity is that some contributions are omitted. In the light of the preceding discussion, we will be rather specific about cardinal arithmetic.

- Building on work of Abraham [1], Cummings and Foreman [2] showed that consistently $2^{\aleph_n} = \aleph_{n+2}$ and $\aleph_{n+2}$ has the tree property for all $n < \omega$ simultaneously. They also showed that the tree property can hold at κ^{++} where κ is strong limit of cofinality ω and $2^\kappa = \kappa^{++}$.
- Neeman [16] showed that the tree property can hold at $\aleph_{n+2}$ for $n < \omega$ and at $\aleph_{\omega+1}$. In this model $2^{\aleph_n} = \aleph_{n+2}$ for $n < \omega$ and $2^{\aleph_\omega} = \aleph_{\omega+1}$. Unger [25] showed it can hold for all regular cardinals in the interval $[\aleph_2, \aleph_{\omega+\omega})$. In this model $2^{\aleph_0} = \aleph_2$, $2^{\aleph_i} = \aleph_{\omega+2}$ for $1 \leq i < \omega$, and $2^{\aleph_{\omega+i}} = \aleph_{\omega+i+2}$ for $i < \omega$.
- Building on work of Gitik and Sharon [5], Neeman [15] showed that the tree property can hold at κ^+ where κ is strong limit of cofinality ω and $2^\kappa = \kappa^{++}$.
- Sinapova [19] produced a model of GCH where $\aleph_{\omega+1}$ has the tree property, using different methods from those of Magidor and Shelah together with weaker hypotheses.
- Sinapova [18] produced a model where $\aleph_{\omega^2}$ is strong limit, $2^{\aleph_{\omega^2}} = \aleph_{\omega^2+2}$, and $\aleph_{\omega^2+1}$ has the tree property. Sinapova and Unger [20] produced a model where $\aleph_{\omega^2}$ is strong limit, $2^{\aleph_{\omega^2}} = \aleph_{\omega^2+2}$, and both $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$ have the tree property.

- Unger [26] proved a result closely related to Theorem 1.1, producing a model where there is no *special* κ -Aronszajn tree for regular κ with $\aleph_2 \leq \kappa \leq \aleph_{\omega^2+3}$. In this model none of the cardinals $\aleph_{\omega \cdot n}$ for n finite is strong limit, in fact $2^{\aleph_{(\omega \cdot n)+4}} = \aleph_{\omega \cdot (n+1)+3}$. However $\aleph_{\omega^2}$ is strong limit, and $2^{\aleph_{\omega^2}} = \aleph_{\omega^2+3}$.

The proof of Theorem 1.1 has several steps, which we outline here with many technicalities omitted.

- (Section 5.1) We start with a model V_0 such that θ and κ are the first two supercompact cardinals, there are θ^+ supercompact cardinals above κ , and if δ is the supremum of the first θ^+ supercompact cardinals then there is $j_0 : V_0 \rightarrow M_0$ which witnesses that κ is δ^+ -supercompact and is such that supercompact cardinals up to δ are supercompact in M_0 .
- (Section 5.1) We build a generic extension V of V_0 , in which θ is the continuum and exhibits a strong form of generic supercompactness. We lift j_0 to obtain $j : V \rightarrow M$ with similar properties.
- (Section 5.2) Working in V , we use the generic supercompactness of θ to show that for every supercompact cardinal λ with $\kappa < \lambda < \delta$, there exist an ω -successor cardinal $\rho < \theta$ and a forcing poset uniformly defined from ρ and λ , forcing (among other things) the following conclusions: ρ^+ is $\aleph_1$, the successor of the supremum of the first ω supercompact cardinals above λ is $\aleph_{\omega+1}$, and $\aleph_{\omega+1}$ has the tree property. We will never actually force with this forcing, rather we will use it as a device to show that certain cardinals in our final model have the tree property. Informally we can think of ρ as being “good for λ ”.
- (Section 5.3) Still working in V , we select cardinals $\rho, \lambda^a, \lambda^b$ such that $\rho < \theta < \kappa < \lambda^a < \lambda^b < \delta$ and ρ is good for both λ^a and λ^b in the sense described above. In the final model ρ^+ will become $\aleph_1$, θ will become $\aleph_2$, and κ will be $\aleph_{\omega^2}$.

Using the supercompactness of κ , we argue that there are many triples $(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$ where $\tau < \Lambda^a(\tau) < \Lambda^b(\tau) < \kappa$ and $(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$ reflects the properties of $(\kappa, \lambda^a, \lambda^b)$. In particular ρ is good for both $\Lambda^a(\tau)$ and $\Lambda^b(\tau)$.

- (Section 6) We build a generic extension $V[L]$ of V in which κ is still highly supercompact and certain cardinals above λ^a are collapsed: in particular the cardinals $(\lambda^b)^{+n}$ for $n \in \omega \cup \{\omega + 2, \omega + 3\}$ as computed in $V[L]$ were all supercompact in V . A similar situation holds below κ in $V[L]$ for the reflected cardinals $\Lambda^b(\tau)$. Working in $V[L]$ we carefully choose an embedding j^* witnessing that κ is $< (\lambda^b)^{+\omega+3}$ -supercompact, and derive supercompactness measures U_n on $P_\kappa(\lambda^b)^{+n}$ for large enough $n < \omega$.
- (Section 8) Working in $V[L]$ we define a forcing poset $\mathbb{A}^{gg} * \bar{\mathbb{P}}$ where $\mathbb{A}^{gg}$ is a highly distributive auxiliary forcing, and $\bar{\mathbb{P}}$ is a diagonal supercompact Prikry forcing with some complex forcing posets interleaved between successive points of the generic ω -sequence. Our final model is the extension

of $V[L]$ by $\mathbb{A}^{gg} * \bar{\mathbb{P}}$.

The definition of $\bar{\mathbb{P}}$ uses the measures U_n , and a “guiding generic” K whose definition involves $\mathbb{A}^{gg}$. $\bar{\mathbb{P}}$ has the effect of making $\rho^+ = \aleph_1$, $\theta = \aleph_2$, and $\kappa = \aleph_{\omega^2}$. Above κ all cardinals up to and including $(\lambda_b)^{+\omega}$ are collapsed to have cardinality κ , while cardinals above this point are preserved, so that $(\lambda^b)^{+\omega+n}$ becomes $\aleph_{\omega^2+n}$.

- (Section 9) We verify that in our final model all regular cardinals in $[\aleph_2, \aleph_{\omega^2})$ have the tree property.
- (Section 10) We verify that in our final model the cardinals $\aleph_{\omega^2+1}$, $\aleph_{\omega^2+2}$ and $\aleph_{\omega^2+3}$ all have the tree property.

Our notational conventions are fairly standard. When p and q are forcing conditions we write “ $q \leq p$ ” when q is stronger than p . A poset is τ -closed if every decreasing τ -sequence has a lower bound, and $<\tau$ -closed if every decreasing $<\tau$ -sequence has a lower bound: note that some authors call these properties τ^+ -closed and τ -closed respectively. Our convention for directed closure is similar, so that a poset is $<\tau$ -directed closed if every directed subset of size less than τ has a lower bound. When the decreasing sequences have greatest lower bounds we describe posets as being *canonically closed*: in particular a poset is *canonically τ -closed* if every decreasing τ -sequence has a greatest lower bound, and *canonically $<\tau$ -closed* if every decreasing $<\tau$ -sequence has a greatest lower bound. Of course the Cohen poset $\text{Add}(\tau, \rho)$ and the Levy collapse posets $\text{Coll}(\tau, \rho)$ and $\text{Coll}(\tau, <\rho)$ are examples of canonically $<\tau$ -closed posets. When p_0 and p_1 are compatible conditions we will sometimes abuse notation and write “ $p_0 \wedge p_1 \Vdash \phi$ ”, when we should more properly write “ $p \Vdash \phi$ for every common refinement $p \leq p_0, p_1$ ”. Most of the forcing posets appearing in this paper have a top element, but we do not demand this.

In general we will name forcing posets with blackboard bold letters (for example $\mathbb{A}$) and the associated generic objects with the corresponding upper case italic letter (for example A). When this naming convention would cause confusion we may call the $\mathbb{A}$ -generic object $G_{\mathbb{A}}$. If $\dot{\tau}$ is an $\mathbb{A}$ -name then $\dot{\tau}[A]$ is the interpretation of $\dot{\tau}$ by A . For x in the ground model, $\check{x}$ is the canonical name for x , where the forcing for which $\check{x}$ is a name should always be clear from the context.

When κ is inaccessible and $\lambda \geq \kappa$ we abuse notation and write $P_\kappa \lambda$ for the set of $x \subseteq \lambda$ with $x \cap \kappa \in \kappa$ and $|x| < \kappa$. When $x, y \in P_\kappa \lambda$ we write $x \prec y$ for the relation “ $x \subseteq y$ and $\text{ot}(x) < y \cap \kappa$ ”. Of course we will also use $\prec$ for the relation “is an elementary substructure of” but in practice there is no possibility of confusion.

Once the main construction begins at the start of Section 5, we will begin to introduce many objects which are then fixed for the whole duration of the construction. To help the reader keep track, all these “global” objects will be flagged as they appear and will correspond to entries in the “Index of Notation” section.

2. Preliminaries

2.1. A fact about κ -cc forcing

The following Lemma is often useful.

Lemma 2.1. *Let κ be regular and uncountable, let $\mathbb{P}$ be κ -cc and let $(p_\alpha)_{\alpha < \kappa}$ be a κ -sequence of conditions in $\mathbb{P}$. Then there exists $\alpha < \kappa$ such that p_α forces $\{\beta < \kappa : p_\beta \in P\}$ to be unbounded in κ .*

Proof. If not, for each α we choose $r_\alpha \leq p_\alpha$ such that r_α forces $\{\beta < \kappa : p_\beta \in P\}$ to be bounded in κ , and then use κ -cc to find an ordinal $\eta_\alpha < \kappa$ such that r_α forces $\{\beta < \kappa : p_\beta \in P\} \subseteq \eta_\alpha$. Let $C = \{\delta : \forall \gamma < \delta \ \eta_\gamma < \delta\}$, so that C is club in γ . If $\gamma, \delta \in C$ with $\gamma < \delta$ then r_γ forces $p_\delta \notin P$, so that r_γ forces $r_\delta \notin P$, which is to say that r_γ is incompatible with r_δ . So $\{r_\gamma : \gamma \in C\}$ is an antichain in $\mathbb{P}$, contradicting κ -cc for $\mathbb{P}$. $\square$

Remark 2.2. An easy variation on this argument shows that there is α such that p_α forces $\{\beta < \kappa : p_\beta \in P\}$ to be stationary in $V[P]$, but this is more than we need.

2.2. Laver functions and Laver indestructibility

Recall that if κ is supercompact, there is a *Laver function* $f : \kappa \rightarrow V_\kappa$, that is to say a function such that for all x and all λ there is a λ -supercompactness embedding $j : V \rightarrow M$ such that $\text{crit}(j) = \kappa$ and $j(f)(\kappa) = x$.

Definition 2.3. *Let f be a partial function defined on ordinals. A closure point of f is an ordinal γ such that $f(\alpha) \in V_\gamma$ for all $\alpha \in \text{dom}(f) \cap \gamma$.*

Thinning the domain of a Laver function f , we may assume that $\text{dom}(f)$ consists of inaccessible closure points of f .

Given a Laver function f the *Laver iteration* is an Easton support iteration $\mathbb{L}$ of length κ , where we force with $f(\alpha)[L_\alpha]$ whenever $f(\alpha)$ is a $\mathbb{L}_\alpha$ -name for a $< \alpha$ -directed closed forcing poset: the poset $\mathbb{L}$ is κ -cc and has cardinality κ . Laver [10] showed that if κ is supercompact then a Laver function exists, and that the Laver iteration forces the supercompactness of κ to be indestructible by subsequent $< \kappa$ -directed closed forcing. With our conventions the Laver iteration defined from f preserves the inaccessibility of all points in $\text{dom}(f)$.

Definition 2.4. *Let I be an interval of cardinals, then a partial function f on I is a universal Laver function on I if and only if $\text{dom}(f) \subseteq \text{sup}(I)$, and $f \restriction \kappa$ is a Laver function on κ for every supercompact $\kappa \in I$.*

Adapting the standard argument for the existence of a Laver function, it is easy to see that every interval has a universal Laver function. Since the construction of a Laver function f proceeds by choosing $f(\alpha)$ as the least counterexample to $f \restriction \alpha$ being a Laver function, we may (and will) assume that the domain of a universal Laver function contains no supercompact cardinals. If f is a universal Laver function

on I then the *standard Laver iteration defined from f* is the Easton support iteration $\mathbb{L}$ which runs from $\min(I)$ to $\sup(I)$, forcing as before with $f(\alpha)[L_\alpha]$ whenever $f(\alpha)$ is a $\mathbb{L}_\alpha$ -name for a $< \alpha$ -directed closed forcing poset. The poset $\mathbb{L}$ is $< \min(\text{dom}(f))$ -directed closed and makes every supercompact cardinal in I indestructible.

We will need a strengthening of the concept of Laver indestructibility due to Neeman [16].

Definition 2.5. *Let κ be a supercompact cardinal. An indestructible Laver function for κ is a partial function ϕ from κ to V_κ such that for every $x \in V$, $\lambda \geq \kappa$ and $< \kappa$ -directed closed forcing extension $V[E]$, there is an elementary embedding $\pi : V[E] \rightarrow N$ such that:*

- (1) *The embedding π is defined in $V[E]$, and witnesses that κ is λ -supercompact in $V[E]$.*
- (2) *$\pi \restriction ON$ is definable in V .*
- (3) *$\kappa \in \text{dom}(\pi(\phi))$ and $\pi(\phi)(\kappa) = x$.*
- (4) *The first point in $\text{dom}(\pi(\phi))$ past κ is greater than λ .*

Note that an indestructible Laver function for κ can only exist when κ is indestructibly supercompact. Adapting the arguments of [16] to use a universal Laver function, one can readily get a *universal indestructible Laver function*.

Fact 2.6. *Let I be an interval of cardinals. Then there is a forcing poset $\mathbb{L}$ such that in the extension by $\mathbb{L}$, there exists a partial function ϕ such that $\phi \restriction \kappa$ is an indestructible Laver function for every V -supercompact cardinal $\kappa \in I$.*

Proof. We do a straightforward adaptation of the argument from the beginning of [16, Section 4]. Let f be a universal Laver function on I , and derive functions f_0 and f_1 from f such that $f(\alpha) = (f_0(\alpha), f_1(\alpha))$ when $f(\alpha)$ is an ordered pair and the values $f_i(\alpha)$ are undefined otherwise. Let $\mathbb{L}$ be the standard Laver iteration defined from f_0 , and let L be $\mathbb{L}$ -generic over V . Define $\phi(\alpha) = f_1(\alpha)[L_\alpha]$ at every point α such that $f_1(\alpha)$ is an $\mathbb{L}_\alpha$ -name. $\square$

Remark 2.7. The poset $\mathbb{L}$ does not create any new instances of supercompactness, and by convention the domain of a universal Laver function does not include any supercompact cardinals. It follows that in the extension by $\mathbb{L}$, $\kappa \notin \text{dom}(\phi)$ and $\phi \restriction \kappa$ is an indestructible Laver function for every supercompact $\kappa \in I$.

Unfortunately the property of Laver indestructibility is quite fragile:

Fact 2.8 (Hamkins [6]). *If κ is supercompact and $\mathbb{Q}$ is a non-trivial forcing poset with $|\mathbb{Q}| < \kappa$, then κ is not indestructible in the extension by $\mathbb{Q}$. In fact κ becomes “superdestructible”, that is to say its supercompactness (even its weak compactness) is destroyed by any further $< \kappa$ -closed forcing which adds a new subset of κ .*

Since indestructibility plays a central role in our arguments, we will need to make repeated appeals to Fact 2.6.

Fact 2.9. *Let $\mathbb{Q}$ be a forcing poset, let $|\mathbb{Q}| < \mu$ and let f be a universal Laver function defined up to μ . Let Q be $\mathbb{Q}$ -generic over V and define $F \in V[Q]$ by setting $F(\alpha) = f(\alpha)[Q]$ for all α such that $f(\alpha)$ is a $\mathbb{Q}$ -name. Then F is a universal Laver function on the interval $(|\mathbb{Q}|, \mu)$ in $V[Q]$.*

Proof. Let $|\mathbb{Q}| < \kappa < \mu$ with κ supercompact in $V[Q]$, so that κ is supercompact in V and $f \upharpoonright \kappa$ is a Laver function. Let $\gamma > \kappa$, let $x \in H_\gamma^{V[Q]}$ and let $x = \dot{x}[Q]$ for some $\dot{x} \in H_\gamma$. Choose $j : V \rightarrow M$ witnessing κ is γ -supercompact in V with $j(f)(\kappa) = \dot{x}$, then j lifts to an embedding $j : V[Q] \rightarrow M[Q]$ such that j witnesses κ is γ -supercompact in $V[Q]$ and $j(F)(\kappa) = x$. $\square$

Remark 2.10. Note that the Laver functions ϕ and F from Facts 2.6 and 2.9 are derived from an initial Laver function f in such a way that $\text{rk}(\phi(\alpha)), \text{rk}(F(\alpha)) \leq \text{rk}(f(\alpha))$. It follows that closure points of f are automatically closure points of its derived Laver functions.

2.3. Trees and systems

We will sometimes be in a situation where T is a tree, we know that T has a branch in some generic extension, and we want to conclude that T has a branch in V . In this situation we will often use one of the following *preservation lemmas* or *branch lemmas*.

Fact 2.11 (Unger [24]). *Let κ be regular and uncountable. If $\mathbb{P} \times \mathbb{P}$ is κ -cc, then $\mathbb{P}$ has the κ -approximation property. In particular forcing with $\mathbb{P}$ cannot add a branch through a tree of height κ .*

Fact 2.12 (Unger [23, Lemma 6]). *Let κ and η be regular and uncountable with $\kappa < \eta \leq 2^{<\kappa}$. Let $\mathbb{P}$ be $< \kappa$ -closed in V and let V' be a κ -cc extension of V . Then forcing with $\mathbb{P}$ over V' cannot add a branch through an η -tree in V' .*

If the universe is a κ -cc generic extension of a submodel in which $\mathbb{P}$ is $< \kappa$ -closed, we sometimes say that $\mathbb{P}$ is *formerly $< \kappa$ -closed*.

Fact 2.13 (Magidor and Shelah [12, Theorem 2.1]). *Suppose that $\mu < \nu$ where μ is an infinite cardinal and ν is a singular cardinal of cofinality ω . Let $V[G]$ be a μ -closed generic extension, and let E be generic over $V[G]$ for a poset in V of size μ . If $T \in V[E]$ and T is a ν^+ -tree, then any branch through T in $V[E][G]$ is already in $V[E]$.*

The concepts of *system* and a *system of branches* will play a central role. Typically a system arises from a name for a tree T in some generic extension, and a system of branches arises from a name for a branch of T in a further generic extension.

Definition 2.14. *Let D be a set of ordinals and τ be a cardinal. A system on $D \times \tau$ is an indexed collection $(R_i)_{i \in I}$ of transitive reflexive relations on $D \times \tau$ such that:*

- $(\alpha, \eta)R_i(\beta, \zeta)$ and $(\alpha, \eta) \neq (\beta, \zeta)$ implies $\alpha < \beta$.
- $(\alpha, \eta)R_i(\beta, \zeta)$ and $(\alpha', \eta')R_i(\beta, \zeta)$ implies that (α, η) and (α', η') are R_i -comparable.
- For $\alpha < \beta$ both in D there exist $\eta, \zeta < \tau$ and $i \in I$ such that $(\alpha, \eta)R_i(\beta, \zeta)$.

A system of branches through such a system is an indexed collection $(b_j)_{j \in J}$ of partial functions from D to τ such that:

- b_j is a branch through R_i for some i , that is for every $\beta \in \text{dom}(b_j)$ and every $\alpha \in D \cap \beta$, $\alpha \in \text{dom}(b_j)$ if and only if there is η with $(\alpha, \eta)R_i(\beta, b_j(\beta))$ and in this case $b_j(\alpha)$ is the unique such η .
- For every $\alpha \in D$ there is j such that $\alpha \in \text{dom}(b_j)$.

We will need the following technical fact about systems and systems of branches, which appears in a slightly different form as [16, Remark 3.4].

Fact 2.15. *Let $(R_i)_{i \in I}$ be a system on $D \times \tau$ and let ν be a cardinal such that D is a cofinal subset of ν^+ . Let $\mathbb{P}$ be a poset which adds a system of branches $(b_j)_{j \in J}$ through the system $(R_i)_{i \in I}$, and let λ be a regular cardinal such that:*

- $\max(|I|, |J|, \tau) < \lambda < \nu$.
- *There is a forcing $\mathbb{Q}$ which adds λ mutually generic filters for $\mathbb{P}$, without collapsing λ or forcing that $\text{cf}(\nu^+) \leq \lambda$.*

Then there is $j \in J$ such that $b_j \in V$ and $\text{dom}(b_j)$ is cofinal in ν^+ .

Since the proof is quite short we sketch it here.

Proof sketch. Towards a contradiction, we may assume without loss of generality that $\mathbb{P}$ forces “ $\text{dom}(b_j)$ cofinal implies $b_j \notin V$ ” for all j . Force with $\mathbb{Q}$ and let b_j^α be the realization of b_j by the α^{th} $\mathbb{P}$ -generic filter. If $\alpha \neq \beta$ and both $\text{dom}(b_j^\alpha)$ and $\text{dom}(b_j^\beta)$ are cofinal, then by mutual genericity $b_j^\alpha \neq b_j^\beta$. Since $\text{cf}(\nu^+) > \lambda$ we may choose $\eta < \nu^+$ so large such that $\text{dom}(b_j^\alpha)$ bounded implies $\text{dom}(b_j^\alpha) \subseteq \eta$ for all $j \in J$ and $\alpha < \lambda$, and also $\text{dom}(b_j^\alpha)$ and $\text{dom}(b_j^\beta)$ both cofinal and $\alpha \neq \beta$ implies $b_j^\alpha \restriction \eta \neq b_j^\beta \restriction \eta$ for all $j \in J$ and distinct $\alpha, \beta < \lambda$. Let $\gamma \in D \setminus \eta$, then for all $\alpha < \lambda$ there exist $j \in J$, $i \in I$ and $\zeta < \tau$ such that $\gamma \in \text{dom}(b_j^\alpha)$ (in particular $\text{dom}(b_j^\alpha)$ is cofinal), b_j^α is a branch through R_i and $b_j^\alpha(\gamma) = \zeta$. Since λ is a cardinal we may choose $\alpha \neq \beta$ which give the same values for (j, i, ζ) , but then $b_j^\alpha(\gamma) = b_j^\beta(\gamma)$ and both b_j^α, b_j^β are branches through R_i , so that $b_j^\alpha \restriction \gamma = b_j^\beta \restriction \gamma$ in contradiction to the choice of η . $\square$

2.4. A branch lemma

In Fact 2.15 it is important that the “width” τ of the system is considerably less than the “height” ν^+ . In Section 10.2 we are forced to consider systems where the

height is the successor of the width, and to handle these we will use an alternative branch lemma (due to Unger) whose proof is similar to that of Fact 2.12.

Let $V \subseteq W$. Let $\delta < \nu < \mu < \lambda$ be cardinals in W where μ and λ are regular. Assume that $2^\delta \geq \lambda$ in V , and $W = V[E]$ where $\mathbb{E}$ is μ -cc in V . Let $\mathbb{P}$ be $< \mu$ -closed in V , where we note that by Easton's Lemma $\text{cf}(\lambda) \geq \mu$ in $W[P]$.

Let $\mathcal{R}$ be a system on $\lambda \times \mu$ in W , with relations R_i for $i < \nu$. Assume that forcing with $\mathbb{P}$ over W adds a system of branches $(b_i)_{i \in \nu}$ where b_i is a branch through R_i . Then there is i such that $b_i \in W$ and $\text{dom}(b_i)$ is cofinal in λ .

Proof. We work in V until further notice. For each $i < \nu$ we fix $\dot{R}_i$ an $\mathbb{E}$ -name for R_i and $\dot{b}_i$ an $\mathbb{E} \times \mathbb{P}$ -name for b_i . Assume for a contradiction that $\mathbb{E} \times \mathbb{P}$ forces that $b_i \notin W$ for every i with $\text{dom}(b_i)$ cofinal. Since $\mathbb{P}$ is $< \mu$ -closed and $\mathbb{E}$ is μ -cc in $V[P]$, it is easy to find $e^* \in \mathbb{E}$, $p^* \in \mathbb{P}$ and $\eta < \lambda$ such that $(e^*, p^*) \Vdash_{\mathbb{E} \times \mathbb{P}}^V$ “ $\text{dom}(b_i)$ bounded implies $\text{dom}(b_i) \subseteq \eta$ ” for all $i < \nu$. Going forward we work below (e^*, p^*) .

Let $e \in \mathbb{E}$, $p_0, p_1 \in \mathbb{P}$, $i < \nu$ and $\gamma \in [\eta, \lambda)$. Then (e, p_0, p_1) forces divergence for b_i at γ if both (e, p_0) and (e, p_1) decide “ $\text{dom}(b_i)$ is cofinal” and one of the following holds:

- At least one of (e, p_0) and (e, p_1) forces “ $\text{dom}(b_i)$ is bounded”.
- Both (e, p_0) and (e, p_1) force “ $\text{dom}(b_i)$ is cofinal”, and one of the following holds:
 - $(e, p_0) \Vdash \gamma \in \text{dom}(b_i)$, $(e, p_1) \Vdash \gamma \notin \text{dom}(b_i)$.
 - $(e, p_0) \Vdash \gamma \notin \text{dom}(b_i)$, $(e, p_1) \Vdash \gamma \in \text{dom}(b_i)$.
 - $(e, p_j) \Vdash b_i(\gamma) = \zeta_j$ for $j \in 2$, and $\zeta_0 \neq \zeta_1$.

Claim 2.17. *Let $e \in \mathbb{E}$, $p_0, p_1 \in \mathbb{P}$ and $i < \nu$. Then there exist $\gamma \in [\eta, \lambda)$ and $(e', p'_0, p'_1) \leq (e, p_0, p_1)$ such that (e', p'_0, p'_1) forces divergence for b_i at γ .*

Proof. Extending if necessary, we may as well assume that both (e, p_0) and (e, p_1) decide “ $\text{dom}(b_i)$ is cofinal”. There is nothing to do unless (e, p_0) and (e, p_1) force “ $\text{dom}(b_i)$ is cofinal”. In this case force with $\mathbb{E}$ below e , and then force over $V[E]$ with the formerly closed forcing $\mathbb{P} \times \mathbb{P}$ below (p_0, p_1) to obtain $P_{\text{left}} \times P_{\text{right}}$. Now since b_i is forced over $V[E]$ by $\mathbb{P}$ not to lie in $V[E]$, $b_i[E \times P_{\text{left}}]$ and $b_i[E \times P_{\text{right}}]$ are distinct partial functions with cofinal domains, and it is easy to choose $(e', p'_0, p'_1) \in \mathbb{E} \times \mathbb{P} \times \mathbb{P}$ and γ as required. $\square$

Claim 2.18. *Let $p_0, p_1 \in \mathbb{P}$ and $i < \nu$. Then there exist $p'_0 \leq p_0$, $p'_1 \leq p_1$, $\gamma^* \in [\eta, \lambda)$ and a maximal antichain A in $\mathbb{E}$ such that for all $e \in A$, (e, p'_0, p'_1) forces divergence for b_i at γ for some $\gamma \in [\eta, \gamma^*)$.*

Proof. We construct pairwise incompatible $e^\alpha \in \mathbb{E}$, decreasing p_0^α below p_0 and p_1^α below p_1 , and $\gamma_\alpha < \lambda$. Since $\mathbb{E}$ is μ -cc the construction halts before μ steps. If $(e^\alpha)_{\alpha < \beta}$ does not enumerate a maximal antichain in $\mathbb{E}$, then we choose $f^\beta \in \mathbb{E}$ incomparable with all e^α for $\alpha < \beta$ and lower bounds q_0^β and q_1^β in $\mathbb{P}$ for

the sequences $(p_0^\alpha)_{\alpha < \beta}$ and $(p_1^\alpha)_{\alpha < \beta}$, and then apply Claim 2.17 to the condition $(f^\beta, q_0^\beta, q_1^\beta)$. This gives $(e^\beta, p_0^\beta, p_1^\beta) \leq (f^\beta, q_0^\beta, q_1^\beta)$ and $\gamma_\beta \in [\eta, \lambda)$ such that $(e^\beta, p_0^\beta, p_1^\beta)$ forces divergence at γ_β . Once the construction terminates after β stages, we let $A = \{e_\alpha : \alpha < \beta\}$, let p'_0 and p'_1 be lower bounds for $(p_0^\alpha)_{\alpha < \beta}$ and $(p_1^\alpha)_{\alpha < \beta}$ respectively, and let $\gamma^* = \sup_{\alpha < \beta} \gamma_\alpha$. $\square$

Claim 2.19. *Let $p_0, p_1 \in \mathbb{P}$. Then there exist $p'_0 \leq p_0$, $p'_1 \leq p_1$ and $\gamma^* \in [\eta, \lambda)$ such that for all $i < \nu$, there is a maximal antichain A_i in $\mathbb{E}$ such that for all $e \in A_i$, (e, p'_0, p'_1) forces divergence for b_i at γ for some $\gamma \in [\eta, \gamma^*)$.*

Proof. Apply Claim 2.18 to each $i < \nu$ in turn, using the $< \mu$ -closure of $\mathbb{P}$ and the regularity of λ to find p'_0 , p'_1 and γ^* that work for all i . $\square$

Claim 2.20. *Assuming that δ is minimal with $2^\delta \geq \lambda$, there exist a binary tree of decreasing sequences $(p_\sigma)_{\sigma \in {}^{<\delta}2}$ and an ordinal $\gamma^* \in [\eta, \lambda)$ with the following property: for all $\sigma \in {}^{<\delta}2$ and all $i < \nu$, there is a maximal antichain A_i^σ such that for all $e \in A_i^\sigma$, $(e, p_{\sigma \smallfrown 0}, p_{\sigma \smallfrown 1})$ forces divergence for b_i at γ for some $\gamma < \gamma^*$.*

Proof. For each $\sigma \in {}^{<\delta}2$ we appeal to Claim 2.19 with $p_0 = p_1 = p_\sigma$ to find $p_{\sigma \smallfrown 0}$, $p_{\sigma \smallfrown 1}$, antichains A_i^σ for $i < \nu$, and an ordinal γ_σ as in the conclusion. At limit stages we take lower bounds. Since $2^{<\delta} < \lambda$ and λ is regular, we may set $\gamma^* = \sup_\sigma \gamma_\sigma$. $\square$

For each $f \in {}^\delta 2$, let p_f be a lower bound for $(p_{f \restriction j})_{j < \delta}$. Now force with $\mathbb{E}$ and start to work in $V[E]$, so that $\mathcal{R} = \dot{\mathcal{R}}[E]$ is a system with relations $R_i = \dot{R}_i[E]$, and $(\dot{b}_i[E])_{i < \nu}$ is a $\mathbb{P}$ -name for a system of branches with b_i a branch through R_i .

For each $f \in {}^\delta 2$, let $q_f \leq p_f$ decide a value of i such that $\gamma^* \in \text{dom } b_i[E]$, and let q_f also decide the value $b_i[E](\gamma^*)$ for this i . Since λ is still a cardinal in $V[E]$, there exist $f \neq g$ and values $i < \nu$ and $\zeta < \mu$ such that q_f and q_g both force that $b_i[E](\gamma^*) = \zeta$. Let σ be the longest common initial segment of f and g , so that without loss of generality $q_f \leq p_{\sigma \smallfrown 0}$ and $q_g \leq p_{\sigma \smallfrown 1}$. By construction there exist a condition $e \in E$ and $\gamma < \gamma^*$ such that:

- $(e, p_{\sigma \smallfrown 0}, p_{\sigma \smallfrown 1})$ forces divergence for b_i at γ .
- e forces that both q_f and q_g force $b_i[E](\gamma^*) = \zeta$.

This is impossible as both q_f and q_g force that $b_i[E](\gamma) = \zeta'$ for the unique ζ' such that $(\gamma, \zeta') R_i(\gamma^*, \zeta)$. $\square$

2.5. Another branch lemma

We will require a branch lemma with the same general flavor as Fact 2.13. This will be used in Section 10.1 to help establish the tree property at $\aleph_{\omega^2+1}$ in our final model. This branch lemma is quite general and has some independent interest, so we prove it here axiomatizing the needed assumptions. It is a descendant of a branch lemma due to Sinapova and Unger [20].

Let $\mathbb{P}$ and $\mathbb{R}$ be forcing posets and let μ and ν be cardinals. We assume that:

- (1) ν is a cardinal of cofinality ω and $\mu = \nu^+$.
- (2) There is a cardinal $\kappa \leq \nu$ such that $\Vdash_{\mathbb{P}}^V \mu = \kappa^+$ and $\Vdash_{\mathbb{P} \times \mathbb{R}}^V \mu = \kappa^+$.
- (3) $\mathbb{R}$ is $< \mu$ -distributive and countably closed.
- (4) Every condition p in $\mathbb{P}$ has a *stem* (we write it as $\text{stem}(p)$) and there are at most ν stems.
- (5) If $\text{stem}(p) = \text{stem}(p') = h$ then there is $q \leq p, p'$ with $\text{stem}(q) = h$.
- (6) If $(p_n)_{n \in \omega}$ is a decreasing sequence of conditions with stem h then there is a lower bound with stem h .

We note for the record that by assumptions 1, 4 and 5 the poset $\mathbb{P}$ is μ -cc.

We say that stem h' *extends* stem h if there are conditions $p, p' \in \mathbb{P}$ such that $\text{stem}(p) = h$, $\text{stem}(p') = h'$ and $p' \leq p$.

The motivating idea is that $\mathbb{P}$ is some type of Prikry forcing, and $\mathbb{R}$ is a “mild” forcing poset. Our assumptions on $\mathbb{P}$ are quite weak, in particular we do not need to assume any form of the Prikry lemma. In the intended application $\mathbb{P}$ will be a complex Prikry-type forcing where taking a direct extension can change the stem, and the direct extension ordering is not countably closed.

Lemma 2.21. *Let $P \times R$ be $\mathbb{P} \times \mathbb{R}$ -generic and let $T \in V[P]$ be a μ -tree. If T has a cofinal branch in $V[P][R]$, then T has a cofinal branch in $V[P]$.*

Proof. Suppose that $\dot{T} \in V$ is a $\mathbb{P}$ -name of a μ -tree. As usual, for each $\alpha < \mu$ we assume that level α in the tree consists of pairs in $\{\alpha\} \times \kappa$. We refer to elements of $\mu \times \kappa$ as *nodes*, and if u is a node often we call u_0 the *level* of u and write it as $\text{lev}(u)$. Of course $\dot{T} \in V[R]$ and can be viewed as a $\mathbb{P}$ -name for a μ -tree in this model.

A note on notation: α and β will typically be levels of nodes. h will typically be a stem. p and q will typically be conditions in $\mathbb{P}$. r and s will typically be conditions in $\mathbb{R}$. u and v will typically be nodes. Of course these letters may be decorated with subscripts and superscripts as needed.

Without loss of generality, let $\dot{b}$ be an $\mathbb{R} \times \mathbb{P}$ -name which is forced by the empty condition to be a cofinal branch through $\dot{T}$. Let $\dot{b}[R]$ be the $\mathbb{P}$ -name in $V[R]$ for such a branch obtained by partially realizing $\dot{b}$.

Claim 2.22. *Let $p \in \mathbb{P}$. Let u and v be nodes with $\text{lev}(u) < \text{lev}(v)$. If $p \Vdash_{\mathbb{P}}^{V[R]} u, v \in \dot{b}[R]$ then $p \Vdash_{\mathbb{P}}^V u <_{\dot{T}} v$.*

Proof. Let r force that $p \Vdash_{\mathbb{P}}^{V[R]} u, v \in \dot{b}[R]$, so that $(r, p) \Vdash_{\mathbb{R} \times \mathbb{P}}^V u, v \in \dot{b}$. If there is $p' \leq p$ such that $p' \Vdash_{\mathbb{P}}^V u \not<_{\dot{T}} v$, then $(r, p') \Vdash_{\mathbb{R} \times \mathbb{P}}^V u \not<_{\dot{T}} v$ and $(r, p') \Vdash_{\mathbb{R} \times \mathbb{P}}^V u, v \in \dot{b}$, which is impossible as $\dot{b}$ is forced to be a branch. So $p \Vdash_{\mathbb{P}}^V u <_{\dot{T}} v$ as claimed. $\square$

The following definition takes place in $V[R]$.

Definition 2.23. *For a stem h , we say that $\dagger_h$ holds if there are an unbounded $J \subseteq \mu$, $\xi < \kappa$ and $\langle p_\alpha \mid \alpha \in J \rangle$ such that for all $\alpha \in J$:*

- *The condition p_α has stem h .*

- $p_\alpha \Vdash_{\mathbb{P}}^{V[R]} \langle \alpha, \xi \rangle \in \dot{b}[R]$.

Before reading the following remark, the reader should recall our convention that when we write “ $p_0 \wedge p_1 \Vdash \phi$ ” we mean only that p_0 and p_1 are compatible and every common lower bound $r \leq p_0, p_1$ is such that $r \Vdash \phi$. We are not asserting that p_0 and p_1 have a greatest lower bound.

Remark 2.24. Let J , ξ and $\langle p_\alpha \mid \alpha \in J \rangle$ witness $\dagger_h$ as in Definition 2.23 and let $\alpha, \beta \in J$ with $\alpha < \beta$. Then p_α and p_β are compatible in $\mathbb{P}$ by item 5 of our hypotheses, and $p_\alpha \wedge p_\beta \Vdash_{\mathbb{P}}^V \langle \alpha, \xi \rangle <_{\dot{T}} \langle \beta, \xi \rangle$ by Claim 2.22.

Claim 2.25. *In $V[R]$ every stem h can be extended to a stem h' such that $\dagger_{h'}$ holds.*

Proof. Work in $V[R]$. Let h be a stem, and let $p \in \mathbb{P}$ be a condition with stem h . For each $\alpha < \mu$, let $p_\alpha \leq p$ and $u_\alpha = \langle \alpha, \xi_\alpha \rangle$ be such that $p_\alpha \Vdash_{\mathbb{P}} u_\alpha \in \dot{b}[R]$. Since the number of stems is less than μ , there exist an unbounded set $J \subseteq \mu$, an ordinal $\xi < \kappa$ and a stem h' such that p_α has stem h' and $\xi_\alpha = \xi$ for all $\alpha \in J$. Then h' extends h and $\dagger_{h'}$ holds. $\square$

If $\dagger_h$ holds and h' extends h , it does not follow in general that $\dagger_{h'}$ will hold. The issue is that in general not every condition with stem h can be extended to a condition with stem h' . However we do have the following in $V[R]$:

Claim 2.26. *If $\dagger_h$ holds then there is a condition p with stem h such that $\{p' : \dagger_{\text{stem}(p')} \text{ holds}\}$ is dense below p .*

Proof. Let J , ξ and $(p_\alpha)_{\alpha \in J}$ witness $\dagger_h$. As $\mathbb{P}$ is μ -cc, it follows from Lemma 2.1 that there is α such that p_α forces the set of $\beta \in J$ with $p_\beta \in P$ to be unbounded. Set p equal to p_α and let $\bar{p} \leq p$. Then $\bar{p}$ is compatible with p_β for every β in some unbounded $J' \subseteq J$, and we may choose $p'_\beta \leq \bar{p}, p_\beta$ for all $\beta \in J'$. Thinning out J' we may assume that for some stem h' , $\text{stem}(p'_\beta) = h'$ for all $\beta \in J'$. Then the conditions $(p'_\beta)_{\beta \in J'}$ together with ξ and J' witness $\dagger_{h'}$, and for any $\beta \in J'$ we have that $p'_\beta \leq \bar{p}$ and $\dagger_{\text{stem}(p'_\beta)}$ holds. $\square$

The following definition takes place in V .

Definition 2.27. *Let h be a stem and let $s \in \mathbb{R}$. There is an (h, s) -splitting if there are a condition $p \in \mathbb{P}$ with stem h , conditions s^0, s^1 in $\mathbb{R}$, and nodes $u^0, u^1 \in \mu \times \kappa$ such that:*

- (1) $s^0, s^1 \leq s$.
- (2) $(s^k, p) \Vdash_{\mathbb{R} \times \mathbb{P}}^V u^k \in \dot{b}$ for $k \in 2$.
- (3) $p \Vdash_{\mathbb{P}}^V “u^0 \text{ and } u^1 \text{ are incomparable in } \dot{T}.”$

We note that the witnessing conditions s^0 and s^1 for an (h, s) -splitting must be incompatible. The issue is that if $s^* \leq s^0, s^1$ then (s^*, p) forces that both u^0 and u^1 lie on $\dot{b}$, while p forces them to be incomparable in $\dot{T}$.

Definition 2.28. $\dot{b}$ is h -new below s if and only if the set of s' such that there is an (h, s') -splitting is dense below s .

The following key claim takes place in V .

Claim 2.29. Suppose that s forces “ $\dot{\uparrow}_h$ holds”, and $\dot{b}$ is h -new below s . There are sequences $\langle s_i \mid i < \nu \rangle$, $\langle p_i \mid i < \nu \rangle$, and $\langle v_i \mid i < \nu \rangle$, such that

- (1) For all $i < \nu$, $s_i \leq s$ and the stem of p_i is h .
- (2) For all $i < \nu$, $(s_i, p_i) \Vdash_{\mathbb{R} \times \mathbb{P}}^V v_i \in \dot{b}$.
- (3) For $i < j < \nu$, $p_i \wedge p_j \Vdash_{\mathbb{P}}^V$ “ v_i and v_j are incomparable in $\dot{T}$ ”.

Proof. Suppose that s forces $\dot{\uparrow}_h$ as witnessed by ξ , $\dot{J}$, and $\dot{p}_\alpha$ for $\alpha \in \dot{J}$. Forcing below s we pass to a generic extension $V[R]$ where ξ , $\dot{J}$ and $\langle p_\alpha : \alpha \in \dot{J} \rangle$ witness $\dot{\uparrow}_h$.

The following subclaim takes place in $V[R]$.

Subclaim 2.30. For every $\gamma \in \dot{J}$ there exist $p \in \mathbb{P}$ with stem h , conditions r^0, r^1 in $\mathbb{R}$ below s and nodes $v^0, v^1 \in \mu \times \kappa$ such that:

- $(r^0, p) \Vdash_{\mathbb{R} \times \mathbb{P}}^V (\gamma, \xi) \in \dot{b}$.
- For $k \in 2$, $(r^k, p) \Vdash_{\mathbb{R} \times \mathbb{P}}^V v^k \in \dot{b}$.
- $p \Vdash_{\mathbb{P}}^V$ “ v^0 and v^1 are incomparable elements above (γ, ξ) in $\dot{T}$ ”.
- $r^0 \in R$.

Proof. We will do a density argument in V to show that suitable values for r^0 are dense below s . Let $r \leq s$ force that $\gamma \in \dot{J}$, and decide the value of p_γ as q . Then q has stem h and $(r, q) \Vdash_{\mathbb{R} \times \mathbb{P}}^V (\gamma, \xi) \in \dot{b}$. Since $r \leq s$ and $\dot{b}$ is h -new below s , we may extend r if needed and assume that there is an (h, r) -splitting.

Fix $r^0, r^1 \leq r$, nodes v^0, v^1 and a condition q' with stem h such that:

- (1) $(r^k, q') \Vdash_{\mathbb{R} \times \mathbb{P}}^V v^k \in \dot{b}$ for $k \in 2$.
- (2) $q' \Vdash_{\mathbb{P}}^V$ “ v^0 and v^1 are incomparable in $\dot{T}$ ”.

Since $\text{stem}(q) = \text{stem}(q') = h$, by item 5 of our hypotheses we may find $p \leq q, q'$ with $\text{stem}(p) = h$. Since (r^k, p) forces that both (γ, ξ) and v^k are in $\dot{b}$, p forces that (γ, ξ) and v^k are comparable in $\dot{T}$. Since p also forces that v^0 and v^1 are incomparable, it follows that they are both on levels above γ . $\square$

Still working in $V[R]$, choose a club $C \subseteq \mu$ such that for all $\beta \in C$ and all $\gamma \in \dot{J} \cap \beta$, the conclusion of Subclaim 2.30 holds with witnessing nodes v^0, v^1 having levels below β .

We select increasing sequences γ_i and β_i for $i < \nu$ such that

- (1) $\beta_i \in C$,
- (2) $\gamma_i \in \dot{J}$,
- (3) $\gamma_i < \beta_i \leq \gamma_{i+1}$.

Now for each γ_i the conclusion of Subclaim 2.30 holds, with witnessing nodes on levels below γ_{i+1} . We record the witnesses to this splitting as p_i, r_i^k and v_i^k . Let $u_i = (\gamma_i, \xi)$.

For $i < j$ we claim that $p_i \wedge p_j$ forces that v_i^1 and v_j^1 are incomparable. The point is that we can choose a lower bound r^* for r_i^0, r_j^0 as both are in R . Now $(r^*, p_i \wedge p_j) \Vdash_{\mathbb{R} \times \mathbb{P}}^V v_i^0, u_j \in \dot{b}$, so $p_i \wedge p_j \Vdash_{\mathbb{P}}^V v_i^0 <_{\dot{T}} u_j <_{\dot{T}} v_j^1$. Since v_i^0, v_i^1 are on levels below γ_j and $p_i \wedge p_j$ forces that v_i^0 is incomparable with v_i^1 , $p_i \wedge p_j$ forces that v_i^1 is incomparable with v_j^1 .

Now let $v_i = v_i^1$ and $s_i = r_i^1$. By the distributivity of $\mathbb{R}$, the sequence $\langle s_i, p_i, v_i \mid i < \nu \rangle$ is in V and by construction it satisfies the desired properties. $\square$

Claim 2.31. *If s forces $\dot{\uparrow}_h$, then $\dot{b}$ is not h -new below s .*

Proof. Assume for a contradiction that s forces $\dot{\uparrow}_h$ and $\dot{b}$ is h -new below s . Note that these properties also hold for conditions below s .

Using Claim 2.29, we will construct a tree of conditions $\langle (r_\sigma, p_\sigma) \mid \sigma \in \nu^{<\omega} \rangle$ in $\mathbb{R} \times \mathbb{P}$ and nodes $\langle v_\sigma \mid \sigma \in \nu^{<\omega} \rangle$ such that:

- (1) For all σ , p_σ has stem h .
- (2) If $\sigma' \supset \sigma$, then $(r_{\sigma'}, p_{\sigma'}) \leq (r_\sigma, p_\sigma)$.
- (3) For all σ , $(r_\sigma, p_\sigma) \Vdash_{\mathbb{R} \times \mathbb{P}}^V v_\sigma \in \dot{b}$.
- (4) For all σ and all $i \neq j$ in ν , $p_{\sigma \frown i} \wedge p_{\sigma \frown j}$ forces that $v_{\sigma \frown i}$ and $v_{\sigma \frown j}$ are incomparable in $\dot{T}$.

Given r_σ and s_σ , we appeal to Claim 2.29 with s_σ in place of s to produce s_i, v_i and p_i . We then set $r_{\sigma \frown i} = s_i$ and $v_{\sigma \frown i} = v_i$. Finally we set $p_{\sigma \frown i} = p_\sigma \wedge p_i$.

When the construction is done we choose $\gamma < \mu$ such that all the nodes v_σ have levels below γ . We use the countable closure of $\mathbb{R}$ and item 6 of our hypotheses on $\mathbb{P}$ to choose (r_f, p_f) for $f \in \nu^\omega$ such that p_f has stem h , and $(r_f, p_f) \leq (r_{f \upharpoonright n}, p_{f \upharpoonright n})$ for all $n < \omega$. We then choose $(r'_f, p'_f) \leq (r_f, p_f)$ so that (r'_f, p'_f) determines the node on level γ in the branch $\dot{b}$ as u_f .

Since $\nu^\omega > \nu \geq \kappa$, there exist $f \neq g$ such that $u_f = u_g = u^*$ and $\text{stem}(p'_f) = \text{stem}(p'_g) = h^*$ for some node u^* and stem h^* . Let n be least such that $f(n) \neq g(n)$. Let $\sigma = f \upharpoonright n = g \upharpoonright n$, $i = f(n)$ and $j = g(n)$, so that $f \upharpoonright n + 1 = \sigma \frown i$ and $g \upharpoonright n + 1 = \sigma \frown j$.

By construction $p'_f \wedge p'_g$ forces that $v_{\sigma \frown i}$ and $v_{\sigma \frown j}$ are incomparable in $\dot{T}$. Also $(r_f, p'_f \wedge p'_g)$ forces that both $v_{\sigma \frown i}$ and u^* are in $\dot{b}$, so $p'_f \wedge p'_g \Vdash_{\mathbb{P}}^V v_{\sigma \frown i} < u^*$. Similarly $p'_f \wedge p'_g \Vdash_{\mathbb{P}}^V v_{\sigma \frown j} < u^*$. This is a contradiction. $\square$

For each stem h , let D_h be the set of s such that either s forces $\neg \dot{\uparrow}_h$ or there is no (h, s) -splitting. It is easy to see that D_h is open, and we claim that it is also dense. To see this let s be arbitrary, where by extending s we may assume that s decides $\dot{\uparrow}_h$. If s forces $\neg \dot{\uparrow}_h$ then $s \in D_h$ by definition. If s forces $\dot{\uparrow}_h$ then $\dot{b}$ is not h -new below s by Claim 2.31, in which case by definition there is $s' \leq s$ with no (h, s') -splitting and $s' \in D_h$.

Since $\mathbb{R}$ is ν -distributive, $\bigcap_h D_h$ is dense and open. Let $s^* \in \bigcap_h D_h$. By Claims 2.25 and 2.26, and extending s^* if necessary, we may assume that:

- For some stem h , s^* forces that $\dagger_h$ holds.
- There is a condition p such that s^* forces that $\{p' : \dagger_{\text{stem}(p')}\}$ is dense below p .

Now we force below p to obtain a $\mathbb{P}$ -generic filter P with $p \in P$. Working in $V[P]$, let

$$d = \{u \in \mu \times \kappa : \exists q \in P \exists s' \leq s^* (s', q) \Vdash_{\mathbb{R} \times \mathbb{P}}^V u \in \dot{b}\}.$$

Claim 2.32. d is a cofinal branch in T .

Proof. Since $\mathbb{R} \times \mathbb{P}$ forces that $\dot{b}$ is a cofinal branch, it is routine to check that d contains nodes with unboundedly high levels. The key remaining point is that d is a chain in T . To see this, suppose for a contradiction that u^0 and u^1 are incomparable members of d .

We may choose $q \in P$ with $q \leq p$, together with $s^0, s^1 \leq s^*$, such that:

- $q \Vdash_{\mathbb{P}}^V$ “ u^0 and u^1 are incomparable in $\dot{T}$.”.
- $(s^i, q) \Vdash_{\mathbb{R} \times \mathbb{P}}^V u^i \in \dot{b}$.

Forcing with $\mathbb{R}$ below s^* over $V[P]$ we obtain R mutually generic with P , such that in $V[R]$ the set $\{p' : \dagger_{\text{stem}(p')}\}$ is dense below p . So we may choose $p' \in P$ with $p' \leq q$ and $s^{**} \leq s^*$ such that s^{**} forces $\dagger_h$ where $h = \text{stem}(p')$. So s^* does not force $\neg \dagger_h$, and since $s^* \in D_h$ it follows that there is no (h, s^*) -splitting. However p' together with s^i and u^i form an example of an (h, s^*) -splitting, which is a contradiction. $\square$

This completes the proof of Lemma 2.21. $\square$

2.6. Term forcing

Let $\mathbb{P}$ be a forcing poset and $\dot{\mathbb{Q}}$ be a $\mathbb{P}$ -name for a forcing poset. Then $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ is the set of $\mathbb{P}$ -names for elements of $\dot{\mathbb{Q}}$, where we identify names $\dot{q}_0$ and $\dot{q}_1$ if $\Vdash_{\mathbb{P}} \dot{q}_0 = \dot{q}_1$. $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ is ordered as follows: $\dot{q}_1 \leq \dot{q}_0$ in $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ if and only if $\Vdash_{\mathbb{P}} \dot{q}_1 \leq_{\dot{\mathbb{Q}}} \dot{q}_0$. Term forcing was introduced by Laver, and the theory was elaborated by Foreman.

The following Lemmas are standard:

Lemma 2.33. *Let $\mathbb{P}$ be a forcing poset, $\dot{\mathbb{Q}}$ be a $\mathbb{P}$ -name for a forcing poset, and let $\mathbb{R} = \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$.*

- (1) *The identity function is a projection from $\mathbb{P} \times \mathbb{R}$ to $\mathbb{P} * \dot{\mathbb{Q}}$.*
- (2) *If $P \times R$ is $\mathbb{P} \times \mathbb{R}$ -generic over V and $Q = \{\dot{q}[P] : \dot{q} \in R\}$, then Q is $\dot{\mathbb{Q}}[G]$ -generic over $V[P]$.*
- (3) *If $P * Q$ is $\mathbb{P} * \dot{\mathbb{Q}}$ -generic and we force over $V[P * Q]$ with $\{\dot{q} \in R : \dot{q}[P] \in Q\}$ using the ordering inherited from $\mathbb{R}$, then we obtain R such that $P \times R$ is $\mathbb{P} \times \mathbb{R}$ -generic and induces $P * Q$.*

- (4) If λ is forced by $\mathbb{P}$ to be a regular uncountable cardinal and $\dot{\mathbb{Q}}$ is forced by $\mathbb{P}$ to be $< \lambda$ -closed (resp $< \lambda$ -directed closed, canonically $< \lambda$ -closed), then $\dot{\mathbb{R}}$ is $< \lambda$ -closed (resp $< \lambda$ -directed closed, canonically $< \lambda$ -closed).

Lemma 2.34. *Let $\mathbb{P}$ be a forcing poset.*

- (1) *If it is forced by $\mathbb{P}$ that $\dot{\mathbb{Q}}$ and $\dot{\mathbb{R}}$ are forcing posets, then $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}} \times \dot{\mathbb{R}})$ is canonically isomorphic to $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}}) \times \mathcal{A}(\mathbb{P}, \dot{\mathbb{R}})$, identifying names for pairs with pairs of names.*
- (2) *If it is forced by $\mathbb{P}$ that $\dot{\mathbb{Q}}_1$ and $\dot{\mathbb{Q}}_0$ are forcing posets and $\dot{\pi}$ is a projection from $\dot{\mathbb{Q}}_1$ to $\dot{\mathbb{Q}}_0$, then $\dot{q} \mapsto \dot{\pi}(\dot{q})$ is a projection from $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}}_1)$ to $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}}_0)$.*

We can view $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ as adding a “universal generic object” for $\dot{\mathbb{Q}}$, which can be realized using any V -generic filter P on $\mathbb{P}$ as a $V[P]$ -generic filter on $\dot{\mathbb{Q}}[P]$. For use later we record some more easy facts about term forcing.

Lemma 2.35. *If κ is weakly compact, $|\mathbb{P}| < \kappa$ and $\Vdash_{\mathbb{P}} \text{“}\dot{\mathbb{Q}} \text{ is } \kappa\text{-cc”}$, then $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ is $\kappa\text{-cc}$.*

Proof. Suppose for a contradiction that $(\tau_i)_{i < \kappa}$ enumerates an antichain in $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$, so that for $i < j$ we have $\not\Vdash_{\mathbb{P}} \text{“}\tau_i \text{ and } \tau_j \text{ are compatible in } \dot{\mathbb{Q}}\text{”}$. Define a coloring of $[\kappa]^2$ in $|\mathbb{P}|$ colors, by coloring (i, j) with some condition $p(i, j)$ such that $p(i, j) \Vdash \text{“}\tau_i \text{ and } \tau_j \text{ are incompatible in } \dot{\mathbb{Q}}\text{”}$.

Since κ is weakly compact, there exist $H \in [\kappa]^\kappa$ and p such that $p(i, j) = p$ for all $(i, j) \in [H]^2$. But then p forces that $(\tau_i)_{i \in H}$ enumerates an antichain in $\dot{\mathbb{Q}}$, contradicting the hypothesis. $\square$

Lemma 2.36. *If $\dot{\mathbb{R}}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$ -name for a forcing poset then $\mathcal{A}(\mathbb{P} * \dot{\mathbb{Q}}, \dot{\mathbb{R}}) \simeq \mathcal{A}(\mathbb{P}, \mathcal{A}^{V[P]}(\dot{\mathbb{Q}}, \dot{\mathbb{R}}))$.*

Proof. This is immediate using the canonical identification between $\mathbb{P} * \dot{\mathbb{Q}}$ -terms for elements of $\dot{\mathbb{R}}$ on the one hand, and $\mathbb{P}$ -terms for $\mathcal{A}$ -terms for elements of $\dot{\mathbb{R}}$ on the other hand. $\square$

It is also useful to analyze $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}} * \dot{\mathbb{R}})$ where $\dot{\mathbb{Q}} * \dot{\mathbb{R}} \in V[P]$.

Lemma 2.37. *There is a projection from $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}}) \times \mathcal{A}(\mathbb{P} * \dot{\mathbb{Q}}, \dot{\mathbb{R}})$ to $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}} * \dot{\mathbb{R}})$*

Proof. In $V[P]$ there is a projection from $\dot{\mathbb{Q}} \times \mathcal{A}(\dot{\mathbb{Q}}, \dot{\mathbb{R}})$ to $\dot{\mathbb{Q}} * \dot{\mathbb{R}}$, and by item 2 of Lemma 2.34 this induces a projection from $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}}) \times \mathcal{A}(\mathbb{P}, \mathcal{A}^{V[P]}(\dot{\mathbb{Q}}, \dot{\mathbb{R}}))$ to $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}} * \dot{\mathbb{R}})$. By Lemma 2.36 the posets $\mathcal{A}(\mathbb{P}, \mathcal{A}^{V[P]}(\dot{\mathbb{Q}}, \dot{\mathbb{R}}))$ and $\mathcal{A}(\mathbb{P} * \dot{\mathbb{Q}}, \dot{\mathbb{R}})$ are canonically isomorphic. $\square$

Remark 2.38. With suitable identifications, the projection map from the proof of Lemma 2.37 is the identity map.

Let $P * Q$ be $\mathbb{P} * \dot{\mathbb{Q}}$ -generic. In a mild abuse of notation, we sometimes denote by “ $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})/P * Q$ ” the forcing from item 3 of Lemma 2.33, which is defined in $V[P * Q]$ to produce an $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ -generic filter R such that $P \times R$ projects to $P * Q$.

We will call this kind of forcing poset a *quotient to term* poset. We will often say “force to remove the dependence of Q on P ” or “force to refine $P * Q$ to $P \times R$ ” as shorthand for “force with the quotient to term poset $\mathcal{A}(\mathbb{P}, \dot{Q})/P * Q$ ”. The forcing $\mathcal{A}(\mathbb{P}, \dot{Q})/P * Q$ is defined in $V[P * Q]$ but we may force with it over generic extensions of this model:

Lemma 2.39. *Let $P * Q$ be $\mathbb{P} * \dot{Q}$ -generic over V , let $\mathbb{K} \in V[P * Q]$ and let K be $\mathbb{K}$ -generic over $V[P * Q]$. Forcing with $\mathcal{A}(\mathbb{P}, \dot{Q})/P * Q$ over $V[P * Q * K]$ produces R such that $P \times R$ induces $P * Q$ and K is $\mathbb{K}$ -generic over $V[P \times R]$.*

Proof. Let A be $\mathcal{A}(\mathbb{P}, \dot{Q})/P * Q$ -generic over $V[P * Q * K]$ and let $V[P * Q * A] = V[P \times R]$. Since $\mathbb{K}$ and $\mathcal{A}(\mathbb{P}, \dot{Q})/P * Q$ are both in $V[P * Q]$, K and A are mutually generic over $V[P * Q]$, so K is $\mathbb{K}$ -generic over $V[P \times R]$. $\square$

Remark 2.40. In the sequel we sometimes replace $\mathcal{A}(\mathbb{P}, \dot{Q})$ by more elaborate posets which have the similar effect of adding a $\mathbb{P}$ -name for a $\dot{Q}[P]$ -generic object: the analogue of Lemma 2.39 is true for such posets by the same argument.

We record some easy but useful equivalences involving quotient to term posets.

Lemma 2.41. *In V , $\mathbb{P} * \mathbb{Q} * \mathcal{A}(\mathbb{P}, \dot{Q})/P * Q$ is equivalent to $\mathbb{P} \times \mathcal{A}(\mathbb{P}, \dot{Q})$. In $V[P]$, $\mathbb{Q} * (\mathcal{A}(\mathbb{P}, \dot{Q})/P * Q)$ is equivalent to $\mathcal{A}(\mathbb{P}, \dot{Q})$.*

The idea of term forcing extends in a natural way to iterations with more than two steps. Suppose that $\langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < j \rangle$ is an iteration with limit $\mathbb{P}_j$. Then we may form a product of term posets $\prod_{\alpha < j} \mathcal{A}(\mathbb{P}_\alpha, \dot{Q}_\alpha)$, using the same supports that were used to form $\mathbb{P}_j$. We note that the poset $\mathbb{P}_0$ is trivial, so the first term poset in the product is equivalent to $\mathbb{Q}_0$.

It is easy to see that:

- the underlying set of $\prod_{\alpha < j} \mathcal{A}(\mathbb{P}_\alpha, \dot{Q}_\alpha)$ is the underlying set of $\mathbb{P}_j$.
- The identity function is a projection from $\prod_{\alpha < j} \mathcal{A}(\mathbb{P}_\alpha, \dot{Q}_\alpha)$ to $\mathbb{P}_j$.
- There is a natural quotient to term forcing defined in $V[P_j]$ to produce a $\prod_{\alpha < j} \mathcal{A}(\mathbb{P}_\alpha, \dot{Q}_\alpha)$ -generic object which projects to $\mathbb{P}_j$.

We will need some lemmas relating Cohen posets computed in different models.

Lemma 2.42. *Let $\kappa^{<\kappa} = \kappa \leq \lambda$ and let $\mathbb{P}$ be a κ -cc forcing poset of cardinality at most κ . Let $\dot{Q}$ be a $\mathbb{P}$ -name for $\text{Add}^{V[\mathbb{P}]}(\kappa, \lambda)$. Then $\mathcal{A}(\mathbb{P}, \dot{Q})$ is equivalent to $\text{Add}^V(\kappa, \lambda)$.*

Proof. We can view $\text{Add}^{V[\mathbb{P}]}(\kappa, \lambda)$ as the $< \kappa$ -support product in $V[P]$ of λ copies of 2, considered as a poset where 0 and 1 are incomparable elements. By the chain condition $\mathcal{A}(\mathbb{P}, \dot{Q})$ is equivalent to the $< \kappa$ -support product of λ copies of $\mathcal{A}(\mathbb{P}, \dot{2})$. An element $\dot{t}$ of $\mathcal{A}(\mathbb{P}, \dot{2})$ is determined by the Boolean value $b_{\dot{t}}$ of “ $\dot{t} = 0$ ”, and easily $\Vdash \dot{t}_0 \leq \dot{t}_1 \iff \Vdash \dot{t}_0 = \dot{t}_1 \iff b_{\dot{t}_0} = b_{\dot{t}_1}$. So $\mathcal{A}(\mathbb{P}, \dot{2})$ is a poset with at most κ pairwise incomparable conditions, and then easily $\mathcal{A}(\mathbb{P}, \dot{Q})$ is equivalent to $\text{Add}^V(\kappa, \lambda)$. $\square$

In the situation of Lemma 2.42, let $\mathbb{R} = \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$. If $P \times R$ induces $P * Q$ then $V[P * Q]$ and $V[P \times R]$ have the same $< \kappa$ -sequences of ordinals: to put it another way the associated quotient to term forcing $\mathbb{R}/P * Q$ is $< \kappa$ -distributive in $V[P * Q]$. Since both $\mathbb{P} * \mathbb{Q}$ and $\mathbb{P} \times \mathbb{R}$ are κ^+ -cc, $\mathbb{R}/P * Q$ is κ^+ -cc in $V[P * Q]$.

For use in the proofs of Claims 9.12 and 9.13 from Section 9.2, we need an easy lemma about $\mathbb{R}/P * Q$. The point of Lemma 2.43 is that chain condition properties of $\mathbb{R}/P * Q$ can be deduced from corresponding properties for $\mathbb{R}$.

Lemma 2.43. *Let the hypotheses of Lemma 2.42 hold and let $\mathbb{R} = \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$. Then in any outer model W of $V[P * Q]$ where $\mathbb{R}$ is $(\kappa^+)^W$ -Knaster, $\mathbb{P} \times (\mathbb{R}/P * Q)$ is also $(\kappa^+)^W$ -Knaster.*

Proof. Since $|\mathbb{P}| = \kappa$ it is enough to show that $\mathbb{R}/P * Q$ is $(\kappa^+)^W$ -Knaster. We use the description of $\mathbb{R}$ from the proof of Lemma 2.42. Let $(r_\alpha)_{\alpha < (\kappa^+)^W}$ be a sequence in W such that $r_\alpha \in \mathbb{R}/P * Q$. Since $\mathbb{R}$ is $(\kappa^+)^W$ -Knaster in W , we may find $B \in W$ such that B is unbounded in $(\kappa^+)^W$ and $(r_\alpha)_{\alpha \in B}$ is a sequence of conditions which are pairwise compatible in $\mathbb{R}$. Let $\alpha, \beta \in B$, then by definition for every $\eta \in \text{dom}(r_\alpha) \cap \text{dom}(r_\beta)$ the same term appears at coordinate η in r_α and r_β . It is easy to see that $(r_\alpha \cup r_\beta)[P] = r_\alpha[P] \cup r_\beta[P] \in Q$, so that $r_\alpha \cup r_\beta \in \mathbb{R}/P * Q$ and is a common lower bound in $\mathbb{R}/P * Q$ for r_α and r_β . $\square$

We also record an easy fact about the closure of quotient-to-term posets.

Lemma 2.44. *Let $\mathbb{P}$ be ρ -distributive and let $\mathbb{P}$ force “ $\dot{\mathbb{Q}}$ is canonically ρ -closed”. Let $P * Q$ be $\mathbb{P} * \dot{\mathbb{Q}}$ -generic over V . Then the quotient-to-term poset $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})/P * Q$ is canonically ρ -closed in $V[P * Q]$.*

More generally, if $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < j \rangle$ is an iteration whose supports are closed under increasing ρ -sequences, $\mathbb{Q}_0$ is ρ -distributive and $\Vdash_{\mathbb{P}_\alpha}$ “ $\dot{\mathbb{Q}}_\alpha$ is ρ -canonically ρ -closed” for $0 < \alpha < j$ then the associated quotient-to-term poset is canonically ρ -closed in $V[P_j]$.

Proof. Let $(\tau_i)_{i < \rho} \in V[P * Q]$ be a decreasing sequence in $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})/P * Q$. Since $\mathbb{P} * \dot{\mathbb{Q}}$ is ρ -distributive, $(\tau_i)_{i < \rho} \in V$. By the definition of $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})/P * Q$, $\tau_i[P] \in Q$ for all i and $\Vdash_{\mathbb{P}} \tau_j \leq \tau_i$ for $i < j < \rho$. Let τ be a name for a greatest lower bound for $(\tau_i)_{i < \rho}$ in $\dot{\mathbb{Q}}$, then $\tau[P]$ is a greatest lower bound for $(\tau_i[P])_{i < \rho}$ in $\dot{\mathbb{Q}}[P]$, so $\tau[P] \in Q$ and hence $\tau \in \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})/P * Q$. It follows easily that τ is a greatest lower bound for $(\tau_i)_{i < \rho}$ in $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})/P * Q$. The argument for longer iterations is essentially the same. $\square$

2.7. Projection and absorption

We also collect some facts about projections between forcing posets and absorbing forcing posets by collapses which will be used in the sequel. We refer the reader to [12] for a careful discussion of these matters.

Definition 2.45. Let $\mathbb{P}$ and $\mathbb{Q}$ be canonically $< \kappa$ -closed. A projection map $\pi : \mathbb{P} \rightarrow \mathbb{Q}$ is $< \kappa$ -continuous if it preserves the greatest lower bounds assured by the canonical closure. That is to say if $(p_i)_{i < \alpha}$ is decreasing in $\mathbb{P}$ for some $\alpha < \kappa$, and p is the greatest lower bound in $\mathbb{P}$ for $(p_i)_{i < \alpha}$, then $\pi(p)$ is the greatest lower bound in $\mathbb{Q}$ for $(\pi(p_i))_{i < \alpha}$.

Facts 2.46 and 2.47 both form part of [12, Lemma 2.6].

Fact 2.46. Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are canonically $< \kappa$ -closed and $\pi : \mathbb{P} \rightarrow \mathbb{Q}$ is a $< \kappa$ -continuous projection. If $\mathbb{Q}$ is $\mathbb{Q}$ -generic, then in $V[\mathbb{Q}]$ the quotient forcing $\mathbb{P}/\mathbb{Q}$ is canonically $< \kappa$ -closed.

Fact 2.47. Suppose that $\kappa < \mu$ are inaccessible cardinals. Suppose that $\mathbb{Q}$ is a canonically $< \kappa$ -closed forcing of size at most μ . Then there is a $< \kappa$ -continuous projection from $\text{Coll}(\kappa, \mu)$ to $\mathbb{Q}$.

Definition 2.48. Let $\kappa < \lambda$ where κ is inaccessible and λ is Mahlo. Let E be a set of inaccessible cardinals such that $\kappa, \lambda \in E$, and $E \cap [\kappa, \lambda)$ is the intersection of a club subset of $[\kappa, \lambda)$ with the set of inaccessible cardinals in this interval. For each $\alpha \in E$, let $\alpha^* = \min(E \setminus (\alpha + 1))$.

Let $\text{East}^E(\kappa, < \lambda)$ be the collection of partial functions f with $\text{dom}(f) \subseteq E \cap [\kappa, \lambda)$ such that

- (1) $\text{dom}(f)$ is an Easton set, that is to say it is bounded in every inaccessible cardinal.
- (2) For all $\alpha \in \text{dom}(f)$, $f(\alpha) \in \text{Coll}(\alpha, < \alpha^*)$.

$\text{East}^E(\kappa, < \lambda)$ is ordered coordinatewise.

Note that by the hypotheses on E and λ , $E \cap [\kappa, \lambda)$ is stationary in λ , and the Easton support condition for f is equivalent to demanding that $\text{dom}(f)$ is bounded in every cardinal in $E \cup \{\lambda\}$.

Lemma 2.49. $\text{East}^E(\kappa, < \lambda)$ is canonically $< \kappa$ -closed and λ -Knaster.

Proof. The closure is immediate since each component is canonically $< \kappa$ -closed and the union of fewer than κ Easton subsets of $[\kappa, \lambda)$ is Easton. Given $(p_i)_{i < \lambda}$ we may find a stationary set $E' \subseteq E \cap [\kappa, \lambda)$ such that $p_i \restriction i$ is constant for $i \in E'$, and then a stationary $E'' \subseteq E'$ such that $\text{dom}(p_i) \subseteq j$ for $i, j \in E''$ with $i < j$. The conditions p_i for $i \in E''$ are pairwise compatible. $\square$

Lemma 2.50. With the same hypotheses as in Definition 2.48, let $(\mathbb{U}(\alpha))_{\alpha \in E \cap [\kappa, \lambda)}$ be such that $\mathbb{U}(\alpha)$ is a canonically $< \alpha$ -closed poset (which may be trivial) in V_{α^*} , and let $\mathbb{U}$ be the Easton support product of the $\mathbb{U}(\alpha)$'s. Then there is a $< \kappa$ -continuous projection from $\text{East}^A(\kappa, < \lambda)$ to $\mathbb{U}$.

Proof. By Fact 2.47, for every $\alpha \in A \cap [\kappa, \lambda)$ there is a $< \alpha$ -continuous projection $\pi_\alpha : \text{Coll}(\alpha, < \alpha^*) \rightarrow \mathbb{U}(\alpha)$. We define a projection π from $\text{East}^A(\kappa, < \lambda)$ to $\mathbb{U}$

by defining $\text{dom}(\pi(f)) = \text{dom}(f)$ and $\pi(f)(\alpha) = \pi_\alpha(f(\alpha))$ for all α . It is easy to check that the map π is a $< \kappa$ -continuous projection since each of the maps π_α is a $< \alpha$ -continuous projection. $\square$

The point of $\text{East}^E(\kappa, < \lambda)$ is that it can absorb suitable Easton support iterations in a reasonable way. The following lemma is a prototype for the arguments in Section 9.4.

Lemma 2.51. *With the same hypotheses as in Definition 2.48, let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \in E \cap [\kappa, \lambda) \rangle$ be an Easton support iteration, assume that $\Vdash_\alpha$ “ $\dot{\mathbb{Q}}_\alpha \in V_{\alpha^*}$ ” and $\Vdash_\alpha$ “ $\dot{\mathbb{Q}}_\alpha$ is canonically $< \alpha$ -closed” for all α , and let $\mathbb{P}_\lambda$ be the direct limit. Then $\mathbb{P}_\lambda$ can be absorbed into $\text{East}^E(\kappa, \lambda)$ so that the quotient forcing is canonically $< \kappa$ -closed.*

Proof. Let $\mathbb{U}(\alpha) = \mathcal{A}(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha)$ and let $\mathbb{U}$ be the Easton support product of the posets $\mathbb{U}(\alpha)$. Then there is a natural projection from $\mathbb{U}$ to the limit poset $\mathbb{P}_\lambda$, and it routine to check that the projection is $< \kappa$ -continuous. Lemma 2.50 gives a $< \kappa$ -continuous projection from $\text{East}^E(\kappa, \lambda)$ to $\mathbb{U}$. It follows from Fact 2.46 that the quotient forcings for absorbing $\mathbb{P}_\lambda$ into $\mathbb{U}$ and $\mathbb{U}$ into $\text{East}^E(\kappa, \lambda)$ are both canonically $< \kappa$ -closed. $\square$

There is a parallel but simpler fact for the standard Levy collapse.

Lemma 2.52. *Let κ and λ be inaccessible with $\kappa < \lambda$, let $(\mathbb{U}(\alpha))_{\alpha \in [\kappa, \lambda)}$ be a sequence of canonically $< \kappa$ -closed posets of cardinality less than λ and let $\mathbb{U}$ be the product of the $\mathbb{U}(\alpha)$ ’s with $< \kappa$ -supports. Then there is a $< \kappa$ -continuous projection from $\text{Coll}(\kappa, < \lambda)$ to $\mathbb{U}$.*

2.8. Easton sets in Easton extensions

In Section 9.4 we will need to absorb some iterations of the form $\mathbb{L} * \mathbb{P}$, where $\mathbb{L}$ and $\mathbb{P}$ are Easton support iterations done over the same set of cardinals, into a suitable Easton support product of term forcings. To apply the ideas of Section 2.7 we need to analyze the Easton sets in $V[L]$, because they will form the supports of conditions in $\mathbb{P}$.

Let E be a set of inaccessible cardinals with limit order type such that for every inaccessible $\alpha < \sup(E)$ with $\alpha = \sup(E \cap \alpha)$, α is in E . For each $\alpha < \sup(E)$, let $\alpha^* = \min(E \setminus (\alpha + 1))$. Note that a subset of E is an Easton set if and only if it is bounded in every element of E , together with $\sup(E)$ in case this cardinal is inaccessible.

Let $\mathbb{L}$ be an iteration with Easton supports such that:

- The support of $\mathbb{L}$ is contained in E .
- For every $\alpha \in E$, it is forced by $\mathbb{L} \restriction \alpha$ that the iterand at α is $< \alpha$ -closed.
- For every $\alpha \in E$, $|\mathbb{L} \restriction \alpha + 1| < \alpha^*$.

It is easy to see that for every $\beta < \sup(E)$, $|\mathbb{L} \restriction \beta + 1| < \beta^*$. As a consequence it is forced by $\mathbb{L} \restriction \beta + 1$ that the tail iteration above β is $< \beta^*$ -closed. It follows easily that every $\alpha \in E$ remains inaccessible in $V[L]$.

Lemma 2.53. *If $S \in V[L]$ is an Easton subset of E then S is covered by an Easton subset of E which lies in V .*

Proof. Let $\dot{S}$ be an $\mathbb{L}$ -name for an Easton subset of E . We will establish that a stronger statement holds for all triples (p, β, γ) where $p \in \mathbb{L}$, and $\beta < \gamma \leq \sup(E)$:

$R(p, \beta, \gamma)$ is the statement “There exist $p' \leq p$ with $p' \restriction \beta + 1 = p \restriction \beta + 1$ and an Easton set $T \subseteq [\beta, \gamma) \cap E$ such that $p' \Vdash \dot{S} \cap [\beta, \gamma) \subseteq \check{T}$ ”.

Our desired conclusion will follow by setting $\beta = 0$ and $\gamma = \sup(E)$. We prove that $R(p, \beta, \gamma)$ holds for all triples (p, β, γ) by induction on γ .

- Case I: E is bounded in γ , say $\bar{\gamma} = \sup(E \cap \gamma) < \gamma$. If $\beta = \bar{\gamma}$ then $[\beta, \gamma) \cap E = \emptyset$ and there is nothing to do. If $\beta < \bar{\gamma}$ then we appeal to $R(p, \beta, \bar{\gamma})$, which is true by induction.
- Case II: $\gamma = \sup(E \cap \gamma)$ and $\text{cf}(\gamma) = \mu < \gamma$.
 - Subcase IIa: $\mu \leq \beta$. Note that the union of at most μ Easton subsets of $[\beta, \gamma) \cap E$ is Easton. Choose an increasing sequence $(\gamma_i)_{i < \mu}$ of ordinals which is cofinal in (β, γ) . Let $p_0 = p$, and build a decreasing sequence $(p_i)_{i < \mu}$ of conditions in $\mathbb{L}$ with $p_i \restriction \beta + 1 = p_0 \restriction \beta + 1$, together with Easton sets $T_i \subseteq [\beta, \gamma_i) \cap E$, such that $p_{i+1} \Vdash \dot{S} \cap [\beta, \gamma_i) \subseteq \check{T}_i$. At successor steps we choose p_{i+1} by appealing to $R(p_i, \beta, \gamma_i)$, at limits we may take lower bounds because all iterands past β are forced to be μ -closed and (by the remark about unions of Easton sets) there is no problem with the supports. After μ steps we let p' be a lower bound for the conditions p_i such that $p' \restriction \beta + 1 = p_0 \restriction \beta + 1$, and $T = \bigcup_{i < \mu} T_i$, where p' can be chosen as in the choice of p_i for i limit and T is Easton by the remark on unions of Easton sets.
 - Subcase IIb: $\beta < \mu < \gamma$. Start by appealing to $R(p, \beta, \mu)$ to produce $p' \leq p$ and T_0 an Easton subset of $[\beta, \mu)$ such that $p' \restriction \beta + 1 = p \restriction \beta + 1$ and $p' \Vdash \dot{S} \cap [\beta, \mu) \subseteq \check{T}_0$. Then replace β by μ and argue as in Subcase IIa to produce $p'' \leq p'$ and an Easton set $T_1 \subseteq [\mu, \gamma) \cap E$ such that $p'' \restriction \mu + 1 = p' \restriction \mu + 1$ and $p'' \Vdash \dot{S} \cap [\mu, \gamma) \subseteq \check{T}_1$. Clearly p'' and $T_0 \cup T_1$ will serve to witness $R(p, \beta, \gamma)$.
- Case III: $\gamma = \sup(E \cap \gamma)$ and γ is inaccessible, in particular $\gamma \in E$ or $\gamma = \sup(E)$.

It is forced that $\dot{S}$ is bounded in γ . Since $|\mathbb{L} \restriction \beta + 1| < \beta^* < \gamma$ we may build $p' \leq p$ such that $p' \restriction \beta + 1 = p \restriction \beta + 1$, and for every $q \leq p'$ such that q decides $\sup(\dot{S} \cap \gamma)$ we have that $q \restriction \beta + 1 \restriction p' \restriction (\beta + 1, \sup(A))$ decides it: the key points are that all iterands past β are $< \beta^*$ -closed, $(\beta + 1, \gamma) \cap E = [\beta^*, \gamma) \cap E$, and the union of fewer than β^* Easton subsets

of $[\beta^*, \gamma) \cap E$ is Easton. If we let $\gamma' = \sup\{\eta : \exists r \in \mathbb{L} \upharpoonright \beta + 1 \text{ } r \frown p' \upharpoonright (\beta + 1, \gamma) \Vdash \sup(\dot{S} \cap \gamma) = \eta\}$ then $\gamma' < \gamma$ and $p' \Vdash \dot{S} \cap \gamma \subseteq \gamma'$. Appealing to $R(p', \beta, \gamma')$ we find $p'' \leq p'$ and an Easton set $T \subseteq [\beta, \gamma')$ such that $p'' \upharpoonright \beta + 1 = p' \upharpoonright \beta + 1$ and $p'' \Vdash \dot{S} \cap [\beta, \gamma) \subseteq \check{T}$.

□

2.9. Robustness of chain condition

In the sequel we will often force over V using the Cohen poset $\text{Add}(\kappa, \lambda)$ defined over some inner model of V . This idea is often useful in the situation where $\kappa = \mu^+$ and $2^\mu > \mu^+$: forcing with $\text{Add}(\mu^+, \lambda)$ as defined in V will collapse μ^+ , so instead we force with $\text{Add}(\mu^+, \lambda)$ defined in some inner model where $2^\mu = \mu^+$. The following Lemma shows that the chain condition (really the Knaster property) of Cohen forcing is quite robust. We will also need that the distributivity of Cohen forcing is robust, but we will typically establish this by *ad hoc* arguments using Easton's Lemma and term forcing, see for instance Lemma 4.9 below.

Lemma 2.54. *Let κ be regular and let $\mathbb{P} = \text{Add}(\kappa, \lambda)$. If $\eta^{<\kappa} < \mu$ for every $\eta < \mu$, and W is an outer model in which μ is regular and every set of ordinals of size less than κ in W is covered by a set of size less than κ in V , then $W \models \text{"}\mathbb{P} \text{ is } \mu\text{-Knaster"}$.*

Proof. We work in W , noting that our hypotheses imply that κ is still regular in W . Let $(p_i)_{i < \mu}$ be a μ -sequence of conditions in $\mathbb{P}$. Let $X = \bigcup_{i < \mu} \text{dom}(p_i)$, so that $X \subseteq \kappa \times \lambda$ with $|X| \leq \mu$, and enumerate X as $(x_i)_{i < \mu}$. Let $d_i = \{j < \mu : x_j \in \text{dom}(p_i)\}$.

Let $S = \mu \cap \text{cof}(\kappa)$, and for $i \in S$ define $f(i) = \sup(d_i \cap i)$. Since f is regressive we may fix $S_0 \subseteq S$ stationary and $\eta < \mu$ such that $f(i) = \eta$ for all $i \in S_0$. Thinning out S_0 if necessary, we may assume that if $i, j \in S_0$ with $i < j$ then $\sup d_i < j$. Let $D_i \in V$ be such that $d_i \cap \eta \subseteq D_i \subseteq \eta$ and $|D_i| < \kappa$. Since $\eta^{<\kappa} < \mu$ in V , we may find $S_1 \subseteq S_0$ stationary and D such that $D_i = D$ for every $i \in S_1$.

Now let $z = \{x_j : j \in D\}$, and use the covering hypothesis again to find $Z \in V$ such that $z \subseteq Z \subseteq \kappa \times \lambda$ and $|Z| < \kappa$. Since $2^{|Z|} < \mu$ in V , we may find $S_2 \subseteq S_1$ stationary and a partial function p from Z to 2 such that $p_i \upharpoonright Z = p$ for all $i \in S_2$.

We claim that the conditions p_i for $i \in S_2$ are compatible. Let $i < j$ with $i, j \in S_2$, and let $(\alpha, \beta) \in \text{dom}(p_i) \cap \text{dom}(p_j)$. Since $(\alpha, \beta) \in X$, we find k with $(\alpha, \beta) = x_k$, so that by definition $k \in d_i \cap d_j$. Since $\sup(d_i) < j$, $k \in d_j \cap j$, so $k < f(j) = \eta$.

It follows that $k \in d_i \cap \eta$, so that $k \in D_i = D$. By definition $x_k = (\alpha, \beta) \in z \subseteq Z$, and since $(\alpha, \beta) \in \text{dom}(p_i) \cap \text{dom}(p_j)$ we have $p_i(\alpha, \beta) = p(\alpha, \beta) = p_j(\alpha, \beta)$. □

Remark 2.55. Similar lemmas with similar purposes appear in papers by Abraham [1, Lemma 2.16] and Cummings and Foreman [2, Lemma 2.6]

To streamline the process of applying Lemma 2.54, we encapsulate some of the hypotheses in a definition.

Let κ and μ be regular cardinals. Then an outer model $W \supseteq V$ is (κ, μ) -good if and only if every set of ordinals of size less than κ in W is covered by a set of size less than κ in V , and μ is regular in W .

Remark 2.57. If W_1 is a (κ, μ) -good outer model of V and W_2 is a (κ, μ) -good outer model of W_1 , then W_2 is a (κ, μ) -good outer model of V . If W is a (κ, μ) -good outer model of V and W' is an intermediate model, then W' is also a (κ, μ) -good outer model of V .

2.10. A technical fact

We will need a version of a technical fact from [16]. The exact statement is slightly different but the proof will be essentially the same, see the discussion following the statement.

Fact 2.58 (essentially [16, Lemma 3.10]). *Let $(\kappa_m)_{2 \leq m < \omega}$ be an increasing sequence of regular cardinals and let $\nu = \sup_m \kappa_m$. Let $\text{Index} \subseteq \kappa_2$, let $N < \omega$ and let $\mathbb{M}(\rho)$ for $\rho \in \text{Index}$ be forcing posets such that $|\mathbb{M}(\rho)| \leq \kappa_N$ for all ρ . Let $R = V_\zeta$ where $\zeta > \nu^+$ and R satisfies a large enough finite fragment of ZFC. Assume that:*

- *For all sufficiently large $m < \omega$, there exist posets $\mathbb{P}$ and $\mathbb{Q}$ such that:*
 - $\mathbb{P}$ *adds a generic embedding $\pi : V \rightarrow V^*$ such that $\text{crit}(\pi) > \kappa_m$ and π is discontinuous at ν^+ .*
 - $\mathbb{Q}$ *adds κ_m mutually generic filters for $\mathbb{P}$.*
 - $\mathbb{Q}$ *preserves cardinals up to and including κ_m , and forces $\text{cf}(\nu^+) > \kappa_m$.*
- *There are stationarily many $X \prec R$ such that for some ν^+ -Knaster poset $\mathbb{P}_X$:*
 - $\nu^+ \subseteq X$ *and $|X| = \nu^+$.*
 - *Letting M be the transitive collapse of X , $\mathbb{P}_X$ adds a generic embedding $\pi : M \rightarrow M^*$ such that $\text{crit}(\pi) = \kappa_2$, $\pi(\kappa_2) > \nu^+$, and π is discontinuous at ν^+ .*
 - $\nu \in \pi(\text{Index})$, *and $\mathbb{P}_X$ adds L which is $\pi(\mathbb{M})(\nu)$ -generic over M^* .*

Then there exists $\rho \in \text{Index}$ such that $\mathbb{M}(\rho)$ forces “ ν^+ has the tree property”.

The only differences between the proof here and in [16, Lemma 3.10] are that:

- The forcing posets $\mathbb{M}(\rho)$ are potentially larger (cardinality κ_N rather than κ_2) which does not materially affect the argument for choosing D and n .
- Only a tail of the cardinals κ_m for $m > 2$ are assumed to have the necessary properties, but we can still choose a suitable $m > n, N$.

3. $\mathbb{A}$, $\mathbb{B}$, $\mathbb{U}$, $\mathbb{C}$ and $\mathbb{S}$

We will use several versions of the main forcing construction from Neeman’s paper [16]. To minimize repetition we describe here the simplest version that we will need,

then later in the paper we modify the construction as needed. See Section 3.3 for a discussion of how we modify the construction.

3.1. The basic forcing

The version we describe in detail here is basically the forcing of [16] with the minor simplification that the cardinal μ_1 is fixed from the start rather than chosen generically. The initial setup involves an increasing ω -sequence of regular cardinals $(\mu_n)_{n < \omega}$, where $\mu_0^{<\mu_0} = \mu_0$, $\mu_1^{<\mu_1} = \mu_1$, and the μ_n 's are indestructibly supercompact for $n \geq 2$, together with a universal indestructible Laver function ϕ defined on (μ_1, μ_ω) , where $\mu_\omega = \sup_n \mu_n$. We will define forcing posets $\mathbb{A}$, $\mathbb{B}$, $\mathbb{U}$, $\mathbb{C}$ and $\mathbb{S}$ along with a number of auxiliary forcing posets.

The basic idea is that we will define $\mathbb{A}, \mathbb{B}, \mathbb{C} \in V$ and then project $\mathbb{A} \times \mathbb{B} \times \mathbb{C}$ to an iteration $\mathbb{A} * \mathbb{U} * \mathbb{S}$. Forcing with $\mathbb{A} * \mathbb{U} * \mathbb{S}$ will produce an extension in which $2^{\mu_n} = \mu_{n+2}$ for all n , $\mu_{n+1} = \mu_n^+$ for all $n > 0$, and μ_n enjoys a highly indestructible version of the tree property for $n \geq 2$. Very roughly speaking $\mathbb{A}$ is responsible for making $2_n^\mu = \mu_{n+2}$ for all n , $\mathbb{S}$ is responsible for collapsing cardinals so that μ_{n+1} becomes the successor of μ_n for $n > 0$, and $\mathbb{U}$ is responsible for making the tree property at μ_n indestructible for $n \geq 2$. We discuss these points in more detail after defining $\mathbb{A} * \mathbb{U} * \mathbb{S}$.

Remark 3.1. Readers of [16] will notice that the definitions of $\mathbb{A}_n$ and $\mathbb{A}$ are slightly different here. This makes the definitions more uniform, and is possible because the value of μ_1 is fixed.

- $\mathbb{A}$: Conditions in $\mathbb{A}_n$ are partial functions from the interval $[\mu_{n+1}, \mu_{n+2})$ to 2 with supports of size less than μ_n , ordered by extension. We will sometimes write $\mathbb{A}_n$ as $\text{Add}(\mu_n, [\mu_{n+1}, \mu_{n+2}))$. Of course $\mathbb{A}_n$ is equivalent to the standard Cohen poset $\text{Add}(\mu_n, \mu_{n+2})$. The poset $\mathbb{A}_n \restriction \alpha$ is defined in the obvious way.

$\mathbb{A}$ is the full support product of the posets $\mathbb{A}_n$ for $n < \omega$. Whenever it is convenient we will regard conditions in $\mathbb{A}$ as partial functions p from $[\mu_1, \mu_\omega)$ to 2, such that $p \restriction [\mu_{n+1}, \mu_{n+2})$ has support of size less than μ_n . Intuitively $\mathbb{A}$ is set up so that we finish adding Cohen subsets of μ_n before we begin to add Cohen subsets of μ_{n+1} .

For $\alpha < \mu_\omega$, $\mathbb{A} \restriction \alpha$ is equivalent to $\prod_{i < n} \mathbb{A}_i \times \mathbb{A}_n \restriction \alpha$ for the least n such that $\alpha \leq \mu_{n+2}$. We let A be some $\mathbb{A}$ -generic object, and define A_n and $A \restriction \alpha$ in the obvious way.

- $\mathbb{B}$ and $\mathbb{U}$: $\mathbb{B}$ and $\mathbb{U}$ are two posets with the same set of conditions but different orderings, with $\mathbb{B} \in V$ and $\mathbb{U} \in V[A]$. In a sense that we make precise later $\mathbb{B}$ is a term forcing for $\mathbb{U}$, but its definition involves a kind of “self-reference” not present in the simple term forcing of Section 2.6. Conditions in $\mathbb{B}$ will be certain functions with domains contained in (μ_1, μ_ω) , and $\mathbb{B} \restriction \alpha$ is the set of $b \in \mathbb{B}$ with $\text{dom}(b) \subseteq \alpha$; more generally if I is an interval then $\mathbb{B} \restriction I$ is the set of $b \in \mathbb{B}$ with $\text{dom}(b) \subseteq I$, and in all cases we will view

$\mathbb{B} \restriction I$ as a poset with the ordering inherited from $\mathbb{B}$. $\mathbb{U} \restriction \alpha$ has the same conditions as $\mathbb{B} \restriction \alpha$.

Formally speaking we will define $\mathbb{B} \restriction \alpha$ and $\mathbb{U} \restriction \alpha$ by simultaneous induction on α , in such a way that $\mathbb{U} \restriction \alpha \in V[A \restriction \alpha]$. A condition $b \in \mathbb{B} \restriction \alpha$ is a function such that:

- $\text{dom}(b)$ is an Easton subset of the set of $\alpha' \in \text{dom}(\phi) \cap \alpha$ such that $\phi(\alpha')$ is a $\mathbb{A} \restriction \alpha' * \dot{\mathbb{U}} \restriction \alpha'$ -name for a $< \alpha'$ -directed closed forcing poset.
- For every $\alpha' \in \text{dom}(b)$, $b(\alpha')$ is an $\mathbb{A} \restriction \alpha' * \dot{\mathbb{U}} \restriction \alpha'$ -name for a condition in $\phi(\alpha')$.

$\mathbb{B} \restriction \alpha$ and $\mathbb{U} \restriction \alpha$ are ordered as follows:

- $b_1 \leq b_0$ in $\mathbb{B} \restriction \alpha$ if and only if $\text{dom}(b_0) \subseteq \text{dom}(b_1)$ and $(0, b_1 \restriction \alpha') \Vdash_{\mathbb{A} \restriction \alpha' * \dot{\mathbb{U}} \restriction \alpha'} b_1(\alpha') \leq b_0(\alpha')$ for all $\alpha' \in \text{dom}(b_0)$.
- $u_1 \leq u_0$ in $\mathbb{U} \restriction \alpha$ if and only if $\text{dom}(u_0) \subseteq \text{dom}(u_1)$ and there is $a \in A \restriction \alpha$ such that $(a \restriction \alpha', u_1 \restriction \alpha') \Vdash_{\mathbb{A} \restriction \alpha' * \dot{\mathbb{U}} \restriction \alpha'} u_1(\alpha') \leq u_0(\alpha')$ for all $\alpha' \in \text{dom}(u_0)$.

By going to a dense subset we may view $\mathbb{A} * \dot{\mathbb{U}}$ as consisting of pairs (a, u) where $a \in \mathbb{A}$ and $u \in \mathbb{B}$, ordered as follows: $(a_1, u_1) \leq (a_0, u_0)$ if and only if $a_1 \leq a_0$ in $\mathbb{A}$, $\text{dom}(u_0) \subseteq \text{dom}(u_1)$, and $(a_1 \restriction \alpha, u_1 \restriction \alpha) \Vdash u_1(\alpha) \leq u_0(\alpha)$ for all $\alpha \in \text{dom}(u_0)$. A similar remark applies to initial segments $\mathbb{A} \restriction \beta * \dot{\mathbb{U}} \restriction \alpha$ where $\alpha \leq \beta \leq \mu_\omega$.

Remark 3.2.

- (1) We see from the definition that $\mathbb{U} \in V[A]$, and that $\mathbb{U}$ may be viewed as some type of iteration in $V[A]$, where at every α in the domain of $\mathbb{B}$ we use the $\mathbb{U} \restriction \alpha$ -name $\phi(\alpha)[A \restriction \alpha]$
- (2) The construction of $\mathbb{B}$ is also iterative, so that in particular for $\alpha < \beta \leq \mu_\omega$ the poset $\mathbb{B} \restriction \beta$ is *not* isomorphic to $\mathbb{B} \restriction \alpha \times \mathbb{B} \restriction [\alpha, \beta)$. However standard term forcing arguments show that the natural concatenation map is a projection from $\mathbb{B} \restriction \alpha \times \mathbb{B} \restriction [\alpha, \beta)$ to $\mathbb{B} \restriction \beta$.

Let $\alpha \leq \beta \leq \mu_\omega$ and let $F \subseteq \mathbb{A} \restriction \beta * \dot{\mathbb{U}} \restriction \alpha$ be a filter, which we assume to be generated by pairs (a, u) with $a \in \mathbb{A} \restriction \beta$ and $u \in \mathbb{U} \restriction \alpha$. The reader is warned that F may only exist in a generic extension of V .

- $\mathbb{B}^{+F} \restriction [\alpha, \beta)$: The underlying set of the poset $\mathbb{B}^{+F} \restriction [\alpha, \beta)$ is $\mathbb{B} \restriction [\alpha, \beta)$, and it is ordered by feeding in information from F . Formally $b_1 \leq b_0$ if and only if $\text{dom}(b_0) \subseteq \text{dom}(b_1)$ and there is $(a, u) \in F$ such that $(a \restriction \alpha', u \cup b_1 \restriction \alpha') \Vdash b_1(\alpha') \leq b_0(\alpha')$ for all $\alpha' \in \text{dom}(b_0)$. Note that the definition makes sense because $\text{dom}(u) \subseteq \alpha$ and $\text{dom}(b_1) \subseteq [\alpha, \beta)$, so that $u \cup b_1 \restriction \alpha' \in \mathbb{B} \restriction \alpha'$. Note also that F being a filter generated by pairs (a, u) as above is sufficient to show that the ordering on $\mathbb{B}^{+F} \restriction [\alpha, \beta)$ is transitive.

A couple of examples may help to clarify this definition, where throughout $\alpha \leq \beta < \mu_\omega$:

- $F = 0$ (the trivial filter): $\mathbb{B}^{+0} \restriction [\alpha, \beta) = \mathbb{B} \restriction [\alpha, \beta)$.
- $\alpha = 0$ and $F = A \restriction \beta * 0$: $\mathbb{B}^{+A \restriction \beta * 0} \restriction [0, \beta) = \mathbb{U} \restriction \beta$.
- $F = A \restriction \beta * U \restriction \alpha$: $\mathbb{B}^{+A \restriction \beta * U \restriction \alpha} \restriction [\alpha, \beta)$ is equivalent to the natural forcing for prolonging $A \restriction \beta * U \restriction \alpha$ to $A \restriction \beta * U \restriction \beta$. We let $\mathbb{U} \restriction [\alpha, \beta) = \mathbb{B}^{+A \restriction \beta * U \restriction \alpha} \restriction [\alpha, \beta)$.

We quote without proof some facts from [16]: the proofs are in every case the same, or slightly easier because here we fixed a value for μ_1 in advance.

Fact 3.3 ([16, Claim 4.5]). *Let $F_0 \subseteq F_1$ be two filters on $\mathbb{A} \restriction \beta * \dot{\mathbb{U}} \restriction \alpha$, and let G_0 be generic for $\mathbb{B}^{+F_0} \restriction [\alpha, \beta)$ over a universe $W \supseteq V$ with $F_0, F_1 \in W$. Let G_1 be the upwards closure of G_0 in $\mathbb{B}^{+F_1} \restriction [\alpha, \beta)$. Then G_1 is generic for $\mathbb{B}^{+F_1} \restriction [\alpha, \beta)$ over W .*

Remark 3.4. Fact 3.3 explains our comment above that $\mathbb{B}$ is a kind of term forcing. As an instructive example let G_0 be $\mathbb{B}^{+A \restriction \alpha * U \restriction \alpha} \restriction [\alpha, \beta)$ -generic over $V[A \restriction \alpha * U \restriction \alpha]$. If we force over $V[A \restriction \alpha * U \restriction \alpha][G_0]$ with $\mathbb{A} \restriction [\alpha, \beta)$ and prolong $A \restriction \alpha * U \restriction \alpha$ to $A \restriction \beta * U \restriction \alpha$, then in $V[A \restriction \beta * U \restriction \alpha][G_0]$ we may induce G_1 which is $\mathbb{U} \restriction [\alpha, \beta) = \mathbb{B}^{+A \restriction \beta * U \restriction \alpha} \restriction [\alpha, \beta)$ -generic over $V[A \restriction \beta * U \restriction \alpha][G_0]$. So $\mathbb{B}^{+A \restriction \alpha * U \restriction \alpha} \restriction [\alpha, \beta)$ serves as a kind of term poset, adding an $\mathbb{A} \restriction [\alpha, \beta)$ -name for a $\mathbb{U} \restriction [\alpha, \beta)$ -generic object.

Fact 3.3 has a kind of reversal: if G_1 is generic for $\mathbb{B}^{+F_1} \restriction [\alpha, \beta)$ over W then we can force over $W[G_1]$ with a suitable factor forcing to obtain G_0 which induces G_1 as above: the factor forcing is just G_1 with the ordering of $\mathbb{B}^{+F_0} \restriction [\alpha, \beta)$, and is a version of the “quotient to term” forcing discussed in Section 2.6.

Fact 3.5 ([16, Claim 4.7]). *If $\alpha' \leq \alpha$ and F' is $\mathbb{A} \restriction \alpha' * \mathbb{U} \restriction \alpha'$ -generic over V , then $\mathbb{B}^{+F'} \restriction [\alpha, \beta)$ is $< \alpha$ -directed closed in $V[F']$.*

Remark 3.6. As a useful special case of Claim 3.5, we may set $\alpha' = 0$ and $F' = 0$ to see that $\mathbb{B} \restriction [\alpha, \beta)$ is $< \alpha$ -directed closed in V .

To lend some insight into what the forcing $\mathbb{A} * \mathbb{U}$ is doing, we quote a fact from [16]. We will not be appealing to this fact directly, but the ideas in its proof will be used heavily in the proof of Lemma 4.5 below.

Fact 3.7 ([16, Claim 4.12]). *Let $A * U \restriction \mu_{n+2}$ be $\mathbb{A} * \mathbb{U} \restriction \mu_{n+2}$ -generic over V . Then in $V[A * U \restriction \mu_{n+2}]$ the cardinal μ_{n+2} is indestructibly generically supercompact for $< \mu_{n+2}$ -directed closed posets lying in $V[A \restriction \mu_{n+2} * U \restriction \mu_{n+2}]$, where the generic embeddings π witnessing the generic supercompactness are added by posets of the form $\text{Add}^V(\mu_n, \pi(\mu_{n+2})) \times \text{Add}^V(\mu_{n+1}, \pi(\mu_{n+3}))$.*

Now we define more posets $\mathbb{C} \in V$, $\mathbb{C}^{+F}$ for a filter F on $\mathbb{A} * \mathbb{U}$, and $\mathbb{S}$:

- $\mathbb{C}$: The forcing poset $\mathbb{C}$ is the full support product of forcing posets $\mathbb{C}_n$ for $n < \omega$. Conditions in $\mathbb{C}_n$ are functions whose domains are subsets of (μ_{n+1}, μ_{n+2}) with domains of size less than μ_{n+1} . If $c \in \mathbb{C}$ and $\alpha \in \text{dom}(c) \cap (\mu_{n+1}, \mu_{n+2})$ then $c(\alpha)$ is an $\mathbb{A} \restriction \alpha * \dot{\mathbb{U}} \restriction \mu_{n+1}$ -name for a condition in

$\text{Add}(\mu_{n+1}, 1)^{V[A \restriction \alpha * U \restriction \mu_{n+1}]}$. $\mathbb{C}$ is ordered like a pure term forcing, that is to say $c_1 \leq c_0$ if and only if $\text{dom}(c_0) \subseteq \text{dom}(c_1)$ and $\Vdash c_1(\alpha) \leq c_0(\alpha)$ for all $\alpha \in \text{dom}(c_0)$.

- $\mathbb{C}^{+F}$: Let $F \subseteq \mathbb{A} * \mathbb{U}$ be a filter, and define a forcing poset $\mathbb{C}^{+F}$ with the same set of conditions as $\mathbb{C}$ but a richer ordering: $c_1 \leq c_0$ if and only if $\text{dom}(c_0) \subseteq \text{dom}(c_1)$ and there is $(a, u) \in F$ such that for all n and all $\alpha \in \text{dom}(c_0) \cap (\mu_{n+1}, \mu_{n+2})$, $(a \restriction \alpha, u \restriction \mu_{n+1}) \Vdash c_1(\alpha) \leq c_0(\alpha)$.
- $\mathbb{S} = \mathbb{C}^{+A * U}$.

$\mathbb{C}$ serves as a term forcing for $\mathbb{S}$ in roughly the same way that $\mathbb{B}$ serves as a term forcing for $\mathbb{U}$. Restrictions of the posets $\mathbb{C}$ and $\mathbb{S}$ to intervals are defined in the natural way, and there is an analogous version of Fact 3.3 for $\mathbb{C}$ and $\mathbb{S}$.

We let $\mathbb{B}_n = \mathbb{B} \restriction [\mu_{n+1}, \mu_{n+2})$, $\mathbb{U}_n = \mathbb{U} \restriction [\mu_{n+1}, \mu_{n+2})$, and $\mathbb{S}_n = \mathbb{S} \restriction [\mu_{n+1}, \mu_{n+2})$. In connection with this we note that $\mathbb{A}_n = \mathbb{A} \restriction [\mu_{n+1}, \mu_{n+2})$ and $\mathbb{C}_n = \mathbb{C} \restriction [\mu_{n+1}, \mu_{n+2})$. It is easy to see that $\mathbb{U} \restriction \mu_n \in V[A \restriction \mu_n]$ and $\mathbb{S} \restriction \mu_n \in V[A \restriction \mu_n][U \restriction \mu_{n-1}]$ for all $n > 1$.

Remark 3.8. Each of the posets $\mathbb{A}, \mathbb{B}, \mathbb{C}$ consists of partial functions with domains contained in $[\mu_1, \mu_\omega)$. It is useful to note that we are using different supports in each of these posets on the interval $[\mu_{n+1}, \mu_{n+2})$, which corresponds to the factors with index n : supports of size less than μ_n for $\mathbb{A}_n$, Easton supports for $\mathbb{B}_n$, supports of size less than μ_{n+1} for $\mathbb{C}_n$.

In the current setting, $\mathbb{S}$ is just a product in $V[A * U]$ of the posets $\mathbb{S}_n$. We emphasize that $\mathbb{U}$ is not a product but an iteration. We may view $\mathbb{A} * \mathbb{U} * \mathbb{S}$ as a projection of $\mathbb{A} \times \mathbb{B} \times \mathbb{C}$ in the natural way. Much as in Remark 3.2 we may also view $\mathbb{B}$ as a projection of $\prod_n \mathbb{B}_n$, and so may view $\mathbb{A} * \mathbb{U} * \mathbb{S}$ as a projection of $\prod_n \mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n$. See Lemma 3.13 below for more on this.

One small difference with [16] is that here the definitions are valid for $n = 0$, because we fixed the value of μ_1 in advance. The definitions for $n = 0$ have some special features that will be useful later, and which we record in the following remarks.

Remark 3.9. $\mathbb{B}_0 \restriction \mu_1$ and $\mathbb{U}_0 \restriction \mu_1$ are trivial. $\mathbb{U}_0 = (\mathbb{B}_0)^{+A_0} \in V[A_0]$. Since $U_0 \restriction \mu_1$ is trivial, U_0 is irrelevant to the definition of $\mathbb{S}_0$, and $\mathbb{S}_0 = \mathbb{C}_0^{+A_0} \in V[A_0]$. $V[A_0 * U_0 * S_0] = V[A_0 * (U_0 * S_0)]$, and we may view $\mathbb{A}_0 * \mathbb{U}_0 * \mathbb{S}_0$ as a projection of $\mathbb{A}_0 \times \mathbb{B}_0 \times \mathbb{C}_0$. $\mathbb{A}_0 * \mathbb{S}_0$ is essentially Mitchell forcing [13], and $\mathbb{C}_0$ is essentially the term forcing from Abraham's product analysis of Mitchell forcing [1].

Remark 3.10. The natural forcing to add a $\mathbb{B}_0 \times \mathbb{C}_0$ -generic object $B_0 \times C_0$ such that $A_0 \times (B_0 \times C_0)$ induces $A_0 * U_0 * S_0$ is $< \mu_0$ -closed in $V[A_0 * U_0 * S_0]$. The argument is essentially the same as that for Lemma 2.44: conditions are pairs (b, c) with $(b, c)[A_0] \in U_0 \times S_0$, $\mathbb{A}_0 * \mathbb{U}_0 * \mathbb{S}_0$ is $< \mu_0$ -distributive so that decreasing $< \mu_0$ -sequences lie in V , hence it is easy to find a lower bound.

We quote more facts from [16].

Fact 3.11 ([16, Claim 4.15]).

- (1) Let $\alpha \leq \mu_{n+1}$ and let $F = A \restriction \alpha * U \restriction \alpha$. Then in $V[F]$ the poset $\mathbb{C}^{+F} \restriction [\mu_{n+1}, \mu_\omega)$ is $< \mu_{n+1}$ -directed closed.
- (2) Let $\alpha \in (\mu_{n+1}, \mu_{n+2})$ and let $F = A \restriction \alpha * U \restriction \mu_{n+1}$. Then in $V[F]$ the poset $\mathbb{C}^{+F} \restriction [\alpha, \mu_{n+2})$ is $< \mu_{n+1}$ -directed closed.

Fact 3.12 ([16, Claim 4.30]). Let $F = A \restriction \mu_{n+2} * U \restriction \mu_{n+2}$, let $\mathbb{P}_1$ be the poset to refine $U \restriction [\mu_{n+2}, \mu_\omega)$ to a generic object for $\mathbb{B}^{+F} \restriction [\mu_{n+2}, \mu_\omega)$, and let $\mathbb{P}_2$ be the poset to refine $S \restriction [\mu_{n+2}, \mu_\omega)$ to a generic object for $\mathbb{C}^{+F} \restriction [\mu_{n+2}, \mu_\omega)$. Then both $\mathbb{P}_1$ and $\mathbb{P}_2$ are $< \mu_{n+1}$ -closed in $V[A][U][S \restriction [\mu_{n+1}, \mu_\omega)]$.

As we already mentioned, if we force with $\mathbb{A} * \mathbb{U} * \mathbb{S}$ we obtain an extension in which $2^{\mu_n} = \mu_{n+2}$ for all n , and $\mu_{n+1} = \mu_n^+$ for all $n > 0$.

- $\mathbb{A}$ is responsible for blowing up the powersets of the μ_n 's.
- $\mathbb{U}$ is responsible for ensuring that μ_{n+2} has the indestructible generic supercompactness property from Fact 3.7 in $V[A * U \restriction \mu_{n+2}]$.
- $\mathbb{S}$ is responsible for collapsing cardinals in the interval (μ_{n+1}, μ_{n+2}) to have cardinality μ_{n+1} .

$\mathbb{A} * \mathbb{U} * \mathbb{S}$ is a descendant of Mitchell's original forcing [13] for collapsing a large cardinal while preserving the tree property. Exactly as in that forcing the $\mathbb{S}$ -coordinate is collapsing cardinals between μ_{n+1} and μ_{n+2} “in parallel” with the $\mathbb{A}$ coordinate adding subsets of μ_n , so that there is no inner model where $2^{\mu_n} = \mu_{n+1}$ and $\mu_{n+2} = \mu_{n+1}^+$ and we do not run afoul of Specker's result from [22].

We record some information about $\mathbb{A} * \mathbb{U} * \mathbb{S}$ for use later.

Lemma 3.13.

- (1) $\mathbb{A} \restriction \mu_{n+2}$ is μ_{n+1} -Knaster and $\mathbb{A} \restriction [\mu_{n+2}, \mu_\omega)$ is $< \mu_{n+1}$ -directed closed.
- (2) $\mathbb{B} \restriction \mu_{n+1}$ is μ_{n+1} -Knaster.
- (3) $\mathbb{C} \restriction \mu_{n+1}$ is μ_{n+1} -Knaster.
- (4) $\mathbb{A} \restriction \mu_{n+2} * \mathbb{U} \restriction \mu_{n+1} * \mathbb{S} \restriction \mu_{n+1}$ is μ_{n+1} -Knaster.
- (5) $\mathbb{B} \restriction [\mu_{n+1}, \gamma)$ is $< \mu_{n+1}$ -directed closed for all γ , in particular $\mathbb{B}_n$ is $< \mu_{n+1}$ -directed closed.
- (6) $\mathbb{C}_n$ is $< \mu_{n+1}$ -directed closed, as is $\mathbb{C} \restriction [\mu_{n+1}, \mu_\omega)$.
- (7) For each n , the forcing poset $\mathbb{A} * \mathbb{U} * \mathbb{S}$ is the projection of $\mathbb{P}_0 \times \mathbb{P}_1$, where $\mathbb{P}_0 = \mathbb{A} \restriction \mu_{n+2} * \mathbb{U} \restriction \mu_{n+1} * \mathbb{S} \restriction \mu_{n+1}$ and $\mathbb{P}_1 = \mathbb{A} \restriction [\mu_{n+2}, \mu_\omega) \times \mathbb{B} \restriction [\mu_{n+1}, \mu_\omega) \times \mathbb{C} \restriction [\mu_{n+1}, \mu_\omega)$. $\mathbb{P}_0$ is μ_{n+1} -Knaster and $\mathbb{P}_1$ is $< \mu_{n+1}$ -directed closed.
- (8) It is forced by $\mathbb{P}_1$ that $\mathbb{P}_0$ is μ_{n+1} -cc.
- (9) $\prod_{n < \omega} \mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n$ adds no $< \mu_0$ -sequences of ordinals, and preserves the cardinals μ_n for $n < \omega$ together with μ_ω^+ . Since $\mathbb{A} * \mathbb{U} * \mathbb{S}$ is a projection of $\prod_{n < \omega} \mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n$, the same holds for $\mathbb{A} * \mathbb{U} * \mathbb{S}$.

Proof.

- (1) This is immediate since $\mathbb{A} \restriction \mu_{n+2} = \prod_{i \leq n} \mathbb{A}_i$ and $\mathbb{A} \restriction [\mu_{n+2}, \omega) = \prod_{i > n} \mathbb{A}_i$.
- (2) For $n = 0$, $\mathbb{B} \restriction \mu_1$ is trivial forcing. For $n > 0$ the supports of conditions in $\mathbb{B} \restriction \mu_{n+1}$ are Easton subsets of the Mahlo cardinal μ_{n+1} , $|\mathbb{B} \restriction \alpha| < \mu_{n+1}$ for all $\alpha < \mu_{n+1}$, and the μ_{n+1} -Knaster property for $\mathbb{B} \restriction \mu_{n+1}$ follows by standard arguments in iterated forcing.
- (3) For $n = 0$, $\mathbb{C} \restriction \mu_1$ is trivial forcing. For $n > 0$, $\mathbb{C} \restriction \mu_{n+1} = (\mathbb{C} \restriction \mu_n) \times \mathbb{C}_{n-1}$, and $|\mathbb{C} \restriction \mu_n| < \mu_{n+1}$ so this factor is trivially μ_{n+1} -Knaster. $\mathbb{C}_{n-1}$ is the product taken with $< \mu_n$ -supports of μ_{n+1} posets each with cardinality less than μ_{n+1} , and μ_{n+1} is inaccessible, so the μ_{n+1} -Knaster property for $\mathbb{B} \restriction \mu_{n+1}$ follows by standard arguments in product forcing.
- (4) $\mathbb{A} \restriction \mu_{n+2} * \mathbb{U} \restriction \mu_{n+1} * \mathbb{S} \restriction \mu_{n+1}$ is a projection of $\mathbb{A} \restriction \mu_{n+2} \times \mathbb{B} \restriction \mu_{n+1} \times \mathbb{C} \restriction \mu_{n+1}$, which is a product of μ_{n+1} -Knaster posets. We note that the projection is the identity map between two posets with the same underlying set but different orderings.
- (5) This follows from Remark 3.6.
- (6) $\mathbb{C}_n$ is the product taken with $< \mu_{n+1}$ -supports of $< \mu_{n+1}$ -closed term forcing posets, and $\mathbb{C} \restriction [\mu_{n+1}, \mu_\omega) = \prod_{i \geq n} \mathbb{C}_i$.
- (7) It is routine to verify that the natural map from $\mathbb{P}_0 \times \mathbb{P}_1$ to $\mathbb{A} * \mathbb{U} * \mathbb{S}$ is a projection. The claims about closure and chain condition follow immediately from what we already proved.
- (8) This is immediate by Easton's lemma.
- (9) The preservation of μ_0 and the claim about $< \mu_0$ -sequences are immediate, as the product is $< \mu_0$ -closed. $\prod_{m \leq n} \mathbb{A}_m \times \prod_{m < n} \mathbb{B}_m \times \prod_{m < n} \mathbb{C}_m$ is μ_{n+1} -Knaster and $\prod_{m > n} \mathbb{A}_m \times \prod_{m \geq n} \mathbb{B}_m \times \prod_{m \geq n} \mathbb{C}_m$ is $< \mu_{n+1}$ -closed, so that μ_{n+1} is preserved by Easton's Lemma. If μ_ω^+ were collapsed we would have $\text{cf}(\mu_\omega^+) < \mu_\omega$ in the extension, but this is impossible by Easton's Lemma.

□

Remark 3.14. Item 8 is immediate in our current setting but will hold and be useful in more general settings, as we discuss in Section 3.3 below.

Corollary 3.15.

- (1) Every $< \mu_{n+1}$ -sequence of ordinals from $V[A * U * S]$ lies in the submodel $V[A \restriction \mu_{n+2}][U \restriction \mu_{n+1}][S \restriction \mu_{n+1}]$.
- (2) Every set of ordinals of cardinality less than μ_{n+1} in $V[A * U * S]$ is covered by such a set lying in V .
- (3) If $\mathbb{Q}$ is μ_ω -closed in V and $\mathbb{Q}$ is $\mathbb{Q}$ -generic over $V[A * U * S]$, every set of ordinals of cardinality less than μ_{n+1} in $V[A * U * S][\mathbb{Q}]$ is covered by such a set lying in V .

Proof. The first two claims are immediate. For the last claim write $\mathbb{Q} \times \mathbb{A} * \mathbb{U} * \mathbb{S}$ as the projection of $\mathbb{Q} \times \mathbb{P}_0 \times \mathbb{P}_1$ where $\mathbb{P}_0$ is μ_{n+1} -cc and $\mathbb{P}_1$ is $< \mu_{n+1}$ -closed, then argue as usual by Easton's Lemma. □

In a similar spirit, we state some more easy projection and absorption facts about $\mathbb{U}_{[n,\omega]}$ and $\mathbb{S}_n$ for use in Section 9.4.

Lemma 3.16. $\mathbb{A} * \mathbb{U}$ can be viewed as the projection of the product of $\mathbb{A} * \mathbb{U}_{[0,n]}$ and the Easton support product of the term forcing posets $\mathcal{A}^V(\mathbb{A} \restriction \alpha, \mathbb{U} \restriction \alpha)$ for $\alpha \in \text{dom}(\mathbb{B}_{[n,\omega]})$.

We note that by Lemma 2.50, the product of term forcing in Lemma 3.16 may be absorbed into a suitable Easton collapse.

Lemma 3.17. $\mathbb{A} * \mathbb{U} * \mathbb{S}_n$ can be viewed as the projection of the product of $\mathbb{A} * \mathbb{U}$ and $\mathbb{C}_n$.

We note that $\mathbb{C}_n$ may be viewed as the $< \mu_{n+1}$ -support product of the term forcing posets $\mathcal{A}^V(\mathbb{A} \restriction \alpha * \mathbb{U} \restriction \mu_{n+1}, \text{Add}(\mu_{n+1}, 1))$ for $\alpha \in (\mu_{n+1}, \mu_{n+2})$. As such, by Lemma 2.52 $\mathbb{C}_n$ may be absorbed into a suitable $< \mu_{n+1}$ -closed Levy collapse.

For use later (notably in Sections 6 and 9.4) we record the fact that $\mathbb{U}_{[n,\omega]}$ and $\mathbb{S}_{[n,\omega]}$ have a very modest degree of closure in the models where they are defined. These results are surely not optimal (and in some special cases we will need and prove more closure) but are all we need for the purposes of Sections 6 and 9.4. The argument is similar to but easier than the proofs of Fact 3.5 or Lemma 7.1.

Lemma 3.18.

$\mathbb{U}_{[n,\omega]}$ is $< \mu_0$ -closed in $V[A * \mathbb{U}_{[0,n]}]$.

$\mathbb{S}_{[n,\omega]}$ is $< \mu_0$ -closed in $V[A * \mathbb{U}]$.

Proof. We only prove the closure of $\mathbb{U}_n$ in $V[A * \mathbb{U}_{[0,n]}]$, which is enough to illustrate the idea. Recall that the underlying set of $\mathbb{U}_n$ is $\mathbb{B}_n = \mathbb{B} \restriction (\mu_{n+1}, \mu_{n+2})$, and the ordering is defined in $V[A_{[0,n]} * \mathbb{U}_{[0,n]}]$. Let $\eta < \mu_0$ and let $\vec{u} = (u_i)_{i < \eta}$ be a decreasing sequence in $\mathbb{U}_n$, where by Lemma 3.13 we have $\vec{u} \in V$. We assume without loss of generality that it is forced by $\mathbb{A}_{[0,n]} * \mathbb{U}_{[0,n]}$ that $\vec{u}$ is decreasing in $\mathbb{U}_n$.

We will construct $b \in \mathbb{B}_n$ inductively, where $b = \bigcup_i \text{dom}(u_i)$ is easily seen to be an Easton set, and arrange that it is forced by $\mathbb{A}_{[0,n]} * \mathbb{U}_{[0,n]}$ that b is a lower bound for $\vec{u}$ in $\mathbb{U}_n$. Suppose that $\mu_{n+1} \leq \alpha < \mu_{n+2}$, and we have defined $b \restriction \alpha$ which is forced by $\mathbb{A}_{[0,n]} * \mathbb{U}_{[0,n]}$ to be a lower bound for $\vec{u} \restriction \alpha$ in $\mathbb{U}_n \restriction \alpha$. We force with $\mathbb{A} \restriction \alpha * \mathbb{U} \restriction \alpha$ below $(0, b \restriction \alpha)$ to obtain $F_\alpha^A * F_\alpha^U$.

Let $d_i = u_i(\alpha)[F_\alpha^A * F_\alpha^U]$. We claim that $(d_i)_{i < \eta}$ forms a decreasing sequence in $\phi(\alpha)[F_\alpha^A * F_\alpha^U]$. To see this let $i < j < \eta$, and force to prolong F_α^A to F^A which is $\mathbb{A}_{[0,n]}$ -generic. By our hypothesis on $\vec{u}$, $V[F^A * F_\alpha^A \restriction \mu_{n+1}] \models u_j \leq_{\mathbb{U}_n} u_i$. By the definition of $\mathbb{U}_n$, there are conditions $p \in F_\alpha^A$ and $q \in F_\alpha^U \restriction \mu_{n+1}$ such that $(p, q \restriction u_j \restriction \alpha) \Vdash u_j(\alpha) \leq u_i(\alpha)$.

Since we forced below $(0, b \restriction \alpha)$, $(0, b \restriction \alpha) \in F_\alpha^A * F_\alpha^U$. By hypothesis $V[F^A * F_\alpha^A \restriction \mu_{n+1}] \models b \restriction \alpha \leq_{\mathbb{U}_n \restriction \alpha} u_j \restriction \alpha$, so extending p and q if necessary we may assume that $(p, q) \Vdash b \restriction \alpha \leq u_j \restriction \alpha$. So $(p, q \restriction u_j \restriction \alpha) \in F_\alpha^A * F_\alpha^U$ and $d_j \leq d_i$ as required.

Now since $\phi(\alpha)$ is forced to be $< \alpha$ -closed we may choose an $\mathbb{A} \restriction \alpha * \mathbb{U} \restriction \alpha$ -name $b(\alpha)$ such that $(0, b \restriction \alpha) \Vdash b(\alpha) \leq d_i$ for all i . Let $F = A_{[0,n]} * \mathbb{U}_{[0,n-1]}$ be an

arbitrary $\mathbb{A}_{[0,n]} * \mathbb{U}_{[0,n-1]}$ -generic object. By induction $b \restriction \alpha \leq u_i \restriction \alpha$ for all $i < \eta$ in the version of $\mathbb{U}_n$ computed in $V[F]$, and we will show that $b \restriction \alpha + 1 \leq u_i \restriction \alpha + 1$ in this poset. This is easy because for any condition $(a, u) \in F$, $(a \restriction \alpha, u \cup b \restriction \alpha)$ forces $b(\alpha) \leq u_i(\alpha)$. $\square$

3.2. Further analysis

As we saw in Section 3.1, $\mathbb{A} * \mathbb{U} * \mathbb{S}$ is naturally a projection of $\prod_n (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n)$, and this latter forcing preserves all cardinals μ_n together with μ_ω^+ . To get more information we will use a style of analysis sometimes called “tail forcing”, which is often useful in the setting of a product of ω many increasingly closed forcing posets.

Let f and g be elements of $\prod_n (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n)$. We say that $f =_{\text{finite}} g$ if and only if $f(n) = g(n)$ for all large n , and $f \leq_{\text{finite}} g$ if and only if $f(n) \leq g(n)$ for all large n . Then $=_{\text{finite}}$ is an equivalence relation on $\prod_n (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n)$.

If we let $\prod_n (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n) / \text{finite}$ be the set of equivalence classes then $\leq_{\text{finite}}$ naturally induces a partial ordering on $\prod_n (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n) / \text{finite}$, and it is easy to see that $f \mapsto [f]_{\text{finite}}$ is a projection from $\prod_n (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n)$ to $\prod_n (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n) / \text{finite}$. If we define $=_{\text{finite}}$ and $\leq_{\text{finite}}$ on $\prod_{n \geq m} (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n)$ in the natural way, then easily $\prod_n (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n) / \text{finite}$ is isomorphic to $\prod_{n \geq m} (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n) / \text{finite}$, and $f \mapsto [f]_{\text{finite}}$ is a projection from $\prod_{n \geq m} (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n)$ to $\prod_{n \geq m} (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n) / \text{finite}$.

Now we may represent $\prod_n (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n)$ as a two-step iteration $\mathbb{E}_0 * \mathbb{E}_1$, where $\mathbb{E}_0 = \prod_n (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n) / \text{finite}$, and $\mathbb{E}_1$ is the set of elements of $\prod_n \mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n$ whose classes modulo finite are in E_0 , where the ordering of $\mathbb{E}_1$ is the the ordering inherited from $\prod_n \mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n$.

Claim 3.19. $\mathbb{E}_0$ is μ_m -strategically closed for every $m < \omega$, so in particular it is μ_ω -distributive.

Proof. By the discussion above, $\mathbb{E}_0$ is isomorphic to the projection via $f \mapsto [f]_{\text{finite}}$ of the $< \mu_m$ -closed poset $\prod_{n \geq m} (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n)$. $\square$

Claim 3.20. $\mathbb{E}_1$ is μ_ω^+ -cc in $V[E_0]$.

Proof. We will show the stronger assertion that $\mathbb{E}_1$ is the union of μ_ω many filters in $V[E_0 * E_1]$. Let R be the $\prod_n (\mathbb{A}_n \times \mathbb{B}_n \times \mathbb{C}_n)$ -generic object added by $\mathbb{E}_0 * \mathbb{E}_1$, so that $V[E_0 * E_1] = V[R]$, E_0 is the set of equivalence classes of elements of R modulo finite, $\mathbb{E}_1$ is the set of conditions which are equal mod finite to some element of R , and $E_1 = R$. The key point is that μ_ω^+ is still an uncountable regular cardinal in $V[R]$.

Now we work in $V[R]$. If $p \in \mathbb{E}_1$ then there are $n < \omega$ and $r \in R$ such that $p \restriction [n, \omega) = r \restriction [n, \omega)$. For all n and $q \in \prod_{i < n} (\mathbb{A}_i \times \mathbb{B}_i \times \mathbb{C}_i)$, let $F_{n,q}$ be the set of $p \in \mathbb{E}_1$ such that $p \restriction n = q$ and $p \restriction [n, \omega) = r \restriction [n, \omega)$ for some $r \in R$. It is easy to see that $F_{n,q}$ is a filter, $\mathbb{E}_1 = \bigcup_{n,q} F_{n,q}$ and there are μ_ω possibilities for (n, q) . $\square$

We have proved:

Lemma 3.21. $V[A * U * S] \subseteq V[E_0 * E_1]$, where $\mathbb{E}_0$ is μ_ω -distributive in V and $\mathbb{E}_1$ is μ_ω^+ -cc in $V[E_0]$.

Remark 3.22. Since $\mathbb{E}_0$ is μ_m -strategically closed for all m , the distributivity of $\mathbb{E}_0$ is quite robust in mild forcing extensions of V . The argument we gave for the chain condition of $\mathbb{E}_1$ shows that the chain condition of $\mathbb{E}_1$ is also robust in mild forcing extensions of $V[E_0]$.

3.3. Modifying the forcing

In the sequel we will need to use some modified forms of $\mathbb{A} * \mathbb{U} * \mathbb{S}$. The main modifications will be:

- We sometimes choose the Cohen forcing $\mathbb{A}_n$ from an inner model $\bar{V}$ (which may depend on n), that is we set $\mathbb{A}_n = \text{Add}^{\bar{V}}(\mu_n, \mu_{n+2})$. When we do this we will make sure to arrange that $\mathbb{A}_{[m,n]}$ is $< \mu_m$ -distributive and μ_{n+1} -Knaster for $m \leq n < \omega$.
- We sometimes weaken the assumptions on the cardinals μ_n for $n \geq 2$ and the function ϕ . The μ_n 's will still be supercompact but may not be indestructibly supercompact, and (relatedly) the function $\phi \upharpoonright \mu_n$ may only be a Laver function rather than an indestructible Laver function. In practice there will typically be an inner model V' such that V is a small generic extension of V' , and ϕ is obtained from an indestructible Laver function in V' using Lemma 2.9.

With these modifications the closure assertions from Facts 3.5, 3.11, and 3.12 will remain true, since they only use chain condition and distributivity properties on the $\mathbb{A}$ -coordinate. Most of the conclusions of Lemmas 3.13 and 3.15 remain true, the only difference is that now $\mathbb{A} \upharpoonright [\mu_{n+2}, \mu_\omega)$ and $\mathbb{P}_1$ are merely $< \mu_{n+1}$ -distributive. The analysis from Section 3.2 needs to be slightly modified but the conclusion is the same: the modified version of $\mathbb{A} * \mathbb{U} * \mathbb{S}$ embeds into a two-step iteration where the first step has a robust form of μ_ω -distributivity, and the second step has a robust form of μ_ω^+ -cc.

Remark 3.23. At certain points in the main construction (see Sections 7.1 and 7.2) we will start with a sequence of cardinal parameters $\mu_0, \mu_1, \mu_2 \dots$, force with $\mathbb{A}_0 * \mathbb{U}_0 * \mathbb{S}_0$ (in Section 7.1) or $\mathbb{A}_0 * \mathbb{U}_0 * \mathbb{L}$ for some preparation forcing $\mathbb{L}$ (in Section 7.2), and then work over the extension to define and force with $\mathbb{A}_{[1,\omega)} * \mathbb{U}_{[1,\omega)} * \mathbb{S}_{[1,\omega)}$ (in Section 7.1) or $\mathbb{A}_{[1,\omega)} * \mathbb{U}_{[1,\omega)} * \mathbb{S}$ (in Section 7.2). The resulting iteration is broadly similar to the $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction defined from $\mu_0, \mu_1, \mu_2 \dots$ but is not equivalent: we will handle this situation by analyzing the two parts of the construction separately. In the sequel we will call this kind of iteration a *two-phase* $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction. Since $\mathbb{A}_0$ forces $2^{\mu_0} = \mu_2$ it will be important to define $\mathbb{A}_1$ in an inner model, so that $\mathbb{A}_1$ does not collapse μ_2 .

Remark 3.24. Readers of this paper and [16] will note a limited family resemblance

between $\mathbb{A} * \mathbb{U} * \mathbb{S}$ and the constructions of Abraham [1] and Cummings and Foreman [2], which also involve forcing posets with an “add coordinate”, a “collapse coordinate” and an “indestructibility coordinate”. The key differences are that in those earlier papers the supercompact cardinals are not assumed to be indestructible, all three coordinates are iterations, and the “indestructibility coordinate” comes last and uses ordinary ground model Laver functions to guess names.

4. Indestructibility results

In the proof of Theorem 1.1 we will produce a model which combines many different instances of the construction of Section 3. Roughly speaking the double successor cardinals below $\aleph_{\omega^2}$ in the final model will be grouped into blocks of length ω , where cardinals in each block will be handled by an instance of that construction. Unfortunately in each block there is interference caused by the instances that handle the neighboring blocks. We will deal with some of this interference by proving general indestructibility results (Lemmas 4.5 and 4.10 below) stating that instances of the tree property produced by the construction of Section 3 are somewhat robust under further mild forcing. All the ideas needed for the indestructibility results are already present in [16], we just need some small adjustments to the proofs.

In some cases we would like to use Lemma 4.5 in situations where the hypotheses do not quite apply, and this issue will be addressed by going to a further generic extension where the hypotheses do apply, and using a mutual genericity argument to finish. See Remarks 4.11 and 4.12 following Lemma 4.5 for more on this. Of course we could have incorporated this idea into the statement and proof of Lemma 4.5, at the cost of further complicating the statement and the proof.

Let $\mathbb{A} * \mathbb{U} * \mathbb{S}$ be a forcing poset of the type described in Section 3, but allowing for the possibility that some of the posets $\mathbb{A}_k$ may be chosen in submodels as discussed in Section 3.3. The poset $\mathbb{A} * \mathbb{U} * \mathbb{S}$ will be constructed in a universe V_{def} : this notation is perhaps a bit cumbersome but makes it easier to specify which universe is to play the role of V_{def} in the sequel. Let $n < \omega$, with a view to showing that the tree property holds at μ_{n+2} in a wide class of generic extensions of $V_{\text{def}}[A * U * S]$, and make the following assumptions:

- There is an inner model V_{inn} of V_{def} , such that V_{def} is a generic extension $V_{\text{inn}}[X]$ and $V_{\text{inn}} \models “\mathbb{X} \text{ is } \omega\text{-distributive and } \mu_1\text{-cc with } |\mathbb{X}| \leq \mu_1”$.
- In V_{inn} , $(\mu_i)_{i < \omega}$ is an increasing sequence of regular cardinals such that μ_i is indestructibly supercompact for $i \geq 2$, and there is a universal indestructible Laver function ψ defined up to μ_ω . The cardinal μ_1 need not be a large cardinal, and in fact is often the successor of a singular cardinal.
- The Laver function ϕ used to define $\mathbb{U}$ is obtained from the universal indestructible Laver function ψ in V_{inn} using Lemma 2.9, that is $\phi(\alpha) = \psi(\alpha)[X]$ whenever $\psi(\alpha)$ is an $\mathbb{X}$ -name in V_{inn} . In V_{def} , ϕ is a universal Laver function on the interval (μ_1, μ_ω) , where $\mu_\omega = \sup_{i < \omega} \mu_i$.
- Each of the posets $\mathbb{A}_k$ may be defined in some inner model of V_{def} , with

the constraints that:

- $\mathbb{A}_k$ is defined in V_{inn} for $k \geq n+2$.
- $\prod_{i \leq m} \mathbb{A}_i$ is μ_{m+1} -Knaster in V_{def} for all m .
- $\prod_{i \geq m} \mathbb{A}_i$ is $< \mu_m$ -distributive in V_{def} for all m .

Remark 4.1. With an eye to future applications, these hypotheses are slightly more general than is needed for our purposes in this paper.

From our hypotheses $\mathbb{A}_{[n+2, \omega)} = (\prod_{n+2 \leq i < \omega} \text{Add}(\mu_i, [\mu_{i+1}, \mu_{i+2}]))^{V_{\text{inn}}}$, so that $\mathbb{A}_{[n+2, \omega)}$ is defined and $< \mu_{n+2}$ -directed closed in V_{inn} . For $k \leq n+1$, $\mathbb{A}_k$ will most often be defined in some model intermediate between V_{inn} and V_{def} , and in this case Lemma 4.2 below will handle most of the work of checking the chain condition and distributivity of products of the $\mathbb{A}_k$'s.

Lemma 4.2. *Suppose that $\mathbb{A}_k = \text{Add}^{V_{\text{int},k}}(\mu_k, [\mu_{k+1}, \mu_{k+2}])$ for $1 \leq k \leq n+1$, where $V_{\text{int},k}$ is intermediate between V_{inn} and V_{def} . Then $\mathbb{A}_{[1,m]} = \prod_{1 \leq i \leq m} \mathbb{A}_i$ is μ_{m+1} -Knaster in V_{def} and $\mathbb{A}_{[m, \omega)} = \prod_{i \geq m} \mathbb{A}_i$ is $< \mu_m$ -distributive in V_{def} for all $m \geq 1$.*

Proof. We set $V_{\text{int},k} = V_{\text{inn}}$ for $k \geq n+2$, so that $\mathbb{A}_k = \text{Add}^{V_{\text{int},k}}(\mu_k, [\mu_{k+1}, \mu_{k+2}])$ for $k \geq 1$. As μ_k is inaccessible in V_{def} for $k \geq 2$, $V_{\text{int},k} \models \text{"}\forall \eta < \mu_{k+1} \ \eta^{< \mu_k} < \mu_{k+1}\text{"}$ for all $k \geq 1$.

Let $V_{\text{int},k} = V_{\text{inn}}[X_k]$ for a forcing poset $\mathbb{X}_k \in V_{\text{inn}}$, and let $\mathbb{A}_k^* = \mathcal{A}^{V_{\text{inn}}}(\mathbb{X}_k, \dot{\mathbb{A}}_k)$, so that $\mathbb{A}_k^*$ is $< \mu_k$ -closed in V_{inn} by Lemma 2.33. For $m \geq n+2$, $\mathbb{A}_{[m, \omega)}$ is $< \mu_m$ -closed in V_{inn} , so it is $< \mu_m$ -distributive in V_{def} by Easton's lemma.

For $1 \leq m \leq n+1$, we may write $V_{\text{def}}[A_{[m, \omega)}] = V_{\text{inn}}[(X * A_{[m, n+2)}) \times A_{[n+2, \omega)}]$, and by a suitable quotient to term forcing we may extend to obtain a generic extension $V_{\text{inn}}[X \times \prod_{m \leq i < n+2} A_i^* \times A_{[n+2, \omega)}]$. Since $\prod_{m \leq i < n+2} \mathbb{A}_i^* \times \mathbb{A}_{[n+2, \omega)}$ is $< \mu_m$ -closed in V_{inn} , by Easton's lemma it is $< \mu_m$ -distributive in V_{def} , so that easily $\mathbb{A}_{[m, \omega)}$ is $< \mu_m$ -distributive in V_{def} .

It is easy to see that V_{def} is a (μ_1, μ_2) -good extension of $V_{\text{int},k}$, so by Lemma 2.54 $\mathbb{A}_1$ is μ_2 -Knaster in V_{def} . Now we show by induction on m that $\mathbb{A}_{[1,m]}$ is μ_{m+1} -Knaster in V_{def} : if $\mathbb{A}_{[1,m]}$ is μ_{m+1} -Knaster in V_{def} then $V_{\text{def}}[A_{[1,m]}]$ is a (μ_{m+1}, μ_{m+2}) -good extension of $V_{\text{int},m+1}$, so that $\mathbb{A}_{m+1}$ is μ_{m+2} -Knaster in $V_{\text{def}}[A_{[1,m]}]$ and hence $\mathbb{A}_{[1,m+1]}$ is μ_{m+2} -Knaster. $\square$

Remark 4.3. Lemma 4.2 leaves us only with the problem of showing that $\mathbb{A}_{[0, \omega]}$ is $< \mu_0$ -distributive and $\mathbb{A}_{[0, m]}$ is μ_{m+1} -cc.

To make the hypotheses of the forthcoming Lemma 4.5 more digestible, we use some notational conventions:

- We will show that the tree property at μ_{n+2} holds in generic extensions of $V_{\text{def}}[A * U * S]$ by products of posets that can be written in the form $\mathbb{D}^{\text{small}} \times \mathbb{D}^0 \times \mathbb{D}^1 \times \mathbb{D}^2 \times \mathbb{D}^3$, where the factors satisfy some hypotheses to be

listed later. We write this product $\mathbb{D}^{s,0,1,2,3}$, and denote subproducts and generic objects for subproducts in the natural way.

- W is $V_{\text{def}}[A * U * S][D^{s,0,1,2,3}]$.
- j is any embedding witnessing the χ -supercompactness of μ_{n+2} in $V_{\text{def}}[H]$ where $H = A_{[n+2,\omega)} \times D^2$ and $\chi = \max(\mu_\omega, |\mathbb{D}^{s,0,1,2,3}|)^+$. Our hypotheses will ensure that μ_{n+2} is supercompact in $V_{\text{def}}[H]$, so that such embeddings j will exist. Note that $V_{\text{def}}[H] \subseteq W$.
- If $\mathbb{Q} \in V_{\text{def}}$ is a μ_{n+2} -cc poset with $|\mathbb{Q}| \leq \chi$ which remains μ_{n+2} -cc in $V_{\text{def}}[H]$, j is an embedding as above (so that in $V_{\text{def}}[H]$, $j \restriction \mathbb{Q}$ is a complete embedding of $\mathbb{Q}$ into $j(\mathbb{Q})$), and Q is $\mathbb{Q}$ -generic over $V_{\text{def}}[H]$, then $j(\mathbb{Q})/j[Q]$ is the natural poset defined in $V_{\text{def}}[H][Q]$ to produce a $j(\mathbb{Q})$ -generic object $\hat{Q}$ with $j[Q] \subseteq \hat{Q}$.

We note that in the proof of Lemma 4.5 we will construct and lift a highly specific embedding j , which is not known in advance and depends on the inputs to the Lemma.

Remark 4.4. The posets $\mathbb{D}^{\text{small}}, \mathbb{D}^0, \mathbb{D}^1, \mathbb{D}^2, \mathbb{D}^3$ are enumerated roughly in order of increasing distributivity. They appear in a different order in the hypotheses of Lemma 4.5 because the hypotheses about $\mathbb{D}^0$ and $\mathbb{D}^1$ mention $\mathbb{D}^2$ and $\mathbb{D}^3$, and the hypothesis about $\mathbb{D}^0$ mentions $\mathbb{D}^1$.

Lemma 4.5. *With the hypotheses on $n, V_{\text{inn}}, V_{\text{def}}$ and $\mathbb{A} * \mathbb{U} * \mathbb{S}$ as above, let $\mathbb{D}^{\text{small}}, \mathbb{D}^0, \mathbb{D}^1, \mathbb{D}^2, \mathbb{D}^3$ be forcing posets such that, setting $W = V_{\text{def}}[A * U * S][D^{s,0,1,2,3}]$ and $H = A_{[n+2,\omega)} \times D^2$:*

- (1) μ_{n+1} and μ_{n+2} are regular cardinals in W .
- (2) $\mathbb{A}_n$ is μ_{n+1} -Knaster in $V_{\text{def}}[H]$, and $\mathbb{A}_{n+1}$ is μ_{n+2} -Knaster in $V_{\text{def}}[H]$.
- (3) $\mathbb{D}^2 \in V_{\text{inn}}$, and $V_{\text{inn}} \models \text{“}\mathbb{D}^2 \text{ is } < \mu_{n+2}\text{-directed closed”}$.
- (4) $\mathbb{D}^3 \in V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+2}]$ and $V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+2}] \models \text{“}\mathbb{D}^3 \text{ is } < \mu_{n+2}\text{-directed closed”}$.
- (5) $\mathbb{D}^1 \in V_{\text{def}}$.
 - (a) $\mathbb{D}^1$ is μ_{n+2} -Knaster in $V_{\text{def}}[H]$.
 - (b) $\mathbb{D}^1$ is $< \mu_{n+1}$ -distributive in $V_{\text{def}}[A * U * S][D^{2,3}]$.
- (6) For any j which is the unique lift^a to $V_{\text{def}}[H]$ of an embedding witnessing the χ -supercompactness of μ_{n+2} in $V_{\text{inn}}[H]$, if $\mathbb{P}_{2b} = j(\mathbb{A}_{n+1} \times \mathbb{D}^1)/j[A_{n+1} \times D^1]$ then:
 - (a) $\mathbb{P}_{2b}$ is μ_{n+2} -Knaster in W .
 - (b) $\mathbb{P}_{2b}$ is $< \mu_{n+1}$ -distributive in W .
- (7) $\mathbb{D}^0 \in V_{\text{def}}$, and $\mathbb{D}^0$ is μ_{n+1} -Knaster in $V_{\text{def}}[A * U * S][D^{1,2,3}][P_{2b}]$.

^aNote that by the indestructibility of μ_{n+2} in V_{inn} and the hypotheses on $\mathbb{A}_{[n+2,\omega)}$ and $\mathbb{D}^2$, μ_{n+2} is χ -supercompact in $V_{\text{inn}}[H]$. Since $V_{\text{def}}[H] = V_{\text{inn}}[H][X]$ and X is generic for forcing of cardinality at most μ_1 , any embedding witnessing χ -supercompactness for μ_{n+2} in $V_{\text{inn}}[H]$ lifts uniquely in a trivial fashion to an embedding witnessing χ -supercompactness for μ_{n+2} in $V_{\text{def}}[H]$.

- (8) For any W' which is an extension of $W[P_{2b}]$ by a forcing which is $< \mu_{n+1}$ -closed in $V_{\text{def}}[A * U * S \restriction [\mu_{n+1}, \mu_\omega)][D^2]$, and any j as in Hypothesis 6, if $\mathbb{P}_{2a} = j(\mathbb{A}_n \times \mathbb{D}^0)/j[A_n \times D^0]$ then $\mathbb{P}_{2a}$ is μ_{n+1} -Knaster in W' .
- (9) $\mathbb{D}^{\text{small}} \in V_{\text{def}}[A * U * S]$, and $V_{\text{def}}[A * U * S] \models |\mathbb{D}^{\text{small}}| \leq \mu_n$.

Then $W \models \text{"}\mu_{n+2} \text{ has the tree property"}$.

Before proving Lemma 4.5, we make some remarks about its hypotheses and show that these hypotheses entail some additional properties.

Remark 4.6.

- In applications $\mathbb{D}^0$ will often be a Cohen poset adding subsets to μ_n defined in some inner model of V_{def} , and similarly $\mathbb{D}^1$ will often be a Cohen poset adding subsets to μ_{n+1} defined in some inner model of V_{def} .
- $\mathbb{P}_{2a}$ and $\mathbb{P}_{2b}$ are so named because they will be used successively in Step 2 of the construction for Lemma 4.5. $\mathbb{P}_{2a}$ and $\mathbb{P}_{2b}$ are defined respectively in the submodels $V_{\text{def}}[H][A_n \times D^0]$ and $V_{\text{def}}[H][A_{n+1} \times D^1]$ of the model W .
- In connection with $\mathbb{D}^{\text{small}}$, we recall that μ_{n+1} is the successor of μ_n in $V_{\text{def}}[A * U * S]$ for $n > 0$. In the intended applications it is often the case that $\mathbb{D}^{\text{small}}$ is defined in a submodel of $V_{\text{def}}[A * U * S]$ where $\mu_n < |\mathbb{D}^{\text{small}}| < \mu_{n+1}$.
- Some cardinals (notably μ_n) may be collapsed in W , for example we could set $\mathbb{D}^{\text{small}} = \text{Coll}(\omega, \mu_n)$.
- Hypotheses 2, 5a and 7 jointly imply that both $\mathbb{A}_n \times \mathbb{D}^0$ and $\mathbb{A}_{n+1} \times \mathbb{D}^1$ are μ_{n+2} -cc in $V_{\text{def}}[H]$. It follows that the posets $\mathbb{P}_{2a}$ and $\mathbb{P}_{2b}$ are guaranteed to be well-defined.
- It is implicit in the hypotheses that $j(\mathbb{A}_n)/j[A_n]$ and $j(\mathbb{A}_{n+1})/j[A_{n+1}]$ have rather robust chain condition and distributivity properties.

The following auxiliary lemma, which we will use in the proof of Lemma 4.5, provides a good example of the use of term forcing and “quotient to term forcing” to analyze complicated generic extensions.

Lemma 4.7. *Under the hypotheses of Lemma 4.5:*

- $V_{\text{def}}[A * U * S] \models \text{"}\mathbb{D}^{2,3} \text{ is } \mu_{n+1}\text{-distributive"}$.
- $V_{\text{def}}[A * U * S] \models \text{"}\mathbb{D}^{1,2,3} \text{ is } < \mu_{n+1}\text{-distributive"}$.

Proof. We begin by analyzing the model $V_{\text{def}}[A * U * S][D^{2,3}] = V_{\text{def}}[A * U * S][D^2 \times D^3]$. Recalling that $V_{\text{def}} = V_{\text{inn}}[X]$ and that $\mathbb{D}^2 \in V_{\text{inn}}$, this model is $V_{\text{inn}}[(X * A * U * S * D^3) \times D^2]$. Since $\mathbb{D}^3 \in V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+2}]$, we may form in V_{inn} the term forcing $\mathbb{T}^3 = \mathcal{A}^{V_{\text{inn}}}(\mathbb{X} * \mathbb{A} \restriction \mu_{n+2} * \mathbb{U} \restriction \mu_{n+2}, \mathbb{D}^3)$. By hypothesis 4 of Lemma 4.5, $\mathbb{D}^3$ is $< \mu_{n+2}$ -directed closed in $V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+2}]$, and it follows from Lemma 2.33 that (just like $\mathbb{D}^2$) the poset $\mathbb{T}^3$ is $< \mu_{n+2}$ -directed closed in V_{inn} .

Now we force over $V_{\text{def}}[A * U * S][D^2 \times D^3]$ with the “quotient to term” forcing $\mathbb{T}^3/(A \restriction \mu_{n+2} * U \restriction \mu_{n+2}) * D^3$, which is computed in the submodel $V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+2} * D^3]$. By Lemma 2.39 we obtain T^3 such that $(X * A \restriction \mu_{n+2} * U \restriction \mu_{n+2}) \times T^3$

induces $X * A \restriction \mu_{n+2} * U \restriction \mu_{n+2} * D^3$, and $D^2 \times T^3$ is generic over $V_{\text{def}}[A * U * S]$ for $\mathbb{D}^2 \times T^3$. In particular $V_{\text{def}}[A * U * S][D^2 \times D^3] \subseteq V_{\text{def}}[A * U * S][D^2 \times T^3]$.

Recall from Lemma 3.13 that in V_{def} we may write $\mathbb{A} * \mathbb{U} * \mathbb{S}$ as the projection of $\mathbb{P}_0^* \times \bar{\mathbb{P}}_1^* \times \mathbb{P}_1^*$, where $\mathbb{P}_0^* = \mathbb{A}_{[0,n+1]} * \mathbb{U}_{[0,n]} * \mathbb{S}_{[0,n]}$, $\bar{\mathbb{P}}_1^* = \mathbb{A}_{[n+2,\omega]}$, and $\mathbb{P}_1^* = \mathbb{B}_{[n+1,\omega]} \times \mathbb{C}_{[n+1,\omega]}$. Under our current hypotheses $\mathbb{P}_0^*$ is defined and μ_{n+2} -cc in V_{def} , $\bar{\mathbb{P}}_1^*$ is defined and $< \mu_{n+2}$ -directed closed in V_{inn} , while $\mathbb{P}_1^*$ is defined and $< \mu_{n+2}$ -directed closed in V_{def} .

Let $\mathbb{T}^* = \mathcal{A}^{V_{\text{inn}}}(\mathbb{X}, \mathbb{P}_1^*)$, so that $\mathbb{T}^*$ is defined and $< \mu_{n+2}$ -directed closed in V_{inn} . With another round of quotient to term forcing and another appeal to Lemma 2.39, we produce a generic extension $V_{\text{inn}}[(X * P_0^*) \times A_{[n+2,\omega]} \times T^* \times D^2 \times T^3] \supseteq V[A * U * S][D^2 \times T^3]$. Now $A_{[n+2,\omega]} \times T^* \times D^2 \times T^3$ is generic over V_{inn} for $< \mu_{n+2}$ -closed forcing and $X * P_0^*$ is generic for μ_{n+2} -cc forcing. Appealing to Easton's Lemma, every μ_{n+1} -sequence of ordinals in $V_{\text{def}}[A * U * S][D^2 \times D^3]$ is in $V_{\text{inn}}[X * P_0^*] = V_{\text{def}}[A_{[0,n+1]} * U_{[0,n]} * S_{[0,n]}]$, in particular $\mathbb{D}^2 \times \mathbb{D}^3$ is μ_{n+1} -distributive in $V_{\text{def}}[A * U * S]$. By hypothesis 5b of Lemma 4.5, $\mathbb{D}^1$ is $< \mu_{n+1}$ -distributive in $V_{\text{def}}[A * U * S][D^2 \times D^3]$, so that $\mathbb{D}^1 \times \mathbb{D}^2 \times \mathbb{D}^3$ is $< \mu_{n+1}$ -distributive in $V_{\text{def}}[A * U * S]$ as required. $\square$

Remark 4.8. For use in Lemma 4.9 below, we note that no hypotheses involving either $\mathbb{D}^0$ or $\mathbb{D}^1$ were used to prove the distributivity of $\mathbb{D}^{2,3}$, and that for the distributivity of $\mathbb{D}^{1,2,3}$ we used only that $\mathbb{D}^1$ is $< \mu_{n+1}$ -distributive in $V_{\text{def}}[A * U * S][D^2 \times D^3]$.

With these preliminaries out of the way, we are now ready to prove Lemma 4.5.

Proof of Lemma 4.5. Recall that $W = V_{\text{def}}[A * U * S][D^{s,0,1,2,3}]$. We will show that the cardinal μ_{n+2} has the tree property in W . This involves constructing a generic embedding with domain W and critical point μ_{n+2} , and then arguing that the forcing which adds the embedding will not add a branch to a μ_{n+2} -tree. The forcing to add the embedding will be constructed in several steps.

Recall that μ_{n+2} is indestructibly supercompact in V_{inn} , and $H = A_{[n+2,\omega]} \times D^2$ which is generic for $< \mu_{n+2}$ -directed closed forcing in V_{inn} , so that μ_{n+2} is supercompact in $V_{\text{inn}}[H]$. We will eventually choose an embedding j defined in $V_{\text{inn}}[H]$ witnessing that μ_{n+2} is sufficiently supercompact, and having some other desirable properties, but we defer this choice for the moment. When we choose j it will trivially lift onto $V_{\text{def}}[H]$, because V_{def} is a small generic extension of V_{inn} .

To help motivate the lifting construction below, we list relevant generic objects which must be added to $V_{\text{def}}[H]$ to obtain W . In the following list the “small” group consists of generic objects for posets of size less than μ_{n+2} where the lifting is essentially trivial.

- Small: $A \restriction \mu_{n+1}$, $U \restriction \mu_{n+1}$, $S \restriction \mu_{n+1}$ and D^{small} .
- Large or potentially large: $A \restriction [\mu_{n+1}, \mu_{n+3})$, $U \restriction [\mu_{n+1}, \mu_\omega)$, $S \restriction [\mu_{n+1}, \mu_\omega)$ and the posets D^i for $i = 0, 1, 3$.

At several steps in the following construction we record some closure and chain condition information about the posets which appear in that step. This information will be used in the proof of the tree property.

- Step 1a: (“Remove dependence of $U \restriction [\mu_{n+2}, \mu_\omega)$ on $A \restriction [\mu_{n+2}, \mu_\omega)$ ”). Recall that $\mathbb{U} \restriction [\mu_{n+2}, \mu_\omega) = \mathbb{B}^{+F'} \restriction [\mu_{n+2}, \mu_\omega)$, where $F' = A * (U \restriction \mu_{n+2})$. Let $F = A \restriction \mu_{n+2} * U \restriction \mu_{n+2}$, and note that $F \subseteq F'$. Let $\mathbb{P}_{1a}$ be the “quotient to term” forcing which adds a filter B_{1a} on $\mathbb{B}_{1a} = \mathbb{B}^{+F} \restriction [\mu_{n+2}, \mu_\omega)$, so that B_{1a} induces $U \restriction [\mu_{n+2}, \mu_\omega)$ as in Fact 3.3. We force over W with $\mathbb{P}_{1a}$. We see that

$$V_{\text{def}}[A * U][P_{1a}] = V_{\text{def}}[F][A \restriction [\mu_{n+2}, \mu_\omega) \times B_{1a}] = V_{\text{def}}[A * U \restriction \mu_{n+2}][B_{1a}]$$

and arguing as in ^b Lemma 2.39 $S * D^{s,0,1,2,3}$ is $\mathbb{S} * \mathbb{D}^{s,0,1,2,3}$ -generic over $V_{\text{def}}[A * U \restriction \mu_{n+2}][B_{1a}]$. We have

$$W[P_{1a}] = V_{\text{def}}[A * U \restriction \mu_{n+2}][B_{1a}][S * D^{s,0,1,2,3}].$$

Now $\mathbb{B}_{1a} \in V_{\text{def}}[F]$, and it follows from Fact 3.5 that $\mathbb{B}_{1a}$ is $< \mu_{n+2}$ -directed closed in this model. On the other hand $\mathbb{P}_{1a} \in V_{\text{def}}[A * U]$, and appealing to Fact 3.12 it is actually $< \mu_{n+1}$ -closed in the larger model $V_{\text{def}}[A * U * S \restriction [\mu_{n+1}, \mu_\omega)]$. This closure still holds in the further extension $V_{\text{def}}[A * U * S \restriction [\mu_{n+1}, \mu_\omega)][D^{1,2,3}]$, since by Lemma 4.7 $\mathbb{D}^{1,2,3}$ is $< \mu_{n+1}$ -distributive in $V_{\text{def}}[A * U * S]$.

- Step 1b: (“Remove dependence of $S \restriction [\mu_{n+2}, \mu_\omega)$ on $A \restriction [\mu_{n+2}, \mu_\omega) * U \restriction [\mu_{n+2}, \mu_\omega)$ ”). Recall that $\mathbb{S} \restriction [\mu_{n+2}, \mu_\omega) = \mathbb{C}^{+F''} \restriction [\mu_{n+2}, \mu_\omega)$ where $F'' = A * U$, and $F = A \restriction \mu_{n+2} * U \restriction \mu_{n+2} \subseteq F''$. Let $\mathbb{P}_{1b}$ be the quotient forcing which adds a filter C_{1b} on $\mathbb{C}_{1b} = \mathbb{C}^{+F} \restriction [\mu_{n+2}, \mu_\omega)$, inducing $S \restriction [\mu_{n+2}, \mu_\omega)$. We force over $W[P_{1a}]$ with $\mathbb{P}_{1b}$, and let $\mathbb{P}_1 = \mathbb{P}_{1a} \times \mathbb{P}_{1b}$. As in Step 1a,

$$W[P_1] = V_{\text{def}}[A * U \restriction \mu_{n+2} * S \restriction \mu_{n+2}][B_{1a}][C_{1b}][D^{s,0,1,2,3}],$$

and $D^{s,0,1,2,3}$ continues to be $\mathbb{D}^{s,0,1,2,3}$ -generic over the slightly larger model $V_{\text{def}}[A * U \restriction \mu_{n+2} * S \restriction \mu_{n+2}][B_{1a}][C_{1b}]$.

Similarly to step 1a, $\mathbb{C}_{1b} \in V_{\text{def}}[F]$, and it follows from Fact 3.11 that $\mathbb{C}_{1b}$ is $< \mu_{n+2}$ -directed closed in this model. On the other hand $\mathbb{P}_{1b} \in V_{\text{def}}[A * U * S \restriction [\mu_{n+2}, \mu_\omega)]$, and appealing to Fact 3.12 it is actually $< \mu_{n+1}$ -closed in the larger model $V[A * U * S \restriction [\mu_{n+1}, \mu_\omega)]$. As in Step 1a, this closure still holds in the further extension $V_{\text{def}}[A * U * S \restriction [\mu_{n+1}, \mu_\omega)][D^{1,2,3}]$.

- Choosing j : Recall that we defined $\mathbb{A} * \mathbb{U}$ using a Laver function ϕ in V_{def} , obtained from a universal indestructible Laver function ψ in V_{inn} , setting $\phi(\alpha) = \psi(\alpha)[X]$ whenever $\psi(\alpha)$ is an $\mathbb{X}$ -name. Working in $V_{\text{inn}}[H]$ we choose j such that

^b $\mathbb{B}_{1a}$ is not literally the termspace forcing $\mathcal{A}^{V_{\text{def}}[F]}(\mathbb{A} \restriction [\mu_{n+2}, \mu_\omega), \mathbb{U} \restriction [\mu_{n+2}, \mu_\omega))$, but it does add an $\mathbb{A} \restriction [\mu_{n+2}, \mu_\omega)$ -name for a filter which is $\mathbb{U} \restriction [\mu_{n+2}, \mu_\omega)$ -generic over $V_{\text{def}}[A * U \restriction \mu_{n+2}]$ and this is sufficient. See Remark 2.40.

- $j \restriction ON$ is definable in V_{inn} .
- j witnesses μ_{n+2} is χ -supercompact where $\chi = \max(\mu_\omega, |\mathbb{D}^{0,1,2,3}|)^+$.
- The next point in $\text{dom}(j(\psi))$ past μ_{n+2} is greater than χ .
- $j(\psi)(\mu_{n+2}) = \check{Z}$, where $\check{Z}$ is an $\mathbb{X}$ -name in V_{inn} for an $\mathbb{A} \restriction \mu_{n+2} * \mathbb{U} \restriction \mu_{n+2}$ -name in V_{def} for $\mathbb{B}_{1a} \times \mathbb{C}_{1b} \times \mathbb{D}^3$.

Since V_{def} is a generic extension of V_{inn} by forcing of size at most μ_1 , we trivially lift j to obtain $j : V_{\text{def}}[H] \rightarrow M_H$. Note that $j(\phi)(\mu_{n+2}) = \check{Z}$, where $\check{Z}$ is a $\mathbb{A} \restriction \mu_{n+2} * \mathbb{U} \restriction \mu_{n+2}$ -name in V_{def} for $\mathbb{B}_{1a} \times \mathbb{C}_{1b} \times \mathbb{D}^3$.

Since $\mathbb{B}_{1a} \times \mathbb{C}_{1b} \times \mathbb{D}^3 \in V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+2}]$ and is $< \mu_{n+2}$ -directed closed in this model, the choice of j implies that μ_{n+2} is in the support of the $\mathbb{U}$ -coordinate of $j(\mathbb{A} * \mathbb{U})$ and the forcing which appears there is $\mathbb{B}_{1a} \times \mathbb{C}_{1b} \times \mathbb{D}^3$.

- Step 2a: (“Stretch $A_n \times D^0$ ”) Let $\mathbb{P}_{2a} = j(\mathbb{A}_n \times \mathbb{D}^0)/j[A_n \times D_0]$. We force over $W[P_1]$ with $\mathbb{P}_{2a}$ and add a $j(\mathbb{A}_n \times \mathbb{D}^0)$ -generic object $\hat{A}_n \times \hat{D}^0$ such that $j[A_n \times D^0] \subseteq \hat{A}_n \times \hat{D}^0$.
- Step 2b: (“Stretch $A_{n+1} \times D^1$ ”) Let $\mathbb{P}_{2b} = j(\mathbb{A}_{n+1} \times \mathbb{D}^1)/j[A_{n+1} \times D^1]$. We force over $W[P_1][P_{2a}]$ with $\mathbb{P}_{2b}$ and add a $j(\mathbb{A}_{n+1} \times \mathbb{D}^1)$ -generic object $\hat{A}_{n+1} \times \hat{D}^1$ such that $j[A_{n+1} \times D^1] \subseteq \hat{A}_{n+1} \times \hat{D}^1$.

We let $\mathbb{P}_2 = \mathbb{P}_{2a} \times \mathbb{P}_{2b}$. Then

$$W[P_1][P_2] = V_{\text{def}}[\hat{A} * U \restriction \mu_{n+2} * S \restriction \mu_{n+2}][B_{1a}][C_{1b}][\hat{D}^{s,0,1,2,3}],$$

where $\hat{A} = A_{[0,n-1]} \times \hat{A}_n \times \hat{A}_{n+1} \times A_{[n+2,\omega]}$ and $\hat{D}^{s,0,1,2,3} = D^s \times \hat{D}^0 \times \hat{D}^1 \times D^2 \times D^3$.

- Step 3 (“Stretch the term forcing for $S \restriction [\mu_{n+1}, \mu_{n+2}]$ ”) Recall that $\mathbb{S} \restriction [\mu_{n+1}, \mu_{n+2}] = \mathbb{C}^{+\bar{F}} \restriction [\mu_{n+1}, \mu_{n+2}]$, where $\bar{F} = A \restriction \mu_{n+2} * U \restriction \mu_{n+1}$. Let $\mathbb{P}_3 = j(\mathbb{C})^{+\bar{F}} \restriction [\mu_{n+2}, j(\mu_{n+2})]$. We force with $\mathbb{P}_3$ over $W[P_1][P_2]$.

Note that $\mathbb{P}_3$ is defined in $V_{\text{def}}[H][\bar{F}] \subseteq V[A * U \restriction \mu_{n+1}][D^2]$. Modifying the proof of [16, Claim 4.31] to account for $\mathbb{D}^2$, $\mathbb{P}_3$ is $< \mu_{n+1}$ -closed in $V[A * U * S \restriction [\mu_{n+1}, \mu_\omega]][D^2]$. By distributivity, $\mathbb{P}_3$ retains this closure in the larger model $V[A * U * S \restriction [\mu_{n+1}, \mu_\omega]][D^{1,2,3}]$.

We now extend $j : V_{\text{def}}[H] \rightarrow M_H$ to a generic embedding with domain W , working (ultimately) in the extension $W' = W[P_{1,2,3}]$ where $P_{1,2,3}$ collects the generic objects we added in the steps above.

- Stage 1: Recall that $B_{1a} \times C_{1b}$ is the generic object added by $\mathbb{P}_1 = \mathbb{P}_{1a} \times \mathbb{P}_{1b}$ for $\mathbb{B}_{1a} \times \mathbb{C}_{1b}$, a poset which is defined and is $< \mu_{n+2}$ -directed closed in $V_{\text{def}}[F]$, where $F = A \restriction \mu_{n+2} * U \restriction \mu_{n+2}$. By our choice of j , $j(\mathbb{A} * \mathbb{U} * \mathbb{S})$ has a “ $\mathbb{U}$ -component” in which $\mathbb{B}_{1a} \times \mathbb{C}_{1b} \times \mathbb{D}^3$ appears at stage μ_{n+2} .

Modifying the proof of Fact 3.7 from [16], we may lift j onto $V_{\text{def}}[A * U \restriction \mu_{n+2}][B_{1a} \times C_{1b}][D^{0,1,2,3}]$. We outline the modified proof, with a focus on where to find the compatible generic objects on the “ j -side”.

- $H = A_{[n+2,\omega]} \times D^2 \in V_{\text{def}}[H]$, so that if $\hat{H} = j(H) = j(A_{[n+2,\omega]}) \times$

- $j(D^2)$ then $\hat{H} \in M_H$ and $M_H = j(V_{\text{def}})[\hat{H}]$.
- $j(A_{[0,n]}) = A_{[0,n]}$.
 - $j(A_n \times D^0)$ is obtained by combining $A_n \times D^0$ and the generic object P_{2a} for the “stretching” poset $\mathbb{P}_{2a}$.
 - $j(D^1 \times A_{n+1})$ is obtained by combining $D^1 \times A_{n+1}$ and the generic object P_{2b} for the “stretching” poset $\mathbb{P}_{2b}$.
 - $j(U \restriction \mu_{n+2})$ is obtained by concatenating $U \restriction \mu_{n+2}$, $B_{1a} \times C_{1b} \times D^3$ (the generic object at μ_{n+2}), and a generic object for $j(\mathbb{U}) \restriction (\mu_{n+2}, j(\mu_{n+2}))$ which is constructed using closure under χ -sequences.
 - $j(B_{1a} \times C_{1b} \times D^3)$ is constructed using closure under χ -sequences and a master condition argument.
- Stage 2: $S \restriction \mu_{n+1}$ is generic for forcing of size less than μ_{n+2} , so we may trivially lift j onto $V_{\text{def}}[A * U \restriction \mu_{n+2}][B_{1a} \times C_{1b}][D^{0,1,2,3}][S \restriction \mu_{n+1}]$.
 - Stage 3: As we noted above in the definition of $\mathbb{P}_3$, $\mathbb{S} \restriction [\mu_{n+1}, \mu_{n+2}) = \mathbb{C}^{+\bar{F}} \restriction [\mu_{n+1}, \mu_{n+2})$ where $\bar{F} = A \restriction \mu_{n+2} * U \restriction \mu_{n+1}$. When we apply j to $\mathbb{S} \restriction [\mu_{n+1}, \mu_{n+2})$ it is only the “ A -component” which gets stretched: more precisely $j(A \restriction \mu_{n+2}) = A_{[0,n]} * \hat{A}_n$, $j(\bar{F}) = (A_{[0,n]} * \hat{A}_n) * U \restriction \mu_{n+1}$ and $j(\mathbb{S} \restriction [\mu_{n+1}, \mu_{n+2})) = j(\mathbb{C})^{+(A_{[0,n]} * \hat{A}_n) * U \restriction \mu_{n+1}} \restriction [\mu_{n+1}, j(\mu_{n+2}))$.
 Let $S_n = S \restriction [\mu_{n+1}, \mu_{n+2})$. Recall that $\mathbb{P}_3 = j(\mathbb{C})^{+\bar{F}} \restriction [\mu_{n+2}, j(\mu_{n+2}))$, and note that:
 - $\bar{F} \subseteq j(\bar{F}) = (A_{[0,n]} * \hat{A}_n) * U \restriction \mu_{n+1}$.
 - P_3 is generic over $W[P_{1,2}]$ which contains all relevant generic objects.
 - $j(\mathbb{C})^{+\bar{F}} \restriction [\mu_{n+1}, j(\mu_{n+2})) \simeq j(\mathbb{C})^{+\bar{F}} \restriction [\mu_{n+1}, \mu_{n+2}) \times \mathbb{P}_3 \simeq \mathbb{S}_n \times \mathbb{P}_3$.
 We may therefore form the upwards closure $\hat{S}_n$ of $S_n \times P_3$ in $j(\mathbb{S} \restriction [\mu_{n+1}, \mu_{n+2}))$, to produce $\hat{S}_n$ such that $\hat{S}_n$ is generic for $j(\mathbb{S} \restriction [\mu_{n+1}, \mu_{n+2}))$. Since $\text{crit}(j) = \mu_{n+2}$, and conditions in $\mathbb{S}_n$ have supports which are bounded subsets of μ_{n+2} , it is easy to see that $j[S_n] \subseteq \hat{S}_n$ and so we may lift j onto $V_{\text{def}}[A * U \restriction \mu_{n+2}][B_{1a} \times C_{1b}][D^{0,1,2,3}][S \restriction \mu_{n+2}]$.
 - Stage 4: Recall that $\mathbb{P}_{1a}$ added a filter B_{1a} on $\mathbb{B}_{1a} = \mathbb{B}^{+F} \restriction [\mu_{n+2}, \mu_\omega)$, such that B_{1a} induces $U \restriction [\mu_{n+2}, \mu_\omega)$. This used the description of $\mathbb{U} \restriction [\mu_{n+2}, \mu_\omega)$ as $\mathbb{B}^{+F'} \restriction [\mu_{n+2}, \mu_\omega)$, where $F' = A * (U \restriction \mu_{n+2})$.
 Since we have lifted j onto a model which contains both B_{1a} and F' , we may use $j(B_{1a})$ and $j(F')$ to induce a filter $\hat{U}_{[\mu_{n+2}, \mu_\omega)}$ on $j(\mathbb{U} \restriction [\mu_{n+2}, \mu_\omega))$ with $j[U \restriction [\mu_{n+2}, \mu_\omega))] \subseteq \hat{U}_{[\mu_{n+2}, \mu_\omega)}$. This lets us lift j onto $V_{\text{def}}[A * U][B_{1a} \times C_{1b}][D^{0,1,2,3}][S \restriction \mu_{n+2}]$.
 - Stage 5: Similarly to Stage 4, $\mathbb{P}_{1b}$ added a filter C_{1b} on $\mathbb{C}_{1b} = \mathbb{C}^{+F} \restriction [\mu_{n+2}, \mu_\omega)$, inducing $S \restriction [\mu_{n+2}, \mu_\omega)$. This used the description of $\mathbb{S} \restriction [\mu_{n+2}, \mu_\omega)$ as $\mathbb{C}^{+F''} \restriction [\mu_{n+2}, \mu_\omega)$, where $F'' = A * U$.
 Since we have lifted j onto a model which contains both C_{1b} and F'' , we may use $j(C_{1b})$ and $j(F'')$ to induce a filter $\hat{S}_{[\mu_{n+2}, \mu_\omega)}$ on $j(\mathbb{S} \restriction [\mu_{n+2}, \mu_\omega))$ with $j[S \restriction [\mu_{n+2}, \mu_\omega))] \subseteq \hat{S}_{[\mu_{n+2}, \mu_\omega)}$. This lets us lift j onto $V_{\text{def}}[A * U * S][B_{1a} \times C_{1b}][D^{0,1,2,3}]$

- Stage 6: Since $\mathbb{D}^{\text{small}} \in V_{\text{def}}[A * U * S]$ and $|\mathbb{D}^{\text{small}}| \leq \mu_n$, we may trivially lift j onto $V_{\text{def}}[A * U * S][B_{1a} \times C_{1b}][D^{s,0,1,2,3}] = W[B_{1a} \times C_{1b}]$.

To verify the tree property, we need to check that the forcing posets used to extend j onto W can not add a branch to a μ_{n+2} -tree. Recall that the lifting of j is defined in $W' = W[P_{1,2,3}]$. As we already mentioned, μ_{n+1} and μ_{n+2} are preserved in W but it is possible that μ_n has been collapsed.

The proof which follows involves a number of auxiliary models. See the diagram which follows the proof and its legend for a picture of how they are related.

- Let $M_0 = W$ and $M_1 = M_0[P_{2b}]$. By hypothesis 6a of Lemma 4.5, $\mathbb{P}_{2b}$ is μ_{n+2} -Knaster in M_0 , so by Lemma 2.11 no tree of height μ_{n+2} in M_0 has a new branch in M_1 . By hypothesis 6b, $\mathbb{P}_{2b}$ is $< \mu_{n+1}$ -distributive in M_0 , so that both μ_{n+1} and μ_{n+2} are regular in M_1 .
- Let $M_2 = M_1[P_1 \times P_3]$. We claim that no μ_{n+2} -tree in M_1 has a new branch in M_2 .

Recall the closure property which we noted for $\mathbb{P}_{1a}$, $\mathbb{P}_{1b}$ and $\mathbb{P}_3$. They are all $< \mu_{n+1}$ -closed in a certain submodel M_- of M_0 , where $M_- = V_{\text{def}}[A * U][S \restriction [\mu_{n+1}, \mu_\omega]][D^{1,2,3}]$. Our aim is ultimately to make an appeal to Fact 2.12 with $\tau = |\mu_n|$ and $\eta = \mu_{n+2}$. Note that in the model M_- we have $2^{|\mu_n|} \geq \mu_{n+2}$.

Since $M_0 = M_-[(D^0 \times S \restriction \mu_{n+1}) * D^s]$, we have $M_1 = M_-[P_{2b}][(D^0 \times S \restriction \mu_{n+1}) * D^s]$. Now $\mathbb{P}_{2b} \in M_-$ and by hypothesis 6b it is $< \mu_{n+1}$ -distributive in M_0 , so $\mathbb{P}_{2b}$ is $< \mu_{n+1}$ -distributive in M_- and hence $\mathbb{P}_{1a} \times \mathbb{P}_{1b} \times \mathbb{P}_3$ is $< \mu_{n+1}$ -closed in $M_-[P_{2b}]$.

By hypothesis 7, $\mathbb{D}^0$ is μ_{n+1} -Knaster in $V_{\text{def}}[A * U * S][D^{1,2,3}][P_{2b}]$, so it is μ_{n+1} -cc in $M_-[P_{2b}]$. It is easy to see that $\mathbb{S} \restriction \mu_{n+1}$ is μ_{n+1} -Knaster in $M_-[P_{2b}]$, and by hypothesis $|\mathbb{D}^{\text{small}}| \leq \mu_n$. So $(\mathbb{D}^0 \times \mathbb{S} \restriction \mu_{n+1}) * \mathbb{D}^{\text{small}}$ is μ_{n+1} -cc in $M_-[P_{2b}]$.

We are exactly in the situation of Fact 2.12 where the forcing posets live in $M_-[P_{2b}]$:

- (1) Since $M_-[P_{2b}] \subseteq M_1$, μ_{n+1} and μ_{n+2} are regular in $M_-[P_{2b}]$.
- (2) $2^{|\mu_n|} \geq \mu_{n+2}$ in $M_-[P_{2b}]$.
- (3) $\mathbb{P}_1 \times \mathbb{P}_3$ is $< \mu_{n+1}$ -closed in $M_-[P_{2b}]$.
- (4) $M_1 = M_-[P_{2b}][Y]$, where $Y = (D^0 \times S \restriction \mu_{n+1}) * D^s$ and Y is generic for μ_{n+1} -cc forcing over $M_-[P_{2b}]$.

Since $M_2 = M_1[P_{1a} \times P_{1b} \times P_3]$, M_2 is an extension of M_1 by “formerly closed” forcing in the sense of Fact 2.12 and we are done.

- Before the last step we need to analyze the cardinals of M_2 . By Easton’s Lemma, μ_{n+1} is preserved in this model. We claim that in M_2 the cardinal μ_{n+2} is collapsed so that (by Easton’s Lemma again) it has cofinality μ_{n+1} . To see this note that at coordinate μ_{n+2} , conditions in $j(\mathbb{C})$ have $\mathbb{A} \restriction \mu_{n+2} \times \mathbb{U} \restriction \mu_{n+1}$ -terms for conditions in $\text{Add}(\mu_{n+1}, 1)^{V[A \restriction \mu_{n+2} \times \mathbb{U} \restriction \mu_{n+1}]}$.

Since we are augmenting with $A \restriction \mu_{n+2} \times U \restriction \mu_{n+1}$ to form $\mathbb{P}_3$, we add a generic object for $\text{Add}(\mu_{n+1}, 1)^{V[A \restriction \mu_{n+2} \times U \restriction \mu_{n+1}]}$, and collapse μ_{n+2} because $\mu_{n+2} = (2^{\mu_n})^{V[A \restriction \mu_{n+2} \times U \restriction \mu_{n+1}]}$.

- We also need to analyze the chain condition of $\mathbb{P}_{2a}$ in M_2 . $M_2 = M_1[P_1 \times P_3]$, and as we saw above $\mathbb{P}_1 \times \mathbb{P}_3$ is defined and $< \mu_{n+1}$ -closed in M_- , hence it is formerly $< \mu_{n+1}$ -closed in $M_-[S \restriction \mu_{n+1}] = V[A * U * S][D^{1,2,3}]$. By hypothesis 8, $\mathbb{P}_{2a}$ is μ_{n+1} -Knaster in $M_2 = M_1[P_1 \times P_3]$.
- Let $M_3 = W' = M_2[P_{2a}]$. We claim that no tree of height μ_{n+1} in M_2 has a new branch in M_3 . This is immediate by Lemma 2.11 because $\mathbb{P}_{2a}$ is μ_{n+1} -Knaster in M_2 .

□

$$\begin{array}{ccccccc}
 M_- & \xrightarrow{P_{2b}} & M_-[P_{2b}] & & & & \\
 \downarrow Y & & \downarrow Y & & & & \\
 M_0 & \xrightarrow{P_{2b}} & M_1 & \xrightarrow{P_1 \times P_3} & M_2 & \xrightarrow{P_{2a}} & M_3
 \end{array}$$

- $M_- = V_{\text{def}}[A * U][S \restriction [\mu_{n+1}, \mu_\omega]][D^{1,2,3}]$.
- $M_0 = W = V_{\text{def}}[A * U * S][D^{0,1,2,3,s}] = M_-[Y]$, where $Y = (D^0 \times S \restriction \mu_{n+1}) * D^s$.
- $M_1 = M_0[P_{2b}]$.
- $M_2 = M_1[P_1 \times P_3]$.
- $M_3 = M_2[P_{2a}] = W[P_{1,2,3}] = W'$.

The following lemma will enable us to satisfy the hypotheses of Lemma 4.5 in most instances. We are assuming all the background hypotheses listed at the start of this section, notably $\mathbb{A}_{[s,t]}$ is $< \mu_s$ -distributive and μ_{t+1} -cc in V_{def} .

Lemma 4.9. *Let V^a, V^b, V^c, V^d be inner models with $V_{\text{inn}} \subseteq V^x \subseteq V_{\text{def}}$ for $x = a, b, c, d$. Assume that:*

- $n > 0$.
- $\mathbb{D}^{\text{small}}$ is any poset in $V_{\text{def}}[A * U * S]$ with $V_{\text{def}}[A * U * S] \models |\mathbb{D}^{\text{small}}| \leq \mu_n$.
- $\mathbb{D}^0 = \text{Add}^{V^a}(\mu_{\bar{n}}, \sigma)$ for some σ and some $\bar{n} \leq n$.
- $\mathbb{A}_n = \text{Add}^{V^b}(\mu_n, \sigma')$ for some σ' .
- $\mathbb{D}^1 = \text{Add}^{V^c}(\mu_{n+1}, \tau)$ for some τ .
- $\mathbb{A}_{n+1} = \text{Add}^{V^d}(\mu_{n+1}, \tau')$ for some τ' .
- $\mathbb{D}^2$ is any poset in V_{inn} with $V_{\text{inn}} \models \text{“}\mathbb{D}^2 \text{ is } < \mu_{n+2}\text{-directed closed”}$.
- $\mathbb{D}^3$ is any poset in $V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+2}]$ with $V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+2}] \models \text{“}\mathbb{D}^3 \text{ is } < \mu_{n+2}\text{-directed closed”}$.

Then the hypotheses of Lemma 4.5 are satisfied.

Proof. Hypotheses 3, 9 and 4 are immediate. Since $\mathbb{D}^{\text{small}} \in V_{\text{def}}[A * U * S]$ and $|\mathbb{D}^{\text{small}}| \leq \mu_n$, we may assume that $\mathbb{D}^{\text{small}} \in V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+1} * S \restriction \mu_{n+1}]$. As we noted in Remark 4.8, our hypotheses imply that $\mathbb{D}^{2,3}$ is μ_{n+1} -distributive in $V_{\text{def}}[A * U * S]$.

Since H is generic over V_{def} for forcing which is defined and $< \mu_{n+2}$ -closed in V_{inn} , and V_{def} is a μ_1 -cc generic extension of V_{inn} , by Easton's Lemma $V_{\text{def}}[H]$ is a $< \mu_{n+2}$ -distributive extension of V_{def} . Since V_{def} is a μ_1 -cc generic extension of V^b and V^d , and both μ_{n+1} and μ_{n+2} are inaccessible in any submodel of V_{def} , it follows that $V_{\text{def}}[H]$ is a (μ_n, μ_{n+1}) -good extension of V^b and a (μ_{n+1}, μ_{n+2}) -good extension of V^d , so that by Lemma 2.54 $\mathbb{A}_n$ and $\mathbb{A}_{n+1}$ are respectively μ_{n+1} -Knaster and μ_{n+2} -Knaster in $V_{\text{def}}[H]$. We have satisfied Hypothesis 2. Similarly $\mathbb{D}^1$ is μ_{n+2} -Knaster in $V_{\text{def}}[H]$ and we have satisfied Hypothesis 5a.

We need some analysis of $\mathbb{P}_{2a}$ and $\mathbb{P}_{2b}$. Let $\mathbb{X}^x \in V_{\text{inn}}$ and $\mathbb{X}_x \in V^x$ be such that $V^x = V_{\text{inn}}[\mathbb{X}^x]$ and $V_{\text{def}} = V^x[X_x]$ for $x = a, b, c, d$. Note that we may assume that $\mathbb{X}^x$ is μ_1 -cc in V_{inn} and $\mathbb{X}_x$ is μ_1 -cc in V^x . Recall that j is an embedding witnessing that μ_{n+2} is highly supercompact in the model $V_{\text{def}}[A_{[n+2, \omega)} \times D^2]$, and is the trivial lift (keeping in mind that $|\mathbb{X}| \leq \mu_1 < \mu_{n+2}$) of such an embedding defined in $V_{\text{inn}}[A_{[n+2, \omega)} \times D^2]$. It is easy to see that V^a and $j(V^a)$ have the same $< \mu_n$ -sequences of ordinals, so that $j(\mathbb{D}^0) = \text{Add}^{V^a}(\mu_{\bar{n}}, j(\sigma))$ and $j(D_0)/j[D_0] = \text{Add}^{V^a}(\mu_{\bar{n}}, j(\sigma) \setminus j[\sigma])$ and similarly for $\mathbb{A}_n$, $\mathbb{D}^1$ and $\mathbb{A}_{n+1}$.

Now we revisit the argument for Lemma 4.7, but we need a slightly different decomposition for $\mathbb{A} * \mathbb{U} * \mathbb{S}$. $\mathbb{A} * \mathbb{U} * \mathbb{S}$ may be written in V_{def} as a projection of the product

$$(\mathbb{A}_{[0, n]} * \mathbb{U}_{[0, n]} * \mathbb{S}_{[0, n]}) \times \mathbb{A}_{n+1} \times \mathbb{A}_{[n+2, \omega)} \times \mathbb{B}_{[n, \omega)} \times \mathbb{C}_{[n, \omega)}$$

where:

- $\mathbb{A}_{[0, n]} * \mathbb{U}_{[0, n]} * \mathbb{S}_{[0, n]}$ is defined and μ_{n+1} -cc in V_{def} .
- $\mathbb{A}_{n+1}$ is defined and $< \mu_{n+1}$ -closed in $V^d = V_{\text{inn}}[X_d]$.
- $\mathbb{A}_{[n+2, \omega)}$ is defined and $< \mu_{n+2}$ -closed in V_{inn} .
- $\mathbb{B}_{[n, \omega)} \times \mathbb{C}_{[n, \omega)}$ is defined and $< \mu_{n+1}$ -closed in V_{def} .

As in the proof of Lemma 4.7 (and keeping in mind that $\mathbb{D}^1$ is defined and $< \mu_{n+1}$ -closed in $V^c = V_{\text{inn}}[X^c]$) we may force with a series of quotient to term forcings to extend $V_{\text{def}}[A * U * S][D^{1,2,3}]$ to a model of the form $V_{\text{inn}}[(X * P_0'') \times T \times A_{[n+2, \omega)} \times D^2 \times T^3]$, where:

- $\mathbb{P}_0'' = \mathbb{A}_{[0, n]} * \mathbb{U}_{[0, n]} * \mathbb{S}_{[0, n]}$, so that $\mathbb{X} * \mathbb{P}_0''$ is μ_{n+1} -cc in V_{inn} .
- $\mathbb{T} = \mathcal{A}^{V_{\text{inn}}}(\mathbb{X}^d, \mathbb{A}_{n+1}) \times \mathcal{A}^{V_{\text{inn}}}(\mathbb{X}^c, \mathbb{D}_1) \times \mathcal{A}^{V_{\text{inn}}}(\mathbb{X}, \mathbb{B}_{[n, \omega)} \times \mathbb{C}_{[n, \omega)})$, so that $\mathbb{T}$ is $< \mu_{n+1}$ -closed in V_{inn} .
- As before $\mathbb{T}^3 = \mathcal{A}^{V_{\text{inn}}}(\mathbb{X} * \mathbb{A} \restriction \mu_{n+2} * \mathbb{U} \restriction \mu_{n+2}, \dot{D}^3)$, so that $\mathbb{T}^3$ is $< \mu_{n+2}$ -closed in V_{inn} .

By Easton's Lemma all $< \mu_{n+1}$ -sequences of ordinals from $V_{\text{def}}[A * U * S][D^{1,2,3}]$ lie in $V_{\text{inn}}[X * P_0''] = V_{\text{def}}[P_0''] \subseteq V_{\text{def}}[A * U * S]$, so that in particular $\mathbb{D}^1$ is $< \mu_{n+1}$ -distributive in $V_{\text{def}}[A * U * S][D^{2,3}]$ and we satisfied Hypothesis 5b.

By Lemma 3.15 and the hypothesis that V_{def} is a μ_1 -cc extension of V_{inn} , $V_{\text{def}}[A * U * S]$ is both a (μ_n, μ_{n+1}) -good extension of V^a and a (μ_{n+1}, μ_{n+2}) -good extension of V^c . Since $\mathbb{D}^{2,3}$ is $< \mu_{n+2}$ -distributive in $V_{\text{def}}[A * U * S]$, $V_{\text{def}}[A * U * S][D^{2,3}]$ is a (μ_{n+1}, μ_{n+2}) -good extension of V^c , so that $\mathbb{D}^1$ is μ_{n+2} -Knaster in $V_{\text{def}}[A * U * S][D^{2,3}]$. Since $\mathbb{D}^1$ is $< \mu_{n+1}$ -distributive in $V_{\text{def}}[A * U * S][D^{2,3}]$, $V_{\text{def}}[A * U * S][D^{1,2,3}]$ is a (μ_n, μ_{n+1}) -good extension of V^a , so that $\mathbb{D}^0$ is μ_{n+1} -Knaster in $V_{\text{def}}[A * U * S][D^{1,2,3}]$. In fact $\mathbb{D}^0 \times \mathbb{D}^{\text{small}}$ is μ_{n+1} -Knaster in $V_{\text{def}}[A * U * S][D^{1,2,3}]$, from which it follows easily that both μ_{n+1} and μ_{n+2} are regular in W . We have satisfied Hypothesis 1.

The analysis of the last paragraph also shows that W is a (μ_{n+1}, μ_{n+2}) -good extension of V^c and of V^d . From the analysis of $j(\mathbb{A}_{n+1})$ and $j(\mathbb{D}^1)$, it follows readily that $\mathbb{P}_{2b}$ is μ_{n+2} -Knaster in W . We have satisfied Hypothesis 6a.

Now we do another analysis in the same style as Lemma 4.7, but this time we expand the model $W[P_{2b}] = V_{\text{def}}[A * U * S][D^{s,0,1,2,3}][P_{2b}]$ to $V_{\text{inn}}[(X * P_0'' * (D^0 \times D^s)) \times T \times A_{[n+2,\omega)} \times D^2 \times T^3 \times T']$, where $T' = \mathcal{A}^{V_{\text{inn}}}(\mathbb{X}^c, j(\mathbb{D}_1)/j[D_1]) \times \mathcal{A}^{V_{\text{inn}}}(\mathbb{X}^d, j(\mathbb{A}_{n+1})/j[A_{n+1}])$. We recall from our earlier analysis that $\mathbb{D}^{\text{small}} \in V_{\text{def}}[P_0'']$ and that $\mathbb{D}^0$ is μ_{n+1} -cc in $V_{\text{def}}[A * U * S]$, so that easily $\mathbb{X} * \mathbb{P}_0'' * (\mathbb{D}^0 \times \mathbb{D}^{\text{small}})$ is μ_{n+1} -cc in V_{inn} . By Easton's lemma all $< \mu_{n+1}$ -sequences of ordinals in $W[P_{2b}]$ lie in the submodel $V_{\text{inn}}[X * P_0'' * D^{0,s}]$ of W , so that $\mathbb{P}_{2b}$ is $< \mu_{n+1}$ -distributive in W . We have satisfied hypothesis 6b.

We saw already that $V_{\text{def}}[A * U * S][D^{1,2,3}]$ is a (μ_n, μ_{n+1}) -good extension of V^a . Since $\mathbb{P}_{2b}$ is $< \mu_{n+1}$ -distributive in W it has this property in $V_{\text{def}}[A * U * S][D^{1,2,3}]$, so $V_{\text{def}}[A * U * S][D^{1,2,3}][P_{2b}]$ is a (μ_n, μ_{n+1}) -good extension of V^a , and thus $\mathbb{D}^0$ is μ_{n+1} -Knaster in $V_{\text{def}}[A * U * S][D^{1,2,3}][P_{2b}]$. We have satisfied hypothesis 7.

Finally let $\mathbb{Q}$ be defined and $< \mu_{n+1}$ -closed in $V_{\text{def}}[A * U][S \upharpoonright [\mu_{n+1}, \mu_\omega)][D^2]$, and let $W' = W[P_{2b}][\mathbb{Q}] = V_{\text{def}}[A * U * S][D^{1,2,3}][D^{0,s}][\mathbb{Q}][P_{2b}]$. Arguing as before we expand $V_{\text{def}}[A * U * S][D^{1,2,3}][\mathbb{Q}][P_{2b}]$ to a model $V_{\text{inn}}[(X * P_0'') \times T \times A_{[n+2,\omega)} \times D^2 \times T^3 \times T' \times T'']$, where $T'' = \mathcal{A}^{V_{\text{inn}}}(\mathbb{X} * \mathbb{A} * \mathbb{U} * \mathbb{S} \upharpoonright [\mu_{n+1}, \mu_\omega) * \mathbb{D}^2, \dot{\mathbb{Q}})$, and use this to argue that all $< \mu_{n+1}$ -sequences of ordinals from $V_{\text{def}}[A * U * S][D^{1,2,3}][\mathbb{Q}][P_{2b}]$ lie in $V_{\text{def}}[P_0'']$. It follows that $V_{\text{def}}[A * U * S][D^{1,2,3}][\mathbb{Q}][P_{2b}]$ is a (μ_n, μ_{n+1}) -good extension of each model V^x , so that easily $\mathbb{D}^0 \times \mathbb{D}^{\text{small}} \times \mathbb{P}_{2a}$ is μ_{n+1} -Knaster in $V_{\text{def}}[A * U * S][D^{1,2,3}][\mathbb{Q}][P_{2b}]$. So $\mathbb{P}_{2a}$ is μ_{n+1} -Knaster in $V_{\text{def}}[A * U * S][D^{1,2,3}][\mathbb{Q}][P_{2b}][D^{0,s}] = W'$, and we have satisfied hypothesis 8. $\square$

It will be useful later (in the $n = 0$ cases from Sections 9.1 and 9.3, and again in Section 10.2) to know that certain initial segments of $V_{\text{def}}[A * U * S]$ have similar indestructibility properties to those in Lemma 4.5. The following lemma is stated under the same hypotheses as that lemma, and as far as possible with the same notation. Although the construction of the relevant generic embedding is very similar to that for Lemma 4.5, we have given it in some detail as a service to readers of

Section 10.2. We have not stated the Lemma in the maximum possible generality, in particular we have dispensed with $\mathbb{D}^1$ and have only some specific instances of $\mathbb{D}^0$.

Lemma 4.10. *Let $\mu_{n+2} \leq \eta < \mu_\omega$ and let $V' = V_{\text{def}}[A \restriction \eta * U \restriction \mu_{n+2} * S \restriction \mu_{n+2}]$. Let $\mathbb{D}^2, \mathbb{D}^3, \mathbb{D}^0, \mathbb{D}^{\text{small}}$ be forcing posets such that*

- (1) $\mathbb{D}^2 \in V_{\text{inn}}$ and $\mathbb{D}^2$ is $< \mu_{n+2}$ -directed closed in V_{inn} .
- (2) $\mathbb{D}^3 \in V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+2}]$ and $\mathbb{D}^3$ is $< \mu_{n+2}$ -directed closed in $V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+2}]$.
- (3) $\mathbb{D}^0 = \text{Add}^{V_{\text{def}}}(\mu_n, \sigma)$ for some σ , or $\mathbb{D}^0 = \text{Coll}(\omega, \rho)$ for some $\rho < \mu_1$.
- (4) $\mathbb{D}^{\text{small}} \in V'$ and $V' \models |\mathbb{D}^{\text{small}}| \leq \mu_n$.

Then the tree property holds at μ_{n+2} in $V'[D^{s,0,2,3}]$.

Proof. Let $W = V'[D^{s,0,2,3}]$. Let $\bar{A} = A \restriction [\mu_{n+2}, \eta)$, so that $\bar{A}$ is generic for $< \mu_{n+2}$ -directed closed forcing defined in V_{inn} . μ_{n+2} is indestructibly supercompact in V_{inn} , we will construct a supercompactness embedding j defined in $V_{\text{inn}}[D^2 \times \bar{A}]$ and lift it to such an embedding defined in $V_{\text{def}}[D^2 \times \bar{A}]$.

We can dispense with Steps 1a and 1b from the previous construction, so there is no $\mathbb{P}_1$. We choose χ suitably large and then work in $V_{\text{inn}}[D^2 \times \bar{A}]$ to choose j such that $j \restriction ON$ is defined in V_{inn} , j witnesses μ_{n+2} is χ -supercompact, the next point in $\text{dom}(j(\psi))$ past μ_{n+2} is greater than χ , and $j(\psi)(\mu_{n+2})$ is a name in V_{inn} for an $\mathbb{A} \restriction \mu_{n+2} * U \restriction \mu_{n+2}$ -name for $\mathbb{D}^3$. Then after lifting j to $V_{\text{def}}[D^2 \times \bar{A}]$, $j(\phi)(\mu_{n+2})$ is an $\mathbb{A} \restriction \mu_{n+2} * U \restriction \mu_{n+2}$ -name in V_{def} for $\mathbb{D}^3$.

$\mathbb{P}_{2a}$ is chosen as before, with the simplification that it is now just a forcing in V_{def} adding Cohen subsets to μ_n . As before $\mathbb{P}_{2a}$ is μ_{n+1} -Knaster in a robust way. $\mathbb{P}_{2b}$ is also as before, with the simplification that there is no $\mathbb{D}^1$ and so this poset is just “stretching” A_{n+1} : as before $\mathbb{P}_{2b}$ is $< \mu_{n+1}$ -distributive and μ_{n+2} -cc.

$\mathbb{P}_3$ is chosen essentially as before. $\mathbb{P}_3$ is defined in $V_{\text{def}}[A \restriction \eta * U \restriction \mu_{n+1}][D^2]$, and is $< \mu_{n+1}$ -closed in $V_{\text{def}}[A \restriction \eta * U \restriction \mu_{n+2}][S \restriction [\mu_{n+1}, \mu_{n+2}]] [D^{2,3}]$.

In the lifting argument we lift j onto $V_{\text{def}}[A \restriction \eta * U \restriction \mu_{n+2}][D^{0,2,3}]$ (like Stage 1), extend to $V_{\text{def}}[A \restriction \eta * U \restriction \mu_{n+2} * S \restriction \mu_{n+1}][D^{0,2,3}]$ (like Stage 2), extend to $V_{\text{def}}[A \restriction \eta * U \restriction \mu_{n+2} * S \restriction \mu_{n+2}][D^{0,2,3}]$ (like Stage 3), and finally extend to $V_{\text{def}}[A \restriction \eta * U \restriction \mu_{n+2} * S \restriction \mu_{n+2}][D^{s,0,2,3}]$ (like Stage 6). The argument for the tree property is essentially identical, as we still have the relevant cardinal arithmetic and all the posets $\mathbb{P}_i$ are either missing or have the same properties as before. $\square$

Remark 4.11. As we mentioned in the preamble to Lemma 4.5, there are a couple of instances where we would like to apply the Lemma but the hypotheses are not quite satisfied. To be more precise, we want to prove that μ_{n+2} has the tree property in some extension $W' = V_{\text{def}}[A * U * S][D']$ where $\mathbb{D}'$ is a product of posets which does not quite meet the hypotheses of Lemma 4.5. In this case we can sometimes use mutual genericity to our advantage.

More specifically, assume that by forcing over W' with some poset $\mathbb{P}'$, we obtain a generic embedding with domain W' and critical point μ_{n+2} . Let $T \in W'$ be a μ_{n+2} -tree, so that T has a branch $b \in W'[P']$. Assume further that E is mutually generic with P' over W' , and that our previous arguments can be adapted to show that every branch of T from $W'[E][P']$ lies in $W'[E]$. Then $b \in W'[E] \cap W'[P']$, and by the mutual genericity of E and P' we have $b \in W'$ as required.

Remark 4.12. A particular instance of the idea of Remark 4.11 can be used to handle more posets of cardinality μ_{n+1} in the setting of Lemma 4.5. It is clear that in general a forcing poset of size μ_{n+1} can destroy the tree property at μ_{n+2} , for example $\text{Coll}(\omega, \mu_{n+1})$ will always do this. In the language of Lemma 4.5, such a poset may not be a viable choice for $\mathbb{D}^0$ (insufficient chain condition) or $\mathbb{D}^1$ (insufficient distributivity).

Suppose that $\mathbb{D} \in V_{\text{def}}$ and let $W^* = W[D]$ where $W = V_{\text{def}}[A * U * S][D^{s,0,1,2,3}]$ as in Lemma 4.5. Assume that:

- (1) $|\mathbb{D}| = \mu_{n+1}$.
- (2) $\mathbb{D}$ is the projection of a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ where $\mathbb{P}$ forces that $\mathbb{Q}$ is the union of fewer than μ_{n+1} filters, and $|\mathbb{P} * \mathbb{Q}| = \mu_{n+1}$.
- (3) $\mathbb{P}_{2a}$ is μ_{n+1} -Knaster in $W[P_1 \times P_{2b} \times P_3 \times P]$.
- (4) $\mathbb{P}_{2b}$ is $< \mu_{n+1}$ -distributive in $V_{\text{def}}[A * U][S \restriction [\mu_{n+1}, \mu_\omega]][D^{1,2,3}][P]$.
- (5) $\mathbb{P}$ is $< \mu_{n+1}$ -distributive in $V_{\text{def}}[A * U][S \restriction [\mu_{n+1}, \mu_\omega]][D^{1,2,3}]$.

Then we claim that Lemma 4.5 remains true if we add $\mathbb{D}$ as a factor to the product of posets which preserves the tree property at μ_{n+2} , that is to say we claim that μ_{n+2} has the tree property in W^* . As for Lemma 4.5, the proof is followed by a picture with a legend to help the reader keep track of all the models and forcing posets.

Let T be a μ_{n+2} -tree in W^* . We define the embedding j and lift it to W in the model $W[P_{1,2,3}]$ just as in Lemma 4.5. Since $|\mathbb{D}| = \mu_{n+1}$, we may trivially lift the embedding onto $W[D] = W^*$, working in the model $W^*[P_{1,2,3}]$. As usual we obtain a branch b in $W^*[P_{1,2,3}]$.

To cope with the problem that $\mathbb{D}$ is not necessarily μ_{n+1} -cc, force over $W^*[P_{1,2,3}]$ with $\mathbb{P} * \mathbb{Q}/D$ to obtain $P * Q$ which induces D , and is mutually generic over W with $P_{1,2,3}$. Let $W^{**} = W[P * Q]$, so that $W^* \subseteq W^{**}$ and $b \in W^{**}[P_{1,2,3}] = W[P_{1,2,3} \times P * Q]$.

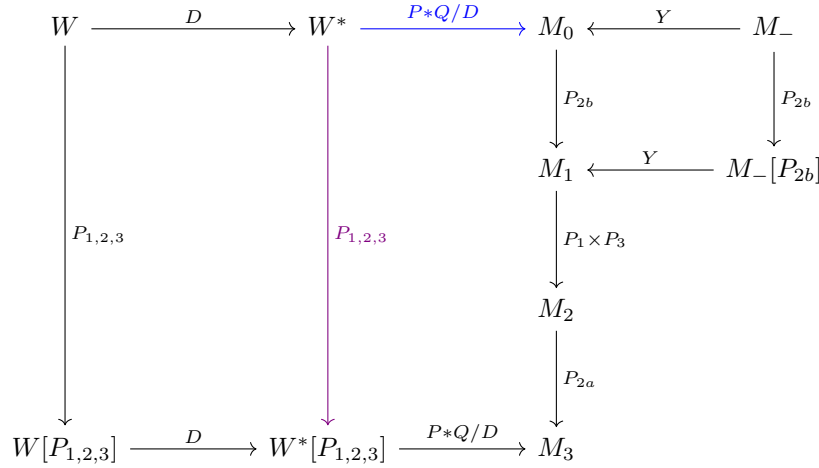
Now let $M_0 = W^{**}$, $M_1 = M_0[P_{2b}]$, $M_2 = M_1[P_1 \times P_3]$, $M_3 = M_2[P_{2a}] = W^{**}[P_{1,2,3}]$. We aim to argue that $b \in M_0$. Since $|\mathbb{P} * \mathbb{Q}| = \mu_{n+1}$, $\mathbb{P}_{2b}$ is μ_{n+2} -Knaster in M_0 , so there is no change in the step from M_1 to M_0 .

Let $M_- = V_{\text{def}}[A * U][S \restriction [\mu_{n+1}, \mu_\omega]][D^{1,2,3}][P]$. By hypothesis $\mathbb{P}$ is $< \mu_{n+1}$ -distributive in $V_{\text{def}}[A * U][S \restriction [\mu_{n+1}, \mu_\omega]][D^{1,2,3}]$, so that $\mathbb{P}_1 \times \mathbb{P}_3$ is $< \mu_{n+1}$ -closed in M_- .

By hypothesis $\mathbb{P}_{2b}$ is $< \mu_{n+1}$ -distributive in M_- . It follows that $\mathbb{P}_1 \times \mathbb{P}_3$ is $< \mu_{n+1}$ -closed in $M_-[P_{2b}]$. Finally $(\mathbb{S} \restriction \mu_{n+1} \times \mathbb{Q} \times \mathbb{D}^0) * \mathbb{D}^{\text{small}}$ is μ_{n+1} -cc in $M_-[P_{2b}]$, where

the factor $\mathbb{Q}$ causes no problems because $\mathbb{Q}$ is the union of fewer than μ_{n+1} filters, so $\mathbb{P}_1 \times \mathbb{P}_3$ is formerly closed in $M_1 = M_-[P_{2b}][(S \restriction \mu_{n+1} \times Q \times D^0) * D^s]$ and we finish the step from M_2 to M_1 as before.

Finally our hypotheses imply that $\mathbb{P}_{2a}$ is μ_{n+1} -Knaster in $W[P_1 \times P_{2b} \times P_3 \times P][Q] = M_2$, and we finish the step from M_3 to M_2 as before. It follows that $b \in W^{**} = W^*[P * Q/D]$. Since $b \in W^*[P_{1,2,3}]$ and $P * Q/D$ is mutually generic with $P_{1,2,3}$, $b \in W^*$ and we are done.



The blue arrow for $P * Q/D$ and the violet arrow for $P_{1,2,3}$ indicate mutually generic objects over W^* .

- $W = V_{\text{def}}[A * U * S][D^{0,1,2,3,s}]$.
- $W^* = W[D]$.
- $M_- = V_{\text{def}}[A * U][S \restriction [\mu_{n+1}, \mu_\omega]][D^{1,2,3}][P]$.
- $M_0 = W^{**} = W^*[P * Q/D] = W[P * Q] = V_{\text{def}}[A * U * S][D^{0,1,2,3,s}][P * Q] = M_-[Y]$, where $Y = (S \restriction \mu_{n+1} \times Q \times D^0) * D^s$.
- $M_1 = M_0[P_{2b}]$.
- $M_2 = M_1[P_1 \times P_3]$.
- $M_3 = M_2[P_{2a}] = W^{**}[P_{1,2,3}]$.

Remark 4.13. The main construction for Theorem 1.1 contains many instances of “quotient to term” posets, for instance in the definitions of $\mathbb{Q}_1(\tau, \tau^*)$ and $\mathbb{Q}_2(\tau, \tau^*)$ in Section 7.1. The role of these quotient to term posets is typically to produce generic objects which fit into one of the indestructibility schemes from the current section.

5. Initial hypotheses

We are now ready to begin the main construction. As we mentioned in the introduction, we will be introducing many pieces of notation which will have fixed meanings for the rest of the paper. Every time we introduce one or more of these important pieces of notation, we will flag it as “Global notation” and add a corresponding entry in the index of notation.

5.1. Preparing V

We start with a model V_0 with the following properties:

- (1) θ is the least supercompact cardinal.
- (2) There exist cardinals κ_α for $\alpha < \theta^+$ such that $\theta < \kappa_0$ and each κ_α is supercompact. We let $\kappa = \kappa_0$ and $\delta = \sup_{\alpha < \theta^+} \kappa_\alpha$.
- (3) There is an elementary embedding $j_0 : V_0 \rightarrow M_0$ such that j_0 witnesses that κ is δ^+ -supercompact, and in addition κ_α is supercompact in M_0 for all $\alpha < \theta^+$.
- (4) There is a universal indestructible Laver function ϕ_0 defined up to δ , in particular θ and the κ_α ’s are all indestructibly supercompact. Every element of $\text{dom}(\phi_0)$ is an inaccessible closure point of ϕ_0 .
- (5) $j_0(\phi_0) \upharpoonright \delta = \phi_0$.

Global notation: $V_0, \theta, \kappa_\alpha, \kappa, \delta, j_0, M_0, \phi_0$.

Remark 5.1. Given a model V'_0 where hypotheses 1-3 hold we may arrange that hypotheses 1-5 hold in a suitable extension V_0 of V'_0 . To see this let 1-3 hold in V'_0 where hypothesis 3 is witnessed by $j'_0 : V'_0 \rightarrow M'_0$. The main point is that in V'_0 we may choose ϕ'_0 a universal Laver function defined up to δ such that $j'_0(\phi'_0) \upharpoonright \delta = \phi'_0$: doing the corresponding Laver preparation will give a model V_0 for hypotheses 1-5.

To see that we can choose a suitable function ϕ'_0 , recall that to define $\phi'_0(\alpha)$ we choose a counterexample to $\phi'_0 \upharpoonright \alpha$ being a Laver function which is minimal with respect to some well-ordering. We will fix a well-ordering $\prec$ of V_κ such that $\prec \upharpoonright V_\alpha$ is an initial segment of $\prec$ for all $\alpha < \kappa$, and define ϕ'_0 using $\prec^* = j'_0(\prec) \upharpoonright V_\delta$: this works because $\prec^* \upharpoonright V_\kappa = \prec$ and $j'_0(\prec^*) \upharpoonright V_\delta = j'_0(\prec^* \upharpoonright V_\kappa) \upharpoonright V_\delta = j'_0(\prec) \upharpoonright V_\delta = \prec^*$.

It will be important later that all Laver functions used during the construction are derived from the initial Laver function ϕ_0 as in 2.9 and 2.6. Let E_0 be defined in V_0 as the set of inaccessible closure points of ϕ_0 in the interval (θ, δ) , and let $\alpha^* = \min(E_0 \setminus (\alpha + 1))$ for $\alpha < \delta$.

Global notation: E_0, α^* .

Our first step is to produce an extension V of V_0 in which the κ_α ’s retain the properties listed above, θ is the continuum, and θ is “generically indestructibly supercompact via Cohen reals”. To be more precise:

Lemma 5.2. *There is a generic extension V of V_0 in which:*

- $2^{\aleph_0} = \theta$.
- For every $< \theta$ -directed closed generic extension $V[H]$ and every $\gamma > \theta$, there exists a generic γ -supercompactness embedding $\pi : V[H] \rightarrow N$ with critical point θ . The embedding π exists in an extension of $V[H]$ obtained by adding $\pi(\theta)$ Cohen reals.
- The embedding π lifts an embedding $i : V_0 \rightarrow N_0$ defined in V_0 , where $\text{crit}(i) = \theta$ and i may be chosen to witness an arbitrarily high degree of supercompactness for θ in V_0 .
- There is a universal indestructible Laver function ϕ defined on (θ, δ) , in particular every supercompact cardinal up to δ is indestructible.
- There is an elementary embedding $j : V \rightarrow M$ such that j witnesses κ is δ^+ -supercompact and in addition each κ_α is supercompact in M .

Global notation: V, j, M, ϕ

Proof. The construction will be reminiscent of that of $\mathbb{A} * \dot{\mathbb{U}}$ from Section 3. This resemblance is not coincidental and will be used later, see Section 7.2. We will freely use the notation and ideas of Section 3. Objects in the current construction typically have names as in Section 3 decorated with a superscript 0

We work in V_0 . Let $\mathbb{A}^0$ be the poset of finite partial functions a from δ to 2 with $\text{dom}(a) \subseteq \theta$. Of course $\mathbb{A}^0$ is equivalent to $\text{Add}(\omega, \theta)$, and $\mathbb{A}^0 \restriction \alpha = \mathbb{A}^0$ for $\theta \leq \alpha < \delta$: defining $\mathbb{A}^0$ in this artificial way just makes the following definitions more uniform. Let A^0 be $\mathbb{A}^0$ -generic. As in Section 3 we define posets $\mathbb{B}^0 \in V$ and $\mathbb{U}^0 \in V[A^0]$ such that:

- $\mathbb{B}^0$ and $\mathbb{U}^0$ have the same set of conditions.
- The support of $\mathbb{B}^0$ consists of $\alpha < \delta$ such that $\phi_0(\alpha)$ is a pair $(\psi_0(\alpha), \psi_1(\alpha))$ with the following properties:
 - $\psi_0(\alpha)$ is an $\mathbb{A}^0 \restriction \alpha * \mathbb{U}^0 \restriction \alpha$ -name for a $< \alpha$ -directed closed forcing poset.
 - $\psi_1(\alpha)$ is an $\mathbb{A}^0 \restriction \alpha * \mathbb{U}^0 \restriction \alpha$ -name.
- An element $b \in \mathbb{B}^0$ is a function such that $\text{dom}(b)$ is an Easton subset of the support of $\mathbb{B}^0$, and $b(\alpha)$ is an $\mathbb{A}^0 \restriction \alpha * \mathbb{U}^0 \restriction \alpha$ -name for an element of $\psi_0(\alpha)$.
- For $b_0, b_1 \in \mathbb{B}^0$, $b_1 \leq b_0$ if and only if $\text{dom}(b_0) \subseteq \text{dom}(b_1)$ and $(0, b_1 \restriction \alpha) \Vdash b_1(\alpha) \leq b_0(\alpha)$ for all $\alpha \in \text{dom}(b_0)$.
- For $u_0, u_1 \in \mathbb{U}^0$, $u_1 \leq u_0$ if and only if $\text{dom}(u_0) \subseteq \text{dom}(u_1)$ and there is $a \in A^0$ such that $(a \restriction \alpha, u_1 \restriction \alpha) \Vdash u_1(\alpha) \leq u_0(\alpha)$ for all $\alpha \in \text{dom}(u_0)$.

Global notation: $\mathbb{A}^0, \mathbb{B}^0, \mathbb{U}^0$

Let $V = V_0[A^0 * U^0]$ where $A^0 * U^0$ is $\mathbb{A}^0 * \mathbb{U}^0$ -generic over V_0 . We record a few remarks:

- $a \restriction \alpha = a \restriction \min(\alpha, \theta)$ for all $a \in \mathbb{A}^0$, and similarly for $\mathbb{A}^0$ and A^0 .

- Since θ is supercompact, $\theta \notin \text{dom}(\phi_0)$.
- We can view $\mathbb{A}^0 * \mathbb{U}^0$ as a two-step iteration, forcing first with $\text{Add}(\omega, \theta) * \mathbb{U}^0 \restriction \theta$ and then with a forcing poset $\mathbb{L}^0$ defined in $V_0[A^0 * U^0 \restriction \theta]$.
- The forcing poset $\mathbb{L}^0$ is essentially a Laver preparation on the interval (θ, δ) , with the minor modification that the guessing function is just guessing names for forcing posets rather than pairs consisting of a name for a forcing poset and an ordinal: in the standard Laver preparation the role of the ordinals is to “space out” the support of the image of the preparation under supercompactness embeddings, and in our context this is handled by the properties of ϕ_0 .
- In V we have a universal indestructible Laver function ϕ on (θ, δ) , given by $\phi : \alpha \mapsto \psi_1(\alpha)[A^0 * U^0 \restriction \alpha]$.
- $2^\omega = \theta$ in V .
- $\mathbb{A}^0 * \mathbb{U}^0 \restriction \theta$ is θ -cc in V_0 .
- The poset $\mathbb{L}^0$ is θ -directed closed in $V_0[A^0 * U^0 \restriction \theta]$.

Global notation: $\mathbb{L}^0$,

The main point is to establish that θ is indestructibly generically supercompact via adding Cohen reals. Since the argument is essentially that for [16, Claim 4.12] with certain simplifications, we have relegated it to Appendix Appendix A. For use in Lemma 5.7, we note that if $\mathbb{Q} \in V$ is $< \theta$ -directed closed and our goal is generic supercompactness for θ via Cohen reals in the extension by $\mathbb{Q}$, then we lift $i : V_0 \rightarrow N_0$ where i witnesses a high degree of supercompactness for θ in V_0 , and the forcing at θ in the second coordinate of $i(\mathbb{A}^0 * \mathbb{U}^*)$ is $\mathbb{L}^0 * \mathbb{Q}$.

It remains to lift j_0 onto V , which is comparatively straightforward. Let $V_1 = V_0[A^0 * U^0 \restriction \theta]$ and $M_1 = M_0[A^0 * U^0 \restriction \theta]$, so that easily j_0 lifts to $j_1 : V_1 \rightarrow M_1$. It is easy to verify that $j_1(\mathbb{L}^0) \restriction \delta = \mathbb{L}^0$. We construct a compatible generic object $L^* \in V_1$ for $j_1(\mathbb{L}^0)$ as follows:

- $L^* \restriction \delta = L^0$.
- $L^* \restriction (\delta, j_0(\kappa))$ is constructed by counting antichains and closure.
- $L^* \restriction (j_0(\kappa), j_0(\delta))$ is constructed by counting antichains and closure, working below a master condition chosen as a lower bound for $j_0[L^0 \restriction (\kappa, \delta)]$.

Now we lift as usual to get $j : V \rightarrow M = M_1[L^*] = M_1[L^0][L^* \restriction (\delta, j(\delta))]$. Each κ_α is (indestructibly) supercompact in M by the Levy-Solovay theorem and the fact that L^* is generic for a Laver preparation over M_1 .

□

Remark 5.3. If $\lambda < \delta$ and λ is supercompact in either V or M , and α is such that $\lambda < \kappa_\alpha$, then (by the agreement between V and M) λ is κ_α -supercompact in both V and M . Since κ_α is supercompact in both V and M , λ is supercompact both in V and in M .

Remark 5.4. The reader may be wondering why we need the κ_α 's to be supercompact in M . The point is that we will eventually be doing a version of Prikry forcing at κ , so that each Prikry point τ comes with reflections of the κ_α 's which are fully supercompact. This is convenient because when τ and τ^* are successive Prikry points, so that τ^* is far above the reflections of the κ_α 's attached to τ , we need these reflections to be supercompact for a long way past τ^* .

Remark 5.5. Our starting hypotheses are consistent relative to the existence of a 2-huge cardinal, and in fact relative to the hypothesis that there is a cardinal which is both supercompact and huge, which we will show is weaker.

- Let κ be 2-huge, and fix $i : V \rightarrow N$ such that $\text{crit}(i) = \kappa$, $i(\kappa) = \lambda$, $i(\lambda) = \mu$ and ${}^\mu N \subseteq N$. Then easily $V_\mu \models \text{"}\kappa \text{ is huge with target } \lambda\text{"}$. Also κ is supercompact up to λ , by elementarity and closure λ is supercompact up to μ , so that κ is supercompact up to μ and hence $V_\mu \models \text{"}\kappa \text{ is supercompact"}$.
- Suppose now that κ is supercompact and also is huge with target λ , as witnessed by $i : V \rightarrow N$ with $\text{crit}(i) = \kappa$, $i(\kappa) = \lambda$, ${}^\lambda N \subseteq N$. By elementarity, λ is supercompact in N . By the agreement between V and N , κ is supercompact up to λ in N , so by reflection there are unboundedly many $\alpha < \kappa$ with α supercompact up to κ . Applying i , in N there are unboundedly many $\beta < \lambda$ which are supercompact up to λ . Let B be the set of such β , where since λ is supercompact in N it follows that every $\beta \in B$ is supercompact in N .

For any η with $\kappa < \eta < \lambda$, let U be the supercompactness measure on $P_\kappa \eta$ derived from i . It is easy to see that every $\beta \in B \cap \eta$ is supercompact in $\text{Ult}(V, U)$. In the universe V_λ every $\beta \in B$ is supercompact, and every $\beta \in B \cap \eta$ is supercompact in the ultrapower by U . It is now easy to get the starting hypotheses.

5.2. $\mathbb{R}_{\text{aux}}(\lambda)$ and $\mathbb{L}_{\text{aux}}(\rho, \lambda)$

We now work in the universe V constructed in the last section, and construct auxiliary posets $\mathbb{R}_{\text{aux}}(\lambda)$ and $\mathbb{L}_{\text{aux}}(\rho, \lambda)$. The subscript “aux” is to underline that we will not actually force with these posets during the main construction. Their role is to help us choose parameters for the main construction, which we will do in Section 5.3.

Let λ be a supercompact cardinal with $\lambda < \delta$. We define:

- $\lambda_0 = \lambda$.
- For $n < \omega$, λ_{n+1} is the least supercompact cardinal greater than λ_n .
- $\lambda_\omega = \sup_{n < \omega} \lambda_n$.
- $\lambda_{\omega+1} = \lambda_\omega^+$.
- For $0 < n < \omega$, $\lambda_{\omega+n+1}$ is the least supercompact cardinal greater than $\lambda_{\omega+n}$.

In V we define a poset $\mathbb{R}_{\text{aux}}(\lambda)$ to be the product of the following three posets:

- (1) $\text{East}^{E_0}(\lambda_{\omega+1}, < \lambda_{\omega+2}) \times \prod_{n < \omega} \text{East}^{E_0}(\lambda_n, < \lambda_{n+1})$. Here East is the Easton collapse defined above in Section 2.7 and E_0 is the set of inaccessible closure points in the interval (θ, δ) of our initial Laver function ϕ_0 : note that elements of E_0 are inaccessible closure points of ϕ , $\text{dom}(\phi) \subseteq E_0$, and E_0 is stationary in every supercompact cardinal up to δ .
- (2) $\text{Coll}(\lambda_{\omega+1}, < \lambda_{\omega+2}) \times \prod_{n < \omega} \text{Coll}(\lambda_n, < \lambda_{n+1})$.
- (3) $\text{Add}(\lambda_{17}, \lambda_{\omega+2}) \times \prod_{n < \omega} \text{Add}(\lambda_n, \lambda_{n+2}) \times \text{Add}(\lambda_{\omega+1}, \lambda_{\omega+2})$.

Global notation: $\lambda_i, \mathbb{R}_{\text{aux}}(\lambda)$

Recall that in V_0 the cardinal θ is supercompact, and $\phi_0 \restriction \theta$ is an indestructible Laver function for θ . We claim that there are many cardinals $\rho < \theta$ such that in V_0 the cardinal ρ is a limit of ω many inaccessible cardinals, and there is an active stage $\bar{\theta} < \rho$ of the preparation forcing from Section 5.1 such that ρ becomes an ω -successor cardinal in $V_0[A^0 \restriction \bar{\theta} * (U^0 \restriction \bar{\theta} + 1)]$. To see this let ρ' be the limit of the first ω V_0 -inaccessible cardinals greater than θ . Use the guessing property of $\phi_0 \restriction \theta$ to anticipate a suitable $< \theta$ -directed closed collapsing forcing defined in $V_0[A^0 * U^0 \restriction \theta]$ which makes ρ' into an ω -successor cardinal.

Let Index be the set of all such ρ . For each $\rho \in \text{Index}$ let $\bar{\theta}(\rho)$ be the least ordinal such that ρ is an ω -successor cardinal in $V_0[A^0 \restriction \bar{\theta}(\rho) * (U^0 \restriction \bar{\theta}(\rho) + 1)]$, let $W(\rho) = V_0[A^0 \restriction \bar{\theta}(\rho) * (U^0 \restriction \bar{\theta}(\rho) + 1)]$, and define $\mathbb{L}_{\text{aux}}(\rho, \lambda)$ to be the poset $\text{Coll}(\omega, \rho) \times \text{Coll}^{W(\rho)}(\rho^+, \lambda_1)$. It is routine to check that $\text{Coll}^{W(\rho)}(\rho^+, \lambda_1)$ is ρ -distributive in any λ -closed extension of V , a fact which will be used in the proof of Lemma 5.7.

Global notation: Index, $\bar{\theta}(\rho)$, $W(\rho)$, $\mathbb{L}_{\text{aux}}(\rho, \lambda)$

Remark 5.6. The proof of Lemma 5.7 uses ideas from unpublished work of Hayut.

Lemma 5.7. *For each supercompact cardinal λ with $\kappa < \lambda < \delta$, there is $\rho \in \text{Index}$ such that $\Vdash_{\mathbb{L}_{\text{aux}}(\rho, \lambda) \times \mathbb{R}_{\text{aux}}(\lambda)}^V$ “the tree property holds at $\lambda_{\omega+1}$ ”*

Proof. For technical reasons we will prove a slightly different (but equivalent) version of the conclusion. Let $\mathbb{R}_{\text{aux}}'(\lambda)$ be the result of replacing $\text{Add}(\lambda_{17}, \lambda_{\omega+2})$ by $\text{Add}(\lambda_{17}, \lambda_{\omega+1})$ in the product that defines $\mathbb{R}_{\text{aux}}(\lambda)$. We will show that for some ρ , $\Vdash_{\mathbb{L}_{\text{aux}}(\rho, \lambda) \times \mathbb{R}_{\text{aux}}'(\lambda)}^V$ “the tree property holds at $\lambda_{\omega+1}$ ”. This is good enough because if $H \times G$ is $\mathbb{L}_{\text{aux}}(\rho, \lambda) \times \mathbb{R}_{\text{aux}}(\lambda)$ -generic over V and $T \in V[H \times G]$ with T a $\lambda_{\omega+1}$ -tree, then by chain condition and homogeneity there is a submodel $V[H \times G']$ where $T \in V[H \times G']$ and $H \times G'$ is $\mathbb{L}_{\text{aux}}(\rho, \lambda) \times \mathbb{R}_{\text{aux}}'(\lambda)$ -generic over V .

Let G be $\mathbb{R}_{\text{aux}}'(\lambda)$ -generic over V . We will use Fact 2.58 in $V[G]$ with the parameters set as follows:

- κ_2 is θ
- κ_n is λ_{n-3} for $n > 2$.
- ν is λ_ω and ν^+ is $\lambda_{\omega+1}$.
- Index is the set Index we just defined.
- $M(\rho)$ is $\mathbb{L}_{\text{aux}}(\rho, \lambda)$.

Once we have verified that the hypotheses of Fact 2.58 hold in $V[G]$ the conclusion will be immediate. For the first hypothesis, let $n \geq 18$ and observe that $\mathbb{R}_{\text{aux}}'(\lambda)$ can be factored as $\mathbb{R}_0 \times \mathbb{R}_1$ where $\mathbb{R}_0$ is λ_{n+2} -cc and $\mathbb{R}_1$ is $< \lambda_{n+2}$ -directed closed. We decompose G accordingly as $G_0 \times G_1$. By indestructibility, there is an embedding j defined in $V[G_1]$ witnessing that λ_{n+2} is ν^+ -supercompact in that model. Forcing over $V[G_1]$ with $\mathbb{P} = j(\mathbb{R}_0)/j[G_0]$ will add a generic object which enables us to lift j onto $V[G]$.

Clearly $\mathbb{R}_0$ is a product of terms of the form $\text{Coll}(\lambda_m, < \lambda_{m+1})$, $\text{East}^{E_0}(\lambda_m, < \lambda_{m+1})$ and $\text{Add}(\lambda_m, \lambda_{m+2})$ for $m \leq n+1$, together with $\text{Add}(\lambda_{17}, \lambda_{\omega+1})$. Since many factors in $\mathbb{R}_0$ are fixed by j , the corresponding factors in $j(\mathbb{R}_0)/G_0$ are trivial. It follows that $j(\mathbb{R}_0)/G_0$ is the product of the factors $\text{Coll}(\lambda_{n+1}, [\lambda_{n+2}, j(\lambda_{n+2})])$, $\text{East}^{j(E_0)}(\lambda_{n+1}, [\lambda_{n+2}, j(\lambda_{n+2})])$, $\text{Add}(\lambda_n, j(\lambda_{n+2}) - \lambda_{n+2})$, $\text{Add}(\lambda_{n+1}, j(\lambda_{n+3}) - j[\lambda_{n+3}])$ and $\text{Add}(\lambda_{17}, j(\lambda_{\omega+1}) - j[\lambda_{\omega+1}])$.

Now we let $\mathbb{Q}$ be the product of λ_n copies of $j(\mathbb{R}_0)/G_0$ with the following supports: full support for the $< \lambda_{n+1}$ -closed components, supports of size less than λ_n for the components of form $\text{Add}(\lambda_n, j(\lambda_{n+2}) - \lambda_{n+2})$ and supports of size less than λ_{17} for the components of form $\text{Add}(\lambda_{17}, j(\lambda_{\omega+1}) - j[\lambda_{\omega+1}])$. It is routine to check that $\mathbb{Q}$ preserves cardinals up to and including λ_{n+1} and forces that $\text{cf}(\lambda_{\omega+1}) = \lambda_{n+1}$, so that $\mathbb{Q}$ is as required.

For the second hypothesis, we will use the indestructible generic supercompactness of θ in V secured by Lemma 5.2 to define a certain generic embedding, and then reflect the existence of this embedding to a well-chosen elementary substructure $X \prec R$ where R is a suitable rank initial segment of $V[G]$.

More precisely, let $\pi : V[G] \rightarrow V^*$ be a generic embedding added by the forcing poset $\text{Add}(\omega, \pi(\theta) - \theta)$ such that:

- $\text{crit}(\pi) = \theta$
- $\pi(\theta) > \lambda_{\omega+1}$
- π is discontinuous at $\lambda_{\omega+1}$.

From the proof of Lemma 5.2 we recall that

- $V = V_0[A^0 * U^0 \restriction \theta * L^0]$, so that $V[G] = V_0[A^0 * U^0 \restriction \theta * L^0 * G]$.
- π is a lift of a supercompactness embedding $i : V_0 \rightarrow N_0$ with critical point θ defined in V_0 , with the property that the forcing at coordinate θ in $i(\mathbb{U}^0)$ is $\mathbb{L}^0 * \mathbb{R}_{\text{aux}}'(\lambda)$.
- The embedding i may witness an arbitrarily high degree of supercompactness for θ in V_0 .

We claim that $\lambda_\omega \in \pi(\text{Index})$: this is easy because λ_ω is a limit of supercompact cardinals in $N_0[A^0 * U^0]$ but becomes $\lambda^{+\omega}$ in $N_0[A^0 * U^0 * L^0 * G]$. By definition $\pi(\mathbb{M})(\lambda_\omega) = \pi(\mathbb{L}_{\text{aux}})(\lambda_\omega, \pi(\lambda)) = \text{Coll}(\omega, \lambda_\omega) \times \text{Coll}^{N_0[A^0 * U^0 * L^0 * G]}(\lambda_{\omega+1}, \pi(\lambda_1))$. Recalling that $V[G] = V_0[A^0 * U^0 * L^0 * G]$ and that i can witness arbitrary levels of supercompactness, we may arrange that $\text{Coll}^{N_0[A^0 * U^0 * L^0 * G]}(\lambda_{\omega+1}, i(\lambda_1)) =$

$\text{Coll}^{V[G]}(\lambda_{\omega+1}, \pi(\lambda_1))$, in particular it is defined and λ_ω -closed in $V[G]$.

In summary, we have shown that there is a generic embedding $\pi : V[G] \rightarrow V^*$ added by $\text{Add}(\omega, \pi(\theta) - \theta)$ such that $\text{crit}(\pi) = \theta$, $\pi(\theta) > \lambda_{\omega+1}$, π is discontinuous at $\lambda_{\omega+1}$, $\lambda_\omega \in \pi(\text{Index})$ and $\pi(\mathbb{L}_{\text{aux}})(\lambda_\omega, \pi(\lambda)) = \text{Coll}(\omega, \lambda_\omega) \times \text{Coll}^{V[G]}(\lambda_{\omega+1}, \pi(\lambda_1))$. Observe that $2^{\lambda_\omega} = \lambda_{\omega+1}$ in $V[G]$ (it was for this reason that we replaced $\mathbb{R}_{\text{aux}}(\lambda)$ by $\mathbb{R}_{\text{aux}}'(\lambda)$). We choose R a long enough rank initial segment of $V[G]$ that for any algebra of finitary functions on R , we may find $X \prec R$ which has size $\lambda_{\omega+1}$, is closed under λ_ω -sequences, and reflects the statement asserting the existence of a suitable generic embedding π .

Let M be the collapse of X , and let $A \times h$ be generic over $V[G]$ for $\mathbb{P}_X = \text{Add}(\omega, \pi(\theta) - \theta) \times \text{Coll}(\omega, \lambda_\omega)$. Using A , we may define in $M[A]$ a generic embedding $\pi_X : M \rightarrow M^* \subseteq M[A]$ such that $\text{crit}(\pi_X) = \theta$, $\pi_X(\theta) > \lambda_{\omega+1}$, π_X is discontinuous at $\lambda_{\omega+1}$, $\lambda_\omega \in \pi_X(\text{Index})$ and $\pi_X(\mathbb{L}_{\text{aux}})(\lambda_\omega, \pi_X(\lambda)) = \text{Coll}(\omega, \lambda_\omega) \times \text{Coll}^M(\lambda_{\omega+1}, \pi_X(\lambda_1))$. Since $|M| = \lambda_{\omega+1}$ and ${}^{\lambda_\omega}M \subseteq M$, we may build a filter $C \in V[G]$ which is generic over M for $\text{Coll}^M(\lambda_{\omega+1}, \pi_X(\lambda_1))$.

Now $M[C] \subseteq V[G]$ and $A \times h$ is generic over $V[G]$, so $A \times h$ is generic over $M[C]$, and since $M^* \subseteq M[A]$ we see that $h \times C$ is generic over M^* . It follows that forcing over $V[G]$ with the $\lambda_{\omega+1}$ -Knaster poset $\mathbb{P}_X = \text{Add}(\omega, \pi_X(\theta) - \theta) \times \text{Coll}(\omega, \lambda_\omega)$ has added the embedding $\pi_X : M \rightarrow M^*$ and a filter $h \times C$ which is $\pi_X(\mathbb{L}_{\text{aux}})(\lambda_\omega, \pi_X(\lambda))$ -generic over M^* . Since we constructed X to be closed under an arbitrary algebra on R , there are stationarily many X and we have fulfilled the second clause in the hypotheses of Fact 2.58. $\square$

5.3. Selecting ρ

Using the fact that there are θ^+ supercompact cardinals above κ , we choose supercompact cardinals λ^a and λ^b above κ such that $\lambda_{\omega+3}^a < \lambda^b$ and the cardinals λ^a, λ^b select the same cardinal ρ from Lemma 5.7. We can assume that $(\rho, \lambda^a, \lambda^b)$ is the lexicographically least such triple with this property: recalling that $j : V \rightarrow M$ is a δ^+ -supercompactness embedding with critical point κ , we see that $(\rho, \lambda^a, \lambda^b)$ is definable from κ in M using the same definition.

Having fixed ρ , we also fix some related parameters. We set $\bar{\theta} = \bar{\theta}(\rho)$ and $\bar{W} = W(\rho)$. It follows that for $\lambda = \lambda^a, \lambda^b$ we have:

- $\mathbb{L}_{\text{aux}}(\rho, \lambda) = \text{Coll}(\omega, \rho) \times \text{Coll}^{\bar{W}}(\rho^+, \lambda_1)$.
- It is forced by $\mathbb{L}_{\text{aux}}(\rho, \lambda) \times \mathbb{R}_{\text{aux}}(\lambda)$ that the tree property holds at $\lambda_{\omega+1}$.

It follows that there is a measure one set of points τ below κ with reflected versions $\Lambda^a(\tau)$ and $\Lambda^b(\tau)$ of the cardinals λ^a and λ^b . To be more specific:

- (1) $j(\Lambda_i^x)(\kappa) = \lambda_i^x$ for $x \in \{a, b\}$.
- (2) $\theta < \tau < \Lambda^a(\tau) < \Lambda^b(\tau) < \kappa$. where $\Lambda^a(\tau)$ and $\Lambda^b(\tau)$ are supercompact.
- (3) Setting $\Lambda_i^x(\tau) = \Lambda^x(\tau)_i$ for $x \in \{a, b\}$ and $i < \omega + \omega$, $\Lambda_{\omega+3}^a(\tau) < \Lambda^b(\tau)$.
- (4) It is forced by $\mathbb{L}_{\text{aux}}(\rho, \Lambda^x(\tau)) \times \mathbb{R}_{\text{aux}}(\Lambda^x(\tau))$ that the tree property holds at $\Lambda_{\omega+1}^x(\tau)$ for $x \in \{a, b\}$.

56 *J. Cummings, Y. Hayut, M. Magidor, I. Neeman, D. Sinapova & S. Unger*

Global notation: $\lambda^a, \lambda^b, \rho, \bar{\theta}, \bar{W}, \Lambda^a(\tau), \Lambda^b(\tau), \Lambda_i^a(\tau), \Lambda_i^b(\tau)$

6. More preparation

Let Y be the set of supercompact cardinals τ less than κ which are such that $\Lambda^a(\tau)$ and $\Lambda^b(\tau)$ are defined, and are closed under the function $\sigma \mapsto \Lambda_{\omega+3}^b(\sigma)$. We define an Easton support iteration $\mathbb{L}$ which is nontrivial only at each $\tau \in Y \cup \{\kappa\}$. For ease of notation we specify the forcing at step κ and note that the forcing at τ can be obtained by replacing κ with τ and λ_i^z by $\Lambda_i^z(\tau)$ (for $z \in \{a, b\}$ and $i \in \omega + \omega$).

Global notation: $Y, \mathbb{L}$

The forcing $\mathbb{L}(\tau)$ at stage τ will be $< \Lambda_{17}^a(\tau)$ -closed. Since $2^\omega = \theta$ in V , and Y is a set of supercompact cardinals, the forcing $\mathbb{L}$ will be much more than θ^+ -closed. This will be important in Section 9.3.

Of course we define the forcing at κ in $V[L \restriction \kappa]$ where $L \restriction \kappa$ is $\mathbb{L}_\kappa$ -generic. The preparation forcing at κ will be defined in stages, and will ultimately have components $\mathbb{L}^b, \mathbb{I}^b$, and $\mathbb{A}_e \times \mathbb{J}^c$.

Global notation: $\mathbb{L}^b, \mathbb{I}^b, \mathbb{A}_e, \mathbb{J}^c$

Note that by Fact 2.8, the cardinals λ_n^b are supercompact but no longer indestructible in $V[L \restriction \kappa]$. Let $\mathbb{L}^b \in V[L \restriction \kappa]$ be a Laver preparation for the interval $(\lambda_{\omega+1}^a, \lambda_\omega^b)$, defined using the Laver function $\alpha \mapsto \phi(\alpha)[L \restriction \kappa]$ on this interval. Let L^b be $\mathbb{L}^b$ -generic over $V[L \restriction \kappa]$, and let ψ be the universal indestructible Laver function added by L^b on the interval $(\lambda_{\omega+1}^a, \lambda_\omega^b)$.

Global notation: ψ

Working in $V[(L \restriction \kappa) * L^b]$, we define posets $\mathbb{A}^b, \mathbb{B}^b, \mathbb{C}^b, \mathbb{S}^b = (\mathbb{C}^b)^{+A^b * U^b}$ following the recipe in Section 3 with the parameters set as follows:

- $\mu_0 = \lambda_{17}^a, \mu_1 = \lambda_{\omega+1}^a, \mu_2 = \lambda_{\omega+2}^a, \mu_{n+3} = \lambda_n^b$ for $n < \omega$.
- The universal indestructible Laver function is the function ψ which we just added using L^b .

Global notation: $\mathbb{A}^b, \mathbb{B}^b, \mathbb{C}^b, \mathbb{S}^b$

Remark 6.1. Since $\mathbb{L}^b$ is defined on the interval $(\lambda_{\omega+1}^a, \lambda_\omega^b)$, it is $< \lambda_{\omega+1}^a$ -closed, so that $\mathbb{A}_0^b = \text{Add}^{V[L \restriction \kappa]}(\mu_0, [\mu_1, \mu_2])$ and $\mathbb{A}_1^b = \text{Add}^{V[L \restriction \kappa]}(\mu_1, [\mu_2, \mu_3])$.

Let I^b be generic over $V[(L \restriction \kappa) * L^b]$ for $\mathbb{I}^b = \mathbb{A}^b * \mathbb{U}^b * \mathbb{S}^b$, where I^b decomposes in the obvious way as $A^b * U^b * S^b$. For the record, in $V[L \restriction \kappa * L^b * I^b]$ we have the following situation:

- The cardinals $\lambda_{\omega+1}^a, \lambda_{\omega+2}^a, \lambda_0^b, \lambda_1^b, \dots, \lambda_\omega^b, \lambda_{\omega+1}^b$ form a block of $\omega + 2$ consecutive cardinals.
- $2^{\lambda_{17}^a} = \lambda_{\omega+2}^a, 2^{\lambda_{\omega+1}^a} = \lambda_0^b, 2^{\lambda_{\omega+2}^a} = \lambda_1^b, 2^{\lambda_n^b} = \lambda_{n+2}^b$ for $n < \omega$.

Global notation: $\mathbb{I}^b, I^b, A^b, U^b, S^b$

Working over the model $V[(L \restriction \kappa) * L^b * I^b]$ (but using some Cohen posets defined in inner models of this model) we will define a poset $\mathbb{A}_e \times \mathbb{J}^c$, where $\mathbb{A}_e = \text{Add}^{V[(L \restriction \kappa) * L^b]}(\lambda_{17}^b, \lambda_{\omega+3}^b)$. We digress briefly to prove that $\mathbb{A}_e$ has reasonable chain condition and distributivity properties in $V[(L \restriction \kappa) * L^b * I^b]$. The point of defining $\mathbb{A}_e$ in the submodel $V[(L \restriction \kappa) * L^b]$ is that after forcing with $\mathbb{I}^b$ we have $2^{\lambda_{16}^b} = \lambda_{18}^b$, so that $\text{Add}^{V[(L \restriction \kappa) * L^b * I^b]}(\lambda_{17}^b, \lambda_{\omega+3}^b)$ collapses λ_{18}^b .

Lemma 6.2. $\mathbb{A}_e$ is $< \lambda_{17}^b$ -distributive and λ_{18}^b -Knaster in $V[(L \restriction \kappa) * L^b * I^b]$.

Proof. By item 7 of Lemma 3.13, we may force to expand $V[(L \restriction \kappa) * L^b * I^b][A_e]$ to $V[(L \restriction \kappa) * L^b][P_0^b \times P_1^b \times A_e]$, where $\mathbb{P}_0^b, \mathbb{P}_1^b \in V[(L \restriction \kappa) * L^b]$, with $\mathbb{P}_0^b$ a λ_{17}^b -cc initial segment of $\mathbb{I}^b$ and $\mathbb{P}_0^b$ being $< \lambda_{17}^b$ -closed in $V[(L \restriction \kappa) * L^b]$. By Easton's Lemma applied to $\mathbb{P}_0^b$ and $\mathbb{P}_1^b \times \mathbb{A}_e$, all $< \lambda_{17}^b$ -sequences of ordinals in $V[(L \restriction \kappa) * L^b * I^b][A_e]$ lie in $V[(L \restriction \kappa) * L^b][P_0^b] \subseteq V[(L \restriction \kappa) * L^b * I^b]$.

Since λ_{18}^b is supercompact in $V[L \restriction \kappa][L^b]$, it follows that in this model $\eta^{< \lambda_{17}^b} < \lambda_{18}^b$ for all $\eta < \lambda_{18}^b$. By item 2 of Lemma 3.15, $V[(L \restriction \kappa) * L^b * I^b]$ is a $(\lambda_{17}^b, \lambda_{18}^b)$ -good extension of $V[(L \restriction \kappa) * L^b]$. Appealing to Lemma 2.54, $\mathbb{A}_e$ is λ_{18}^b -cc in $V[(L \restriction \kappa) * L^b * I^b]$. $\square$

In the sequel, there will be many situations where we use Cohen conditions chosen from inner models, for example the Cohen posets $\mathbb{A}_0^c$ and $\mathbb{A}_1^c$ used below in the definition of $\mathbb{J}^c$. We generally leave the verification of the needed chain condition and distributivity properties, which can all be proved along the lines of the proof of Lemma 6.2, to the reader.

The generic functions added by $\mathbb{A}_e$ will be used below in the lifting arguments of Section 7.3. The poset $\mathbb{J}^c$ will be an initial segment of the kind of “two-phase $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction” discussed in Remark 3.23, using different cardinal parameters from the ones we used for $\mathbb{I}^b$. We first force with a poset $\mathbb{A}_0^c * \mathbb{U}_0^c * \mathbb{S}_0^c$, and then do the rest of the construction over the extension by $\mathbb{A}_0^c * \mathbb{U}_0^c * \mathbb{S}_0^c$: an important new point is that the remainder of the construction now involves $\mathbb{S}_0^c$. This will be used to get some extra closure in Lemma 7.1 below.

To define $\mathbb{J}^c$ we proceed as follows:

- $\mu_0 = \lambda_{17}^b$, $\mu_1 = \lambda_{\omega+1}^b$, $\mu_2 = \lambda_{\omega+2}^b$, $\mu_3 = \lambda_{\omega+3}^b$.
- $\mathbb{J}^c$ will have the form $(\mathbb{A}_0^c * \mathbb{U}_0^c * \mathbb{S}_0^c) * (\mathbb{A}_1^c * \mathbb{U}_1^c * \mathbb{S}_1^c)$, so all its components have supports contained in μ_3 .
- $\mathbb{A}_0^c = \text{Add}^{V[(L \restriction \kappa) * L^b]}(\mu_0, [\mu_1, \mu_2))$
- To define $\mathbb{U}_0^c$ we use a Laver function on (μ_1, μ_2) derived from ϕ as in Fact 2.9, that is the function $\alpha \mapsto \phi(\alpha)[(L \restriction \kappa) * L^b * I^b]$ defined at those α where this makes sense.
- $\mathbb{B}^c \restriction \mu_2$ and $\mathbb{U}_0^c = \mathbb{U}^c \restriction \mu_2$ are defined as in Section 3.
- As we noted in Remark 3.9, $\mathbb{S}_0^c = (\mathbb{C}_0^c)^{+A_0^c * U_0^c \restriction \mu_1} = (\mathbb{C}_0^c)^{+A_0^c}$, so that $\mathbb{A}_0^c * \mathbb{S}_0^c$ is just the standard Mitchell forcing to force that $2^{\mu_0} = \mu_2 = \mu_1^+$ and μ_2 has the tree property.

- For the definition of $\mathbb{A}_1^c * \mathbb{U}_1^c * \mathbb{S}_1^c$, we work in $V[(L \restriction \kappa) * L^b * I^b][A_0^c * U_0^c * S_0^c]$. We use the Laver function $\alpha \mapsto \phi(\alpha)[(L \restriction \kappa) * L^b * I^b][A_0^c * U_0^c * S_0^c]$ on the interval (μ_2, μ_3) . The posets $\mathbb{B}_1^c$ and $\mathbb{C}_1^c$ are defined in $V[(L \restriction \kappa) * L^b * I^b][A_0^c * U_0^c * S_0^c]$, so that for example a condition $b \in B_1^c$ has domain a subset of $[\mu_2, \mu_3]$ lying in $V[(L \restriction \kappa) * L^b * I^b][A_0^c * U_0^c * S_0^c]$, and $b(\alpha)$ is a name which lies in this model.
- $\mathbb{A}_1^c = \text{Add}^{V[(L \restriction \kappa) * L^b]}(\mu_1, [\mu_2, \mu_3])$, and we define $\mathbb{U}_1^c$ and $\mathbb{S}_1^c$ by feeding in information from A_1^c working over the model $V[(L \restriction \kappa) * L^b * I^b][A_0^c * U_0^c * S_0^c]$. In particular $\mathbb{U}_1^c$ and $\mathbb{S}_1^c$ are both defined in the model $V[(L \restriction \kappa) * L^b * I^b][A_{[0,1]}^c * U_0^c * S_0^c]$.

The last stage of the preparation forcing $\mathbb{L}$ at κ is to force with $\mathbb{A}_e \times \mathbb{J}^c$ over $V[L \restriction \kappa * L^b * I^b]$, where $\mathbb{J}^c = (\mathbb{A}_0^c * \mathbb{U}_0^c * \mathbb{S}_0^c) * (\mathbb{A}_1^c * \mathbb{U}_1^c * \mathbb{S}_1^c)$. We write the generic object as $A_e \times J^c$ where $J^c = (A_0^c * U_0^c * S_0^c) * (A_1^c * U_1^c * S_1^c)$. We note that $\mathbb{L}$ is $\lambda_{\omega+3}^b$ -cc.

Again we record some information about cardinals and cardinal arithmetic. In $V[L \restriction \kappa * L^b * I^b * (A_e \times J^c)]$ we have:

- The cardinals $\lambda_{\omega+1}^a, \lambda_{\omega+2}^a, \lambda_0^b, \lambda_1^b, \dots, \lambda_\omega^b, \lambda_{\omega+1}^b, \lambda_{\omega+2}^b, \lambda_{\omega+3}^b$ form a block of $\omega + 4$ consecutive cardinals.
- $2^{\lambda_{17}^a} = \lambda_{\omega+2}^a, 2^{\lambda_{\omega+1}^a} = \lambda_0^b, 2^{\lambda_{\omega+2}^a} = \lambda_1^b, 2^{\lambda_n^b} = \lambda_{n+2}^b$ for $n < 17$, $2^{\lambda_n^b} = \lambda_{\omega+3}^b$ for $17 \leq n \leq \omega + 2$.

Let $V[L]$ be the model obtained after forcing with $\mathbb{L}$. The generic object added by L at a stage $\tau \in Y$ is written as $L(\tau) = L^b(\tau) * I^b(\tau) * (A_e(\tau) \times J^c(\tau))$ with the obvious notation for the components of $I^b(\tau)$ and $J^c(\tau)$.

Global notation: $L(\tau), L^b(\tau), I^b(\tau), A^e(\tau), J^c(\tau)$

We will ultimately do a lifting argument to show that κ is still a large cardinal in $V[L]$. This will enable us to choose some supercompactness measures and other data, which will be ultimately be used to define the Prikry forcing $\mathbb{P}$ in Section 8. The lifting argument involves some objects introduced in Section 7.1, so we defer it until the start of Section 7.3.

Remark 6.3. It follows readily from Lemma 3.18 that $\mathbb{L}(\tau)$ is $< \Lambda_{17}^a(\tau)$ -closed in $V[L \restriction \tau]$. For the purposes of Section 9.4 we note that as a consequence all initial segments of $\mathbb{L}$ are ρ -closed in V .

7. The interleaved forcing posets

7.1. Between successive Prikry points

We now work in $V[L]$ to define the forcing $\mathbb{Q}(\tau, \tau^*)$ which the Prikry-type forcing $\mathbb{P}$ will interleave between successive Prikry points τ and τ^* . A few points to note:

- The points τ and τ^* will be elements of Y .

- The poset $\bar{\mathbb{P}}$ is defined in a certain generic extension $V[L][A^{gg}]$ of $V[L]$, but each poset $\mathbb{Q}(\tau, \tau^*)$ will actually be defined in the extension of V by a certain initial segment of L which we specify shortly.
- The filter on $\mathbb{Q}(\tau, \tau^*)$ added by forcing with $\bar{\mathbb{P}}$ will be generic over $V[L * A^{gg} * E]$, where E is the product of the finitely many generic objects added by $\mathbb{P}$ for the preceding interleaved forcing posets.
- $\mathbb{Q}(\tau, \tau^*)$ will be quite large (bigger than τ^*) and will have an effect on the universe past τ^* , and by the same token E will have an effect on the universe past τ . On the other hand $\mathbb{Q}(\tau, \tau^*)$ does not start to have an effect till some way past τ , so that if τ, τ^*, τ^{**} are successive points on the Prikry sequence then there is a large gap between the intervals where $\mathbb{Q}(\tau, \tau^*)$ and $\mathbb{Q}(\tau^*, \tau^{**})$ are each doing their work: this is crucial to later arguments, particularly in Section 9.
- $\bar{\mathbb{P}}$ will also have to act between ω and the first Prikry point. This will require special treatment, see Section 7.2 below.

Ultimately the Prikry-type forcing of Section 8 will add (mutually) generic objects over $V[L]$ for the posets $\mathbb{Q}(\tau_n, \tau_{n+1})$ where τ_n and τ_{n+1} are successive points on the Prikry sequence, together with a generic object for $\mathbb{Q}^*(\tau_0)$ where $\mathbb{Q}^*(\tau)$ is defined in Section 7.2.

Notation: In the sequel it will be convenient to have a compact notation for certain initial segments of $V[L]$. For $\tau \in Y \cup \{\kappa\}$ we will let:

- $V^l(\tau) = V[L \restriction \tau]$.
- $V^{lb}(\tau) = V[L \restriction \tau][L^b(\tau)]$.
- $V^{lbi}(\tau) = V[L \restriction \tau][L^b(\tau)][I^b(\tau)]$.

Global notation: $V^l(\tau), V^{lb}(\tau), V^{lbi}(\tau)$.

Recall that:

- Part of the the final step of the preparation at stage τ was a forcing $\mathbb{J}^c(\tau)$, which added a generic object $J^c(\tau)$ for an initial segment of the kind of two-phase construction discussed in Remark 3.23. The cardinal parameters were $\Lambda_{17}^b(\tau), \Lambda_{\omega+1}^b(\tau), \Lambda_{\omega+2}^b(\tau), \Lambda_{\omega+3}^b(\tau)$.
- One of the first steps of the preparation at stage τ^* was to add a generic object $I^b(\tau^*)$ for a version of the construction of Section 3 whose first few cardinal parameters were $\Lambda_{17}^a(\tau^*), \Lambda_{\omega+1}^a(\tau^*), \Lambda_{\omega+2}^a(\tau^*), \Lambda_0^b(\tau^*)$: this was computed in $V^{lb}(\tau^*)$. In particular we added a generic object $A_0^b(\tau^*)$ where $\mathbb{A}_0^b(\tau^*) = \text{Add}^{V^{lb}(\tau^*)}(\Lambda_{17}^a(\tau^*), [\Lambda_{\omega+1}^a(\tau^*), \Lambda_{\omega+2}^a(\tau^*)])$.
- $L^b(\tau^*)$ is generic over $V^l(\tau^*)$ for a forcing which is sufficiently closed that actually $\mathbb{A}_0^b(\tau^*) = \text{Add}^{V^l(\tau^*)}(\Lambda_{17}^a(\tau^*), [\Lambda_{\omega+1}^a(\tau^*), \Lambda_{\omega+2}^a(\tau^*)])$.

As we construct $\mathbb{Q}(\tau, \tau^*)$, we will keep track of the models in which its various components are computed. This information will be used later in Lemma 7.5. We will also keep track of some closure properties of the components. This is mostly for

use in Section 9.4, where all we will need is that certain components are ρ -closed. $\mathbb{Q}(\tau, \tau^*)$ is the product of three factors $\mathbb{Q}_i(\tau, \tau^*)$ for $i < 3$.

Global notation: $\mathbb{Q}(\tau, \tau^*)$

The first factor $\mathbb{Q}_0(\tau, \tau^*)$ completes $J^c(\tau)$ to a generic object for a certain forcing poset defined in the model $V^{lbi}(\tau)$, which we now describe:

- The forcing poset is a two-phase $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction of the type discussed in Remark 3.23, and it has the form $(\mathbb{A}_0^c(\tau) * \mathbb{U}_0^c(\tau) * \mathbb{S}_0^c(\tau)) * (\mathbb{A}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1,\omega]}^c(\tau, \tau^*))$, where $\mathbb{A}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1,\omega]}^c(\tau, \tau^*)$ is computed in the extension by $A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)$.
- The first two steps were added as the component $J^c(\tau)$ of $L(\tau)$, explicitly $J^c(\tau) = J_0^c(\tau) * J_1^c(\tau) = (A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)) * (A_1^c(\tau) * U_1^c(\tau) * S_1^c(\tau))$.
- $\mu_0 = \Lambda_{17}^b(\tau)$, $\mu_1 = \Lambda_{\omega+1}^b(\tau)$, $\mu_2 = \Lambda_{\omega+2}^b(\tau)$, $\mu_3 = \Lambda_{\omega+3}^b(\tau)$, then $\mu_{4+n} = \Lambda_n^a(\tau^*)$ for $n < \omega$.
- The forcing $\mathbb{A}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1,\omega]}^c(\tau, \tau^*)$ is computed in the model $V^{lbi}(\tau)[J_0^c(\tau)]$ with parameters set as follows:
 - $\mathbb{A}_1^c(\tau) = \text{Add}^{V^{lb}(\tau)}(\mu_1, [\mu_2, \mu_3])$.
 - $\mathbb{A}_n^c(\tau, \tau^*) = \text{Add}^V(\mu_n, [\mu_{n+1}, \mu_{n+2}])$ for $2 \leq n < \omega$.
 - We define $\mathbb{B}_{[1,\omega]}^c(\tau, \tau^*)$ and $\mathbb{C}_{[1,\omega]}^c(\tau, \tau^*)$ using the Laver function $\alpha \mapsto \phi(\alpha)[L \restriction \tau * L^b(\tau) * I^b(\tau)][A_0^c(\tau) * S_0^c(\tau) * U_0^c(\tau)]$ on the interval (μ_2, μ_ω) .
 - The supports of conditions in $\mathbb{B}_{[1,\omega]}^c(\tau, \tau^*)$ and $\mathbb{C}_{[1,\omega]}^c(\tau, \tau^*)$ are defined in $V^{lbi}(\tau)[J_0^c(\tau)]$.

Global notation: $\mathbb{Q}_0(\tau, \tau^*)$

Keeping in mind that $\mathbb{J}^c(\tau)$ has already added $J^c(\tau) = (A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)) * (A_1^c(\tau) * U_1^c(\tau) * S_1^c(\tau))$, $\mathbb{Q}_0(\tau, \tau^*)$ will add a generic object $Q_0(\tau, \tau^*)$ composed of: $\mathbb{A}_n$ -generic objects $A_n^c(\tau, \tau^*)$ for $n \geq 2$, together with generic objects $U_{[2,\omega]}^c(\tau, \tau^*)$ for $\mathbb{U}_{[2,\omega]}^c$ and $S_{[2,\omega]}^c(\tau, \tau^*)$ for $\mathbb{S}_{[2,\omega]}^c$.

The last claim in the following Lemma is similar to some closure facts from Neeman's paper [16], notably Claim 4.7, but the setting is a bit different and we give a few more details.

Lemma 7.1. $\mathbb{Q}_0(\tau, \tau^*)$ is a forcing poset of cardinality $\Lambda_{\omega+1}^a(\tau^*)$ defined in the model $V^{lbi}(\tau)[J^c(\tau)]$. $\mathbb{A}_{[2,\omega]}^c(\tau, \tau^*)$ is defined and $< \Lambda_{\omega+2}^b(\tau)$ -closed in V , and is $< \Lambda_{\omega+2}^b(\tau)$ -distributive in $V^{lbi}(\tau)[J^c(\tau)]$. $\mathbb{U}_{[2,\omega]}^c(\tau, \tau^*) * \mathbb{S}_{[2,\omega]}^c(\tau, \tau^*)$ is defined and $< \Lambda_{\omega+2}^b(\tau)$ -closed in $V^{lbi}(\tau)[J^c(\tau)][A_{[2,\omega]}^c(\tau, \tau^*)]$.

Proof. It is easy to see that $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{I}^b(\tau) * \mathbb{J}_0^c(\tau) * \mathbb{A}_1^c(\tau)$ is $\Lambda_{\omega+2}^b(\tau)$ -cc in V . In the model $V^{lbi}(\tau)[J_0^c(\tau)]$, $U_1^c(\tau) * S_1^c(\tau)$ is the projection of the $< \Lambda_{\omega+2}^b(\tau)$ -closed poset $\mathbb{B}_1^c(\tau) \times \mathbb{C}_1^c(\tau)$. So by a suitable quotient-to-term forcing we may extend $V^{lbi}(\tau)[J^c(\tau)]$ to $V^{lbi}(\tau)[J_0^c(\tau) * A_1^c(\tau) \times T]$, where T is generic for the term forcing $\mathcal{A}^V(\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{I}^b(\tau) * \mathbb{J}_0^c(\tau), \mathbb{B}_1^c(\tau) \times \mathbb{C}_1^c(\tau))$ which is $< \Lambda_{\omega+2}^b(\tau)$ -closed in V . By

a standard application of Easton's lemma, $\mathbb{A}_{[2,\omega]}^c(\tau, \tau^*)$ is $< \Lambda_{\omega+2}^b(\tau)$ -distributive in $V^{lbi}(\tau)[J^c(\tau)]$.

For the closure of $\mathbb{U}_{[2,\omega]}^c(\tau, \tau^*) * \mathbb{S}_{[2,\omega]}^c(\tau, \tau^*)$, start by noting that by Corollary 3.15 every $< \Lambda_{\omega+2}^b(\tau)$ sequence of ordinals from $V^{lbi}(\tau)[J^c(\tau)][\mathbb{A}_{[2,\omega]}^c(\tau, \tau^*)]$ lies in the submodel $V^{lbi}(\tau)[J_0^c(\tau) \times A_1^c(\tau)]$.

Let $V' = V^{lbi}(\tau)[J_0^c(\tau)]$. To lighten the notation we drop the parameters τ and τ^* , and use the “ μ_i notation” for the cardinal parameters. We will only prove closure for $\mathbb{U}_2^c$, since this proof contains all the ideas. Note that the underlying set $\mathbb{B}^c \restriction (\mu_3, \mu_4)$ of $\mathbb{U}_2^c$ lies in V' , the ordering on $\mathbb{U}_2^c$ is defined in $V'[A_{[1,2]}^c * U_1^c]$, and the relevant decreasing $< \mu_2$ -sequences from $\mathbb{U}_2^c$ lie in $V'[A_1^c]$.

We work for the moment in V' . Let $\eta < \mu_2$ and let $(\dot{b}_i)_{i < \eta}$ be a sequence of $\mathbb{A}_1^c$ -names for elements of $\mathbb{B}^c \restriction (\mu_3, \mu_4)$, where without loss of generality the trivial condition in $\mathbb{A}_{[1,2]}^c * \mathbb{U}_1^c$ forces that the b_i 's form a decreasing sequence in $\mathbb{U}_2^c$.

We will construct $b \in \mathbb{B}^c \restriction (\mu_3, \mu_4)$ which is forced to be a lower bound for the b_i 's. We let $\text{dom}(b)$ be the union over i of the possible values of $\text{dom}(b_i)$, where it is easy to see that this is an Easton subset of (μ_3, μ_4) . Suppose that $\alpha \in \text{dom}(b)$, we have defined $b \restriction \alpha$, and $b \restriction \alpha$ is forced to be a lower bound for the $b_i \restriction \alpha$'s.

Force with $\mathbb{A}^c \restriction (\mu_2, \alpha) * \mathbb{U}^c \restriction (\mu_2, \alpha)$ below the condition $(0, b \restriction \alpha)$ to obtain a generic object $F_\alpha^A * F_\alpha^U$. Let $c_i = \dot{b}_i[F_\alpha^A \restriction (\mu_2, \mu_3)] \in \mathbb{B}^c \restriction (\mu_3, \mu_4)$, and let $d_i = c_i(\alpha)[F_\alpha^A * F_\alpha^U] \in \phi(\alpha)[F_\alpha^A * F_\alpha^U]$ if $\alpha \in \text{dom}(c_i)$. We note that $\text{dom}(c_i)$ increases with i , so that either d_i is never defined or it is defined for all large $i < \eta$.

Let $i < j < \eta$ where d_i and d_j are both defined, we claim that $d_j \leq d_i$. Since it is forced that the b_i 's are decreasing in $\mathbb{U}_2^c$, there is a condition in $F_\alpha^A * F_\alpha^U$ forcing that $c_j(\alpha) \leq c_i(\alpha)$, and so $d_j \leq d_i$ as required. Since $\phi(\alpha)$ is forced to be $< \alpha$ -directed closed, we may choose $b(\alpha)$ as a name such that $(0, b \restriction \alpha)$ forces $b(\alpha) \leq \dot{d}_i$ for all i .

Now let $\bar{F} = A_1^c * A_2^c * U_1^c$ be $\mathbb{A}_{[1,2]}^c * \mathbb{U}_1^c$ -generic over V' , let $\alpha \in \text{dom}(b)$ and let $c_i = \dot{b}_i[A_1^c]$. By the induction hypothesis $b \restriction \alpha \leq c_i \restriction \alpha$ for all i in the version of $\mathbb{U}_2^c$ computed by $V'[\bar{F}]$. If $\alpha \notin \text{dom}(c_i)$ there is nothing to do, so assume that $\alpha \in \text{dom}(c_i)$ and choose a condition $(a_1, a_2, u_1) \in \bar{F}$ where $a_1 \restriction (\mu_2, \mu_3)$ forces $\dot{b}_i = \dot{c}_i$. Consider the condition $(a_1, a_2 \restriction (\mu_3, \alpha), u_1, b \restriction \alpha)$: it forces that $\dot{d}_i = c_i(\alpha)$ by the choice of a_1 , and so forces that $b(\alpha) \leq c_i(\alpha)$ because it refines $(0, b \restriction \alpha)$. So $b \restriction \alpha + 1 \leq c_i \restriction \alpha + 1$, with $(a_1, a_2 \restriction (\mu_3, \alpha), u_1)$ as the witnessing condition at coordinate α . $\square$

As we mentioned earlier in Section 6, Lemma 7.1 depends critically on the definition of $\mathbb{J}^c$ as a two-phase construction where we defined everything past stage zero using S_0^c .

Recall that $\mathbb{A}_0^b(\tau^*) = \text{Add}^{V^l(\tau^*)}(\Lambda_{17}^a(\tau^*), [\Lambda_{\omega+1}^a(\tau^*), \Lambda_{\omega+2}^a(\tau^*)])$. By the discussion in Section 2.6, we may force over $V^l(\tau^*)[A_0^b(\tau^*)]$ to produce a generic object $A_0^V(\tau^*)$ for $\text{Add}^V(\Lambda_{17}^a(\tau^*), \Lambda_{\omega+2}^a(\tau^*))$ so that $V^l(\tau^*) \subseteq V^l(\tau^*)[A_0^b(\tau^*)] \subseteq V[(L \restriction \tau^*) \times A_0^V(\tau^*)]$. $\mathbb{Q}_1(\tau, \tau^*)$ is the “quotient to term” forcing to produce $A_0^V(\tau^*)$ from $A_0^b(\tau^*)$.

Global notation: $\mathbb{Q}_1(\tau, \tau^*)$

Lemma 7.2. $\mathbb{Q}_1(\tau, \tau^*)$ is defined in $V^l(\tau^*)[A_0^b(\tau^*)]$ and has size $\Lambda_{\omega+2}^a(\tau^*)$.

Proof. Clearly the definition of $\mathbb{Q}_1(\tau, \tau^*)$ only needs $L \restriction \tau^*$ and $A_0^b(\tau^*)$, and we can compute its cardinality by counting terms. $\square$

To define $\mathbb{Q}_2(\tau, \tau^*)$, recall that at stage τ in $\mathbb{L}$ we force with $\mathbb{L}(\tau) = \mathbb{L}^b(\tau) * \mathbb{I}^b(\tau) * (\mathbb{A}_e(\tau) \times \mathbb{J}^c(\tau))$ over $V[L \restriction \tau]$. Of course $\mathbb{L} \restriction (\tau, \tau^*) \in V[L \restriction \tau + 1]$, and is easily seen to be τ^* -cc forcing of cardinality τ^* which is at least $< \Lambda_{\omega+3}^b(\tau)$ -directed closed. Working in $V^{lbi}(\tau)[J^c(\tau)]$ we can compute the term forcing $\mathbb{T}(\tau, \tau^*) = \mathcal{A}(\mathbb{A}_e(\tau), \mathbb{L} \restriction (\tau, \tau^*))$. We note that $\mathbb{T}(\tau, \tau^*)$ is a $< \Lambda_{\omega+3}^b(\tau)$ -directed closed and τ^* -cc forcing poset of cardinality τ^* . $\mathbb{Q}_2(\tau, \tau^*)$ is the quotient to term forcing (see Section 2.6) to produce a generic object $T(\tau, \tau^*)$ for the term forcing such that $A_e(\tau) \times T(\tau, \tau^*)$ induces $A_e(\tau) * L \restriction (\tau, \tau^*)$: we write $Q_2(\tau, \tau^*)$ for the generic object for $\mathbb{Q}_2(\tau, \tau^*)$.

Global notation: $\mathbb{Q}_2(\tau, \tau^*) \mathbb{T}(\tau, \tau^*)$

Lemma 7.3. $\mathbb{Q}_2(\tau, \tau^*)$ is a forcing poset of cardinality τ^* defined in $V^l(\tau^*)$.

Proof. Clearly the definition of $\mathbb{Q}_2(\tau, \tau^*)$ only needs $L \restriction \tau^*$, and the cardinality can be calculated by counting terms. $\square$

Remark 7.4. By Remark 6.3 and Lemma 2.44, $\mathbb{Q}_1(\tau, \tau^*)$ and $\mathbb{Q}_2(\tau, \tau^*)$ are both ρ -closed.

It is clear from Lemmas 7.1, 7.2 and 7.3 that $\mathbb{Q}(\tau, \tau^*)$ is a forcing poset of cardinality $\Lambda_{\omega+2}^a(\tau^*)$ defined in $V[L \restriction \tau^* + 1]$. To clarify what $\mathbb{Q}(\tau, \tau^*)$ is doing we record some information about cardinals and cardinal arithmetic after forcing with this poset. Since $\mathbb{Q}(\tau, \tau^*) \in V[L \restriction \tau^*][A_0^b(\tau^*)]$ and it has cardinality less than $\Lambda_0^b(\tau^*)$, to analyze the extension of $V[L]$ by $\mathbb{Q}(\tau, \tau^*)$ it is sufficient to analyze the extension of $V[L \restriction \tau^* + 1]$ by $\mathbb{Q}(\tau, \tau^*)$.

Recall from Section 6 that in $V[L \restriction \tau^* + 1]$:

- $\Lambda_{\omega+1}^a(\tau)$, $\Lambda_{\omega+2}^a(\tau)$, $\Lambda_0^b(\tau)$, $\Lambda_1^b(\tau)$, $\dots$, $\Lambda_\omega^b(\tau)$, $\Lambda_{\omega+1}^b(\tau)$, $\Lambda_{\omega+2}^b(\tau)$, $\Lambda_{\omega+3}^b(\tau)$ form a block of $\omega + 4$ consecutive cardinals, and similarly for τ^* .
- $2^{\Lambda_{17}^a(\tau)} = \Lambda_{\omega+2}^a(\tau)$, $2^{\Lambda_{\omega+1}^a(\tau)} = \Lambda_0^b(\tau)$, $2^{\Lambda_{\omega+2}^a(\tau)} = \Lambda_1^b(\tau)$, $2^{\Lambda_n^b(\tau)} = \Lambda_{n+2}^b(\tau)$ for $n < 17$, $2^{\Lambda_n^b(\tau)} = \Lambda_{\omega+3}^b(\tau)$ for $17 \leq n \leq \omega + 2$, and similarly for τ^* .

After forcing with $\mathbb{Q}(\tau, \tau^*)$ we have that:

- The cardinals $\Lambda_{\omega+3}^b(\tau)$, $\Lambda_0^a(\tau^*)$, $\Lambda_1^a(\tau^*)$, $\Lambda_2^a(\tau^*)$, $\dots$, $\Lambda_\omega^a(\tau^*)$ form a block of ω successive cardinals.
- $2^{\Lambda_{\omega+2}^b(\tau)} = \Lambda_0^a(\tau^*)$, $2^{\Lambda_{\omega+3}^b(\tau)} = \Lambda_1^a(\tau^*)$, $2^{\Lambda_n^a(\tau^*)} = \Lambda_{n+2}^a(\tau^*)$ for $0 \leq n < 17$.

To help analyze $\mathbb{Q}(\tau, \tau^*)$, we embed the generic extension by this poset into something more tractable. This will be useful immediately in the proof of distributivity for $\mathbb{Q}(\tau, \tau^*)$ in Lemma 7.6, and again in Section 7.3. The poset $\mathbb{Q}(\tau, \tau^*)$ is

defined in $V[L \restriction \tau^*][A_0^b(\tau^*)]$, but for our purposes we work over the slightly larger model $V[L \restriction \tau^* + 1]$.

Lemma 7.5. *Let $\tau, \tau^* \in Y$ with $\tau < \tau^*$ and let Q be $\mathbb{Q}(\tau, \tau^*)$ -generic over $V[L \restriction \tau^* + 1]$. Let $A_2^c(\tau, \tau^*)$ be the $\text{Add}^V(\Lambda_{\omega+2}^b(\tau), [\Lambda_{\omega+3}^b(\tau), \Lambda_0^a(\tau^*)])$ -generic filter added by Q as the $\mathbb{A}_2^c(\tau, \tau^*)$ -component. Let $\lambda = \Lambda_{\omega+3}^b(\tau)$. Then in some generic extension of $V[L \restriction \tau^* + 1][Q]$ there exists L' such that:*

- (1) $V[L \restriction \tau^* + 1][Q] \subseteq V[L \restriction \tau + 1 \times A_2^c(\tau, \tau^*) \times L']$.
- (2) $A_2^c(\tau, \tau^*) \times L'$ is generic for the product of $A_2^c(\tau, \tau^*)$ and some $< \lambda$ -closed forcing $\mathbb{L}'$ lying in V , where $\mathbb{L}'$ has cardinality $\Lambda_{\omega+3}^b(\tau^*)$.

Proof. Decompose Q in the natural way as $Q_0 \times Q_1 \times Q_2$. We recall that $\mathbb{Q}_0(\tau, \tau^*) \in V^{lbi}(\tau)[J^c(\tau)]$, $\mathbb{Q}_1(\tau, \tau^*) \in V^l(\tau^*)[A_0^b(\tau^*)]$, and $\mathbb{Q}_2(\tau, \tau^*) \in V^l(\tau^*) = V^{lbi}(\tau)[J^c(\tau)][A_e(\tau) * L \restriction (\tau, \tau^*)]$.

We recall also that $Q_0 = A_{[2, \omega]}^c(\tau, \tau^*) * (U_{[2, \omega]}^c(\tau, \tau^*) * S_{[2, \omega]}^c(\tau, \tau^*))$ where:

- $A_{[2, \omega]}^c(\tau, \tau^*)$ is generic for a product $\mathbb{A}_{[2, \omega]}^c(\tau, \tau^*) = \mathbb{A}_2^c(\tau, \tau^*) \times \mathbb{A}_{[3, \omega]}^c(\tau, \tau^*)$ of Cohen posets defined in V .
- $\mathbb{A}_2^c(\tau, \tau^*) = \text{Add}^V(\Lambda_{\omega+2}^b(\tau), [\Lambda_{\omega+3}^b(\tau), \Lambda_0^a(\tau^*)])$.
- $\mathbb{A}_{[3, \omega]}^c(\tau, \tau^*)$ is defined and $< \lambda$ -closed in V .

We will produce $V[L \restriction \tau + 1 \times A_2^c(\tau, \tau^*) \times L']$ from $V[L \restriction \tau^* + 1][Q]$ by a series of rearrangements and quotient to term forcings. We will be making several appeals to the Product Lemma and Lemma 2.39, but we will not make these explicit: the point is that each generic object will be generic for the forcing which originally introduced it over various larger models than the model where that forcing was originally defined.

- We may rearrange $I^b(\tau^*)$ as $A_0^b(\tau^*) * I_-^b(\tau^*)$ where $I_-^b(\tau^*)$ collects the remaining components of $I^b(\tau^*)$. Recall from Remark 6.1 that $\mathbb{A}_0^b(\tau^*) \in V[L \restriction \tau^*]$. Forcing with an appropriate series of quotient to term forcing posets we extend $V[L \restriction \tau^* + 1][Q]$ to $V[T_0][L \restriction \tau^*][A_0^b(\tau^*)][Q]$ where T_0 is generic for the product $\mathbb{T}_0$ of the following term forcings:

- $\mathcal{A}^V(\mathbb{L} \restriction \tau^*, \mathbb{L}^b(\tau^*)).$
- $\mathcal{A}^V(\mathbb{L} \restriction \tau^* * \mathbb{L}^b(\tau^*) * \mathbb{A}_0^b(\tau^*), \mathbb{I}_-^b(\tau^*)).$
- $\mathcal{A}^V(\mathbb{L} \restriction \tau^* * \mathbb{L}^b(\tau^*), \mathbb{A}_0^c(\tau^*))$
- $\mathcal{A}^V(\mathbb{L} \restriction \tau^* * \mathbb{L}^b(\tau^*), \mathbb{A}_1^c(\tau^*))$
- $\mathcal{A}^V(\mathbb{L} \restriction \tau^* * \mathbb{L}^b(\tau^*), \mathbb{A}_e(\tau^*))$
- $\mathcal{A}^V(\mathbb{L} \restriction \tau^* * \mathbb{L}^b(\tau^*) * \mathbb{I}_-^b(\tau^*) * \mathbb{A}_0^c(\tau^*), \mathbb{U}_0^c(\tau^*) * \mathbb{S}_0^c(\tau^*)).$
- $\mathcal{A}^V(\mathbb{L} \restriction \tau^* * \mathbb{L}^b(\tau^*) * \mathbb{I}_-^b(\tau^*) * \mathbb{J}_0^c(\tau^*) * \mathbb{A}_1^c(\tau^*), \mathbb{U}_1^c(\tau^*) * \mathbb{S}_1^c(\tau^*)).$

- $V[T_0][L \restriction \tau^*][A_0^b(\tau^*)][Q] = V[T_0][(L \restriction \tau^*) * A_0^b(\tau^*) * Q_1][Q_0 \times Q_2] = V[T_0][(L \restriction \tau^*) \times A_0^V(\tau^*)][Q_0 \times Q_2]$, using the definition of $\mathbb{Q}_1$ as a quotient to term forcing. It will be convenient to reorganize $A_0^V(\tau^*)$ as T_1

which is generic for $\mathbb{T}_1 = \mathcal{A}(\mathbb{L} \restriction \tau^*, \mathbb{A}_0^b(\tau^*))$, so our model becomes $V[T_0][T_1][L \restriction \tau^*][Q_0 \times Q_2]$,

- $V[T_0][T_1][L \restriction \tau^*][Q_0 \times Q_2] = V[T_0][T_1][L \restriction \tau][L^b(\tau) * I^b(\tau) * J^c(\tau)][A_e(\tau) * L \restriction (\tau, \tau^*) * Q_2][Q_0] = V[T_0][T_1][L \restriction \tau][L^b(\tau) * I^b(\tau) * J^c(\tau)][A_e(\tau) \times T(\tau, \tau^*)][Q_0] = V[T_0][T_1][L \restriction \tau][L^b(\tau) * I^b(\tau) * J^c(\tau)][T(\tau, \tau^*)][A_e(\tau)][Q_0]$. using the definition of $\mathbb{Q}_2$ as a quotient to term forcing.
- By Lemma 2.36 $\mathcal{A}^V(\mathbb{L} \restriction \tau * L^b(\tau) * I^b(\tau) * J^c(\tau), \mathbb{T}(\tau, \tau^*)) \simeq \mathcal{A}^V(\mathbb{L} \restriction \tau + 1, \mathbb{L} \restriction (\tau, \tau^*))$. So forcing with an appropriate quotient to term forcing we extend to obtain $V[T_0][T_1][T_2][L \restriction \tau][L^b(\tau) * I^b(\tau) * J^c(\tau)][A_e(\tau)][Q_0]$, where T_2 is generic for $\mathbb{T}_2 = \mathcal{A}^V(\mathbb{L} \restriction \tau + 1, \mathbb{L} \restriction (\tau, \tau^*))$.
- By the definition of $\mathbb{Q}_0(\tau, \tau^*)$,

$$\begin{aligned} V[T_0][T_1][T_2][L \restriction \tau][L^b(\tau) * I^b(\tau) * J^c(\tau)][A_e(\tau)][Q_0] = \\ V[T_0][T_1][T_2][L \restriction \tau][L^b(\tau) * I^b(\tau)][A_e(\tau)][A^c(\tau, \tau^*) * U^c(\tau, \tau^*) * S^c(\tau, \tau^*)] \end{aligned}$$

- We defined $\mathbb{A}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1, \omega]}^c(\tau, \tau^*)$ as an $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction as in Section 3, performed in the model $V[L \restriction \tau * L^b(\tau) * I^b(\tau)][A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)]$. In particular the construction involved auxiliary posets $\mathbb{B}_{[1, \omega]}^c(\tau, \tau^*)$ and $\mathbb{C}_{[1, \omega]}^c(\tau, \tau^*)$ constructed in this model. $\mathbb{C}_{[1, \omega]}^c(\tau, \tau^*)$ breaks down as $\mathbb{C}_{[1, \omega]}^c(\tau, \tau^*) = \mathbb{C}_{[1, \omega]}^c(\tau, \tau^*) \times \mathbb{C}_{[2, \omega]}^c(\tau, \tau^*)$ where $\mathbb{C}_{[2, \omega]}^c(\tau, \tau^*)$ is $< \lambda$ -closed in $V[L \restriction \tau * L^b(\tau) * I^b(\tau)][A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)]$. The forcing poset $\mathbb{S}_{[2, \omega]}^c(\tau, \tau^*)$ is defined from $\mathbb{C}_{[2, \omega]}^c(\tau, \tau^*)$ and $A_{[1, \omega]}^c(\tau, \tau^*) * U_{[1, \omega]}^c(\tau, \tau^*)$ as $(\mathbb{C}_{[2, \omega]}^c(\tau, \tau^*))^{+A_{[1, \omega]}^c(\tau, \tau^*) * U_{[1, \omega]}^c(\tau, \tau^*)}$ as in Section 3, so that in $V[L \restriction \tau * L^b(\tau) * I^b(\tau)][A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)]$ we may view $\mathbb{A}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{S}_{[2, \omega]}^c(\tau, \tau^*)$ as a projection of the product $\mathbb{A}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1, \omega]}^c(\tau, \tau^*) \times \mathbb{C}_{[2, \omega]}^c(\tau, \tau^*)$. In $V[L \restriction \tau * L^b(\tau) * I^b(\tau)][A_0^c(\tau) * S_0^c(\tau) * U_0^c(\tau)][A_1^c(\tau) * U_1^c(\tau)]$ we may view $A_{[2, \omega]}^c(\tau, \tau^*) * U_{[2, \omega]}^c(\tau, \tau^*)$ as a projection of $A_{[2, \omega]}^c(\tau, \tau^*) \times (\mathbb{B}_{[2, \omega]}^c(\tau, \tau^*))^{+A_1^c(\tau) * U_1^c(\tau)}$. By Lemma 3.5 for $\mathbb{A}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1, \omega]}^c(\tau, \tau^*)$ with $\alpha = \alpha' = \lambda$ (so that F' in the lemma is $A_1^c(\tau) * U_1^c(\tau)$), $(\mathbb{B}_{[2, \omega]}^c(\tau, \tau^*))^{+A_1^c(\tau) * U_1^c(\tau)}$ is $< \lambda$ -closed in $V[L \restriction \tau * L^b(\tau) * I^b(\tau)][A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)][A_1^c(\tau) * U_1^c(\tau)]$.
- Since $\mathbb{S}_{[1, \omega]}^c(\tau, \tau^*)$ is a product,

$$\begin{aligned} V[T_0][T_1][T_2][L \restriction \tau][L^b(\tau) * I^b(\tau)][A_e(\tau)][A^c(\tau, \tau^*) * U^c(\tau, \tau^*) * S^c(\tau, \tau^*)] \\ = V[T_0][T_1][T_2][L \restriction \tau][L^b(\tau) * I^b(\tau)][A_e(\tau)][A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)] \\ [A_{[1, \omega]}^c(\tau, \tau^*) * U_{[1, \omega]}^c(\tau, \tau^*) * S_{[2, \omega]}^c(\tau, \tau^*)][S_1^c(\tau)] \end{aligned}$$

Forcing with a suitable quotient to term forcing, we may extend to

obtain a model

$$V[T_0][T_1][T_2][L \restriction \tau][L^b(\tau) * I^b(\tau)][A_e(\tau)][A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)] \\ [A_{[1,\omega]}^c(\tau, \tau^*) * U_{[1,\omega]}^c(\tau, \tau^*) \times C_{[2,\omega]}^c(\tau, \tau^*)][S_1^c(\tau)],$$

and then reorganize as

$$V[T_0][T_1][T_2][L \restriction \tau][L^b(\tau) * I^b(\tau)][A_e(\tau)][J^c(\tau)] \\ [A_{[2,\omega]}^c(\tau, \tau^*) * U_{[2,\omega]}^c(\tau, \tau^*) \times C_{[2,\omega]}^c(\tau, \tau^*)].$$

Forcing with another quotient to term forcing, we may extend to obtain a model $V[T_0][T_1][T_2][L \restriction \tau][L^b(\tau) * I^b(\tau)][A_e(\tau)][J^c(\tau)][A_{[2,\omega]}^c(\tau, \tau^*) \times X \times C_{[2,\omega]}^c(\tau, \tau^*)]$ where X is $(\mathbb{B}_{[2,\omega]}^c(\tau, \tau^*))^{+A_1^c(\tau)*U_1^c(\tau)}$ -generic. This model may be rewritten as $V[T_0][T_1][T_2][L \restriction \tau + 1][A_{[2,\omega]}^c(\tau, \tau^*) \times X \times C_{[2,\omega]}^c(\tau, \tau^*)]$.

- Now let

$$\mathbb{T}_3 \\ = \mathcal{A}^V(\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{I}^b(\tau) * \mathbb{J}_0^c(\tau) * \mathbb{A}_1^c(\tau) * \mathbb{U}_1^c(\tau), (\mathbb{B}_{[2,\omega]}^c(\tau, \tau^*))^{+A_1^c(\tau)*U_1^c(\tau)}) \\ \times \mathcal{A}^V(\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{I}^b(\tau) * \mathbb{A}_0^c(\tau) * \mathbb{U}_0^c(\tau) * \mathbb{S}_0^c(\tau), \mathbb{C}_{[2,\omega]}^c(\tau, \tau^*)),$$

where $\mathbb{T}_3$ is $< \lambda$ -closed in V . With one more round of quotient to term forcing we may extend $V[T_0][T_1][T_2][L \restriction \tau + 1][A_{[2,\omega]}^c(\tau, \tau^*) \times X \times C_{[2,\omega]}^c(\tau, \tau^*)]$ to obtain $V[T_0][T_1][T_2][T_3][L \restriction \tau + 1][A_{[2,\omega]}^c(\tau, \tau^*)]$

We set $\mathbb{L}' = \mathbb{A}_{[3,\omega]}^c(\tau, \tau^*) \times \mathbb{T}_0 \times \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3$. It is routine to check that $\mathbb{L}'$ is $< \lambda$ -closed and has cardinality $\lambda_{\omega+3}^b(\tau^*)$. $\square$

Lemma 7.6. *The poset $\mathbb{Q}(\tau, \tau^*)$ is $< \Lambda_{\omega+2}^b(\tau)$ -distributive in $V[L]$.*

Proof. By the agreement between $V[L]$ and $V[L \restriction \tau^* + 1]$, it is enough to show that $\mathbb{Q}(\tau, \tau^*)$ is $< \Lambda_{\omega+2}^b(\tau)$ -distributive in $V[L \restriction \tau^* + 1]$. Let Q be $\mathbb{Q}(\tau, \tau^*)$ -generic over $V[L \restriction \tau^* + 1]$, then by Lemma 7.5 $V[L \restriction \tau^* + 1][Q] \subseteq V[L \restriction \tau + 1 \times A_2^c(\tau, \tau^*) \times L']$, where $A_2^c(\tau, \tau^*) \times L'$ is generic for the product of $\text{Add}^V(\Lambda_{\omega+2}^b(\tau), \Lambda_0^a(\tau^*))$ and some $< \Lambda_{\omega+3}^b(\tau)$ -closed forcing $\mathbb{L}'$ lying in V . The conclusion is now immediate by Easton's lemma. $\square$

A minor elaboration of this argument shows:

Lemma 7.7. *Let $\tau_0 < \dots < \tau_n$ with $\tau_i \in Y$ for all i . Then $\prod_{0 \leq i < n} \mathbb{Q}(\tau_i, \tau_{i+1})$ is $< \Lambda_{\omega+2}^b(\tau_0)$ -distributive in $V[L]$.*

Since $\mathbb{Q}(\tau_n, \tau_{n+1})$ is distributive over the cardinality of $\prod_{0 \leq i < n} \mathbb{Q}(\tau_i, \tau_{i+1})$, we immediately deduce:

Lemma 7.8. *Let $\tau_0 < \dots < \tau_n < \tau_{n+1}$ with $\tau_i \in Y$ for all i , and let E be $\prod_{0 \leq i < n} \mathbb{Q}(\tau_i, \tau_{i+1})$ -generic over $V[L]$. Then $\mathbb{Q}(\tau_n, \tau_{n+1})$ is $< \Lambda_{\omega+2}^b(\tau_n)$ -distributive in $V[L][E]$.*

7.2. The first Prikry point

In this section we define a forcing poset $\mathbb{Q}^*(\tau)$ which will be used in the Prikry forcing $\bar{\mathbb{P}}$ when τ is the first Prikry point.

Recall from Section 5.1 that $V = V_0[A^0 * U^0 \restriction \theta * L^0]$ where $A^0 * U^0 \restriction \theta$ forces that $2^\omega = \theta$ and makes θ indestructibly generically supercompact via Cohen reals, and L^0 is generic over $V_0[A^0 * U^0 \restriction \theta]$ for what is essentially a standard Laver indestructibility iteration in the interval (θ, δ) . Let $V_1 = V_0[A^0 * U^0 \restriction \theta]$. Recall from the discussion preceding Lemma 5.7 that there is a unique stage $\bar{\theta} < \rho$ of the preparation such that ρ is a limit of supercompact cardinals in $V_0[A^0 \restriction \bar{\theta} * U^0 \restriction \bar{\theta}]$ and ρ is an ω -successor in $V_0[A^0 \restriction \bar{\theta} * U^0 \restriction \bar{\theta} + 1]$.

Recall from Remark 6.1 that $\mathbb{A}_0^b(\tau) = \text{Add}^{V[L \restriction \tau]}(\Lambda_{17}^a(\tau), [\Lambda_{\omega+1}^a(\tau), \Lambda_{\omega+2}^a(\tau)])$ and is part of the component of $\mathbb{L}$ at stage τ . The poset $\mathbb{Q}^*(\tau)$ will ultimately be defined in $V[L \restriction \tau][A_0^b(\tau)]$ and will have three components $\mathbb{Q}_i^*(\tau)$ for $i \in \{0, 1, 2\}$.

The idea for defining $\mathbb{Q}_0^*(\tau)$ is that we view the forcing $\mathbb{A}^0 * \mathbb{U}^0 \restriction \theta * \mathbb{L}^0$ which produces V from V_0 as the first phase of a two-phase $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction, and that $\mathbb{Q}_0^*(\tau)$ is defined in V and implements the second phase. Here are the details of the two-phase construction.

- The cardinal parameters are $\mu_0 = \omega$, $\mu_1 = \rho^+$, $\mu_2 = \theta$, $\mu_{3+n} = \Lambda_n^a(\tau)$ for all $n \in \omega$.
- $\mathbb{A}_0 = \mathbb{A}^0 = \mathbb{A}^0 \restriction \theta$, $\mathbb{B}_0 = \mathbb{B}^0 \restriction \theta$ and $\mathbb{U}_0 = \mathbb{U}^0 \restriction \theta$ were already defined in V_0 , and the construction of Section 5.1 already gave us the generic object $A_0 * U_0$.
- $\mathbb{C}_0$ is also defined in V_0 as in Section 3, in particular it adds generic objects for $\text{Add}(\rho^+, 1)^{V_0[A_0 \restriction \alpha * U_0 \restriction \alpha]}$ only for α with $\mu_1 < \alpha < \theta$.
- $\mathbb{A}_1 = \text{Add}(\mu_1, [\mu_2, \mu_3])^{\bar{W}}$, where we recall that $\bar{W} = V_0[A_0 \restriction \bar{\theta} * U_0 \restriction \bar{\theta} + 1]$.
- $\mathbb{A}_n = \text{Add}(\mu_n, [\mu_{n+1}, \mu_{n+2}])^V$ for $2 \leq n < \omega$.
- The Laver function is the universal indestructible function ϕ from Lemma 5.2.
- For $n \geq 1$, $\mathbb{B}_n$ and $\mathbb{C}_n$ are defined over V . To be more precise conditions in $\mathbb{B}_n$ are functions $b \in V$ with supports which are Easton subsets of (μ_{n+1}, μ_{n+2}) , consisting of points α where the Laver function returns an $\mathbb{A} \restriction \alpha * U \restriction \alpha$ name in V for a forcing which is $< \alpha$ -directed closed in $V[A \restriction \alpha * U \restriction \alpha]$. As usual $b(\alpha)$ will name an element of this poset. The definition of $\mathbb{C}_n$ is similar.

The first component $\mathbb{Q}_0^*(\tau)$ of $\mathbb{Q}^*(\tau)$ prolongs $A_0 * U_0 * L^0$ to a generic object for this two-phase construction. The second component $\mathbb{Q}_1^*(\tau)$ is defined over $V[L \restriction \tau][A_0^b(\tau)]$ and adds $A_1^V(\tau)$ which is $\text{Add}^V(\Lambda_{17}^a(\tau), \Lambda_{\omega+2}^a(\tau))$ -generic over $V[L \restriction \tau]$

and is such that $V[L \restriction \tau] \subseteq V[L \restriction \tau][A_0^b(\tau)] \subseteq V[(L \restriction \tau) \times A_0^V(\tau)]$. The third component $\mathbb{Q}_2^*(\tau)$ is $\text{Coll}(\omega, \rho)$. We note that $\mathbb{Q}^*(\tau)$ has cardinality $\Lambda_{\omega+2}^a(\tau)$.

Remark 7.9. By contrast with $\mathbb{Q}(\tau, \tau^*)$, not all components of $\mathbb{Q}^*(\tau)$ are ρ -closed posets defined in V or some ρ -closed extension of V . For use in Section 9.4, we categorize the components of $\mathbb{Q}^*(\tau)$.

- $\mathbb{A}_{[2, \omega]}$ is ρ -closed in V .
- $\mathbb{A}_1$ is ρ -closed in $\bar{W}$ which is a proper submodel of V . By the usual arguments with Easton's lemma, it is ρ -distributive in V .
- $\mathbb{U}_{[1, \omega]} * \mathbb{S}_{[1, \omega]}$ is ρ -closed in $V[A_{[1, \omega]}]$.
- As in the case of $\mathbb{Q}(\tau, \tau^*)$, $\mathbb{Q}_1^*(\tau)$ is ρ -closed in $V[L \restriction \tau][A_0^b(\tau)]$.
- Of course, $\mathbb{Q}_2^*(\tau)$ is not even ω -distributive.

Global notation: $\mathbb{Q}^*(\tau)$, $\mathbb{Q}_0^*(\tau)$, $\mathbb{Q}_1^*(\tau)$, $\mathbb{Q}_2^*(\tau)$

7.3. Some auxiliary computations

Recall that $j : V \rightarrow M$ has critical point κ and witnesses that κ is δ^+ -supercompact. We derive a supercompactness extender E from j witnessing that κ is $< \lambda_{\omega+3}^b$ -supercompact: to be more concrete, for each η with $\kappa \leq \eta < \lambda_{\omega+3}^b$ we let W_η be the supercompactness measure on $P_\kappa \eta$ derived from j , and let E be the system of measures $\langle W_\eta : \kappa \leq \eta < \lambda_{\omega+3}^b \rangle$, with projection maps $\pi_{\eta\zeta} : P_\kappa \zeta \rightarrow P_\kappa \eta$ given by $\pi_{\eta\zeta} : x \mapsto x \cap \eta$.

Let $j_E : V \rightarrow \text{Ult}(V, E)$ be the limit ultrapower by E , so that by standard arguments $\text{crit}(j_E) = \kappa$ and $\text{Ult}(V, E)$ is closed under $< \lambda_{\omega+3}^b$ -sequences. As usual there is an elementary embedding $k_E : \text{Ult}(V, E) \rightarrow M$ such that $k_E \circ j_E = j$ and $\text{crit}(k_E) \geq \lambda_{\omega+3}^b$. Using k_E it is easy to see that $j_E(\Lambda_i^z)(\kappa) = j(\Lambda_i^z)(\kappa) = \lambda_i^z$ for $z \in \{a, b\}$ and $i < \omega + 3$.

Global notation: E , W_η , j_E

We will need to iterate the ultrapower by E , but only for two steps. To simplify the notation let $j_{01} = j_E$ and $M_1 = \text{Ult}(V, E)$. Then as usual $j_{12} : M_1 \rightarrow M_2$ is the ultrapower map computed in M_1 using the extender $j_{01}(E)$, and $j_{02} = j_{12} \circ j_{01}$. Note that by the usual chain condition argument, $V[L] \models {}^{<\lambda_{\omega+3}^b} M_1[L] \subseteq M_1[L]$.

We will use the identity $j_{01} \restriction M_1 \circ j_{01} = j_{02}$. The proof is quite easy: by the elementarity of j_E^V and the fact that j_E^V is defined in V , $j_{01}(j_{01}(x)) = j_E^V(j_E^V(x)) = j_{j_E^V(E)}^{M_1}(j_E^V(x)) = j_{12}(j_{01}(x)) = j_{02}(x)$.

Global notation: j_{01} , j_{12} , j_{02} , M_0 , M_1 , M_2

It is easy to see that:

- $\text{crit}(j_{12}) = j_{01}(\kappa) > \lambda_{\omega+3}^b$.
- $M_1 \models {}^{<j_{01}(\lambda_{\omega+3}^b)} M_2 \subseteq M_2$.

- For any function $g : \kappa \rightarrow \kappa$, $j_{02}(g)(\kappa) = j_{12}(j_{01}(g)(\kappa)) = j_{01}(g)(\kappa)$ and $j_{02}(g)(j_{01}(\kappa)) = j_{01}(j_{01}(g)(\kappa))$. In particular $j_{02}(\Lambda_i^z)(\kappa) = \lambda_i^z$, and also $j_{02}(\Lambda_i^z)(j_{01}(\kappa)) = j_{01}(\lambda_i^z)$, for $z \in \{a, b\}$ and $i < \omega + 3$.

Lemma 7.10. *Let $\mathbb{A}^{gg} = \text{Add}^V(\lambda_{\omega+2}^b, j_{01}(\lambda_0^a))$. There exists $L^* \in V[L]$ and $K \in V[L][A^{gg}]$ such that:*

- (1) L^* is $j_{01}(\mathbb{L})$ -generic over M_1 .
- (2) $j_{01}[L] \subseteq L^*$.
- (3) $L^* \restriction \kappa + 1 = L$.
- (4) *If we lift j_{01} to obtain $j_{01}^* : V[L] \rightarrow M_1^* = M_1[L^*]$, let $j_{12}^* = j_{01}^*(j_{01}^*) : M_1^* \rightarrow M_2^*$ and $j_{02}^* = j_{12}^* \circ j_{01}^*$, and define $\mathbb{Q}_\infty = \mathbb{Q}^{M_2^*}(\kappa, j_{01}(\kappa))$, then K is $\mathbb{Q}_\infty$ -generic over M_1^* .*

Proof. We start with some easy remarks:

- (1) By Easton's Lemma, $\mathbb{A}^{gg}$ is $< \lambda_{\omega+2}^b$ -distributive in $V[L]$.
- (2) Since $L^* \restriction \kappa + 1 = L$, it will follow that M_1^* is closed under $< \lambda_{\omega+3}^b$ -sequences in $V[L]$. By elementarity M_2^* will be closed under $< j_{01}(\lambda_{\omega+3}^b)$ -sequences in M_1^* , in particular M_1^* and M_2^* will agree for a long way past the rank of $\mathbb{Q}_\infty$.

We now appeal to Lemma 7.5 in the model M_2 with $\tau = \kappa$ and $\tau^* = j_{01}(\kappa)$. Using the fact that M_2 is closed under $< j_{01}(\lambda_{\omega+3}^b)$ -sequences in M_1 , we get a projection in M_1 from $\mathbb{L} \times \mathbb{A}^{gg} \times \mathbb{L}'$ to $j_{01}(\mathbb{L}) * \mathbb{Q}(\kappa, j_{01}(\kappa))$, where $\mathbb{A}^{gg} = \text{Add}^{M_2}(\lambda_{\omega+2}^b, j_{01}(\lambda_0^a)) = \text{Add}^V(\lambda_{\omega+2}^b, j_{01}(\lambda_0^a))$ and $\mathbb{L}'$ is the product of various term forcing posets. The most relevant factors in $\mathbb{L}'$ are:

- $\mathbb{L}'_0 = \mathcal{A}^{M_1}(\mathbb{L}, j_{01}(\mathbb{L}) \restriction (\kappa, j_{01}(\kappa)))$.
- $\mathbb{L}'_1 = \mathcal{A}^{M_1}(j_{01}(\mathbb{L} \restriction \kappa), j_{01}(\mathbb{L}^b))$.
- $\mathbb{L}'_2 = \mathcal{A}^{M_1}(j_{01}(\mathbb{L} \restriction \kappa), j_{01}(\mathbb{A}_0^b))$.
- $\mathbb{L}'_3 = \mathcal{A}^{M_1}(j_{01}(\mathbb{L} \restriction \kappa * \mathbb{A}_0^b), \mathbb{I}^b)$, where $\mathbb{I}^b = \mathbb{A}_0^b * \mathbb{I}_-^b$.
- $\mathbb{L}'_4 = \mathcal{A}^{M_1}(j_{01}(\mathbb{L} \restriction \kappa * \mathbb{L}^b), j_{01}(\mathbb{A}_0^c))$.
- $\mathbb{L}'_5 = \mathcal{A}^{M_1}(j_{01}(\mathbb{L} \restriction \kappa * \mathbb{L}^b), j_{01}(\mathbb{A}_1^c))$.
- $\mathbb{L}'_6 = \mathcal{A}^{M_1}(j_{01}(\mathbb{L} \restriction \kappa * \mathbb{L}^b), j_{01}(\mathbb{A}_e))$.
- $\mathbb{L}'_7 = \mathcal{A}^{M_1}(j_{01}(\mathbb{L} \restriction \kappa * \mathbb{L}^b * \mathbb{I}^b * \mathbb{A}_0^c), j_{01}(\mathbb{U}_0^c * \mathbb{S}_0^c))$.
- $\mathbb{L}'_8 = \mathcal{A}^{M_1}(j_{01}(\mathbb{L} \restriction \kappa * \mathbb{L}^b * \mathbb{I}^b * \mathbb{J}_0^c * \mathbb{A}_1^c), j_{01}(\mathbb{U}_1^c * \mathbb{S}_1^c))$.

In the proof of Lemma 7.5 $\mathbb{L}'_2$ corresponds to $\mathbb{T}_1$, $\mathbb{L}'_0$ corresponds to $\mathbb{T}_2$, and the remaining factors correspond to factors in $\mathbb{T}_0$. The projection uses $\mathbb{L}$ and the factors $\mathbb{L}'_i$ listed above in the obvious way to prolong the $\mathbb{L}$ -generic to a $j_{01}(\mathbb{L})$ -generic object.

It is straightforward to verify that the set of maximal antichains of $\mathbb{L}'$ which lie in M_1 has cardinality $\lambda_{\omega+3}^b$ in V . Since $\mathbb{L}'$ is $< \lambda_{\omega+3}^b$ -closed in M_1 , and M_1 is closed under $< \lambda_{\omega+3}^b$ -sequences in V , we may readily work in V to build L' which is $\mathbb{L}'$ -generic over M_1 , but since we will ultimately use L' to build L^* we need to

build L' more carefully. The construction will involve successively lifting j_{01} onto larger and larger initial segments of $V[L]$: to lighten the notation we will denote all the embeddings by “ j_{01} ” and resolve any ambiguity by making the domain and codomain explicit.

To start we choose $L'_0 \in V$ which is $\mathbb{L}'_0$ -generic over M_1 , and combine it with L to construct $L^*_0 \in V[L]$ which is $j_{01}(\mathbb{L}) \restriction (\kappa, j_{01}(\kappa))$ -generic over $M_1[L]$. Note that by the closure of $\mathbb{L}'_0$, $V \models {}^{<\lambda_{\omega+3}^b} M_1[L'_0] \subseteq M_1[L'_0]$. As usual we may lift to obtain $j_{01} : V[L \restriction \kappa] \rightarrow M_1[L^*_0]$. The next stage is slightly harder, because we must choose L'_1 so that it combines with $L * L^*_0$ to produce L^*_1 so that we may lift j_{01} to $V[L \restriction \kappa][L^b]$.

To this end, let H be any filter which is $j_{01}(\mathbb{L} \restriction \kappa)$ -generic over M_1 , so that we may lift to obtain $j_{01} : V[H \restriction \kappa] \rightarrow M_1[H]$. Since $|\mathbb{L}^b| < \lambda_{\omega+3}^b$, it is easy to see that if H' is the generic filter on $\mathbb{L}^b$ added by H then $j_{01}[H'] \in M_1[H]$, and $j_{01}[H']$ has a lower bound in $j_{01}(\mathbb{L}^b)$. Let $\dot{m}$ be a $j_{01}(\mathbb{L} \restriction \kappa)$ -name for such a lower bound, so that we may view $\dot{m}$ as a condition in $\mathbb{L}'_1$ and build $L'_1 \in V$ which is $\mathbb{L}'_1$ -generic over $M_1[L'_0]$ with $\dot{m} \in L'_1$. We combine L'_1 with $L * L^*_0$ to obtain $L^*_1 \in V[L]$ which is $j_{01}(\mathbb{L}^b)$ -generic over $M_1[L^*_0]$. By construction $j_{01}[L^b] \subseteq L^*_1$, so that we may lift and obtain $j_{01} : V[L \restriction \kappa * L^b] \rightarrow M_1[L^*_0 * L^*_1]$.

Similar arguments will handle the other factors of size less than $\lambda_{\omega+3}^b$, but the factors of size $\lambda_{\omega+3}^b$ will need more care because we do not have closure under $\lambda_{\omega+3}^b$ -sequences. We will handle this problem using ideas of Magidor [11].

We will only do the argument for $\mathbb{A}_e$, which has an extra twist: the arguments for $\mathbb{A}_1^c$ and $\mathbb{U}_1^c * \mathbb{S}_1^c$ are similar but simpler. Recall that $\mathbb{A}_e = \text{Add}^{V[L \restriction \kappa][L^b]}(\lambda_{17}^b, \lambda_{\omega+3}^b)$: forcing with $\mathbb{A}_e$ adds $\lambda_{\omega+3}^b$ many generic functions from λ_{17}^b to λ_{17}^b , and for $\alpha < \lambda_{\omega+3}^b$ we let f_α be the function with index α .

As we noted in the previous paragraph, $j_{01}[A_e \restriction \eta] \in M_1[j_{01}(L \restriction \kappa * L^b)]$ for all $\eta < \lambda_{\omega+3}^b$. We will use the following easy remark:

Remark 7.11. For every dense subset D of $\mathcal{A}(\mathbb{L} \restriction \kappa * \mathbb{L}^b, \mathbb{A}_e)$, there is $f : \lambda_{\omega+3}^b \rightarrow \lambda_{\omega+3}^b$ such that if γ is an inaccessible closure point of f and $\Vdash \dot{\sigma} \in \dot{\mathbb{A}}_e \restriction \gamma$, there is $\dot{\tau} \in D$ such that $\Vdash \dot{\tau} \leq \dot{\sigma}$ and $\Vdash \dot{\tau} \in \dot{\mathbb{A}}_e \restriction \gamma$.

Since $|j_{01}(\lambda_{17}^b)| = \lambda_{\omega+3}^b$, we enumerate the elements of $j_{01}(\lambda_{17}^b)$ as γ_j for $j < \lambda_{\omega+3}^b$. Let $\eta = \sup j_{01}[\lambda_{17}^b]$, and note that if $p \in \mathbb{A}_e$ then the support of $j_{01}(p)$ is contained in $j_{01}(\lambda_{\omega+3}^b) \times \eta$. We will arrange the lifting construction so that in the end $j_{01}^* : V[L] \rightarrow M_1^*$ has the property that $j_{01}^*(f_i)(\eta) = \gamma_i$ for every $i < \lambda_{\omega+3}^b$. This idea originates in unpublished work of Woodin, and was used in a construction similar to ours by Gitik and Sharon [5].

We will construct L'_2 as the upwards closure of a decreasing $\lambda_{\omega+3}^b$ -sequence in $\mathbb{L}'_2$. View $\dot{m}$ as a condition in $j_{01}(\mathbb{L} \restriction \kappa * L^b)$, and let H be an arbitrary filter which is $j_{01}(\mathbb{L} \restriction \kappa * L^b)$ -generic over M_1 and contains this condition. Let H' be the $\mathbb{L} \restriction \kappa * L^b$ -generic filter induced by H , so that $j_{01}[H'] \subseteq H$ by the choice of $\dot{m}$ and we may lift to obtain $j_{01} : V[H] \rightarrow M_1[H']$. Much as in the construction for L'_1 , we will use this embedding to define suitable conditions in $\mathbb{L}'_2$.

We will build a decreasing $\lambda_{\omega+3}^b$ -sequence of conditions in $\mathbb{L}'_2$, with the aim of generating a filter which is generic over M_1 , and induces a filter L_2^* which is compatible with j_{01}' and A_e and assigns the right values to $j_{01}^*(f_i)(\eta)$. Suppose that we have reached a stage of the construction where we built a condition $\dot{q} \in \mathbb{L}'_2$ with the following properties:

- $\Vdash \dot{q} \in j_{01}(\mathbb{A}_e \restriction \alpha)$.
- $\Vdash \dot{q} \leq j_{01}[A_e \restriction \alpha]$.
- $\Vdash \dot{q}(j_{01}(i), \eta) = \gamma_i$ for all $i < \alpha$.

Suppose that the next dense set in $\mathbb{L}'_2$ to be handled is $D \in M_1$, and note that (since E is a supercompactness extender) $D = j_{01}(d)(j_{01}[\sigma])$ for some $\sigma < \lambda_{\omega+3}^b$ and function $d \in V$ with $\text{dom}(d) = P_\kappa \sigma$. We may assume that $d(x)$ is a dense subset of $\mathcal{A}(\mathbb{L} \restriction \kappa * \mathbb{L}^b, \mathbb{A}_e)$ for all x : it is now easy to produce a function f which satisfies the conclusion of Remark 7.11 for all the dense sets $d(x)$ simultaneously.

Let $\gamma > \alpha$ be an inaccessible closure point of f . We build a name $\dot{r}$ for a condition extending $\dot{q}$ in stages, making sure that $\dot{r}$ names a condition in $j_{01}(\mathbb{A}_e \restriction \gamma)$:

- Let $\dot{r}_1$ name $q \cup \bigcup j_{01}[A_e \restriction [\alpha, \gamma]]$, so that r_1 names a lower bound for $j_{01}[A_e \restriction \gamma]$.
- Let $\dot{r}_2$ name $r_1 \cup \{(j_{01}(i), \eta, \gamma_i) : \alpha \leq i < \gamma\}$.
- Let $\dot{r} \in D$ with $\Vdash \dot{r} \leq \dot{r}_2$, where it is possible to arrange that $\Vdash \dot{r} \in j_{01}(\mathbb{A}_e \restriction \gamma)$ by the careful choice of γ .

The condition $\dot{r}$ will be the next entry in our descending chain.

By construction, if we induce L_2^* using $L * L_0^* * L_1^*$ then $j_{01}[A_e] \subseteq L_2^*$. We lift to obtain $j_{01} : V[L \restriction \kappa][L^b][A_e] \rightarrow M_1[L * L_0^* * L_1^* * L_2^*]$, where $j_{01}(f_i)(\eta) = \gamma_i$ for all i . Continuing in the same way we build the remainder of L' , induce L^* , and finally lift to get $j_{01}^* : V[L] \rightarrow M_1^* = M_1[L^*]$.

Let A^{gg} be $\mathbb{A}^{gg} = \text{Add}^V(\lambda_{\omega+2}^b, j_{01}(\lambda_0^a))$ -generic over $V[L]$, so that A^{gg} is generic over $M_1[L \times L']$. Using the projection map in M_1 from $\mathbb{L} \times \mathbb{A}^{gg} \times \mathbb{L}'$ to $j_{01}(\mathbb{L}) * \mathbb{Q}(\kappa, j_{01}(\kappa))$, we get $K \in V[L][A^{gg}]$ which is $\mathbb{Q}(\kappa, j_{01}(\kappa))$ -generic over $M_1[L^*]$. $\square$

Global notation: $j_{01}^*, M_1^*, j_{02}^*, M_2^*, \mathbb{Q}_\infty, A^{gg}, K$

Working in $V[L]$ we derive for each $n \geq 17$ a supercompactness measure U_n on $P_\kappa \lambda_n^b$ using the embedding j_{01}^* . We do some computations in $V[L]$ which will be useful when we define the Prikry forcing $\mathbb{P}$ in Section 8. For n with $17 \leq n < \omega$ let $N_n = \text{Ult}(V[L], U_n)$ and $j_n = j_{U_n}^{V[L]}$, so that $j_n : V[L] \rightarrow N_n$ and we obtain as usual a factor map $k_n : N_n \rightarrow M_1^*$ with $j_{01}^* = k_n \circ j_n$.

We will show that k_n has a very large critical point, in fact $\text{crit}(k_n) > j_{01}(\lambda_{17}^b)$. To see this observe that the range of k_n is the set of elements in M_1^* of the form $j_{01}^*(f)(j_{01}[\lambda_n^b])$ where $f \in V[L]$ and $\text{dom}(f) = P_\kappa \lambda_n^b$. If we let $f(x) = f_i(\sup(x \cap \lambda_{17}^b))$ then $j_{01}^*(f)(j_{01}[\lambda_n^b]) = j_{01}^*(f_i)(\eta) = \gamma_i$, so that easily $j_{01}(\mu_1) + 1 \subseteq \text{rge}(k_n)$ and hence $\text{crit}(k_n) > j_{01}(\lambda_{17}^b)$.

We will use the observations that since $\text{crit}(k_n) > j_{01}(\lambda_{17}^b)$ for $n \geq 17$:

- $j_{01}^*(\eta) = j_n(\eta)$ for all $\eta \leq \lambda_{17}^b$.
- $j_{01}^*(\Lambda_k^b)(\kappa) = \lambda_k^b = j_n(\Lambda_k^b)(\kappa)$ for $k \leq 17 \leq n$.

Global notation: U_n, N_n, j_n, k_n

We are interested in comparing the two-step iteration j_{02}^* defined above, and the iteration i_n where we apply j_n and then $j_n(j_{n+1})$. We use the easy equations $i_n = j_n(j_{n+1}) \circ j_n = j_n \circ j_{n+1}$ and $j_{02}^* = j_{01}^* \circ j_{01}^*$.

Global notation: i_n

Lemma 7.12. *For all $n \geq 17$, $\mathbb{Q}_\infty = i_n(\mathbb{Q})(\kappa, j_n(\kappa))$.*

Proof. We will produce a map k such that $k \circ i_n = j_{02}^*$ and $\text{crit}(k) > j_{01}(\lambda_{17}^b)$. This will suffice because $i_n(\mathbb{Q})(\kappa, j_n(\kappa))$ can be coded as a bounded subset of $i_n(\Lambda_{17}^b(j_n(\kappa)))$, and $i_n(\Lambda_{17}^b(j_n(\kappa))) = j_n(j_{n+1}(\Lambda_{17}^b))(j_n(\kappa)) = j_n(j_{n+1}(\Lambda_{17}^b)(\kappa)) = j_n(\lambda_{17}^b) = j_{01}(\lambda_{17}^b)$.

Start by applying the embedding j_n to the equation $j_{01}^* = k_{n+1} \circ j_{n+1}$, to get $j_n(j_{01}^*) = j_n(k_{n+1}) \circ j_n(j_{n+1})$. Here $j_n(j_{01}^*) : N_n \rightarrow j_n(M_1^*)$, $j_n(j_{n+1}) : N_n \rightarrow j_n(N_{n+1})$, and $j_n(k_{n+1}) : j_n(N_{n+1}) \rightarrow j_n(M_1^*)$. By elementarity $\text{crit}(j_n(k_{n+1})) > j_n(j_{01}(\lambda_{17}^b))$.

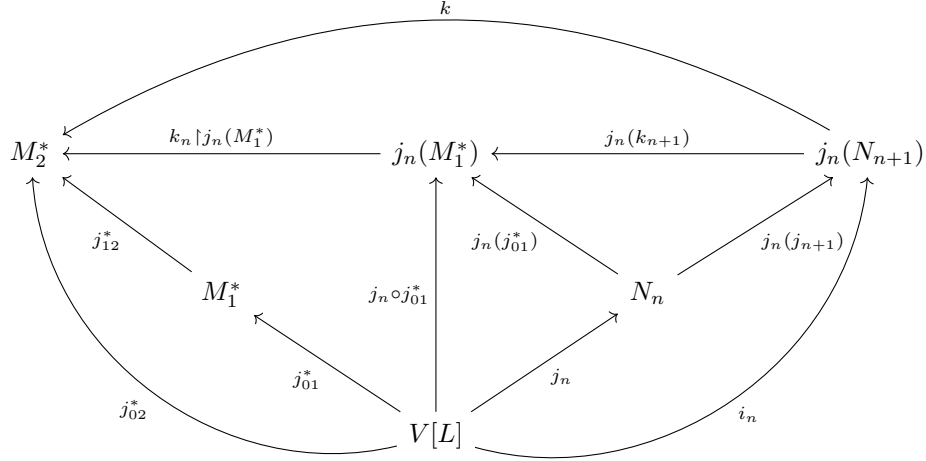
Since M_1^* is a class of $V[L]$, $j_n(M_1^*)$ is a class of N_n and we may form the restriction $k_n \upharpoonright j_n(M_1^*)$. Since $k_n \circ j_n = j_{01}^*$, it is routine to check that $k_n(j_n(M_1^*)) = j_{01}^*(M_1^*) = M_2^*$ and that $k_n \upharpoonright j_n(M_1^*) : j_n(M_1^*) \rightarrow M_2^*$ is elementary.

To finish, we set $k = k_n \circ j_n(k_{n+1})$. To confirm this works, recall first that $j_n(j_{01}^*) = j_n(k_{n+1}) \circ j_n(j_{n+1})$. Now

$$\begin{aligned}
 k \circ i_n &= k_n \circ j_n(k_{n+1}) \circ j_n(j_{n+1}) \circ j_n \\
 &= k_n \circ j_n(j_{01}^*) \circ j_n \\
 &= k_n \circ j_n \circ j_{01}^* \\
 &= j_{01}^* \circ j_{01}^* \\
 &= j_{02}^*
 \end{aligned}$$

where the first equation holds because $k = k_n \circ j_n(k_{n+1})$ and $i_n = j_n(j_{n+1}) \circ j_n$, the second equation holds because $j_n(j_{01}^*) = j_n(k_{n+1}) \circ j_n(j_{n+1})$, the third equation holds because $j_n \circ j_{01}^* = j_n(j_{01}^*) \circ j_n$, the fourth equation holds because $k_n \circ j_n = j_{01}^*$ and the last equation holds because $j_{02}^* = j_{01}^* \circ j_{01}^*$.

As for the critical point, $\text{crit}(k_n) > j_{01}(\lambda_{17}^b)$ for all n , so that $\text{crit}(j_n(k_{n+1})) = j_n(\text{crit}(k_{n+1})) > j_{01}(\lambda_{17}^b)$, and since $k = k_n \circ j_n(k_{n+1})$ we have that $\text{crit}(k) > j_{01}(\lambda_{17}^b)$.



□

The two-step iteration i_n can be viewed as a one-step ultrapower by the measure $U_n \times U_{n+1}$ on $P_\kappa \lambda_n^b \times P_\kappa \lambda_{n+1}^b$, where $A \in U_n \times U_{n+1}$ if and only if $\{x : \{y : (x, y) \in A\} \in U_{n+1}\} \in U_n$. We define product measures $U_n \times U_{n+1} \times \dots \times U_{n+i-1}$ with i factors in a similar way. The following Lemma is an immediate consequence of the normality of the measures U_j .

Lemma 7.13. *$A \in U_n \times U_{n+1} \times \dots \times U_{n+i-1}$ if and only if there exist sets $A_j \in U_j$ for $n \leq j < n+i$ such that every $\prec$ -increasing sequence from $\prod_{n \leq j < n+i} A_i$ lies in A .*

Motivated by the $i = 2$ case of Lemma 7.13 we define a modified version of the Cartesian product.

Definition 7.14. *Let $A \subseteq P_\kappa \lambda_n^b$ and $B \subseteq P_\kappa \lambda_{n+1}^b$, then $A \times^\prec B = \{(x, y) \in A \times B : x \prec y\}$.*

With this definition, the $i = 2$ case of Lemma 7.13 states that $U_n \times U_{n+1} = \{X \subseteq P_\kappa \lambda_n^b \times P_\kappa \lambda_{n+1}^b : \exists A \in U_n \exists B \in U_{n+1} A \times^\prec B \subseteq X\}$.

Global notation: $\times^\prec$

We will need the following version of Rowbottom's theorem, which also follows easily from the normality of the measures U_j .

Lemma 7.15. *Let $m < n < \omega$, let $(A_j)_{m \leq j < n}$ be a sequence of sets with $A_j \in U_j$ and let F be a coloring of the $\prec$ -increasing sequences from $\prod_{m \leq j < n} A_j$ in fewer than κ colors. Then there exists a sequence $(B_j)_{m \leq j < n}$ with $B_j \subseteq A_j$ and $B_j \in U_j$ such that $F \restriction \prod_{m \leq j < n} B_j$ is constant.*

Now that we have defined j_{01}^* , we can define an auxiliary poset that will be useful in Section 8. Recall that Y is a measure one set of cardinals which are less than κ and reflect some properties of κ , which we can think of as the “potential Prikry points”. By the agreement between j_{01}^* and j_n , $j_{01}^*(\mathbb{Q})(\alpha, \kappa) = j_n(\mathbb{Q})(\alpha, \kappa)$ for $\alpha \in Y$. In a mild abuse of notation, we will write $\mathbb{Q}(\alpha, \kappa)$ for this poset.

The following Lemma is immediate from Lemmas 7.6 and 7.7, together with the elementarity of j_{01}^* and the agreement between $V[L]$ and M_{01}^* .

Lemma 7.16. *Let $\tau_0 < \dots < \tau_n$ with $\tau_i \in Y$ for all i . Then $\prod_{0 \leq i < n} \mathbb{Q}(\tau_i, \tau_{i+1}) \times \mathbb{Q}(\tau_n, \kappa)$ is $< \Lambda_{\omega+2}^b(\tau_0)$ -distributive. In particular, in the case $n = 0$, $\mathbb{Q}(\tau_0, \kappa)$ is $< \Lambda_{\omega+2}^b(\tau_0)$ -distributive.*

Global notation: $\mathbb{Q}(\alpha, \kappa)$

8. Prikry forcing

Let $\mathbb{A}^{gg} = \text{Add}^V(\lambda_{\omega+2}^b, j_{01}(\lambda_0^g))$, let A^{gg} be $\mathbb{A}^{gg}$ -generic over $V[L]$, and let $K \in V[L][A^{gg}]$ be the $\mathbb{Q}_\infty$ -generic filter over M_1^* constructed in Section 7.3. Working in $V[L][A^{gg}]$ we will define a Prikry-type forcing $\bar{\mathbb{P}}$. Conditions in $\bar{\mathbb{P}}$ will each lie in $V[L]$, but K will be required to recognize the set of conditions, so that $\bar{\mathbb{P}} \in V[L][A^{gg}]$. Since A^{gg} is generic over $V[L]$ for $< \lambda_{\omega+2}^b$ -distributive forcing, the models $V[L]$ and $V[L][A^{gg}]$ agree on bounded subsets of $\lambda_{\omega+2}^b$. We will use this agreement without comment at several points below.

8.1. Defining the forcing

The definition of $\bar{\mathbb{P}}$ will use the measures U_n for $n \geq 17$. A typical point for U_n is a set $x \in P_\kappa \lambda_n^b$ with $\kappa(x) = x \cap \kappa \in Y$. In a mild abuse of notation we write (for example) “ $\Lambda_n^b(x)$ ” as a shorthand for “ $\Lambda_n^b(\kappa(x))$ ”.

The poset $\bar{\mathbb{P}}$ will add a sequence $\langle x_n : 17 \leq n < \omega \rangle$ where:

- $x_n \in P_\kappa \lambda_n^b$.
- $\kappa(x_n) \in Y$.
- The sequence is $\prec$ -increasing, that is $x_n \subseteq x_{n+1}$ and $\text{ot}(x_n) < \kappa(x_{n+1})$.

We call the x_n ’s the “supercompact Prikry points”, and the associated cardinals $\kappa(x_n)$ “the Prikry points”.

When x and y are successive supercompact Prikry points, the forcing poset $\bar{\mathbb{P}}$ will add a generic object for the poset $\mathbb{Q}(\kappa(x), \kappa(y))$ as defined in Section 7.1. Recall from Section 7.1 that since $\kappa(x), \kappa(y) \in Y$ the preparation forcing $\mathbb{L}$ did some collapsing in a block of cardinals associated with $\kappa(x)$, and some more collapsing at a higher block of cardinals associated with $\kappa(y)$: the point of forcing with $\mathbb{Q}(\kappa(x), \kappa(y))$ is to “close the gap” between these two blocks of cardinals. When x is the first supercompact Prikry point $\bar{\mathbb{P}}$ will add a generic object for the poset $\mathbb{Q}^*(\kappa(x))$ as

defined in Section 7.2. In the sequel we will lighten the notation by writing “ $\mathbb{Q}(x, y)$ ” for $\mathbb{Q}(\kappa(x), \kappa(y))$ and “ $\mathbb{Q}^*(x)$ ” for $\mathbb{Q}^*(\kappa(x))$.

Global notation: $\mathbb{Q}(x, y)$, $\mathbb{Q}^*(x)$

Conditions in $\bar{\mathbb{P}}$ have the form

$$p = \langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1}, f_n, A_n, F_{n+1}, A_{n+1}, F_{n+2}, A_{n+2}, \dots \rangle$$

where:

- (1) $n \geq 17$ (so that for $n = 17$ the condition p is of the form $\langle f_{17}, A_{17}, F_{18}, \dots \rangle$).
- (2) $A_j \in U_j$ for all $j \geq n$.
- (3) For all $i \geq n + 1$, F_i is a function with domain $A_{i-1} \times^\prec A_i$, such that $F_i(x, y) \in \mathbb{Q}(x, y)$ for all $(x, y) \in A_{i-1} \times^\prec A_i$ and $[F_i]_{U_{i-1} \times U_i} \in K$.
- (4) $\langle x_i \mid 17 \leq i < n \rangle$ is a $\prec$ -increasing sequence where $x_i \in \mathcal{P}_\kappa(\lambda_i^b)$ and $\kappa(x_i) \in Y$.
- (5) If $n > 17$, then
 - (a) For all $m \geq n$ and all $y \in A_m$, $x_{n-1} \prec y$.
 - (b) $q_{17} \in \mathbb{Q}^*(x_{17})$.
 - (c) $q_i \in \mathbb{Q}(x_{i-1}, x_i)$ for all i with $17 < i < n$.
 - (d) $\text{dom}(f_n) = A_n$ and $f_n(x) \in \mathbb{Q}(x_{n-1}, x)$ for all $x \in A_n$.
- (6) If $n = 17$, then f_n is a function with $\text{dom}(f_n) = A_n$ such that $f_n(x) \in \mathbb{Q}^*(x)$ for all $x \in A_n$.

Global notation: $\bar{\mathbb{P}}$

The *length* $\text{lh}(p)$ of p is 1 plus the index of the last x_i entry in p , so that $\text{lh}(p) = n$ for the condition displayed above. Note that the length of a condition is the index of the measure one set from which the next “ x point” will be drawn when the condition is extended.

For p as above, the *lower part* of p is the initial segment

$$\langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1} \rangle$$

and the *stem* (written $\text{stem}(p)$) of the condition p is

$$\langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1}, [f_n]_{U_n} \rangle$$

The *length* $\text{lh}(h)$ of a stem h is the length of the corresponding condition, so that the stem displayed above has length n . The *upper part or constraint part* is

$$\langle f_n, A_n, F_{n+1}, A_{n+1}, F_{n+2}, \dots \rangle.$$

Global notation: $\text{lh}(p)$, $\text{stem}(p)$

Remark 8.1. The point of distinguishing stems and lower parts is that the function f_n can be a source of incompatibility between two conditions of the same length.

Remark 8.2. Since $|\mathbb{Q}(\tau, \tau^*)|, |\mathbb{Q}^*(\tau^*)| < \Lambda_0^b(\tau^*)$, there are fewer than λ_0^b possibilities for $[f_n]_{U_n}$. Since $(\lambda_n^b)^{<\kappa} = \lambda_n^b$ for all n , it follows that there are λ_n^b possible stems for conditions of length $n + 1$.

Suppose that

$$p' = \langle q'_{17}, x'_{17}, \dots, q'_{m-1}, x'_{m-1}, f'_m, A'_m F'_{m+1}, A'_{m+1}, \dots \rangle$$

is another condition. Then $p' \leq p$ if:

- (1) $m \geq n$.
- (2) $\vec{x}'$ end-extends $\vec{x}$, that is to say $x_i = x'_i$ for $17 \leq i < n$.
- (3) For all i such that $n \leq i < m$, $x'_i \in A_i$.
- (4) For all $i \geq m$, $A'_i \subseteq A_i$.
- (5) If $m > n$, then
 - (a) $q'_n \leq f_n(x_n)$,
 - (b) for all i such that $n < i < m$, $q'_i \leq F_i(x_{i-1}, x_i)$ and
 - (c) for all $x \in A'_m$, $f'_m(x) \leq F_m(x'_{m-1}, x)$.
- (6) If $m = n$, then for all $x \in A'_m$, $f'_m(x) \leq f_m(x)$.
- (7) For all $i < n$, $q'_i \leq q_i$.
- (8) For all $i \geq m+1$ and all $(x, y) \in A'_{i-1} \times^< A'_i$, $F'_i(x, y) \leq F_i(x, y)$.

Remark 8.3. Since the definition of $\bar{\mathbb{P}}$ includes the demands that $A_i \in U_i$ and $[F_i] \in K$, incompatibility between conditions of the same length can only arise from the stems.

In the case when $q \leq p$ with $\text{lh}(q) = \text{lh}(p)$ we say that q is a *direct extension* of p and write $q \leq^* p$. When $\text{lh}(q) - \text{lh}(p) = t$ we say that q is a *t-step extension* of p . As is typical for Prikry-type forcing posets, when $q \leq p$ we may view q as obtained by first adding the points x_i for $\text{lh}(p) \leq i < \text{lh}(q)$, and then taking a direct extension of the result.

More formally:

Definition 8.4. *Let*

$$p = \langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1}, f_n, A_n, F_{n+1}, A_{n+1}, F_{n+2}, A_{n+2}, \dots \rangle$$

and let $\vec{x} = (x_n, \dots, x_{n+t-1})$ be a $\prec$ -increasing non-empty sequence such that $x_j \in A_j$ for $n \leq j < n+t$ and $x_{n-1} \prec x_n$. Then $p \frown \vec{x}$ (the minimal extension of p by $\vec{x}$) is the condition

$$\langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1}, q_n, x_n, \dots, q_{n+t-1}, x_{n+t-1}, f_{n+t}, A_{n+t}^*, F_{n+t+1}^*, A_{n+t+1}^*, \dots \rangle$$

where $q_n = f_n(x_n)$, $q_{n+k} = F_{n+k}(x_{n+k-1}, x_{n+k})$ for $0 < k < t$, $A_{n+k}^* = \{y \in A_{n+k} : x_{n+t-1} \prec y\}$ for $k \geq t$, $F_{n+k}^* = F_{n+k} \upharpoonright A_{n+k-1}^* \times A_{n+k}^*$ for $k > t$, and $f_{n+t}(y) = F_{n+t}(x_{n+t-1}, y)$ for $y \in A_{n+t}^*$.

Global notation: $p \frown \vec{x}$

By convention $p \frown \vec{x} = p$ for $\vec{x}$ empty, and we abuse notation by writing $p \frown x$ for $p \frown \langle x \rangle$ for sequences of length one. The following Lemma is routine:

Lemma 8.5. $p \frown \vec{x} \leq p$, and if $q \leq p$ then there is a unique $\vec{x}$ such that $q \leq^* p \frown \vec{x}$.

Lemma 8.6. Let $p, q \in \bar{\mathbb{P}}$ with $\text{stem}(p) = \text{stem}(q) = h$. Then there is a lower bound $r \leq p, q$ with $\text{stem}(r) = h$.

Proof. Let the common length of p and q be n . We choose the lower part of r to agree with the common lower part of p and q . The main point is that $[f_n^p]_{U_n} = [f_n^q]_{U_n}$, so that $\{x : f_n^p(x) = f_n^q(x)\} \in U_n$. We may therefore choose f_n^r such that $f_n^r(x) = f_n^p(x) = f_n^q(x)$ for all $x \in \text{dom}(f_n^r)$. It is now easy to choose the remaining entries of r to ensure that $r \leq p, q$. $\square$

Lemma 8.7. In $V[L][A^{gg}]$ the poset $\bar{\mathbb{P}}$ is λ_ω^b -centered, in particular it has the $(\lambda_\omega^b)^+$ -cc.

Proof. It follows from Remark 8.2 that the total number of stems is λ_ω^b . The conclusion is now immediate from Lemma 8.6. $\square$

Essentially the same proof as for Lemma 8.6 shows:

Lemma 8.8. Let h be a stem, let $\nu < \kappa$, and let $p_i \in \bar{\mathbb{P}}$ for $i < \nu$, with $\text{stem}(p_i) = h$ for all i . Then there is r such that $\text{stem}(r) = h$ and $r \leq p_i$ for all i .

Remark 8.9. We only need Lemma 8.8 in the case where $\nu = \omega$. It will be used to verify Hypothesis 6 when we appeal to Lemma 2.21.

We define $\mathbb{P}$ to be the set of p which satisfy all the conditions for membership in $\bar{\mathbb{P}}$, except the condition that $[F_i]_{U_{i-1} \times U_i} \in K$. Note that $\mathbb{P} \in V[L]$. We can view $\mathbb{P}$ as the set of potential elements of $\bar{\mathbb{P}}$.

It will be convenient to factor the forcing poset $\bar{\mathbb{P}} \downarrow p$ for $p \in \bar{\mathbb{P}}$ in various ways. Let

$$p = \langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1}, f_n, A_n, F_{n+1}, A_{n+1}, F_{n+2}, \dots \rangle$$

Let $\tau_j = \kappa(x_j)$ for $17 \leq j < n$, and let $17 \leq m < n-1$. Then $\bar{\mathbb{P}}$ below p is isomorphic to $\mathbb{P}_{\text{low}} \downarrow p_0 \times \mathbb{P}_{\text{high}} \downarrow p_1$ where:

- (1) $\mathbb{P}_{\text{low}} = \mathbb{Q}^*(\tau_{17}) \times \prod_{17 < j \leq m} \mathbb{Q}(x_{j-1}, x_j)$.
- (2) $p_0 = \langle q_{17}, \dots, q_m \rangle$.
- (3) $\mathbb{P}_{\text{high}}$ is defined in a similar way to $\bar{\mathbb{P}}$, with conditions of the form

$$\langle q'_{m+1}, x'_{m+1}, \dots, q'_{n'-1}, x'_{n'-1}, f'_{n'}, A'_{n'}, F'_{n'+1}, A'_{n'+1}, F'_{n'+2}, \dots \rangle$$

ordered in the natural way.

- (4) $p_1 = \langle q_{m+1}, x_{m+1}, \dots, q_{n-1}, x_{n-1}, f_n, A_n, F_{n+1}, A_{n+1}, F_{n+2}, \dots \rangle$

It follows that if G is $\bar{\mathbb{P}}$ -generic and $\langle \tau_j : 17 \leq j < \omega \rangle$ is the Prikry sequence added by G , then for every $m \geq 17$ the generic object G induces a $\mathbb{Q}^*(\tau_{17}) \times \prod_{17 < j \leq m} \mathbb{Q}(x_{j-1}, x_j)$ -generic filter.

Remark 8.10. Formally the posets $\mathbb{P}_{\text{low}}, \mathbb{P}_{\text{high}}$ and conditions p_0, p_1 depend on the choice of m . When we use this kind of factorization in the sequel, the value of m should always be clear from the context.

8.2. The Prikry lemma

Recall from section 7.3 that in $V[L]$ we derived measures U_n on $P_\kappa \lambda_n^b$ for $17 \leq n < \omega$ from the embedding $j_{01}^* : V[L] \rightarrow M_1^*$, and formed ultrapower maps $j_n : V[L] \rightarrow N_n = \text{Ult}(V[L], U_n)$. We arranged that if $k_n : N_n \rightarrow M_1^*$ is the natural factor map with $j_{01}^* = k_n \circ j_n$, then $\text{crit}(k_n) > j_{01}(\lambda_{17}^b)$. It follows that for $\alpha \leq \lambda_{17}^b$ we have $j_{01}(\alpha) = j_n(\alpha)$.

Recall also that $i_n = j_n(j_{n+1}) \circ j_n$, and that $\mathbb{Q}_\infty = i_n(\mathbb{Q})(\kappa, j_n(\kappa))$ for all $n \geq 17$. Now $\mathbb{Q}_\infty \in M_{01}^*$, and in M_{01}^* we have $|\mathbb{Q}_\infty| = j_{01}(\lambda_{\omega+2}^a)$ and $2^{j_{01}(\lambda_{\omega+2}^a)} = j_{01}(\lambda_1^b)$. It follows that $\mathbb{Q}_\infty \in N_n$ and K is $\mathbb{Q}_\infty$ -generic over N_n for all $n \geq 17$. By similar arguments, if we let $N_n^+ = \text{Ult}(N_n, j_n(U_{n+1}))$, so that $i_n : V[L] \rightarrow N_n^+$, then $\mathbb{Q}_\infty \in N_n^+$ and K is $\mathbb{Q}_\infty$ -generic over N_n^+ .

Global notation: N_n^+

For each n with $17 \leq n < \omega$, $|P_\kappa \lambda_n^b| = \lambda_n^b$ in $V[L]$, so that U_n is still a supercompactness measure on $P_\kappa \lambda_n^b$ in the $< \lambda_{\omega+2}^b$ -distributive extension $V[L][A^{gg}]$. It follows that j_n lifts to the ultrapower map computed from U_n in $V[L][A^{gg}]$, and we write $j_n^A : V[L][A^{gg}] \rightarrow N_n^A = N_n[j_n^A(A^{gg})]$. Similarly i_n lifts, and we obtain $i_n^A : V[L][A^{gg}] \rightarrow N_n^{A+} = N_n^+[i_n^A(A^{gg})]$. By distributivity it is easy to see that K is still $\mathbb{Q}_\infty$ -generic over the models N_n^A and N_n^{A+} . We also note that $\mathbb{Q}_\infty$ is still $< \lambda_{\omega+2}^b$ -distributive in each of the models N_n, N_n^+, N_n^A and N_n^{A+} .

Global notation: $j_n^A, i_n^A, N_n^A, N_n^{A+}$

Fix E a dense open subset of $\bar{\mathbb{P}}$ with $E \in V[L][A^{gg}]$, and let $E^{(k)}$ be the dense open set of conditions whose every k -step extension lies in E . We describe a series of steps to “canonize” membership in E .

Global notation: $E^{(k)}$

For each $n > 17$ we define $\mathbf{F}_n$ to be the set of functions of two variables F such that $\text{dom}(F) = A \times \prec B$ for some $A \in U_{n-1}$ and $B \in U_n$, and $F(x, y) \in \mathbb{Q}(\kappa(x), \kappa(y))$ for all $(x, y) \in \text{dom}(F)$: that is to say, $\mathbf{F}_n$ is the set of functions which can appear as F_n^p for some $p \in \mathbb{P}$. In this situation, for each $x \in A$ we define $F(x, -)$ to be the function with domain $\{y \in B : x \prec y\}$ given by $F(x, -)(y) = F(x, y)$.

Global notation: $\mathbf{F}_n, F(x, -)$

We define L_n to be the set of lower parts s of the form $q_{17} \dots x_{n-1}$. When $n > 17$ and $s = q_{17} \dots x_{n-1} \in L_n$ we let $\kappa(s) = \kappa(x_{n-1})$, and for $x \in P_\kappa \lambda_n^b$ we write $s \prec x$ for $x_{n-1} \prec x$. By convention L_{17} is the singleton set containing the empty sequence, $\langle \rangle \prec x$ for all $x \in P_\kappa \lambda_{17}^b$, and $\mathbb{Q}(\langle \rangle, x) = \mathbb{Q}^*(\kappa(x))$ for all $x \in P_\kappa \lambda_{17}^b$.

Global notation: L_n

It is easy to see that if $L \subseteq L_n$ and $(A_s)_{s \in L}$ is an L -indexed family of sets in U_n , then $\{x \in P_\kappa \lambda_n^b : \forall s \in L \ s \prec x \implies x \in A_s\} \in U_n$. In the sequel we use this form of normality for U_n without comment.

Lemma 8.11. *There exist functions $(F_n^0)_{n \geq 17}$ and sets $(A_n^0)_{n \geq 17}$ such that:*

- $A_n^0 \in U_n$.
- $\text{dom}(F_n^0) = A_{n-1}^0 \times^\prec A_n^0$.
- $[F_n^0]_{U_{n-1} \times U_n} \in K$.
- For every k , every $n \geq 17$, every $x \in A_n^0$, every lower part $s \in L_n$ with $s \prec x$, and every condition $q \in \mathbb{Q}(\kappa(s), \kappa(x))$, one of the two mutually exclusive conditions holds:

- There is a condition in $E^{(k)}$ with an initial segment of the form

$$s \frown q \frown x \frown F_{n+1}^0(x, -).$$

- There is no condition in $E^{(k)}$ with an initial segment of the form

$$s \frown q \frown x \frown f_{n+1}$$

$$\text{where } f_{n+1} \leq F_{n+1}^0(x, -).$$

Proof. Fix n for the moment. Recall that $j_n^A : V[L][A^{gg}] \rightarrow N_n^A$ is the ultrapower map computed from U_n in $V[L][A^{gg}]$, and j_n^A is a lift of j_n . Let $x_n^1 = j_n[\lambda_n^b]$, so that $U_n = \{X \subseteq P_\kappa \lambda_n^b : x_n^1 \in j_n(X)\}$. Observe that $\{t \in j_n(L_n) : t \prec x_n^1\} = j_n[L_n] \in N_n$.

The key point is now to observe that if $F \in \mathbf{F}_{n+1}$ then $j_n(F)(x_n^1, -) \in N_n$, and is a function which can be integrated in N_n with respect to $j_n(U_{n+1})$ to obtain $[F]_{U_n \times U_{n+1}} \in \mathbb{Q}_\infty$. For each k , each $s \in L_n$ and each $Q \in \mathbb{Q}^{N_n}(\kappa(s), \kappa)$ we define in N_n^A a dense open set of conditions in $\mathbb{Q}_\infty$, namely the set of conditions $r \in \mathbb{Q}_\infty$ such that one of the following mutually exclusive conditions holds:

- There is a condition in $j_n(E^{(k)})$ with an initial segment of the form

$$j_n(s) \frown Q \frown x_n^1 \frown f_{n+1}$$

$$\text{where } [f_{n+1}]_{j_n(U_{n+1})} = r.$$

- There is no condition in $j_n(E^{(k)})$ with an initial segment of the form

$$j_n(s) \frown Q \frown x_n^1 \frown f_{n+1}$$

$$\text{where } [f_{n+1}]_{j_n(U_{n+1})} \leq r.$$

By the genericity of K over N_n^A and the distributivity of $\mathbb{Q}_\infty$ in N_n^A , there is $r_n \in K$ which is in the dense set defined above for every s and Q . We choose $F'_{n+1} \in \mathbf{F}_{n+1}$ such that $[F'_{n+1}]_{U_n \times U_{n+1}} = r_n$, that is $[j_n(F'_{n+1})(x_n^1, -)]_{j_n(U_{n+1})} = r_n$.

Working in $V[L][A^{gg}]$, let A'_n be the set of $x \in P_\kappa \lambda_n^b$ such that for every k , every $s \in L_n$ with $s \prec x$, and every $q \in \mathbb{Q}(\kappa(s), \kappa(x))$ one of the following mutually exclusive conditions holds:

1_x) There is a condition in $E^{(k)}$ with an initial segment of the form

$$s \smallfrown q \smallfrown x \smallfrown f_{n+1}$$

where $[f_{n+1}]_{U_{n+1}} = [F'_{n+1}(x, -)]_{U_{n+1}}$.

2_x) There is no condition in $E^{(k)}$ with initial segment of the form

$$s \smallfrown q \smallfrown x \smallfrown f_{n+1}$$

where $[f_{n+1}]_{U_{n+1}} \leq [F'_{n+1}(x, -)]_{U_{n+1}}$.

By Loś's theorem $A'_n \in U_n$.

For each k , $x \in A'_n$, $s \prec x$, and $q \in \mathbb{Q}(\kappa(s), \kappa(x))$ such that 1_x holds, let $B'_{n+1}(k, s, q, x) \in U_{n+1}$ be such that there is a condition in $E^{(k)}$ with initial segment

$$s \smallfrown q \smallfrown x \smallfrown F'_{n+1}(x, -) \restriction B'_{n+1}(k, s, q, x).$$

Let B'_{n+1} be the set of $y \in P_\kappa \lambda_{n+1}^b$ such that $y \in B'_{n+1}(k, s, q, x)$ for every k , every $x \in A'_n$ with $x \prec y$ and every relevant s and q , so that $B'_{n+1} \in U_{n+1}$ by normality.

Now we choose $A_n^0 \in U_n$ so that $A_n^0 \subseteq A'_n \cap B'_n$ for every relevant n , let $F_n^0 = F'_n \restriction A_{n-1}^0 \times^\prec A_n^0$, and verify that this satisfies the desired property. Let $n \geq 17$ and suppose that $k < \omega$, $x \in A_n^0$, $s \in L_n$ with $s \prec x$, and $q \in \mathbb{Q}(\kappa(s), \kappa(x))$. By construction $x \in A'_n$.

Suppose first that 1_x holds, so that we defined $B'_{n+1}(k, s, q, x)$. By definition $\text{dom}(F_{n+1}^0(x, -)) \subseteq \text{dom}(F'_{n+1}(x, -)) \subseteq \{y \in B'_{n+1} : x \prec y\} \subseteq B'_{n+1}(k, s, q, x)$, so that $F_{n+1}^0(x, -) \leq F'_{n+1}(x, -) \restriction B'_{n+1}(k, s, q, x)$ and hence there is a condition in $E^{(k)}$ with initial segment

$$s \smallfrown q \smallfrown x \smallfrown F_{n+1}^0(x, -).$$

If alternatively 2_x holds then *a fortiori* there is no condition in $E^{(k)}$ with an initial segment of the form

$$s \smallfrown q \smallfrown x \smallfrown f_{n+1}$$

where $f_{n+1} \leq F_{n+1}^0(x, -)$, because in this case we have

$$[f_{n+1}]_{U_{n+1}} \leq [F'_{n+1}(x, -)]_{U_{n+1}} = [F_{n+1}^0(x, -)]_{U_{n+1}}.$$

□

Lemma 8.12. *There exist functions $(F_n^1)_{n \geq 17}$ and sets $(A_n^1)_{n \geq 17}$ such that:*

- $A_n^1 \in U_n$ with $A_n^1 \subseteq A_n^0$.
- $\text{dom}(F_n^1) = A_{n-1}^1 \times^\prec A_n^1$ with $F_n^1 \leq F_n^0$.
- $[F_n^1]_{U_{n-1} \times U_n} \in K$.
- For every k , every $n \geq 17$, every $x \in A_n^1$, and every $t \in L_{n+1}$ with $t = s \smallfrown q \smallfrown x$, if there is a condition in $E^{(k)}$ with initial segment

$$t \smallfrown F_{n+1}^0(x, -)$$

then

$$t \smallfrown F_{n+1}^0(x, -) \smallfrown (A_{m-1}^1, F_m^*)_{m>n+1} \in E^{(k)},$$

$$\text{where } F_m^* = F_m^1 \upharpoonright \{(y, z) \in A_{m-1}^1 \times A_m^1 : x \prec y \prec z\}.$$

Proof. Fix n for the moment. For every k , $x \in A_n^0$, $s \in L_n$ with $s \prec x$, and $q \in \mathbb{Q}(\kappa(s), \kappa(x))$, let $t = s \smallfrown q \smallfrown x$ (so that $t \in L_{n+1}$) and if there is a condition in $E^{(k)}$ with initial segment $t \smallfrown F_{n+1}^0(x, -)$ then choose such a condition $p^{t,k}$. To lighten the notation let $F_m^{t,k} = F_m^{p^{t,k}}$ for $m > n+1$.

For all k , t , and m , $[F_m^{t,k}]_{U_{m-1} \times U_m} \in K$. Since K is generic over M_1^* , it follows from the closure properties of M_1^* and the distributivity of $\mathbb{Q}_\infty$ that there exists a sequence $(G_m^n)_{m>n+1}$ such that $G_m^n \in \mathbf{F}_m$, $[G_m^n]_{U_{m-1} \times U_m} \in K$ and $[G_m^n]_{U_{m-1} \times U_m} \leq [F_m^{t,k}]_{U_{m-1} \times U_m}$ for all t and k . Using closure and distributivity again there exists a sequence (G_m) such that $G_m \in \mathbf{F}_m$, $[G_m]_{U_{m-1} \times U_m} \in K$, $[G_m]_{U_{m-1} \times U_m} \leq [F_m^0]_{U_{m-1} \times U_m}$, and $[G_m]_{U_{m-1} \times U_m} \leq [G_m^n]_{U_{m-1} \times U_m}$ for all $n > m+1$.

By taking appropriate diagonal intersections to define the sets A_m^1 and setting $F_m^1 = G_m \upharpoonright A_{m-1}^1 \times_{\prec} A_m^1$, we may arrange that for every k , n , $t \in L_{n+1}$, $m > n+1$ and $(y, z) \in \text{dom}(F_m^1)$ with $t \prec y \prec z$, we have $F_m^1(y, z) = G_m(y, z) \leq G_m^n(y, z) \leq F_m^{t,k}(y, z)$ and $F_m^1(y, z) \leq F_m^0(y, z)$. To verify that this works, let $t = s \smallfrown q \smallfrown x \in L_{n+1}$ with $x \in A_n^1$ and assume that there is a condition in $E^{(k)}$ with initial segment $t \smallfrown F_{n+1}^0(x, -)$, so that we chose $p^{t,k} \in E^{(k)}$. The desired conclusion is immediate. $\square$

Lemma 8.13. *There exist functions $(F_n^2)_{n>17}$ and sets $(A_n^2)_{n\geq 17}$ such that:*

- $A_n^2 \in U_n$ with $A_n^2 \subseteq A_n^1$.
- $\text{dom}(F_n^2) = A_{n-1}^2 \times_{\prec} A_n^2$ with $F_n^2 \leq F_n^1$.
- $[F_n^2]_{U_{n-1} \times U_n} \in K$.
- For every k , every $n \geq 17$, every $(x, y) \in \text{dom}(F_{n+1}^2)$, every $s \in L_n$ with $s \prec x$, and every $q \in \mathbb{Q}(\kappa(s), \kappa(x))$, one of the following mutually exclusive statements holds:

– There is a condition in $E^{(k)}$ with initial segment

$$s \smallfrown q \smallfrown x \smallfrown F_{n+1}^2(x, y) \smallfrown y \smallfrown F_{n+2}^1(y, -).$$

– There is no condition in $E^{(k)}$ with initial segment of the form

$$s \smallfrown q \smallfrown x \smallfrown \bar{r} \smallfrown y \smallfrown F_{n+2}^1(y, -)$$

$$\text{where } \bar{r} \leq F_{n+1}^2(x, y).$$

Proof. As in the proof of Lemma 8.11, let $x_n^1 = j_n[\lambda_n^b]$. Let $x_n^2 = j_n(j_{n+1})(x_n^1) = i_n[\lambda_n^b]$, and let $y_{n+1}^2 = j_n(j_{n+1})[j_n(\lambda_{n+1}^b)]$. By a routine calculation $U_n \times U_{n+1} = \{X \subseteq P_\kappa \lambda_n^b \times P_\kappa \lambda_{n+1}^b : (x_n^2, y_{n+1}^2) \in i_n(X)\}$. It is easy to see that $i_n[L_n] = \{t \in i_n(L_n) : t \prec x_n^2\}$.

Let $k < \omega$, $s \in L_n$, and $Q \in \mathbb{Q}^{N_n^+}(\kappa(s), \kappa)$. Working in N_n^{+A} define the dense open set of conditions $r \in \mathbb{Q}_\infty$ such that one of the following mutually exclusive conditions holds:

- There is a condition in $i_n(E^{(k)})$ with initial segment

$$i_n(s) \frown Q \frown x_n^2 \frown r \frown y_{n+1}^2 \frown i_n(F_{n+2}^1)(y_{n+1}^2, -).$$

- There is no condition in $i_n(E^{(k)})$ with initial segment of the form

$$i_n(s) \frown Q \frown x_n^2 \frown \bar{r} \frown y_{n+1}^2 \frown i_n(F_{n+2}^1)(y_{n+1}^2, -)$$

where $\bar{r} \leq r$.

Using the genericity of K over N_n^{A+} and the distributivity of $\mathbb{Q}_\infty$ in this model, we find $r_n \in K$ which lies in this dense open set for every k, s , and Q , and fix $F_{n+1}'' \in \mathbf{F}_{n+1}$ such that $[F_{n+1}'']_{U_n \times U_{n+1}} = i_n(F_{n+1}'')(x_n^2, y_n^2) = r_n$.

By Łoś's theorem there is a set $C_{n+1} \in U_n \times U_{n+1}$ such that for every k , $(x, y) \in C_{n+1}$, $s \in L_n$ with $s \prec x$, and every $q \in \mathbb{Q}(\kappa(s), \kappa(x))$ one of the following mutually exclusive conditions holds:

- There is a condition in $E^{(k)}$ with initial segment

$$s \frown q \frown x \frown F_{n+1}''(x, y) \frown y \frown F_{n+2}^1(y, -).$$

- There is no condition in $E^{(k)}$ with an initial segment of the form

$$s \frown q \frown x \frown \bar{r} \frown y \frown F_{n+2}^1(y, -)$$

where $\bar{r} \leq F_{n+1}''(x, y)$.

Now we choose $F_n^2 \leq F_n'', F_n^1$ and $A_n^2 \subseteq A_n^1$ so that $\text{dom}(F_{n+1}^2) = A_n^2 \times^\prec A_{n+1}^2 \subseteq C_{n+1}$. Clearly this satisfies the requirements. $\square$

Lemma 8.14. *There exist sets $(A_n^3)_{n \geq 17}$ such that:*

- $A_n^3 \in U_n$ with $A_n^3 \subseteq A_n^2$.
- For every k , every $n \geq 17$, and every $t \in L_{n+1}$, one of the following mutually exclusive conditions holds:

- For every $y \in A_{n+1}^3$ with $t \prec y$, there is a condition in $E^{(k)}$ with initial segment

$$t \frown F_{n+1}^2(x, y) \frown y \frown F_{n+2}^2(y, -).$$

- For every $y \in A_{n+1}^3$ with $t \prec y$, there is no condition in $E^{(k)}$ with initial segment

$$t \frown F_{n+1}^2(x, y) \frown y \frown F_{n+2}^2(y, -).$$

Proof. For every k, n and $t = s \frown q \frown x \in L_{n+1}$ partition $\{y \in A_{n+1}^2 : t \prec y\}$ as follows:

- $A_{n+1}^+(t, k)$ is the set of $y \in A_{n+1}^2$ such that $t \prec y$ and there is a condition in $E^{(k)}$ with initial segment

$$t \frown F_{n+1}^2(x, y) \frown y \frown F_{n+2}^2(y, -).$$

- $A_{n+1}^-(t, k)$ is the set of $y \in A_{n+1}^2$ such that $t \prec y$ and there is no condition in $E^{(k)}$ with initial segment

$$t \frown F_{n+1}^2(x, y) \frown y \frown F_{n+2}^2(y, -).$$

Let $A_{n+1}^3(t, k)$ be whichever of the sets $A_{n+1}^+(t, k)$ and $A_{n+1}^-(t, k)$ lies in U_{n+1} , and then let $A_{n+1}^3 = \{y \in A_{n+1}^2 : \forall k \forall t \in L_{n+1} \ t \prec y \implies y \in A_{n+1}^3(t, k)\}$. Clearly this satisfies the requirements. $\square$

To keep the indices in step, we define $F_m^3 = F_m^2 \upharpoonright A_{m-1}^3 \times^\prec A_m^3$.

Lemma 8.15. *Let $n \geq 17$, let $x \in P_\kappa \lambda_n^b$, and let f_{n+1} be a function such that $A_{n+1} = \text{dom}(f_{n+1}) \in U_{n+1}$, where $x \prec y$ and $f_{n+1}(y) \in \mathbb{Q}(\kappa(x), \kappa(y))$ for all $y \in A_{n+1}$. Then there exist $B_{n+1} \subseteq A_{n+1}$ and f'_{n+1} with domain B_{n+1} such that:*

- $f'_{n+1}(y) \leq f_{n+1}(y)$ for all $y \in B_{n+1}$.
- For every k and every $t \in L_{n+1}$ of the form $s \frown q \frown x$, one of the two following mutually exclusive conditions holds:

- For every $y \in B_{n+1}$, there is a condition in $E^{(k)}$ with initial segment

$$t \frown f'_{n+1}(y) \frown y \frown F_{n+2}^3(y, -).$$

- For every $y \in B_{n+1}$ and every $r \leq f'_{n+1}(y)$, there is no condition in $E^{(k)}$ with initial segment

$$t \frown r \frown y \frown F_{n+2}^3(y, -).$$

Proof. Shrinking A_{n+1} if necessary, we may assume that $A_{n+1} \subseteq A_{n+1}^3$. Note that $\mathbb{Q}(\kappa(x), \kappa(y))$ is $< \Lambda_{\omega+2}^b(\kappa(x))$ -distributive, and the set of elements of L_{n+1} of form $s \frown q \frown x$ is of cardinality at most $\Lambda_n^b(\kappa(x))$. It follows that for each $y \in A_{n+1}$ there is $r \leq f_{n+1}(y)$ such that for every k and every t in L_{n+1} of the form $s \frown q \frown x$ one of the two following mutually exclusive conditions holds:

- 1_{t,k}) There is a condition in $E^{(k)}$ with initial segment

$$t \frown r \frown y \frown F_{n+2}^3(y, -).$$

- 2_{t,k}) There is no condition in $E^{(k)}$ with an initial segment of the form

$$t \frown \bar{r} \frown y \frown F_{n+2}^3(y, -)$$

where $\bar{r} \leq r$.

For each $y \in A_{n+1}$ choose $f'_{n+1}(y)$ be some $r \leq f_{n+1}(y)$ as above. For each t and k , let $A_{n+1}(t, k)$ be whichever of the sets $\{y \in A_{n+1} : f_{n+1}(y) \text{ satisfies } 1_{t,k}\}$ and $\{y \in A_{n+1} : f_{n+1}(y) \text{ satisfies } 2_{t,k}\}$ is measure one for U_{n+1} . Let $B_{n+1} = \bigcap_{t,k} A_{n+1}(t, k)$. $\square$

Remark 8.16. Lemma 8.15 is only useful for conditions of length $n > 17$, because for a condition $\langle f_{17}, A_{17}, F_{18} \rangle$ of length 17, $f_{17}(x) \in \mathbb{Q}^*(x)$ for all $x \in A_{17}$. This

explains why the following Lemma 8.17 is restricted to conditions of length greater than 17.

Lemma 8.17 (Strong Prikry Lemma). *For every dense open subset E of $\bar{\mathbb{P}}$ and every condition $p \in \bar{\mathbb{P}}$ of length greater than 17, there exist q a direct extension of p and $k \in \omega$ such that $q \in E^{(k)}$.*

Proof. Let the condition p be $\langle q_{17}, x_{17} \dots x_{n-1}, f_n, A_n, F_{n+1}, \dots \rangle$ where $n > 17$, and as usual $\text{dom}(f^n) = A_n$ and $\text{dom}(F_m) = A_{m-1} \times^< A_m$ for $n < m < \omega$.

Appealing to Lemma 8.15, we refine f_n to $f'_n \leq f_n$ with $\text{dom}(f'_n) = A'_n \subseteq A_n$ such that for every k and every $t \in L_n$ with last entry x_{n-1} , one of the following holds:

- For every $y \in A'_n$ there is a condition in $E^{(k)}$ with initial segment

$$t \frown f'_n(y) \frown y \frown F_{n+1}^3(y, -).$$

- For every $y \in A'_n$ there is no condition in $E^{(k)}$ with an initial segment of the form

$$t \frown r \frown y \frown F_{n+1}^3(y, -) \text{ where } r \leq f'_n(y).$$

We then form a direct extension p' of p , where p' has the form

$$\langle q_{17}, x_{17} \dots x_{n-1}, f'_n, A'_n, F'_{n+1}, \dots \rangle$$

with $F'_m \leq F_m, F_m^3$ for all $m > n$. Since E is a dense open set, there is a condition $p'' \leq p'$ such that $p'' \in E$. Let p'' be a k -step extension of p' . If $k = 0$ we are done setting $q = p''$, so assume that $k > 0$.

The condition p'' has the form

$$\langle q''_{17}, x_{17} \dots x_{n-1}, q''_n, x_n \dots q''_{m-1}, x_{m-1}, f''_m, A''_m, F''_{m+1}, \dots \rangle$$

where $m = n + k > n$. We note that:

- $q''_n \leq f'_n(x_n)$.
- $x_j \in A_j^3$ for $n \leq j < m$.
- $f''_m \leq F_m^3(x_{m-1}, -)$.
- $F''_j \leq F_j^3$ for $j > m$.

Claim 8.18. *If p^{**} is the condition*

$$\langle q''_{17}, x_{17} \dots x_{n-1}, q''_n, x_n, F'_{n+1}(x_n, -), A'_{n+1}, F'_{n+2}, F'_{n+3}, \dots \rangle$$

*then $p^{**} \in E^{(k-1)}$.*

Proof. We will show by induction on i that for $0 \leq i \leq k-1$, if p^* is the condition

$$\langle q''_{17}, x_{17} \dots q''_{m-i-1}, x_{m-i-1}, F'_{m-i}(x_{m-i-1}, -), A'_{m-i}, F'_{m-i+1}, \dots \rangle$$

then $p^* \in E^{(i)}$.

(Base case) $i = 0$: Since $f_m'' \leq F_m'(x_{m-1}) \leq F_m^0(x_{m-1})$ and $x_{m-1} \in A_{m-1}^0$, it follows from Lemma 8.11 that there is a condition in E with initial segment

$$\langle q_{17}'', x_{17} \dots q_{m-1}'', x_{m-1}, F_m^0(x_{m-1}, -) \rangle$$

Since E is open, it follows from the choice of the functions F_j' and Lemma 8.12 that

$$p^* = \langle q_{17}'', x_{17} \dots q_{m-1}'', x_{m-1}, F_m'(x_{m-1}), A_{m+1}', F_{m+1}', \dots \rangle \in E = E^{(0)}.$$

(Successor step) $i = i_0 + 1$ for $0 \leq i_0 < k - 1$. By the induction hypothesis if p^{*-} is the condition

$$\langle q_{17}'', x_{17} \dots q_{m-i}'', x_{m-i}, F_{m-i+1}'(x_{m-i}, -), A_{m-i+1}', F_{m-i+2}', \dots \rangle$$

then $p^{*-} \in E^{(i-1)}$.

Since $q_{m-i}'' \leq F_{m-i}'(x_{m-i-1}, x_{m-i}) \leq F_{m-i}^2(x_{m-i-1}, x_{m-i})$, $x_{m-i-1} \in A_{m-i-1}^2$, and $x_{m-i} \in A_{m-i}^2$, it follows from Lemma 8.13 that there is a condition in $E^{(i-1)}$ with initial segment

$$\langle q_{17}'', x_{17} \dots x_{m-i-1}, F_{m-i}^2(x_{m-i-1}, x_{m-i}), x_{m-i}, F_{m-i+1}^1(x_{m-i}, -) \rangle.$$

Since $E^{(i-1)}$ is open there is a condition in $E^{(i-1)}$ with initial segment

$$\langle q_{17}'', x_{17} \dots x_{m-i-1}, F_{m-i}^2(x_{m-i-1}, x_{m-i}), x_{m-i}, F_{m-i+1}^2(x_{m-i}, -) \rangle.$$

Since $x_{m-i} \in A_{m-i}^3$, it follows from Lemma 8.14 that for every $y \in A_{m-i}^3$ with $x_{m-i-1} \prec y$, there is a condition in $E^{(i-1)}$ with initial segment

$$\langle q_{17}'', x_{17} \dots x_{m-i-1}, F_{m-i}^3(x_{m-i-1}, y), y, F_{m-i+1}^3(y) \rangle.$$

It follows from the choice of the functions F_j' that for every such y

$$\begin{aligned} & \langle q_{17}'', x_{17} \dots x_{m-i-1}, F_{m-i}'(x_{m-i-1}, y), y, F_{m-i+1}'(y), A_{m-i+1}', F_{m-i+2}', \dots \rangle \\ & \in E^{(i-1)}. \end{aligned}$$

So every 1-step extension of p^* lies in $E^{(i-1)}$, so by definition $p^* \in E^{(i)}$. $\square$

Since $q_n'' \leq f_n'(x_n)$, it follows from the choice of f_n' that for every $y \in A_n'$, there is a condition in $E^{(k-1)}$ with initial segment

$$\langle q_{17}'', x_{17} \dots x_{n-1}, f_n'(y), y, F_{n+1}^3(y, -) \rangle.$$

Let q be the condition

$$\langle q_{17}'', x_{17} \dots x_{n-1}, f_n', A_n', F_{n+1}', \dots \rangle$$

By the choice of the functions F_j' and Lemma 8.15, it follows that $q \smallfrown y \in E^{(k-1)}$ for all $y \in A_n'$, that is to say $q \in E^{(k)}$. $\square$

Lemma 8.19 (Prikry Lemma). *Let b be a Boolean value for $\bar{\mathbb{P}}$ and let $p \in \bar{\mathbb{P}}$ be a condition of length greater than 18. Then there is $s \leq^* p$ such that s decides b .*

Proof. Let E be the dense open set of conditions which decide b and let $q \leq^* p$ and k be as in the conclusion of Lemma 8.17. For each appropriate k -tuple $\vec{x}$, define $F(\vec{x}) = 0$ if $q \cap \vec{x} \Vdash \neg b$ and $F(\vec{x}) = 1$ if $q \cap \vec{x} \Vdash b$. By Lemma 7.15, we may find $r \leq^* q$ such that all k -step extensions of r decide b the same way: since every extension of r is compatible with some k -step extension, r decides b . $\square$

Recall from the end of Section 8.1 that given a condition

$$p = \langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1}, f_n, A_n, F_{n+1}, A_{n+1}, F_{n+2}, \dots \rangle$$

and $m < n - 1$, we factored $\bar{\mathbb{P}} \downarrow p$ as $\mathbb{P}_{\text{low}} \downarrow p_0 \times \mathbb{P}_{\text{high}} \downarrow p_1$, where $\mathbb{P}_{\text{high}}$ is defined in a very similar way to $\bar{\mathbb{P}}$ with the associated Prikry sequence starting at x_{m+1} . The proofs of Lemmas 8.17 and 8.19 can easily be adapted to prove the parallel assertions for $\mathbb{P}_{\text{high}}$.

Lemma 8.20. *Let p , m , $\mathbb{P}_{\text{low}}$, $\mathbb{P}_{\text{high}}$, p_0 and p_1 be as above. Let $\tau = \kappa(x'_m)$ and $\lambda = \Lambda_{\omega+2}^b(\tau)$. Then:*

- (1) *Forcing with $\mathbb{P}_{\text{high}} \downarrow p_1$ adds no new bounded subsets of λ .*
- (2) *Forcing with $\bar{\mathbb{P}} \downarrow p$, all bounded subsets of λ are in the intermediate extension by $\mathbb{P}_{\text{low}}$.*

Proof. To show the first claim, let $\gamma < \lambda$ and let $\dot{X}$ name a subset of γ . Let p' be a condition in $\mathbb{P}_{\text{high}} \downarrow p_1$, and let

$$p' = \langle q'_{m+1}, x'_{m+1}, \dots, q'_{n'-1}, x'_{n'-1}, f'_{n'}, A'_{n'}, F'_{n'+1}, A'_{n'+1}, F'_{n'+2}, \dots \rangle,$$

where $x'_j = x_j$ for $m < j < n$.

Let $\tau_j = \kappa(x'_j)$ for $m < j < n'$. For each $\alpha < \gamma$ we will define a subset D_α of $\mathbb{Q}(\tau, \tau_{m+1}) \times \prod_{m+1 < j < n'} \mathbb{Q}(\tau_{j-1}, \tau_j) \times \mathbb{Q}(\tau_{n'-1}, \kappa)$ as follows: D_α is the set of tuples $(q''_{m+1}, \dots, q''_{n'-1}, q''_{n'})$ such that there is a direct extension $p'' \leq^* p'$ deciding $\alpha \in \dot{X}$ where

$$p'' = \langle q''_{m+1}, x'_{m+1}, \dots, q''_{n'-1}, x'_{n'-1}, f''_{n'}, A''_{n'}, F''_{n'+1}, A''_{n'+1}, F''_{n'+2}, \dots \rangle,$$

and $q''_{n'} = [f''_{n'}]$.

Clearly D_α is open. It follows by Lemma 8.19 for $\mathbb{P}_{\text{high}}$ that D_α is dense below $(q'_{m+1}, \dots, q'_{n'-1}, [f'_{n'}])$ for each α . By Lemma 7.16 $\bigcap_{\alpha < \gamma} D_\alpha$ is dense below $(q'_{m+1}, \dots, q'_{n'-1}, [f'_{n'}])$, so we find $(q''_{m+1}, \dots, q''_{n'-1}, q''_{n'}) \leq (q'_{m+1}, \dots, q'_{n'-1}, [f'_{n'}])$ with $(q''_{m+1}, \dots, q''_{n'-1}, q''_{n'}) \in \bigcap_{\alpha < \gamma} D_\alpha$.

For each $\alpha < \gamma$, choose a condition

$$p^\alpha = \langle q''_{m+1}, x'_{m+1}, \dots, q''_{n'-1}, x'_{n'-1}, f_n^\alpha, A_{n'}^\alpha, F_{n'+1}^\alpha, A_{n'+1}^\alpha, F_{n'+2}^\alpha, \dots \rangle$$

witnessing that $(q''_{m+1}, \dots, q''_{n'-1}, [f'_{n'}]) \in D_\alpha$. Since $[f_n^\alpha] = q''_{n'}$ for each α , by κ -completeness there is a large set where all the functions f_n^α agree, so refining their domains we may as well assume that there is a fixed function f_n^* with $f_n^\alpha = f_n^*$ for all $\alpha < \gamma$.

For each $j > n'$ and $\alpha < \gamma$, $[F_j^\alpha]_{U_{j-1} \times U_j} \in K$. Since K is generic over a highly closed inner model, we may find F_j^* for $j > n$ such that $[F_j^*]_{U_{j-1} \times U_j} \in K$ and $[F_j^*] \leq [F_j^\alpha]$ for all α . Since $\gamma < \kappa$, by κ -completeness we may refine the domains of the functions F_j^* and assume that $F_j^* \leq F_j^\alpha$ for all α . In summary we have constructed a condition

$$p^* = \langle q''_{m+1}, x'_{m+1}, \dots, q''_{n'-1}, x'_{n'-1}, f_{n'}^*, A_{n'}^*, F_{n'+1}^*, A_{n'+1}^*, F_{n'+2}^*, \dots \rangle$$

which refines p' and decides $\alpha \in \dot{X}$ for all $\alpha < \gamma$.

For the second claim we observe that $|\mathbb{P}_{\text{low}}| < \lambda$, so that all $\mathbb{P}_{\text{low}}$ -names for bounded subsets of λ are coded by bounded subsets of λ , and we are done by the factorization of $\bar{\mathbb{P}} \downarrow p$ and the first claim. $\square$

The following corollary is immediate.

Corollary 8.21. *Let G be $\bar{\mathbb{P}}$ -generic and let $\langle \tau_j : 17 \leq j < \omega \rangle$ be the Prikry sequence added by G . Let $\gamma < \kappa$, and let $m > 17$ be least such that $\gamma < \Lambda_{\omega+2}^b(\tau_m)$, and let $X \in P(\gamma)^{V[L][A^{gg}][G]}$. Then $X \in V[L][G_0]$ where G_0 is the $\mathbb{Q}^*(\tau_{17}) \times \prod_{17 < j \leq m} \mathbb{Q}(x_{j-1}, x_j)$ -generic filter induced by G .*

For the purposes of the analysis in Section 9, we record some more refined information about how much of the various generic objects we need to define some bounded subsets of κ . We remind the reader that $\mathbb{Q}(\tau, \tau^*)$ is $< \Lambda_{\omega+2}^b(\tau)$ -distributive, and that only the $\mathbb{A}_0^b$ component of $\mathbb{I}^b(\tau)$ adds any subsets of $\Lambda_\omega^a(\tau)$. The proof of the following easy Lemma uses these facts and Corollary 8.21.

Lemma 8.22. *In the generic extension by $\bar{\mathbb{P}}$:*

- *If τ and τ^* are successive Prikry points, the cardinals between τ and τ^* are $\Lambda_j^a(\tau)$ for $j < \omega + 3$ and $\Lambda_k^b(\tau)$ for $k < \omega + 4$. If E is the generic object added by the interleaved forcing between Prikry points up to τ , and $Q(\tau, \tau^*)$ is the generic object added by the forcing between τ and τ^* , then:*
 - *All bounded subsets of $\Lambda_\omega^b(\tau)$ lie in $V^{lbi}(\tau)[E][A_e(\tau) \times J_0^c(\tau)]$.*
 - *All subsets of $\Lambda_{\omega+1}^b(\tau)$ lie in $V^{lbi}(\tau)[E][J_0^c(\tau)][A_e(\tau) \times A_1^c(\tau)]$.*
 - *All bounded subsets of $\Lambda_\omega^a(\tau^*)$ lie in $V[L \restriction \tau^*][E][Q(\tau, \tau^*)][A_0^b(\tau^*)]$.*
 - *All subsets of $\Lambda_{\omega+1}^a(\tau^*)$ lie in*

$$V[L \restriction \tau^*][E][Q(\tau, \tau^*)][L^b(\tau^*) \restriction \lambda_{\omega+2}^a(\tau^*)][A_{[0,1]}^b(\tau^*) * U_0^b(\tau^*) * S_0^b(\tau^*)].$$
- *If τ is the first Prikry point then the infinite cardinals below τ are ω , $\rho^+ = \omega_1$, $\theta = \omega_2$. If $Q^*(\tau)$ is the generic object added by the first interleaved forcing then:*
 - *All bounded subsets of $\Lambda_\omega^a(\tau)$ lie in $V[L \restriction \tau][Q^*(\tau)][A_0^b(\tau)]$.*
 - *All subsets of $\Lambda_{\omega+1}^a(\tau)$ lie in*

$$V[L \restriction \tau][Q^*(\tau)][L^b(\tau) \restriction \lambda_{\omega+2}^a(\tau)][A_{[0,1]}^b(\tau) * U_0^b(\tau) * S_0^b(\tau)].$$

Lemma 8.23. $\kappa = (\aleph_{\omega_2})^{V[L][A^{gg}][\bar{P}]}$ and $(\lambda_\omega^b)^+ = (\kappa^+)^{V[L][A^{gg}][\bar{P}]}$.

Proof. It follows immediately from 8.22 that $\kappa = \aleph_{\omega^2}$ in $V[L][A^{gg}][\bar{P}]$. By Lemma 8.7, $(\lambda_{\omega}^b)^+$ is a cardinal in this model. An easy density argument shows that $\lambda_{\omega}^b = \bigcup_{n \geq 17} x_n$ where the x_n 's are the supercompact Prikry points added by $\bar{P}$, and it follows immediately that λ_{ω}^b is collapsed to have cardinality κ in $V[L][A^{gg}][\bar{P}]$. $\square$

The following Lemma gives an analysis of names for sequences of ordinals, in a similar spirit to Lemma 8.20 and Corollary 8.21.

Lemma 8.24. *Let $p \in \bar{\mathbb{P}}$, where*

$$p = \langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1}, f_n, A_n, F_{n+1}, A_{n+1}, F_{n+2}, \dots \rangle.$$

Let $\tau_j = \kappa(x_j)$, let $\gamma < \Lambda_{\omega+2}^b(\tau_{n-1})$, and let $\dot{f}$ be a $\mathbb{P}$ -name for a function from γ to ON . Let $\mathbb{P}_{\text{low}} = \mathbb{Q}^(\tau_{17}) \times \prod_{17 < j < n} \mathbb{Q}(x_{j-1}, x_j)$, and let $p_0 = (q_{17}, \dots, q_{n-1})$. Then there are a direct extension*

$$p' = \langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1}, f'_n, A'_n, F'_{n+1}, A'_{n+1}, F'_{n+2}, \dots \rangle.$$

of p , conditions $(p_0^\alpha)_{\alpha < \gamma}$ in $\mathbb{P}_{\text{low}}$ and natural numbers $(k_\alpha)_{\alpha < \gamma}$ such that for all $\alpha < \gamma$:

- $p_0^\alpha \leq p_0$.
- If $p_0^\alpha = (q_{17}^\alpha, \dots, q_{n-1}^\alpha)$, then every k_α -step extension of

$$p^\alpha = \langle q_{17}^\alpha, x_{17}, \dots, q_{n-1}^\alpha, x_{n-1}, f'_n, A'_n, F'_{n+1}, A'_{n+1}, F'_{n+2}, \dots \rangle$$

decides $\dot{f}(\alpha)$.

Proof. For each α , let D_α be the set of $q \in \mathbb{Q}(\tau_{n-1}, \kappa)$ such that there exist a direct extension $\bar{p}$ of p with $[f_n^{\bar{p}}] = q$ and $k < \omega$ such that every k -step extension of $\bar{p}$ decides $\dot{f}(\alpha)$. Clearly D_α is open, and by Lemma 8.17 D_α is dense below $[f_n]$. Since $\gamma < \Lambda_{\omega+2}^b(\tau_{n-1})$ and $\mathbb{Q}(\tau_{n-1}, \kappa)$ is $< \Lambda_{\omega+2}^b(\tau_{n-1})$ -distributive, we may find $q \leq [f_n]$ with $q \in \bigcap_{\alpha < \gamma} D_\alpha$.

For each $\alpha < \gamma$ we choose $\bar{p}^\alpha \leq^* p$ witnessing that $q \in D_\alpha$. Arguing exactly as in the proof of Lemma 8.20, we may assume that all the entries of p^α past x_{n-1} are independent of α . This defines a suitable condition p' . $\square$

9. The tree property below $\aleph_{\omega^2}$ in the final model

We now establish the various instances of the tree property below $\aleph_{\omega^2}$ needed to prove Theorem 1.1. The instances above $\aleph_{\omega^2}$ require different techniques and will be discussed in Section 10.

Let $\langle \tau_i : 17 \leq i < \omega \rangle$ be the Prikry-sequence added by $\bar{\mathbb{P}}$, that is $\tau_i = \kappa(x_i)$ in the notation of Section 8. As we noted in Lemma 8.22, it follows from the Prikry lemma that bounded subsets of κ in the final model live in extensions of $V[L]$ by posets of the form $\mathbb{Q}^*(\tau_{17}) \times \mathbb{Q}(\tau_{17}, \tau_{18}) \times \dots \times \mathbb{Q}(\tau_i, \tau_{i+1})$ for some $i < \omega$.

Global notation: τ_i

The general plan is based on the fact that all the cardinals of interest are either double successor cardinals or successors of singular cardinals. To handle the double successors we will use the fact that the forcing posets $\mathbb{Q}^*(\tau_0)$, $\mathbb{I}^b(\tau_i)$ and $\mathbb{J}^c(\tau_i) * \mathbb{Q}_0(\tau_i, \tau_{i+1})$ establish instances of the tree property at all the cardinals which concern us, but only in submodels of our final model. We will use Lemma 4.5 to show that these instances of the tree property persist into our final model. To handle the successors of singulars we will exploit the fact that all the cardinals of concern have the form $\lambda_{\omega+1}$ where $\lambda = \Lambda^{a,b}(\tau_i)$, and that it is forced over V by $\mathbb{L}_{\text{aux}}(\rho, \lambda) \times \mathbb{R}_{\text{aux}}(\lambda)$ that the tree property holds at $\lambda_{\omega+1}$: the extension by $\mathbb{L}_{\text{aux}}(\rho, \lambda) \times \mathbb{R}_{\text{aux}}(\lambda)$ collapses so many cardinals that $\lambda_{\omega+1} = \aleph_{\omega+1}$, the point is that this extension absorbs enough of our final model to argue that the tree property also holds at $\lambda_{\omega+1}$ in our final model.

Let τ_i and τ_{i+1} be successive Prikry points. To lighten notation we make the following definitions:

- $\tau = \tau_i$.
- $\tau^* = \tau_{i+1}$.
- $\sigma_n^a = \Lambda_n^a(\tau)$, $\sigma_n^b = \Lambda_n^b(\tau)$.
- $\sigma_n^{a*} = \Lambda_n^a(\tau^*)$, $\sigma_n^{b*} = \Lambda_n^b(\tau^*)$.

We will discuss the cardinals in groups, roughly corresponding to the various instances of the $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction which are used below κ . Recall that one instance of this construction was done entirely by the $\mathbb{I}^b(\tau)$ component of $\mathbb{L}(\tau)$ at stage τ with cardinal parameters $\mu_0 = \sigma_{17}^a$, $\mu_1 = \sigma_{\omega+1}^a$, $\mu_2 = \sigma_{\omega+2}^a$, $\mu_{n+3} = \sigma_n^b$ for $n < \omega$. Another “two -phase” version was done partly by the $\mathbb{J}^c(\tau)$ component of $\mathbb{L}(\tau)$ and partly by $\mathbb{Q}(\tau, \tau^*)$ with cardinal parameters $\mu_0 = \sigma_{17}^b$, $\mu_1 = \sigma_{\omega+1}^b$, $\mu_2 = \sigma_{\omega+2}^b$, $\mu_3 = \sigma_{\omega+3}^b$, $\mu_{n+4} = \sigma_n^{a*}$ for $n < \omega$. When τ is the first Prikry point yet a third version was done partly by the construction of V and partly by the forcing $\mathbb{Q}^*(\tau)$, this time with cardinal parameters $\mu_0 = \omega$, $\mu_1 = \rho^+$, $\mu_2 = \theta$, $\mu_{3+n} = \sigma_n^a$ for $n < \omega$.

We let $\mathbb{E} = \mathbb{Q}^*(\tau_{17}) \times \prod_{17 \leq k < i} \mathbb{Q}(\tau_k, \tau_{k+1})$, so that $\mathbb{E}$ accounts for the forcing posets interleaved between the Prikry points up to and including τ . Let E be the $\mathbb{E}$ -generic object added to $V[L]$ by $\bar{P}$.

9.1. Group I: $\sigma_{\omega+2}^b$, $\sigma_{\omega+3}^b$, σ_n^{a*} for $n < \omega$

Recall from Section 6 that $L(\tau) = L^b(\tau) * I^b(\tau) * (A_e(\tau) \times J^c(\tau))$ and is generic over $V[L \restriction \tau]$. Recall also from Section 7.1 that $\mathbb{Q}(\tau, \tau^*) \in V[L \restriction \tau^*][A_0^b(\tau^*)]$ and adds $Q(\tau, \tau^*) = \prod_{i < 3} Q_i(\tau, \tau^*)$ which is generic over $V[L][E]$. The generic objects $J^c(\tau)$ and $Q_0(\tau, \tau^*)$ combine as described in Section 7.1. The cardinal parameters are $\mu_0 = \sigma_{17}^b$, $\mu_1 = \sigma_{\omega+1}^b$, $\mu_2 = \sigma_{\omega+2}^b$, $\mu_3 = \sigma_{\omega+3}^b$, $\mu_{n+4} = \sigma_n^{a*}$ for $n < \omega$.

To help the reader keep track, here is a picture of some of the most relevant cardinals for this group under the various names that they go by in the proof. Cardinals

in each row are equal, cardinals in each column are strictly increasing.

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \Lambda_1^a(\tau^*) & \sigma_1^{a*} & \mu_5 \\
\vdots & \Lambda_0^a(\tau^*) & \sigma_0^{a*} & \mu_4 \\
\tau_{i+1} & \tau^* & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \Lambda_{\omega+3}^b(\tau) & \sigma_{\omega+3}^b & \mu_3 \\
\vdots & \Lambda_{\omega+2}^b(\tau) & \sigma_{\omega+2}^b & \mu_2 \\
\vdots & \Lambda_{\omega+1}^b(\tau) & \sigma_{\omega+1}^b & \mu_1 \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \Lambda_{17}^b(\tau) & \sigma_{17}^b & \mu_0 \\
\vdots & \vdots & \vdots & \vdots \\
\tau_i & \tau & \vdots & \vdots
\end{array}$$

As we noted in Lemma 8.22 above, all the relevant trees for cardinals in Group I exist in the model $W_I = V[L \restriction \tau^*][A_0^b(\tau^*)][E][Q(\tau, \tau^*)]$. So to cover the cardinals in Group I it will suffice to prove:

Lemma 9.1. *For all $n < \omega$, μ_{n+2} has the tree property in W_I .*

Proof. Expanding $L \restriction \tau^*$,

$$W_I = V^{lbi}(\tau)[A_e(\tau) \times J^c(\tau)][L \restriction (\tau, \tau^*)][A_0^b(\tau^*)][E][Q(\tau, \tau^*)].$$

With a view to rearranging W_I we note that:

- By similar considerations as for $\mathbb{Q}(\tau, \tau^*)$, $\mathbb{E} \in V^l(\tau)[A_0^b(\tau)]$.
- $\mathbb{A}_e(\tau) \in V^{lb}(\tau)$.
- $\mathbb{J}^c(\tau) \in V^{lbi}(\tau)$.
- $\mathbb{Q}_0(\tau, \tau^*) \in V^{lbi}(\tau)[J^c(\tau)]$.
- $\mathbb{Q}_1(\tau, \tau^*) \in V^{lbi}(\tau)[A_e(\tau) \times J^c(\tau)][L \restriction (\tau, \tau^*)][A_0^b(\tau^*)]$.
- $\mathbb{Q}_2(\tau, \tau^*) \in V^{lbi}(\tau)[I^b(\tau)][A_e(\tau) \times J^c(\tau)][L \restriction (\tau, \tau^*)]$.

So we may rearrange W_I as

$$V^{lbi}(\tau)[E][J^c(\tau)][A_e(\tau)][L \restriction (\tau, \tau^*)][Q_0(\tau, \tau^*) \times Q_2(\tau, \tau^*)][A_0^b(\tau^*) * Q_1(\tau, \tau^*)].$$

By the definition of $\mathbb{Q}_1(\tau, \tau^*)$, we may rearrange $A_0^b(\tau^*) * Q_1(\tau, \tau^*)$ as $A_0^V(\tau^*)$ which is generic for $\mathbb{A}_0^V(\tau^*) = \text{Add}^V(\sigma_{17}^{a*}, \sigma_{\omega+2}^{a*})$. We note that $\sigma_{17}^{a*} = \mu_{21}$ in our list of cardinal parameters.

So W_I is

$$V^{lbi}(\tau)[E][J^c(\tau)][A_e(\tau)][L \restriction (\tau, \tau^*)][Q_0(\tau, \tau^*) \times Q_2(\tau, \tau^*)][A_0^V(\tau^*)].$$

Since $\mathbb{A}_0^V(\tau^*) \in V$ we may rearrange W_I as

$$V^{lbi}(\tau)[E][J^c(\tau)][Q_0(\tau, \tau^*)][A_0^V(\tau^*)][A_e(\tau)][L \restriction (\tau, \tau^*)][Q_2(\tau, \tau^*)].$$

Recalling that $Q_2(\tau, \tau^*)$ adds a term generic $T(\tau, \tau^*)$ such that $A_e(\tau) \times T(\tau, \tau^*)$ projects to $A_e(\tau) * L \restriction (\tau, \tau^*)$, we may rearrange W_I as

$$V^{lbi}(\tau)[E][J^c(\tau)][Q_0(\tau, \tau^*)][A_0^V(\tau^*)][A_e(\tau) \times T(\tau, \tau^*)],$$

and then as

$$V^{lbi}(\tau)[J^c(\tau) * Q_0(\tau, \tau^*)][T(\tau, \tau^*)][A_0^V(\tau^*)][A_e(\tau)][E]$$

We recall that $J^c(\tau) = (A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)) * (A_1^c(\tau) * U_1^c(\tau) * S_1^c(\tau))$, while $Q_0(\tau, \tau^*)$ adds $A_{[2, \omega]}^c(\tau, \tau^*)$, $U_{[2, \omega]}^c(\tau, \tau^*)$ and $S_{[2, \omega]}^c(\tau, \tau^*)$. The reader is advised to keep in mind that μ_j for $j \leq 3$ depends on τ while μ_j for $j \geq 4$ depends on τ^* , so there is a “seam” between μ_3 and μ_4 .

Bearing in mind that $\mathbb{A}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1, \omega]}^c(\tau, \tau^*)$ is defined in the extension by $\mathbb{J}_0^c(\tau) = \mathbb{A}_0^c(\tau) * \mathbb{U}_0^c(\tau) * \mathbb{S}_0^c(\tau)$, we reorganize $J^c(\tau) * Q_0(\tau, \tau^*)$ as $J_0^c(\tau) * (A_{[1, \omega]}^c(\tau, \tau^*) * U_{[1, \omega]}^c(\tau, \tau^*) * S_{[1, \omega]}^c(\tau, \tau^*))$. So W_I is

$$V^{lbi}(\tau)[J_0^c(\tau) * (A_{[1, \omega]}^c(\tau, \tau^*) * U_{[1, \omega]}^c(\tau, \tau^*) * S_{[1, \omega]}^c(\tau, \tau^*))][T(\tau, \tau^*)][A_0^V(\tau^*)][A_e(\tau)][E].$$

The general idea is now to use the indestructibility guaranteed by Lemma 4.5, but there are a couple of obstacles:

- Since $\mathbb{A}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1, \omega]}^c(\tau, \tau^*)$ was defined in an extension by $\mathbb{A}_0^c(\tau) * \mathbb{U}_0^c(\tau) * \mathbb{S}_0^c(\tau)$, we need to treat μ_2 separately.
- For $n = 2$, $\mathbb{T}(\tau, \tau^*) \times \mathbb{A}_0^V(\tau^*) \times \mathbb{A}_e(\tau) \times \mathbb{E}$ does not fit perfectly into the hypotheses of Lemma 4.5 as applied to μ_4 and $\mathbb{A}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1, \omega]}^c(\tau, \tau^*)$, and extra arguments are required.

With a view to applying Lemma 4.5 to $\mathbb{A}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1, \omega]}^c(\tau, \tau^*)$ in $V^{lbi}(\tau)[J_0^c(\tau)]$, recall that:

- E is generic for a poset $\mathbb{E} \in V^{lbi}(\tau)$, where $|\mathbb{E}| < \mu_0$.
- $A_e(\tau)$ is generic for a Cohen poset adding μ_3 subsets of μ_0 , defined in $V^{lb}(\tau)$.
- $A_0^V(\tau^*)$ is generic for a Cohen poset adding many subsets of μ_{21} , defined in V .
- $T(\tau, \tau^*)$ is generic for a $< \mu_3$ -directed closed poset $\mathbb{T}(\tau, \tau^*)$ defined in $V^{lbi}(\tau)[J^c(\tau)]$, with $\mu_3 < |\mathbb{T}(\tau, \tau^*)| < \mu_4$.
- $\mathbb{A}_{[2, \omega]}^c(\tau, \tau^*)$ is a product of Cohen posets defined in V .
- The cardinals μ_j for $j \geq 3$ are indestructibly supercompact in V , and ϕ is an indestructible Laver function there. They remain supercompact in $V^{lbi}(\tau)[J_0^c(\tau)]$, and $\mathbb{A}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1, \omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1, \omega]}^c(\tau, \tau^*)$ was defined using a Laver function derived from ϕ .
- $V^{lbi}(\tau)[J_0^c(\tau)]$, is an extension of V by a poset of size μ_2 .

Now we verify that μ_{n+2} has the tree property in W_I for all n . For most n we can directly apply Lemma 4.5 to $\mathbb{A}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1,\omega]}^c(\tau, \tau^*)$ in $V^{lbi}(\tau)[J_0^c(\tau)]$. In this context $V_{\text{def}} = V^{lbi}(\tau)[J_0^c(\tau)]$, and $V_{\text{inn}} = V$. The reader is warned that since we are working with indices in the interval $[1, \omega)$, μ_{k+1} in our current context plays the role of μ_k in Lemma 4.5. In most cases, our appeals to Lemma 4.5 are justified by Lemma 4.9.

The proof involves various auxiliary models, which we have sought to name in a consistent way. The models W_I^x where $x = i, ii, iii$ are submodels of W_I which isolate some families of trees, and W_I^{x*} is a generic extension of W_I^x obtained by some form of quotient to term forcing.

Claim 9.2. μ_{n+2} has the tree property in W_I for all $n \geq 21$.

Proof. Use Lemma 4.5 with $\mathbb{D}^0 = \mathbb{A}_0^V(\tau^*)$, $\mathbb{D}^{\text{small}} = \mathbb{T}(\tau, \tau^*) \times \mathbb{A}_e(\tau) \times \mathbb{E}$, and the other factors trivial. The hypotheses of Lemma 4.5 are satisfied by appealing to Lemma 4.9, where we note that $\mathbb{A}_0^V(\tau^*)$ is a Cohen poset defined in V (which is V_{inn}) adding subsets of μ_{21} , so that it is a reasonable value for $\mathbb{D}^0$. $\square$

Claim 9.3. μ_{n+2} has the tree property in W_I for $n = 20$

Proof. Use Lemma 4.5 with $\mathbb{D}^1 = \mathbb{A}_0^V(\tau^*)$, $\mathbb{D}^{\text{small}} = \mathbb{T}(\tau, \tau^*) \times \mathbb{A}_e(\tau) \times \mathbb{E}$, and the other factors trivial. Again we use Lemma 4.9 to justify the appeal to Lemma 4.5, where this time $\mathbb{A}_0^V(\tau^*)$ is a Cohen poset defined in V adding subsets of μ_{n+1} , so that it is a reasonable value for $\mathbb{D}^1$. $\square$

Claim 9.4. μ_{n+2} has the tree property in W_I for $3 \leq n \leq 19$.

Proof. Use Lemma 4.5 with $\mathbb{D}^2 = \mathbb{A}_0^V(\tau^*)$, $\mathbb{D}^{\text{small}} = \mathbb{T}(\tau, \tau^*) \times \mathbb{A}_e(\tau) \times \mathbb{E}$, and the other factors trivial. In this range of values of n , $\mathbb{A}_0^V(\tau^*)$ is $< \mu_{n+2}$ -directed closed forcing defined in V , hence it is a reasonable value for $\mathbb{D}^2$. $\square$

Claim 9.5. μ_{n+2} has the tree property in W_I for $n = 2$.

Proof. This case is slightly harder because we need the factor $T(\tau, \tau^*)$, but this doesn't fit smoothly into Lemma 4.5. We will use the mutual genericity idea from Remark 4.11.

All the relevant μ_4 -trees lie in the model $W_I^i = V^{lbi}(\tau)[J_0^c(\tau)][\mathbb{A}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1,\omega]}^c(\tau, \tau^*)][E \times \mathbb{A}_e(\tau) \times T(\tau, \tau^*)]$. Let $T \in W_I^i$ be a μ_4 -tree.

Now while $|\mathbb{E}| < \mu_0$ (so $\mathbb{E}$ would be a reasonable value for $\mathbb{D}^{\text{small}}$), and $\mathbb{A}_e(\tau)$ is Cohen forcing defined in a model between V_{inn} and V_{def} adding Cohen subsets to μ_0 (so would be a reasonable value for $\mathbb{D}^0$), the poset $\mathbb{T}(\tau, \tau^*)$ does not fit into our indestructibility scheme.

Proceeding exactly as in the proof of Lemma 4.5, we construct a generic embedding j with critical point μ_4 in an extension $W_I^i[P_{1-3}]$. Since $|\mathbb{T}(\tau, \tau^*)| < \mu_4$, no additional forcing is needed to handle $T(\tau, \tau^*)$. Using j we obtain a branch b of the tree T with $b \in W_I^i[P_{1-3}]$, and aim to show that $b \in W_I^i$.

To this end we force over $W_I^i[P_{1-3}]$ with a “quotient to term” forcing $\mathbb{QTT}$ to remove the dependence of $T(\tau, \tau^*)$ on S_1^c , obtaining a generic object TT for the term forcing poset $\mathbb{TT} = \mathcal{A}^{V^{lbi}(\tau)[J_0^c(\tau)][A_1^c(\tau)*U_1^c(\tau)]}(\mathbb{S}_1^c(\tau), \mathbb{T}(\tau, \tau^*))$ such that $S_1^c(\tau) \times TT$ induces $S_1^c(\tau) * T(\tau, \tau^*)$.

Note that:

- (1) TT is generic for the term forcing poset $\mathbb{TT}$ which is $< \mu_3$ -closed in the model $V^{lbi}(\tau)[J_0^c(\tau)][A_1^c(\tau) * U_1^c(\tau)]$, and by routine calculations this term poset is still $< \mu_3$ -distributive in $V^{lbi}(\tau)[J_0^c(\tau)][A_{[1,\omega]}^c(\tau, \tau^*) * U_{[1,\omega]}^c(\tau, \tau^*) * S_{[1,\omega]}^c(\tau, \tau^*)]$.
- (2) It follows from Lemma 2.44 that $\mathbb{QTT}$ is defined and $< \mu_2$ -closed in the model $V^{lbi}(\tau)[A_{[0,1]}^c(\tau) * U_{[0,1]}^c(\tau) * S_{[0,1]}^c(\tau)][T(\tau, \tau^*)]$, and by the usual distributivity arguments $\mathbb{QTT}$ remains $< \mu_2$ -closed in the model $V^{lbi}(\tau)[A^c(\tau, \tau^*) * U^c(\tau, \tau^*) * S^c(\tau, \tau^*)][T(\tau, \tau^*)]$.

Let $W_I^{i*} = W_I^i[QTT] = V^{lbi}(\tau)[A^c(\tau, \tau^*) * U^c(\tau, \tau^*) * S^c(\tau, \tau^*)][E \times A_e(\tau) \times TT]$, so that $b \in W_I^{i*}[P_{1-3}]$. We now proceed to argue that $b \in W_I^{i*}$ by a similar line of argument to that in Lemma 4.5.

Let $M_0 = W_I^{i*}$, $M_1 = M_0[P_{2b}]$, $M_2 = M_1[P_{1a} \times P_{1b} \times P_3]$, $M_3 = M_2[P_{2a}]$. The arguments that $b \in M_1 \implies b \in M_0$ and $b \in M_3 \implies b \in M_2$ work exactly as before. To complete the argument we need only to argue that M_2 is an extension of M_1 by “formerly $< \mu_3$ -closed” forcing in the sense of Fact 2.12.

Arguing as before, $\mathbb{P}_{1a} \times \mathbb{P}_{1b} \times \mathbb{P}_3$ is $< \mu_3$ -closed in $V^{lbi}(\tau)[A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)][A_{[1,\omega]}^c(\tau, \tau^*) * U_{[1,\omega]}^c(\tau, \tau^*) * S^c(\tau, \tau^*) \restriction [\mu_3, \mu_\omega]]$, and it remains $< \mu_3$ -closed in $M_- = V^{lbi}(\tau)[A_0^c(\tau) * U_0^c(\tau) * S_0^c(\tau)][A_{[1,\omega]}^c(\tau, \tau^*) * U_{[1,\omega]}^c(\tau, \tau^*) * S^c(\tau, \tau^*) \restriction [\mu_3, \mu_\omega]][TT]$. Now $M_0 = M_-[S^c(\tau, \tau^*) \restriction [\mu_2, \mu_3] \times E \times A_e(\tau)]$ and $M_1 = M_-[P_{2b}][S^c(\tau, \tau^*) \restriction [\mu_2, \mu_3] \times E \times A_e(\tau)]$. Since $S^c(\tau, \tau^*) \restriction [\mu_2, \mu_3] \times E \times A_e(\tau)$ is μ_3 -cc in $M_-[P_{2b}]$, $\mathbb{P}_{1a} \times \mathbb{P}_{1b} \times \mathbb{P}_3$ is formerly $< \mu_3$ -closed, and we see that $b \in M_2 \implies b \in M_1$.

We have shown that $b \in M_0 = W_I^{i*} = W_I^i[QTT]$. Since QTT is mutually generic with P_{1-3} and $b \in W_I^i[P_{1-3}]$, $b \in W_I^i$ and we are done. $\square$

Claim 9.6. μ_{n+2} has the tree property in W_I for $n = 1$.

Proof. Again this case needs a slightly different argument, using some of the ingredients from the proof of Claim 9.5, but appealing directly to Lemma 4.5 and avoiding the use of mutual genericity.

As in the preceding case, all the relevant μ_3 -trees lie in the model $W_I^{ii} = V^{lbi}(\tau)[A^c(\tau, \tau^*) * U^c(\tau, \tau^*) * S^c(\tau, \tau^*)][E \times A_e(\tau) \times T(\tau, \tau^*)]$, and the troublesome factor is $T(\tau, \tau^*)$. Let $T \in W_I^{ii}$ be a μ_3 -tree.

Exactly as in the proof of Claim 9.5, we force over W_I^{ii} with the quotient to term forcing $\mathbb{QTT}$ to obtain a term forcing generic TT such that $S_1^c(\tau, \tau^*) \times TT$ induces $S_1^c(\tau, \tau^*) * T(\tau, \tau^*)$. Since TT is generic for a term forcing poset $\mathbb{TT}$ which is defined and $< \mu_3$ -directed closed in $V^{lbi}(\tau)[A_{[0,1]}^c(\tau, \tau^*) * U_{[0,1]}^c(\tau, \tau^*) * S_0^c(\tau, \tau^*)]$, we may

appeal to Lemma 4.5 for $\mathbb{A}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{U}_{[1,\omega]}^c(\tau, \tau^*) * \mathbb{S}_{[1,\omega]}^c(\tau, \tau^*)$ with $\mathbb{D}^{\text{small}} = E$, $\mathbb{D}^0 = \mathbb{A}_e(\tau)$, and $\mathbb{D}^3 = \mathbb{T}\mathbb{T}$. As before, Lemma 4.9 ensures that we satisfied the hypotheses of Lemma 4.5. We conclude that μ_3 has the tree property in $W_I^{ii}[QTT]$, so that our tree has a branch b in $W_I^{ii}[QTT]$.

Since $\mathbb{Q}\mathbb{T}\mathbb{T}$ is $< \mu_2$ -closed in $V^{lbi}(\tau)[A^c(\tau, \tau^*) * U^c(\tau, \tau^*) * S^c(\tau, \tau^*)][T(\tau, \tau^*)]$, and $\mathbb{E} \times \mathbb{A}_e(\tau)$ is μ_2 -cc in this model, $\mathbb{Q}\mathbb{T}\mathbb{T}$ is formerly $< \mu_2$ -closed in W_I^{ii} and so $b \in W_I^{ii}$ by Fact 2.12. $\square$

Claim 9.7. μ_{n+2} has the tree property in W_I for $n = 0$.

Proof. Routine calculation shows that all the relevant μ_2 -trees lie in W_I^{iii} , where $W_I^{iii} = V^{lbi}(\tau)[J^c(\tau)][A_2^c(\tau, \tau^*) \times E \times A_e(\tau)]$. The key point is that $A_{[0,2]}^c(\tau, \tau^*) * U_0^c(\tau) * S_0^c(\tau)$ is a forcing poset which is in the scope of Lemma 4.10.

We need to extend W_I^{iii} before applying Lemma 4.10. Let

$$\mathbb{T}\mathbb{B}\mathbb{C} = \mathcal{A}^{V^{lbi}(\tau)[A_0^c(\tau) * U_0^c(\tau)]}(\mathbb{S}_0^c(\tau), \mathbb{B}_1^c(\tau) \times \mathbb{C}_1^c(\tau)),$$

so by Lemma 2.33 $\mathbb{T}\mathbb{B}\mathbb{C}$ is $< \mu_2$ -directed closed in $V^{lbi}(\tau)[A_0^c(\tau) * U_0^c(\tau)]$.

Forcing over $V^{lbi}(\tau)[J^c(\tau)]$ with an appropriate quotient forcing $\mathbb{Q}\mathbb{T}\mathbb{T}_0$, we may obtain an extension of the form $V^{lbi}(\tau)[A_{[0,1]}^c(\tau) * U_0^c(\tau) * S_0^c(\tau)][B_1^c(\tau) \times C_1^c(\tau)]$. Since $\mathbb{A}_1^c(\tau) * \mathbb{U}_1^c(\tau) * \mathbb{S}_1^c(\tau)$ is the first stage of an $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction defined in $V^{lbi}(\tau)[J_0^c]$, it follows from Remark 3.10 that $\mathbb{Q}\mathbb{T}\mathbb{T}_0$ is $< \mu_1$ -closed in $V^{lbi}(\tau)[J^c(\tau)]$. Forcing with another quotient forcing $\mathbb{Q}\mathbb{T}\mathbb{T}_1$, we may further extend to obtain a model $V^{lbi}(\tau)[A_{[0,1]}^c(\tau) * U_0^c(\tau) * S_0^c(\tau)][TBC]$: by Lemma 2.44 we see that $\mathbb{Q}\mathbb{T}\mathbb{T}_1$ is $< \mu_1$ -closed in $V^{lbi}(\tau)[J_0^c(\tau)][B_1^c(\tau) \times C_1^c(\tau)]$. By the distributivity of $\mathbb{A}_1^c(\tau)$, $\mathbb{Q}\mathbb{T}\mathbb{T}_1$ is $< \mu_1$ -closed in $V^{lbi}(\tau)[A_{[0,1]}^c(\tau) * U_0^c(\tau) * S_0^c(\tau)][B_1^c(\tau) \times C_1^c(\tau)]$, so that if we set $\mathbb{Q}\mathbb{T}\mathbb{T} = \mathbb{Q}\mathbb{T}\mathbb{T}_0 * \mathbb{Q}\mathbb{T}\mathbb{T}_1$ then $\mathbb{Q}\mathbb{T}\mathbb{T}$ is $< \mu_1$ -closed in $V^{lbi}(\tau)[J^c(\tau)]$.

Forcing with $\mathbb{Q}\mathbb{T}\mathbb{T}$ over W_I^{iii} , we get

$$\begin{aligned} W_I^{iii} &= V^{lbi}(\tau)[J^c(\tau)][A_2^c(\tau, \tau^*) \times E \times A_e(\tau)] \\ &\subseteq W_I^{iii*} = V^{lbi}(\tau)[A_{[0,2]}^c(\tau, \tau^*) * U_0^c(\tau) * S_0^c(\tau)][TBC \times E \times A_e(\tau)]. \end{aligned}$$

Now we use Lemma 4.10 to show that μ_2 has the tree property in W_I^{iii*} . To save the reader some work we record how the parameters from that Lemma should be set:

- n is 0.
- η is μ_4 , so that $A \restriction \eta$ is $A_0^c(\tau) \times A_1^c(\tau) \times A_2^c(\tau, \tau^*)$.
- V_{inn} is V .
- V_{def} is $V^{lbi}(\tau)$.
- $\mathbb{D}^{\text{small}}$ is E .
- $\mathbb{D}^3$ is $\mathbb{T}\mathbb{B}\mathbb{C}$.
- $\mathbb{D}^0$ is $\mathbb{A}_e(\tau)$.

We claim that the quotient-to-term forcing which we used to obtain W_I^{iii*} from W_I^{iii} is formerly $< \mu_1$ -closed in W_I^{iii} . To see this note that W_I^{iii} is obtained from $V^{lbi}(\tau)[J^c(\tau)]$ by adding $A_2^c(\tau, \tau^*)$ (which is generic for highly distributive forcing and preserves the closure) and then $E \times A_e(\tau)$ (which is generic for μ_1 -cc forcing). It follows that μ_2 has the tree property in W_I^{iii} . $\square$

This concludes the proof of Lemma 9.1 $\square$

9.2. Group II: $\sigma_{\omega+2}^a, \sigma_n^b$ for $n < \omega$.

Recall from Section 6 that $L(\tau) = L^b(\tau) * I^b(\tau) * (A_e(\tau) \times J^c(\tau))$ and is generic over $V[L \restriction \tau]$. Here $\mathbb{L}^b(\tau)$ is making the cardinals $\Lambda_n^b(\tau) = \sigma_n^b$ for $n < \omega$ indestructible, and $\mathbb{I}^b(\tau)$ is a forcing of the form $\mathbb{A} * \mathbb{U} * \mathbb{S}$ defined in $V[L \restriction \tau][L^b(\tau)]$ with parameters set as follows: $\mu_0 = \sigma_{17}^a$, $\mu_1 = \sigma_{\omega+1}^a$, $\mu_2 = \sigma_{\omega+2}^a$, $\mu_{3+n} = \sigma_n^b$. The poset $\mathbb{I}^b(\tau)$ uses the indestructible Laver function added by $L^b(\tau)$.

$$\begin{array}{cccc}
 \vdots & \vdots & \vdots & \vdots \\
 \vdots & \Lambda_1^b(\tau) & \sigma_1^b & \mu_4 \\
 \vdots & \Lambda_0^b(\tau) & \sigma_0^b & \mu_3 \\
 \vdots & \Lambda_{\omega+2}^a(\tau) & \sigma_{\omega+2}^a & \mu_2 \\
 \vdots & \Lambda_{\omega+1}^a(\tau) & \sigma_{\omega+1}^a & \mu_1 \\
 \vdots & \vdots & \vdots & \vdots \\
 \vdots & \Lambda_{17}^a(\tau) & \sigma_{17}^a & \mu_0 \\
 \vdots & \vdots & \vdots & \vdots \\
 \tau_i & \tau & \vdots & \vdots
 \end{array}$$

By the design of $\mathbb{I}^b(\tau)$, all the cardinals in Group II have the tree property in the model $V[L \restriction \tau][L^b(\tau)][I^b(\tau)]$. As in Group I, to see that the cardinals in Group II have the tree property in our final model $V[L][A^{gg}][\bar{P}]$ we have to account for various generic objects added by $L \restriction [\tau, \kappa)$ and by $\bar{P}$. The objects of potential concern are:

- E , which we recall is added by the interleaved posets between Prikry points up to and including τ . This object is slightly more troublesome in this group because it is generic for forcing of size $\mu_2 = \sigma_{\omega+2}^a$.
- $A_e(\tau) \times J^c(\tau)$, where $A_e(\tau)$ is adding subsets to $\sigma_{17}^b = \mu_{20}$ and $J^c(\tau)$ is doing the first two steps of an $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction whose first few cardinal parameters are $\sigma_{17}^b, \sigma_{\omega+1}^b, \sigma_{\omega+2}^b$: since we only care about trees up to σ_ω^b , the only relevant part of $J^c(\tau)$ is A_0^c .
- $Q(\tau, \tau^*)$, which we can safely ignore since it is generic for σ_ω^b -distributive forcing.

So all the relevant trees lie in the model $W_{II} = V^{lbi}(\tau)[E][A_e(\tau) \times A_0^c(\tau)]$.

We need a slightly finer analysis of E :

- If τ is not the first Prikry point, let τ^- be the preceding Prikry point. Then $E = E_0 \times Q(\tau^-, \tau)$ where $\mathbb{E}_0$ represents the product of interleaved forcing posets up to τ^- . It is easy to see that $\mathbb{E}_0 \in V[L \restriction \tau]$ and $|\mathbb{E}_0| < \tau < \mu_0$. Furthermore $Q(\tau^-, \tau) = \prod_{i < 3} Q_i(\tau^-, \tau)$ where:
 - $Q_0(\tau^-, \tau)$ is defined in $V^{lbi}(\tau^-)[J^c(\tau^-)]$ (which is a submodel of $V[L \restriction \tau]$) and $Q_0(\tau^-, \tau)$ is generic for a version of $\mathbb{A} * \mathbb{U} * \mathbb{S}$ forcing of size $\sigma_{\omega+1}^a = \mu_1$ defined in $V^{lbi}(\tau^-)[J^c(\tau^-)]$. Appealing to Lemma 3.21, in $V^{lbi}(\tau^-)[J^c(\tau^-)]$ the poset $Q_0(\tau^-, \tau)$ embeds into a two-step iteration where the first step is $< \mu_1$ -distributive and the second step is μ_1 -cc.
 - $Q_1(\tau^-, \tau)$ is a “quotient to term” forcing defined in $V[L \restriction \tau][A_0^b(\tau)]$, refining $A_0^b(\tau)$ to $A_0^V(\tau)$ which is a generic object for $\text{Add}^V(\sigma_{17}^a, \sigma_{\omega+2}^a) = \text{Add}^V(\mu_0, \mu_2)$, so that $V[L \restriction \tau][A_0^b(\tau)][Q_1(\tau, \tau^*)] = V[(L \restriction \tau) \times A_0^V(\tau)]$.
 - $Q_2(\tau^-, \tau)$ is generic for a forcing of cardinality $\tau < \mu_0$, defined in $V[L \restriction \tau]$.
- If τ is the first Prikry point then $E = \prod_{i < 3} Q_i^*(\tau)$ where:
 - $Q_0^*(\tau)$ is again generic for a version of $\mathbb{A} * \mathbb{U} * \mathbb{S}$ forcing of size μ_1 , which embeds into an iteration where $< \mu_1$ -distributive forcing is followed by μ_1 -cc forcing.
 - $Q_1^*(\tau)$ is generic for a quotient-to-term forcing defined exactly as above.
 - $Q_2^*(\tau)$ is generic for $\text{Coll}(\omega, \rho)$, where we note that $\rho < \tau < \mu_0$.

In summary, many of the factors in $\mathbb{E}$ are forcing posets which lie in $V^{lb}(\tau)$ and have size less than μ_0 .

Now we argue that all the cardinals in Group II have the tree property in W_{II} . We mostly do this by applying Lemma 4.5 to $\mathbb{I}^b(\tau)$, with $V_{\text{inn}} = V_{\text{def}} = V^{lb}(\tau)$. As before, most appeals to Lemma 4.5 are justified by Lemma 4.9.

Lemma 9.8. *For all $n < \omega$, μ_{n+2} has the tree property in W_{II} .*

Proof. As before we break the proof into several claims.

Claim 9.9. *μ_{n+2} has the tree property in W_{II} for $n \geq 20$.*

Proof. Appeal to Lemma 4.5 with $\mathbb{D}^{\text{small}} = \mathbb{E}$, $\mathbb{D}^0 = \mathbb{A}_e(\tau) \times \mathbb{A}_0^c(\tau)$, and the remaining factors trivial. $\square$

Claim 9.10. *μ_{n+2} has the tree property in W_{II} for $n = 19$.*

Proof. Appeal to Lemma 4.5 with $\mathbb{D}^{\text{small}} = \mathbb{E}$, $\mathbb{D}^1 = \mathbb{A}_e(\tau) \times \mathbb{A}_0^c(\tau)$, remaining factors trivial. $\square$

Claim 9.11. *μ_{n+2} has the tree property in W_{II} for $2 \leq n \leq 18$.*

Proof. Appeal to Lemma 4.5 with $\mathbb{D}^{small} = \mathbb{E}$, $\mathbb{D}^2 = \mathbb{A}_e(\tau) \times \mathbb{A}_0^c(\tau)$, remaining factors trivial. $\square$

Before we handle the last two cases, we need to discuss some issues involving the $\mathbb{Q}_0$ and $\mathbb{Q}_1$ factors in the posets $\mathbb{Q}(\tau^-, \tau)$ and $\mathbb{Q}^*(\tau)$.

- The $\mathbb{Q}_0$ factor: Let $\mathbb{Q}_0 = \mathbb{Q}_0^*(\tau)$ if τ is the first Prikry point, and $\mathbb{Q}_0 = \mathbb{Q}_0(\tau^-, \tau)$ otherwise. In either case $\mathbb{Q}_0$ is defined in some model $V[L']$ intermediate between V and $V[L \restriction \tau]$. By the analysis from Section 3.2, in $V[L']$ we may write $\mathbb{Q}_0$ as the projection of a two-step iteration $\mathbb{Q}_0^{\text{dist}} * \mathbb{Q}_0^{\text{cc}}$, where:
 - $|\mathbb{Q}_0^{\text{dist}} * \mathbb{Q}_0^{\text{cc}}| = \mu_1 = \sigma_{\omega+1}^a$.
 - For all large $n < \omega$, $\mathbb{Q}_0^{\text{dist}}$ is the projection of some σ_n^a -closed forcing poset (so that $\mathbb{Q}_0^{\text{dist}}$ is $< \mu_1$ -distributive).
 - It is forced by $\mathbb{Q}_0^{\text{dist}}$ that $\mathbb{Q}_0^{\text{cc}}$ is the union of σ_ω^a filters, so that it is μ_1 -cc in any outer model of $V[L'][\mathbb{Q}_0^{\text{dist}}]$ where μ_1 is still a cardinal.
- The $\mathbb{Q}_1$ factor: Let $\mathbb{Q}_1 = \mathbb{Q}_1^*(\tau)$ if τ is the first Prikry point, and $\mathbb{Q}_1 = \mathbb{Q}_1(\tau^-, \tau)$ otherwise. In either case $\mathbb{Q}_1$ is a quotient to term poset defined in $V[L \restriction \tau][A_0^b(\tau)]$, refining $A_0^b(\tau)$ to $A_0^V(\tau)$ which is generic for $\text{Add}^V(\mu_0, \mu_2)$: in the notation from Lemma 2.42 $\mathbb{P}$ is $\mathbb{L} \restriction \tau$ and $\mathbb{Q}$ is $\mathbb{A}_0^b(\tau)$. We would like to set $\mathbb{D}^0$ equal to $\mathbb{Q}_1$ but there are some obstacles:
 - The definition of the poset involves $A_0^b(\tau)$, so it is not in V_{def} (which is $V[L \restriction \tau][L^b(\tau)]$) and thus hypothesis 7 of Lemma 4.5 is definitely not satisfied.
 - Hypotheses 7 and 8 require that $\mathbb{D}^0$ and some related posets have quite a robust chain condition, which in our context should be the μ_1 -cc since we plan to deal with μ_2 as well as μ_3 . We need to verify versions of these hypotheses, appropriately modified to handle the dependence on $A_0^b(\tau)$, for the poset $\mathbb{Q}_1$.

The cure for the first of these issues is to modify the statement and proof of Lemma 4.5 to permit some dependence of $\mathbb{D}^0$ on $\mathbb{A}$. In the version appropriate for Claim 9.12 (resp. 9.13) we modify the hypotheses concerning $\mathbb{D}^0$ as follows:

- (1) $\mathbb{D}^0 \in V_{\text{def}}[A_{n-1}]$ (resp. $\mathbb{D}^0 \in V_{\text{def}}[A_n]$), and $\mathbb{D}^0$ is μ_{n+1} -Knaster in $V_{\text{def}}[A * U * S][D^{1,2,3}][P_{2b}]$.
- (2) For any W' which is an extension of $W[P_{2b}]$ by a forcing which is $< \mu_{n+1}$ -closed in $V_{\text{def}}[A * U * S \restriction [\mu_{n+1}, \mu_\omega]][D^2]$, and any j as in hypotheses 6 of Lemma 4.5, if $\mathbb{P}_{2a} = j(\mathbb{A}_{n-1} * \mathbb{D}^0)/j[A_{n-1} \times D^0]$ (resp. $\mathbb{P}_{2a} = j(\mathbb{A}_n \times \mathbb{D}^0)/j[A_n \times D^0]$) then $\mathbb{P}_{2a}$ is μ_{n+1} -Knaster in W' .

It is straightforward to modify the proof of Lemma 4.5 for these slightly more general hypotheses.

In our specific context we can use the following observations to satisfy these hypotheses:

- By Lemmas 2.43 and 2.42, to verify that $\mathbb{Q}_1$ is μ_1 -Knaster in some outer model it is sufficient to verify that $\mathbb{A}_0^V(\tau) = \text{Add}^V(\mu_0, \mu_2)$ is μ_1 -Knaster in the same model.
- By Lemma 2.41, $\mathbb{A}_0^b(\tau) * \mathbb{Q}_1$ is equivalent to $\mathbb{A}_0^V(\tau)$ in $V[L \restriction \tau]$, so as for $\mathbb{Q}_1$ it is enough to check that $\mathbb{A}_0^V(\tau)$ is μ_1 -Knaster.
- $|\mathbb{A}_0^b(\tau) * \mathbb{Q}_1| = \mu_2$, so that in the case $n = 1$ this poset is fixed by j and the technical hypothesis involving stretching by j is vacuously true.
- In the case $n = 0$, $j(\mathbb{A}_0^b(\tau) * \mathbb{Q}_1) / j[\mathbb{A}_0^b(\tau) * \mathbb{Q}_1]$ is easily seen to be equivalent to $\text{Add}^V(\mu_0, j(\mu_2) \setminus \mu_2)$, so that again verifying μ_1 -Knasterness amounts to verifying this property for a Cohen poset adding subsets of μ_0 and defined in V .

Claim 9.12. μ_{n+2} has the tree property in W_{II} for $n = 1$.

Proof. Appeal to Lemma 4.5 (modified as above to permit $\mathbb{D}^0$ to depend on A) where $\mathbb{D}^2$ is $\mathbb{A}_e(\tau) \times \mathbb{A}_0^c(\tau)$, $\mathbb{D}^0$ is $\mathbb{Q}_1(\tau^-, \tau)$ or $\mathbb{Q}_1^*(\tau)$, and $\mathbb{D}^{\text{small}}$ is the product of the remaining factors in $\mathbb{E}$. $\square$

Claim 9.13. μ_{n+2} has the tree property in W_{II} for $n = 0$.

Proof. Appeal to Lemma 4.5 in the more general version from Remark 4.12. Here $\mathbb{D}^2 = \mathbb{A}_e(\tau) \times \mathbb{A}_0^c(\tau)$, the factor $\mathbb{D}$ from Remark 4.12 is $\mathbb{Q}_0(\tau^-, \tau)$ or $\mathbb{Q}_0^*(\tau)$, $\mathbb{D}^0$ is $\mathbb{Q}_1(\tau^-, \tau)$ or $\mathbb{Q}_1^*(\tau)$, and $\mathbb{D}^{\text{small}}$ is the product of the remaining factors in $\mathbb{E}$.

To see that this is legitimate we need to verify that the hypotheses from Remark 4.12 are satisfied. Since $\mathbb{D}^1$ is trivial and $|\mathbb{D}^0| < \mu_0$, $\mathbb{P}_{2a}$ and $\mathbb{P}_{2b}$ are Cohen posets computed in V_{def} adding subsets to μ_0 and μ_1 respectively, so that by the usual arguments we can establish the necessary Knasterness and distributivity hypotheses. As we already discussed, $\mathbb{Q}_0^{\text{cc}}$ has a very robust form of μ_1 -cc, so it remains to show that $\mathbb{Q}_0^{\text{dist}}$ is $< \mu_1$ -distributive in $V^{lb_i}(\tau)[A_e(\tau) \times \mathbb{A}_0^c(\tau)]$.

Since $|\mathbb{Q}_0^{\text{dist}}| = \mu_1$ and $A_e(\tau) \times \mathbb{A}_0^c(\tau)$ is generic for highly distributive forcing, it is enough to verify the distributivity in $V^{lb_i}(\tau)$. In fact by article 7 of Lemma 3.13 it will be enough to verify it in $V^{lb}(\tau)[A_{[0,1]}^b(\tau) * U_0^b(\tau) * S_0^b(\tau)]$.

Recall from Remark 6.1 that since $\mathbb{L}^b(\tau)$ is μ_1 -closed, $\mathbb{A}_0^b(\tau)$ and $\mathbb{A}_1^b(\tau)$ are Cohen posets defined in $V[L \restriction \tau]$, moreover $\mathbb{A}_0^b(\tau)$ is μ_0^+ -cc (where $\mu_0 = \sigma_{17}^a(\tau)$) and $\mathbb{A}_1^b(\tau)$ is $< \mu_1$ -closed. In $V^{lb}(\tau)$, $\mathbb{A}_{[0,1]}^b(\tau) * \mathbb{U}_0^b(\tau) * \mathbb{S}_0^b(\tau)$ is the projection of $\mathbb{A}_{[0,1]}^b(\tau) \times \mathbb{B}_0^b(\tau) \times \mathbb{C}_0^b(\tau)$, where $\mathbb{B}_0^b(\tau) \times \mathbb{C}_0^b(\tau)$ is $< \mu_1$ -closed.

By the usual methods we may extend $V^{lb}(\tau)[A_{[0,1]}^b(\tau) * U_0^b(\tau) * S_0^b(\tau)]$ to $V[L \restriction \tau][A_0^b(\tau) \times T]$, where $A_0^b(\tau)$ is generic over $V[L \restriction \tau]$ for μ_0^+ -cc forcing, and T is generic over $V[L \restriction \tau]$ for some $< \mu_1$ -closed term forcing. By Easton's lemma, for all large n we have that $\mathbb{Q}_0^{\text{dist}}$ is σ_n^a -distributive in $V[L \restriction \tau][A_0^b(\tau) \times T]$. So $\mathbb{Q}_0^{\text{dist}}$ is $< \mu_1$ -distributive in $V^{lb_i}(\tau)[A_e(\tau) \times \mathbb{A}_0^c(\tau)]$, as required for an appeal to Remark 4.12.

□

This concludes the proof of Lemma 9.8

□

9.3. Group III (At the first Prikry point): θ, σ_n^a for $n < \omega$

Let τ' be the first Prikry point, and define $\sigma_n^a = \Lambda_n^a(\tau')$ and so on as usual. We recall some salient facts and definitions from Section 7.2.

- $V = V_0[A_0 * U_0 * L^0]$, $V_1 = V_0[A_0 * U_0]$.
- $L \restriction \tau'$ is generic over V for θ^+ -directed τ' -cc forcing of cardinality τ' .
- $L^0 * L \restriction \tau'$ is generic for θ^+ -directed closed forcing defined in V_1 .
- $\bar{\theta} < \rho < \theta < \tau' < \sigma_0^a$.
- ρ is a limit of supercompact cardinals in $V_0[A_0 \restriction \bar{\theta} * U_0 \restriction \bar{\theta}]$, but becomes an ω -successor cardinal in $V_0[A_0 \restriction \bar{\theta} * U_0 \restriction \bar{\theta} + 1]$.
- $A_0^b(\tau')$ is generic for $\text{Add}^{V[L \restriction \tau']}(\sigma_{17}^a, [\sigma_{\omega+1}^a, \sigma_{\omega+2}^a])$. This is added as part of $L(\tau')$.
- $\mathbb{Q}_0^*(\tau') \in V$. $A_0 * U_0$ combines with $\mathbb{Q}_0^*(\tau')$ to give us a generic object $A * U * S$ for a two-phase $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction with cardinal parameters $\mu_0 = \omega$, $\mu_1 = \rho^+$, $\mu_2 = \theta$, and $\mu_{3+n} = \sigma_n^a$ for $n < \omega$. So $\mathbb{Q}_0^*(\tau')$ adds $A_{[1,\omega]}$, $U_{[1,\omega]}$ and $S = S_0 * S_{[1,\omega]}$.
- $\mathbb{C}_0$ is defined in V_0 , while $\mathbb{B}_n$ and $\mathbb{C}_n$ for $n \geq 1$ are defined in V . The definition of $\mathbb{S}_0$ does not depend on U^0 .
- θ is indestructibly supercompact in V_0 . $A_0 * U_0 * S_0$ may be viewed as generic for the first stage of an $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction defined over V_0 with parameters ω, ρ^+, θ and using the indestructible Laver function ϕ_0 from V_0 .
- The cardinals μ_k for $k \geq 3$ are indestructibly supercompact in V , with an indestructible Laver function ϕ which was added by L^0 . It was ϕ which was used to define $\mathbb{A}_{[1,\omega]} * \mathbb{U}_{[1,\omega]} * \mathbb{S}_{[1,\omega]}$. So $A_{[1,\omega]} * U_{[1,\omega]} * S_{[1,\omega]}$ may be viewed as generic for a version of the $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction defined in V with parameters $\mu_1, \mu_2, \mu_3 \dots$ and ϕ .
- $\mathbb{A}_1$ is defined in $V_0[A_0 \restriction \bar{\theta} * U_0 \restriction \bar{\theta} + 1]$, $\mathbb{A}_n$ for $n \geq 2$ is defined in V .
- $\mathbb{Q}_1^*(\tau') \in V[L \restriction \tau']$. $\mathbb{Q}_1^*(\tau')$ is a “quotient to term” forcing poset, and refines $A_0^b(\tau')$ to $A_0^V(\tau')$ which is $\mathbb{A}_0^V(\tau')$ -generic where $\mathbb{A}_0^V(\tau') = \text{Add}^V(\sigma_{17}^a, \sigma_{\omega+2}^a)$ -generic.
- $\mathbb{Q}_2^*(\tau') = \text{Coll}(\omega, \rho)$, we call the generic object h .

$$\begin{array}{cccc}
& \vdots & \vdots & \vdots \\
& \vdots & \Lambda_1^a(\tau') & \sigma_1^a \mu_4 \\
& \vdots & \Lambda_0^a(\tau') & \sigma_0^a \mu_3 \\
\tau_0 & \tau' & \vdots & \vdots \\
& \vdots & \theta & \mu_2 \\
& \vdots & \rho^+ & \mu_1 \\
& \vdots & \omega & \mu_0
\end{array}$$

Arguing as in Sections 9.1 and 9.2, all the relevant trees for Group III lie in W_{III} where $W_{III} = V[L \restriction \tau][A_{[1,\omega)} * U_{[1,\omega)} * S][A_0^V(\tau')][h]$.

Lemma 9.14. *For all $n < \omega$, μ_{n+2} has the tree property in W_{III} .*

Proof. We will break the problem of establishing the tree property at μ_{n+2} into two parts. For $n \geq 1$ we will apply Lemma 4.5 to $\mathbb{A}_{[1,\omega)} * \mathbb{U}_{[1,\omega)} * \mathbb{S}_{[1,\omega)}$, working over V , which makes sense because $\mathbb{A}_{[1,\omega)} * \mathbb{U}_{[1,\omega)} * \mathbb{S}_{[1,\omega)}$ is an $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction defined in V which establishes the tree property from μ_3 onwards. In this setting we have to account for the effects of $L \restriction \tau$, S_0 , $A_0^V(\tau')$ and h .

Claim 9.15. *μ_{n+2} has the tree property in W_{III} for $n \geq 20$.*

Proof. Appeal to Lemma 4.5 with $\mathbb{D}^{\text{small}} = \mathbb{L} \restriction \tau \times S_0 \times h$, $\mathbb{D}^0 = \mathbb{A}_0^V(\tau')$. □

Claim 9.16. *μ_{n+2} has the tree property in W_{III} for $n = 19$.*

Proof. Appeal to Lemma 4.5 with $\mathbb{D}^{\text{small}} = \mathbb{L} \restriction \tau \times S_0 \times h$, $\mathbb{D}^1 = \mathbb{A}_0^V(\tau')$. □

Claim 9.17. *μ_{n+2} has the tree property in W_{III} for $2 \leq n \leq 18$.*

Proof. Appeal to Lemma 4.5 with $\mathbb{D}^{\text{small}} = \mathbb{L} \restriction \tau' \times S_0 \times h$, $\mathbb{D}^2 = \mathbb{A}_0^V(\tau')$. □

Claim 9.18. *μ_{n+2} has the tree property in W_{III} for $n = 1$.*

Proof. Note that A_0^V is generic for highly distributive forcing, so adds no μ_3 -trees and can be ignored in this context. Appeal to Lemma 4.5 with $\mathbb{D}^{\text{small}} = \text{Coll}(\omega, \rho)$, $\mathbb{D}^0 = \mathbb{S}_0$, $\mathbb{D}^1 = L \restriction \tau'$. □

Claim 9.19. *μ_{n+2} has the tree property in W_{III} for $n = 0$.*

Proof. We are concerned with $\mu_2 = \theta$. By the usual distributivity arguments, the only relevant part of $A_{[1,\omega)} * U_{[1,\omega)} * S_{[1,\omega)}$ is $(A_1 * U_1 * S_1) \times A_2$. Note that $A_2 = \text{Add}(\mu_2, [\mu_3, \mu_4))^V = \text{Add}(\mu_2, [\mu_3, \mu_4))^{V_1}$, where the last equality holds by the distributivity of L^0 .

We aim to apply Lemma 4.10 but we need to be careful because $\mathbb{A}_1$ is only defined in the model $V_0[A^0 \restriction \bar{\theta} * U^0 \restriction \bar{\theta} + 1]$, so we view this as the ground model V_{def} in our appeal to the lemma. This is not a problem because the remaining part

of $A^0 * U^0$ is $A^0 \upharpoonright [\bar{\theta}, \theta] * U^0 \upharpoonright (\bar{\theta}, \theta)$, and by the definition of $\mathbb{Q}_0^*(\tau)$ the definition of $\mathbb{S}_0$ only uses $A^0 \upharpoonright \alpha$ where $\alpha > \mu_1 > \rho > \bar{\theta}$.

Working over $V_0[A^0 * U^0 * S_0]$, we have to account for the effects of L^0 (which prolongs V_1 to V) plus $L \upharpoonright \tau$, h and $A_{[1, \omega]} * U_{[1, \omega]} * S_{[1, \omega]}$.

As usual we may force over $V[A_1 * U_1 * S_1]$ with a suitable quotient to term forcing to remove the dependence of $U_1 * S_1$ on A_1 . We obtain $B_1 \times C_1$ which is $\mathbb{B}_1 \times \mathbb{C}_1$ -generic over $V[A_1]$, so that $A_1 \times B_1 \times C_1$ induces $A_1 * U_1 * S_1$ and $V[A_1 * U_1 * S_1] \subseteq V[A_1 \times B_1 \times C_1]$. $B_1 \times C_1$ is generic for $< \mu_2$ -directed closed forcing defined in V , and by Remark 3.10 the forcing which produces $B_1 \times C_1$ is generic for $< \mu_1$ -closed forcing defined in $V[A_1 * U_1 * S_1]$.

At this point we recall that S_0 is added as part of $Q^*(\tau)$, is generic for $\mathbb{S}_0 \in V_0[A_0]$, and is mutually generic with $U_0 * L^0 * L \upharpoonright \tau * (A_{[1, 2]} * U_1 * S_1 \times h)$. We will use Lemma 4.10 with the non-trivial parameters set as follows:

- $V_{\text{def}} = V_{\text{inn}} = V_0[A_0 \upharpoonright \bar{\theta} * U_0 \upharpoonright \bar{\theta} + 1]$.
- $\eta = \mu_3$
- $\mathbb{D}^3 = \mathbb{L}^0 * (\mathbb{L} \upharpoonright \tau \times \mathbb{A}_2 \times \mathbb{B}_1 \times \mathbb{C}_1)$
- $\mathbb{D}^0 = \text{Coll}(\omega, \rho)$.

It follows from Lemma 4.10 that θ has the tree property in

$$\begin{aligned} & V_0[A_0 * U_0 * S_0][A_1][L^0][L \upharpoonright \tau \times A_2 \times B_1 \times C_1][h] \\ &= V[L \upharpoonright \tau][S_0][A_1 \times A_2 \times B_1 \times C_1][h]. \end{aligned}$$

The forcing which produces $B_1 \times C_1$ is $< \mu_1$ -closed in $V[A_1 * U_1 * S_1]$, and retains this closure in $V[L \upharpoonright \tau][A_1 * U_1 * S_1 \times A_2]$. Since $\mathbb{A}_0 * \mathbb{S}_0$ is the projection of $\mathbb{A}_0 \times \mathbb{C}_0$ where $\mathbb{C}_0$ is $< \mu_1$ -closed in V_0 , the usual arguments show that $\mathbb{S}_0$ is $< \mu_1$ -distributive in $V[L \upharpoonright \tau][S_0][A_1 * U_1 * S_1 \times A_2]$, so that the forcing which produces $B_1 \times C_1$ is $< \mu_1$ -closed in $V[L \upharpoonright \tau'][S_0][A_1 * U_1 * S_1 \times A_2]$, and so is formerly $< \mu_1$ -closed in $V[L \upharpoonright \tau'][S_0][A_1 * U_1 * S_1 \times A_2][h]$ because $\text{Coll}(\omega, \rho)$ trivially has μ_1 -cc. So by Fact 2.12 μ_2 has the tree property in $V[L \upharpoonright \tau][S_0][A_1 * U_1 * S_1 \times A_2][h]$ and we are done. $\square$

This concludes the proof of Lemma 9.14. $\square$

9.4. Group IV (Successors of singulars)

: $\sigma_{\omega+1}^a, \sigma_{\omega+1}^b$

Finally we treat the cardinals below κ which are successors of singular cardinals. Such cardinals are either of the form $\sigma_{\omega+1}^a = \Lambda_{\omega+1}^a(\tau)$ or $\sigma_{\omega+1}^b = \Lambda_{\omega+1}^b(\tau)$ for some Prikry point τ . The case of $\Lambda_{\omega+1}^a(\tau')$ for τ' the first Prikry point, which becomes $\aleph_{\omega+1}$ in the final model, requires special attention.

Recall from Section 5.3 that we chose the cardinals, ρ , λ^a and λ^b so that for $\lambda \in \{\lambda_a, \lambda_b\}$ the cardinal $\lambda_{\omega+1}$ has the tree property in the extension of V by

$\mathbb{L}_{\text{aux}}(\rho, \lambda) \times \mathbb{R}_{\text{aux}}(\lambda)$, where $\lambda_{\omega+1}$ becomes $\aleph_{\omega+1}$. We defined the reflected versions $\Lambda^a(\tau)$ and $\Lambda^b(\tau)$ of λ^a and λ^b to secure the following key property: for every potential Prikry point τ , if $\lambda \in \{\Lambda^a(\tau), \Lambda^b(\tau)\}$ then $\lambda_{\omega+1}$ is forced to have the tree property in the extension of V by $\mathbb{L}_{\text{aux}}(\rho, \lambda) \times \mathbb{R}_{\text{aux}}(\lambda)$.

The key idea in this section is that for every relevant λ , we can establish the tree property at $\lambda_{\omega+1}$ in our final model $V[L][\bar{P}]$ by embedding an appropriate submodel of $V[L][\bar{P}]$ into the extension of V by $\mathbb{L}_{\text{aux}}(\rho, \lambda) \times \mathbb{R}_{\text{aux}}(\lambda)$. It is important that the quotient to term posets that accomplish this embedding are ρ -closed, and this consideration played a role in the design of our construction. The posets $\mathbb{L}_{\text{aux}}(\rho, \lambda)$ and $\mathbb{R}_{\text{aux}}(\lambda)$ were designed to absorb the many different posets which will appear in these embedding arguments. We will make repeated use of the term forcing and absorption arguments from Sections 2.6 and Section 2.7.

At this point it becomes important that all of the Laver functions used in our construction were derived from the initial Laver function ϕ_0 . It is for this reason that various products of term forcings which will be used in the absorption arguments fit the hypotheses of Lemma 2.50.

9.4.1. The cardinal $\sigma_{\omega+1}^b$

: By the usual arguments, all relevant trees in the final model lie in the submodel $V^{lb_i}(\tau)[E][J_0^c(\tau)][A_e(\tau) \times A_1^c(\tau)]$. Since any particular $\sigma_{\omega+1}^b$ -tree only involves at most $\sigma_{\omega+1}^b$ coordinates in the (highly homogeneous) generic object $A_e(\tau) \times A_1^c(\tau)$, it will suffice to establish the tree property in M , where $M = V^{lb_i}(\tau)[E][J_0^c(\tau)][A_e(\tau) \restriction \sigma_{\omega+2}^b \times A_1^c(\tau)']$ and $A_1^c(\tau)'$ is $\text{Add}^{V[L \restriction \tau][L^b(\tau)]}(\sigma_{\omega+1}^b, \sigma_{\omega+2}^b)$ -generic.

For the purposes of absorbing M into an extension of V by $\mathbb{L}_{\text{aux}}(\rho, \sigma_0^b) \times \mathbb{R}_{\text{aux}}(\sigma_0^b)$, we note that:

- $V = V_0[A_0 * U_0 * L^0] = \bar{W}[A_0 \restriction [\bar{\theta}, \theta) * U_0 \restriction (\bar{\theta} + 1, \theta)][L^0]$, where we note that the first element in the support of $U_0 \restriction (\bar{\theta} + 1, \theta)$ is much larger than ρ .
- E breaks down as $h \times A_1 \times E'$, where:
 - $h \times A_1$ is part of the generic object for the forcing at the first Prikry point τ' as described in Section 7.2. h is $\text{Coll}(\omega, \rho)$ -generic and A_1 is $\text{Add}^{\bar{W}}(\rho^+, \mu')$ -generic, where $\mu' = \Lambda_0^a(\tau')$.
 - E' collects the rest of E , that is the remainder of $Q^*(\tau')$ together with generic objects for the interleaved forcing posets up to and including τ . E' is generic for a poset of cardinality $\sigma_{\omega+2}^a$.
- $I^b(\tau)$ is generic for an $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction defined in $V^{lb}(\tau)$ with parameters $\sigma_{17}^a, \sigma_{\omega+1}^a, \sigma_{\omega+2}^a, \sigma_0^b, \dots$
- $J_0^c(\tau)$ is the first phase of a two-phase $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction defined in $V^{lb_i}(\tau)$ where the relevant parameters are $\sigma_{17}^b, \sigma_{\omega+1}^b$ and $\sigma_{\omega+2}^b$. $A_0^c(\tau)$ is generic for $\text{Add}^{V^{lb}(\tau)}(\sigma_{17}^b, [\sigma_{\omega+1}^b, \sigma_{\omega+2}^b])$, and the support of $U_0^c(\tau) * S_0^c(\tau)$ is contained in the interval $(\sigma_{\omega+1}^b, \sigma_{\omega+2}^b)$.

To help the reader keep track of the indices, we note that σ_n^b is playing the role of μ_{n+3} in $\mathbb{L}^b(\tau)$, and that the components of this forcing with index n have supports in the interval $[\mu_{n+1}, \mu_{n+2})$ as usual.

As a first step we set aside h (which will eventually be absorbed by the $\text{Coll}(\omega, \rho)$ component of $\mathbb{L}(\rho, \sigma_0^b)$), to obtain a model $\bar{M} = V^{lbi}(\tau)[A_1][E'][J_0^c(\tau)][A_e(\tau) \restriction \sigma_{\omega+2}^b \times A_1^c(\tau)']$. We note that $\bar{M}$ is a ρ -distributive extension of V . Then we isolate the generic objects which we plan to absorb into $\mathbb{R}_{\text{aux}}(\sigma_0^b)$: these are $L^b(\tau) \restriction [\sigma_0^b, \sigma_\omega^b)$, $A_{[3,\omega)}^b(\tau)$, $U_{[2,\omega)}^b(\tau) = U^b(\tau) \restriction [\sigma_0^b, \sigma_\omega^b)$, $S_{[2,\omega)}^b(\tau) = S^b(\tau) \restriction [\sigma_0^b, \sigma_\omega^b)$, $A_0^c(\tau)$, $U_0^c(\tau)$, $S_0^c(\tau)$, $A_e(\tau) \restriction \sigma_{\omega+2}^b$ and $A_1^c(\tau)'$.

We now need to specify how these various objects are to be absorbed into $\mathbb{R}_{\text{aux}}(\sigma_0^b)$ by doing a series of quotient to term forcings, which in every case will be ρ -closed in the models where they are defined. The closure of these forcings will follow by appealing to Fact 2.47, Lemma 2.50 and Lemma 2.52.

- $\mathbb{L}^b(\tau) \restriction [\sigma_0^b, \sigma_\omega^b)$ is a Laver-type iteration defined in the model $V[L \restriction \tau][L^b(\tau) \restriction \sigma_0^b]$.

We claim that $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \sigma_0^b * \mathbb{L}^b(\tau) \restriction [\sigma_0^b, \sigma_\omega^b)$ can be written as a projection of the product of $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \sigma_0^b$ and an Easton support product of term posets of the form $\mathcal{A}^V(\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \alpha, \phi(\alpha))$, taken over $\alpha \in (\sigma_0^b, \sigma_\omega^b)$ such that $\alpha \in \text{dom}(\phi)$ and $\phi(\alpha)$ is an appropriate name. The only tricky point is that since $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \sigma_0^b$ has cardinality σ_0^b , taking Easton supports in V suffices.

Since $\text{dom}(\phi)$ consists of inaccessible closure points of ϕ , and $\phi(\alpha)$ names a $< \alpha$ -directed closed poset for all relevant α , it follows from Lemma 2.50 that the Easton product of term forcing posets can be absorbed by the component $\prod_{n < \omega} \text{East}^{E_0}(\sigma_n^b, < \sigma_{n+1}^b)$ of $\mathbb{R}_{\text{aux}}(\sigma_0^b)$.

- $\mathbb{A}_{[3,\omega)}^b = \prod_{n \geq 3} \mathbb{A}_n^b$,

where $\mathbb{A}_n^b = \text{Add}^{V[L \restriction \tau * L^b(\tau)]}(\sigma_n^b, [\sigma_{n+1}^b, \sigma_{n+2}^b))$, and so by closure of tails of $L^b(\tau)$ in fact $\mathbb{A}_n^b = \text{Add}^{V[L \restriction \tau * L^b(\tau) \restriction \sigma_n^b]}(\sigma_n^b, [\sigma_{n+1}^b, \sigma_{n+2}^b))$.

For each n , $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \sigma_n^b * \mathbb{A}_n^b$ is the projection of the product of $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \sigma_n^b$ and $\mathcal{A}^V(L \restriction \tau * L^b(\tau) \restriction \sigma_n^b, \text{Add}(\sigma_n^b, [\sigma_{n+1}^b, \sigma_{n+2}^b))$. By Lemma 2.42 the term poset at n is equivalent to $\text{Add}^V(\sigma_n^b, \sigma_{n+2}^b)$, and can be absorbed by the component $\text{Add}(\sigma_n^b, \sigma_{n+2}^b)$ of $\mathbb{R}_{\text{aux}}(\sigma_0^b)$.

- We claim that $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{A}^b(\tau) * \mathbb{U}^b(\tau)$ is the projection of the product of $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{A}^b(\tau) * \mathbb{U}_{[0,1]}^b(\tau)$ and an Easton support product of term posets of the form $\mathcal{A}^V(\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \gamma_\alpha * \mathbb{A}^b(\tau) \restriction \alpha * \mathbb{U}^b(\tau) \restriction \alpha, \dot{\mathbb{U}}^b(\tau)(\alpha))$, where $\sigma_0^b < \alpha < \sigma_\omega^b$, $\gamma_\alpha < \alpha^*$ and γ_α is chosen large enough that $\mathbb{A}^b(\tau) \restriction \alpha * \mathbb{U}^b(\tau) \restriction \alpha + 1$ can be defined in $V[L \restriction \tau * L^b(\tau) \restriction \gamma_\alpha]$.

This is a variation on Lemma 3.16, adjusted to take account of the fact that $\mathbb{L}^b(\tau)$ is defined in the generic extension $V[L \restriction \tau][L^b(\tau)]$ of V . The only subtle points here are that we restricted $L^b(\tau)$ to make the term forcing small enough, and that we used Lemma 2.53 to ensure that we may take

an Easton support product in V .

This product of term forcing posets can be absorbed by the component $\prod_{n < \omega} \text{East}^{E_0}(\sigma_n^b, < \sigma_{n+1}^b)$ of $\mathbb{R}(\sigma_0^b)$.

- $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{A}^b(\tau) * \mathbb{U}^b(\tau) * \mathbb{S}^b(\tau)_{[2, \omega)}$ is the projection of the product of $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{A}^b(\tau) * \mathbb{U}^b(\tau)$ and a full support product of term posets defined for $n < \omega$, where at n we take the product with $< \sigma_n^b$ supports over $\alpha \in (\sigma_n^b, \sigma_{n+2}^b)$ of posets of the form $\mathcal{A}^V(\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \gamma_\alpha * \mathbb{A}^b(\tau) \restriction \alpha * \mathbb{U}^b(\tau) \restriction \sigma_n^b, \text{Add}(\sigma_n^b, 1))$ and $\delta_\alpha < \alpha^*$ is chosen large enough that $\mathbb{A}^b(\tau) \restriction \alpha * \mathbb{U}^b(\tau) \restriction \sigma_{n+1}^b * \text{Add}(\sigma_{n+1}^b, 1)$ can be defined in $V[L \restriction \tau * L^b \restriction \delta_\alpha]$.

The issue about supports is easier here than for $\mathbb{U}^b(\tau)_{[2, \omega)}$. Given a set of ordinals of size less than ρ_n^b in $V[L \restriction \tau][L^b(\tau)]$, we use closure to cover it by a small set in $V[L \restriction \tau][L^b(\tau) \restriction \rho_n^b]$, and then chain condition to cover the covering set by a small set in V .

Here the product of term forcing posets can be absorbed by the component $\prod_{n < \omega} \text{Coll}(\sigma_n^b, \sigma_{n+1}^b)$ of $\mathbb{R}_{\text{aux}}(\sigma_0^b)$.

- $\mathbb{A}_0^c(\tau) = \text{Add}^{V[L \restriction \tau][L^b(\tau)]}(\sigma_{17}^b, [\sigma_{\omega+}^b, \sigma_{\omega+2}^b))$, and by the closure of tails of $L^b(\tau)$, $\mathbb{A}_0^c(\tau) = \text{Add}^{V[L \restriction \tau][L^b(\tau) \restriction \sigma_{17}^b]}(\sigma_{17}^b, [\sigma_{\omega+}^b, \sigma_{\omega+2}^b))$.

$\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \sigma_{17}^b * \mathbb{A}_0^c(\tau)$ is a projection of the product of $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \rho_{17}^b$ and $\mathcal{A}^V(\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \sigma_{17}^b, \text{Add}(\sigma_{17}^b, [\sigma_{\omega+}^b, \sigma_{\omega+2}^b)))$, which is equivalent to $\text{Add}^V(\sigma_{17}^b, \sigma_{\omega+2}^b)$ by Lemma 2.42. The term forcing poset can be absorbed by the component $\text{Add}^V(\sigma_{17}^b, \sigma_{\omega+2}^b)$ of $\mathbb{R}_{\text{aux}}(\sigma_0^b)$.

- $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{I}^b(\tau) * \mathbb{A}_0^c(\tau) * \mathbb{U}_0^c(\tau)$ is the projection of the product of $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{I}^b(\tau) * \mathbb{A}_0^c(\tau)$ and an Easton support product of term forcing posets of the form $\mathcal{A}^V(\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{I}^b(\tau) * \mathbb{A}_0^c(\tau) \restriction \alpha * \mathbb{U}_0^c(\tau) \restriction \alpha, \dot{\mathbb{U}}_0^c(\tau)(\alpha))$ for relevant $\alpha \in (\sigma_{\omega+1}^b, \sigma_{\omega+2}^b)$. By similar arguments to those above, this product of term forcing posets can be absorbed by the component $\text{East}^{E_0}(\sigma_{\omega+1}^b, < \sigma_{\omega+2}^b)$ of $\mathbb{R}_{\text{aux}}(\sigma_0^b)$.
- $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{I}^b(\tau) * \mathbb{A}_0^c(\tau) * \mathbb{S}_0^c(\tau)$ is the projection of the product of $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{I}^b(\tau) * \mathbb{A}_0^c(\tau)$ with $< \sigma_{\omega+1}^b$ supports of term posets of the form $\mathcal{A}^V(\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{I}^b(\tau) * \mathbb{A}_1^c(\tau) \restriction \alpha, \text{Add}(\sigma_{\omega+1}^b, 1))$ for $\alpha \in (\sigma_{\omega+1}^b, \sigma_{\omega+2}^b)$. By similar arguments to those above, this product of term forcing posets can be absorbed by the component $\text{Coll}(\sigma_{\omega+1}^b, < \sigma_{\omega+2}^b)$ of $\mathbb{R}_{\text{aux}}(\sigma_0^b)$.
- $\mathbb{A}_e(\tau) \restriction \sigma_{\omega+2}^b = \mathbb{A}_0^c(\tau)$, and in exactly the same way it may be written as the projection of the product of $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) \restriction \rho_{17}^b$ and a term forcing, and the term forcing may be absorbed into the component $\text{Add}^V(\sigma_{17}^b, \sigma_{\omega+2}^b)$ of $\mathbb{R}_{\text{aux}}(\sigma_0^b)$.
- $\mathbb{A}_1^c(\tau)' = \text{Add}^{V[L \restriction \tau * L^b(\tau)]}(\sigma_{\omega+1}^b, \sigma_{\omega+2}^b)$, and $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau) * \mathbb{A}_1^c(\tau)'$ is the projection of the product of $\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau)$ and $\mathcal{A}^V(\mathbb{L} \restriction \tau * \mathbb{L}^b(\tau), \text{Add}(\sigma_{\omega+1}^b, \sigma_{\omega+2}^b))$. By Lemma 2.42 this term poset is equivalent to $\text{Add}^V(\sigma_{\omega+1}^b, \sigma_{\omega+2}^b)$, and so can be absorbed by the component $\text{Add}^V(\sigma_{\omega+1}^b, \sigma_{\omega+2}^b)$ of $\mathbb{R}_{\text{aux}}(\sigma_0^b)$.

Remark 9.20. Note that we have used some components of $\mathbb{R}_{\text{aux}}(\sigma_0^b)$ to absorb

multiple term forcing posets. This is not problematic, we may easily write (for example) $\text{East}^{E_0}(\sigma_n^b, < \sigma_{n+1}^b)$ as the product of two copies of itself and use it to absorb two Easton iterations in the interval $[\sigma_n^b, \sigma_{n+1}^b)$.

Forcing over $\bar{M}$ with an appropriate iteration of quotient to term forcings, we may absorb all the generic objects for these forcing posets which we isolated above into a generic object R_{aux} for $\mathbb{R}_{\text{aux}}(\sigma_0^b)$. Since $\bar{M}$ is a ρ -distributive extension of V and each step in our iteration is ρ -closed in the model where it is defined, the whole iteration is ρ -closed. We produce $M_0^* = V[L \restriction \tau][L^b(\tau) \restriction \sigma_0^b][A_{[0,2]}^b(\tau) * U_{[0,1]}^b(\tau) * S_{[0,1]}^b(\tau)][A_1][E'][R_{\text{aux}}]$, where M_0^* is a ρ -closed extension of $\bar{M}$. Let $M^* = M_0^*[h]$, so that M^* is an extension of $M = \bar{M}[h]$: by Lemma 2.13 the passage from M to M^* does not add branches to $\sigma_{\omega+1}^b$ -trees in M .

We need to absorb the generic objects other than R_{aux} used to obtain M_0^* . The generic objects $L \restriction \tau$, $L^b(\tau) \restriction \sigma_0^b$, $A_{[0,2]}^b * U_{[0,1]}^b * S_{[0,1]}^b$ and E' are generic for ρ -closed forcing posets of cardinality at most σ_1^b , each defined in some (possibly trivial) generic extensions of V . We note that all these posets actually exist in some generic extension of $V' = V_0[A^0 * U^0 * L^0 \restriction \sigma_1^b] = \bar{W}[Q]$, where $\mathbb{Q} = \mathbb{A}^0 \restriction [\bar{\theta}, \theta) * \mathbb{U}^0 \restriction (\bar{\theta} + 1, \theta) * \mathbb{L}^0 \restriction \rho_1^b$, and that in $\bar{W}$ the poset $\mathbb{Q}$ is σ_1^b -cc with cardinality σ_1^b .

Let $M_1^* = V'[L \restriction \tau][L^b(\tau) \restriction \sigma_0^b][A_{[0,2]}^b(\tau) * U_{[0,1]}^b(\tau) * S_{[0,1]}^b(\tau)][A_1][E'] = \bar{W}[Q][L \restriction \tau][L^b(\tau) \restriction \sigma_0^b][A_{[0,2]}^b(\tau) * U_{[0,1]}^b(\tau) * S_{[0,1]}^b(\tau)][A_1][E']$. We may perform a series of quotient to term forcings to embed M_1^* into a model of the form $\bar{W}[Q \times T \times A_1]$, where T is generic for some ρ -closed product of term forcings defined in $\bar{W}$, and we may assume that T is generic for forcing of size σ_1^b (it was for this reason that we truncated V to V'). Since A_1 is generic for small ρ -closed forcing defined in $\bar{W}$, we may do more forcing to embed M_1^* into a model $\bar{W}[Q \times L_{\text{aux}}^{\text{coll}}]$, where $L_{\text{aux}}^{\text{coll}}$ is generic for $\text{Coll}^{\bar{W}}(\rho^+, \sigma_1^b)$ which forms part of $\mathbb{L}_{\text{aux}}(\rho, \sigma_0^b)$. The quotient to term forcing used to produce $\bar{W}[Q \times L_{\text{aux}}^{\text{coll}}]$ from M_1^* also has cardinality at most σ_1^b .

Performing the same quotient to term forcing over the larger model M^* , we obtain a model $M^{**} = V[L_{\text{aux}}^{\text{coll}} \times h \times R_{\text{aux}}] = V[L_{\text{aux}} \times R_{\text{aux}}]$, where $L_{\text{aux}} = L_{\text{aux}}^{\text{coll}} \times h$ is $\mathbb{L}_{\text{aux}}(\rho, \sigma_0^b)$ -generic. By construction $\sigma_{\omega+1}^b$ has the tree property in M^{**} . By Lemma 2.11 the passage from M^* to M^{**} does not add branches to $\sigma_{\omega+1}^b$ -trees in M^* . So $\sigma_{\omega+1}^b$ has the tree property in M and we are done.

9.4.2. The cardinal $\sigma_{\omega+1}^{a*}$ above the first Prikry point

:

Let τ and τ^* be successive Prikry points, we will establish the tree property at $\sigma_{\omega+1}^{a*} = \Lambda_{\omega+1}^a(\tau^*)$ in the final model. We can do a similar analysis to that in Section 9.1 to find a submodel M of the final model in which all the relevant trees will lie. As in Section 9.4.1 we decompose E as $h \times A_1 \times E'$.

We need slightly more of the generic object $L^b(\tau^*) * I^b(\tau^*)$ than we did in Section 9.1, because there we only considered σ_n^{a*} -trees for n finite. Recalling that $\mathbb{I}^b(\tau^*)$ is an $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction whose first few parameters are σ_{17}^{a*} , $\sigma_{\omega+1}^{a*}$, $\sigma_{\omega+2}^{a*}$ we

see that the relevant parts of $L^b(\tau^*) * I^b(\tau^*)$ are $A_{[0,1]}^b(\tau^*)$, $U_0^b(\tau^*)$, $S_0^b(\tau^*)$ and $L^b(\tau^*) \restriction \sigma_{\omega+2}^{a*}$ (which is enough of $L^b(\tau^*)$ to define $A_{[0,1]}^b(\tau^*) * U_0^b(\tau^*) * S_0^b(\tau^*)$). By the same considerations as in Section 9.4.1, we may replace $A_1^b(\tau^*)$ by $A_1^b(\tau^*)'$ which is generic for $\text{Add}^{V[L \restriction \tau^*]}(\sigma_{\omega+1}^{a*}, \sigma_{\omega+2}^{a*})$.

We see that all the relevant trees lie in the model

$$M = V[E][L \restriction \tau][A_{[2,\omega]}^c(\tau, \tau^*) * U_{[2,\omega]}^c(\tau, \tau^*) * S_{[2,\omega]}^c(\tau, \tau^*)][A_0^V(\tau^*) \times T(\tau, \tau^*)][L^b(\tau^*) \restriction \sigma_{\omega+2}^{a*}][A_1^b(\tau^*) \restriction \sigma_{\omega+2}^{a*} \times U_0^b(\tau^*) * S_0^b(\tau^*)]$$

To help keep track of the indices, we note that σ_n^{a*} plays the role of μ_{n+4} in the construction $\mathbb{A}^c(\tau, \tau^*) * \mathbb{U}^c(\tau, \tau^*) * \mathbb{S}^c(\tau, \tau^*)$.

As in Section 9.4.1, we start by breaking out the generic objects which may be absorbed by $\mathbb{R}_{\text{aux}}(\sigma_0^{a*})$. In this case they are $A_{[4,\omega]}^c$, $U_{[3,\omega]}^c$, $S_{[3,\omega]}^c$, $A_0^V(\tau^*)$, $L^b(\tau^*) \restriction \sigma_{\omega+2}^{a*}$, $A_1^b(\tau^*)'$ and $U_0^b(\tau^*) * S_0^b(\tau^*)$. The argument that these generic objects may be absorbed using a ρ -closed quotient to term forcing into an $\mathbb{R}_{\text{aux}}(\sigma_0^{a*})$ -generic object R_{aux} are exactly as in Section 9.4.1. After the absorption process we obtain a model

$$M^* = V[E][L \restriction \tau][A_{[2,3]}^c(\tau^*) * U_2^c(\tau^*) * S_2^c(\tau^*)][T(\tau, \tau^*)][R_{\text{aux}}]$$

and just as before the passage from M to M^* does not add branches to $\sigma_{\omega+1}^{a*}$ -trees.

Arguing as in Section 9.4.1, the generic objects $E' \times A_1$, $L \restriction \tau$, $A_{[2,3]}^c(\tau^*) * U_2^c(\tau^*) * S_2^c(\tau^*)$ and $T(\tau, \tau^*)$ may be absorbed into a generic object $L_{\text{aux}}^{\text{coll}}$ for $\text{Coll}^W(\rho^+, \sigma_1^{a*})$ where the quotient forcing has cardinality σ_1^{a*} . Exactly as before we obtain a model $M^{**} = V[L_{\text{aux}}^{\text{coll}} \times h][R_{\text{aux}}] = V[L_{\text{aux}} \times R_{\text{aux}}]$, where the tree property holds at $\sigma_{\omega+1}^{a*}$, and the passage from M^* to M^{**} does not add branches to $\sigma_{\omega+1}^{a*}$ -trees. So $\sigma_{\omega+1}^{a*}$ has the tree property in M and we are done.

9.4.3. The cardinal $\sigma_{\omega+1}^a$ at the first Prikry point

:

Let τ' be the first Prikry point, so that $\sigma_{\omega+1}^a = \Lambda_{\omega+1}^a(\tau')$. We use the same notation as in Section 9.3. By the usual analysis, all the relevant trees lie in the model

$$M = V[h][L \restriction \tau'][A_{[1,\omega]} * U_{[1,\omega]} * S][A_0^V(\tau')][L^b(\tau') \restriction \sigma_{\omega+2}^a][A_1^b(\tau') \restriction \sigma_{\omega+2}^a * U_0^b(\tau') * S_0^b(\tau')]$$

To help keep track of the indices, recall that $A_{[1,\omega]} * U_{[1,\omega]} * S$ comes from an $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction with parameters $\omega, \rho^+, \theta, \sigma_0^a, \dots$. So the generic objects that we will absorb into $\mathbb{R}_{\text{aux}}(\sigma_0^a)$ are $A_{[3,\omega]}$, $U_{[2,\omega]}$, $S_{[2,\omega]}$, $A_0^V(\tau')$, $L^b(\tau') \restriction \sigma_{\omega+2}^a$, $A_1^b(\tau') \restriction \sigma_{\omega+2}^a$, $U_0^b(\tau')$, and $S_0^b(\tau')$. Forcing with a suitable quotient to term forcing

to absorb these generic objects into an $\mathbb{R}_{\text{aux}}(\sigma_0^a)$ -generic object R_{aux} , we obtain a model

$$M^* = V[h][L \restriction \tau][A_{[1,2]} * U_1 * S_{[0,1]}][R_{\text{aux}}]$$

such that the passage from M to M^* adds no branches to $\sigma_{\omega+1}^a$ -trees from M .

We then force to absorb $L \restriction \tau$, $A_{[1,2]}$, U_1 , S_0 and S_1 into a $\text{Coll}^{\bar{W}}(\rho^+, \sigma_1^a)$ -generic object $L_{\text{aux}}^{\text{coll}}$. Most of these objects are generic for ρ -closed forcing posets defined in V or generic extensions of V . The exception is S_0 , which we may absorb because (by the careful choice of $\mathbb{C}_0$ in Section 7.2) it is the projection of a ρ -closed term forcing defined in $\bar{W}$. As usual we have absorbed M^* into

$$M^{**} = V[L_{\text{aux}}^{\text{coll}} \times h \times R_{\text{aux}}] = V[L_{\text{aux}} \times R_{\text{aux}}],$$

without adding branches to $\sigma_{\omega+1}^a$ -trees from M^* , and $\sigma_{\omega+1}^a$ has the tree property in M^{**} . So $\sigma_{\omega+1}^a$ has the tree property in M and we are done.

10. The tree property above $\aleph_{\omega^2}$ in the final model

10.1. The tree property at $\aleph_{\omega^2+1}$

We argue that in our final model the tree property holds at $\aleph_{\omega^2+1}$. The arguments are parallel to those in Sinapova's paper [18, Section 4], and also use ideas from work of Sinapova and Unger [20], but there are some additional complications: the relevant generic supercompactness embeddings are added by a more complex forcing poset, and there are extra issues with the constraint functions in the Prikry-type poset $\bar{\mathbb{P}}$.

Recall that $\bar{\mathbb{P}} \in V[L][A^{gg}]$ where A^{gg} is generic over $V[L]$ for the auxiliary forcing $\mathbb{A}^{gg} = \text{Add}^V(\lambda_{\omega+2}^b, j_{01}(\lambda_0^a))$, and $\mathbb{A}^{gg}$ is $< \lambda_{\omega+2}^b$ -distributive in $V[L]$. We defined $\bar{\mathbb{P}}$ using a filter $K \in V[L][A^{gg}]$ which is $\mathbb{Q}_\infty$ -generic over M_1^* .

Before starting the proof, we derive some auxiliary filters $\mathcal{F}_n$ from K , working in $V[L][A^{gg}]$. We recall from the discussion at the end of Section 7.3 that for $\alpha \in Y$, $\mathbb{Q}(\alpha, \kappa) = j_{01}^*(\mathbb{Q})(\alpha, \kappa)$.

Definition 10.1. Let $B_n = \{(x, q) : x \in P_\kappa(\lambda_n^b), q \in \mathbb{Q}(\kappa(x), \kappa)\}$.

Note that $|B_n| = \lambda_n^b$.

Definition 10.2. Let C_n be the set of functions F such that $\text{dom}(F) \in U_{n-1} \times U_n$, $F(x, y) \in \mathbb{Q}(x, y)$ for all x and y and $[F]_{U_{n-1} \times U_n} \in K$.

If $F \in C_{n+1}$, and $x \in P_\kappa \lambda_n^b$ is such that $\{y : (x, y) \in \text{dom}(F)\} \in U_{n+1}$, then $[F(x, -)]_{U_{n+1}} \in \mathbb{Q}(\kappa(x), \kappa)$.

Remark 10.3. In the sequel we will often drop the subscript for the measure in expressions like $[F(x, -)]_{U_{n+1}}$, where the relevant measure should be clear from the context.

Note that C_n includes the functions which can appear as F_n^p for some $p \in \bar{\mathbb{P}}$.

Definition 10.4. A subset E of B_n is downwards closed if whenever $(x, q) \in E$ and $q' \leq q$ then $(x, q') \in E$.

Definition 10.5. Let $\mathcal{F}_n$ be the filter on B_n defined as follows. $E \in \mathcal{F}_n$ if and only there exist a set $D \in U_n$ and a function $F \in C_{n+1}$ such that $(x, q) \in E$ for all $x \in D$ and all $q \leq [F(x, -)]_{U_{n+1}}$.

Global notation: $\mathcal{F}_n$

The following Lemma is immediate from the definition and the agreement between j_n and j_{01}^* :

Lemma 10.6. $E \in \mathcal{F}_n$ if and only if there is a condition $r \in K$ such that $(j_{01}[\lambda_n^b], q) \in j_{01}^*(E)$ for all $q \leq r$.

The following Lemma should be viewed as expressing an “ultrafilter-like” property of $\mathcal{F}_n$.

Lemma 10.7. Let $E \subseteq B_n$ and let E be downwards closed. Then $E \in \mathcal{F}_n$ or $B_n \setminus E \in \mathcal{F}_n$.

Proof. Let D be the dense set of conditions r in $\mathbb{Q}_\infty$ such that either $(j_{01}[\lambda_n^b], r) \in j_{01}^*(E)$ or there is no $q \leq r$ with $(j_{01}[\lambda_n^b], q) \in j_{01}^*(E)$. Since $D \in M_1^*$ and K is $\mathbb{Q}_\infty$ -generic over M_1^* , there is $r \in K \cap D$, and the conclusion follows. $\square$

It is easy to see that $\mathcal{F}_n$ is a κ -complete filter on B_n . We will also need a version of normality for families of $\mathcal{F}_n$ -large sets indexed by lower parts.

Recall that for $p = \langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1}, f_n, A_n, F_{n+1}, A_{n+1}, F_{n+2}, A_{n+2}, \dots \rangle$ a condition in $\mathbb{P}$, the stem of p is $\text{stem}(p) = \langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1}, [f_n]_{U_n} \rangle$, the lower part of p is $\langle q_{17}, x_{17}, \dots, q_{n-1}, x_{n-1} \rangle$, the length of p is $\text{lh}(p) = n$, and L_n is the set of lower parts of conditions with length n .

Definition 10.8. Let $(E_s)_{s \in L}$ be a family of subsets of B_n indexed by some set $L \subseteq L_n$ of lower parts. Then the diagonal intersection $\Delta_{s \in L} E_s$ is $\{(x, q) \in B_n : \forall s \in L (s \prec x \implies (x, q) \in E_s)\}$.

Lemma 10.9. Let $(E_s)_{s \in L}$ be such that $E_s \in \mathcal{F}_n$ for all $s \in L$. Then $\Delta_{s \in L} E_s \in \mathcal{F}_n$.

Proof. By the characterization of $\mathcal{F}_n$, for each $s \in L$ we choose $r_s \in K$ such that $(j_{01}[\lambda_n^b], q) \in j_{01}^*(E_s)$ for all $q \leq r_s$. Since $(r_s)_{s \in L} \in M_1^*$ by closure, and K is generic over M_1^* , there is $r \in K$ such that $r \leq r_s$ for all s . As usual $j_{01}^*[L] = \{t \in j_{01}(L) : t \prec j_{01}[\lambda_n^b]\}$. For $q \leq r$ we have $(j_{01}[\lambda_n^b], q) \in j_{01}^*(E)_t$ for all $t \in j_{01}^*[L]$, so that $(j_{01}[\lambda_n^b], q) \in j_{01}^*(\Delta_{s \in L} E_s)$. It follows that $\Delta_{s \in L} E_s \in \mathcal{F}_n$. $\square$

Definition 10.10. Let h be a stem. $h \Vdash^* \phi$ if and only if there is a condition p such that $\text{stem}(p) = h$ and $p \Vdash \phi$.

Global notation: $\Vdash^*$

Let $\nu = \lambda_\omega^b$, $\mu = \lambda_{\omega+1}^b$. The strategy of the proof will be to introduce an auxiliary forcing $\mathbb{R} \in V[L][A^{gg}]$, show that every μ -tree in $V[L][A^{gg}][\bar{P}]$ has a branch in $V[L][A^{gg}][R][\bar{P}]$, and use Lemma 2.21 to show there is a branch in $V[L][A^{gg}][\bar{P}]$. Let $\bar{T}$ in $V[L][A^{gg}]$ be a $\bar{\mathbb{P}}$ -name for a μ -tree. We assume that $T \subseteq \mu \times \kappa$, and that level α is a subset of $\{\alpha\} \times \kappa$: this makes sense because $\mu = \kappa^+$ in $V[L][A^{gg}][\bar{P}]$.

For the following lemma we work in $V[L][A^{gg}]$. Before stating and proving the lemma we make a remark on compatibility of conditions in $\bar{\mathbb{P}}$ which explains some complications in the proof.

Remark 10.11. If p and q are compatible conditions in $\bar{\mathbb{P}}$ with $\text{lh}(p) = \text{lh}(q) = n$, it does not follow in general that p and q have a common lower bound r with $\text{lh}(r) = n$. The issue is that possibly $[f_n^p]_{U_n}$ and $[f_n^q]_{U_n}$ are incompatible, so there is no reasonable choice for f_n^r : compatibility of p and q only guarantees that $f^p(x)$ and $f^q(x)$ are compatible for at least one value of x . If p and q have the same stem h , then they are compatible and there is $r \leq p, q$ with $\text{stem}(r) = h$.

Lemma 10.12. *Let $p \in \bar{\mathbb{P}}$. There are $n < \omega$ and cofinal $I \subseteq \mu$, such that for all $\alpha < \beta$ both in I , there are a condition $p' \leq p$ in $\bar{\mathbb{P}}$ of length n , and $\xi, \delta < \kappa$ such that: $p' \Vdash \langle \alpha, \xi \rangle <_{\bar{T}} \langle \beta, \delta \rangle$.*

Proof. Recall from Section 7.3 that we defined an embedding j_{01}^* in $V[L]$ witnessing that κ is $< \lambda_{\omega+3}^b$ -supercompact, and used this to derive the supercompactness measures U_n on $P_\kappa \lambda_n^b$. Let U_μ be the supercompactness measure on $P_\kappa \mu$ derived from j_{01}^* .

Observe that $|P_\kappa \mu| = \mu$ [21]. Since A^{gg} is generic for $< \lambda_{\omega+2}^b$ -distributive forcing, it is easy to see that U_μ is still a supercompactness measure on $P_\kappa \mu$ in $V[L][A^{gg}]$, and that taking the ultrapower of $V[L][A^{gg}]$ by U_μ gives an embedding $j_\mu^* : V[L][A^{gg}] \rightarrow N$ which lifts the ultrapower of $V[L]$ by U_μ . It follows that j_μ^* induces the measure U_n for every n .

Let $p \in \bar{\mathbb{P}}$ have length m , so that the first measure one set appearing in p is $A_m^p \in U_m$. By the choice of j_μ^* we have that $j_\mu^*[\lambda_m^b] \in j_\mu^*(A_m^p)$, so we may form in $j_\mu^*(\bar{\mathbb{P}})$ a one point extension q of $j_\mu^*(p)$ which forces that κ is the Prikry point with index m .

Let $\tau_{17}, \dots, \tau_m$ be the Prikry points determined by q , so that $\tau_m = \kappa$, and let $\mathbb{Q}_{\text{low}} = \prod_{17 \leq i < m} \mathbb{Q}^N(\tau_i, \tau_{i+1})$. We observe that $|\mathbb{Q}_{\text{low}}| = \lambda_{\omega+2}^a < \mu$, and that every extension r of q determines a condition $r_{\text{low}} \in \mathbb{Q}_{\text{low}}$.

Now let $u = (\sup(j_\mu^*[\mu]), 0)$ and for all $\alpha < \mu$ let ξ_α name the unique ordinal $\xi_\alpha < j_\mu^*(\kappa)$ such that $(j_\mu^*(\alpha), \xi_\alpha) <_{j_\mu^*(\bar{T})} u$.

By elementarity and Lemma 8.24, we may find $r \leq^* q$ together with $(s_\alpha)_{\alpha < \mu}$, $(k_\alpha)_{\alpha < \mu}$ such that:

- $r_{\text{low}} = q_{\text{low}}$.
- $s_\alpha \in \mathbb{Q}_{\text{low}}$ with $s_\alpha \leq r_{\text{low}}$.
- $k_\alpha < \omega$.

- If r_α is the condition obtained from r by extending r_{low} to s_α , then every k_α -step extension of r_α decides $\dot{\xi}_\alpha$.

Let $I \subseteq \mu$ be an unbounded set such that for some $k < \omega$ and $s \in \mathbb{Q}_{\text{low}}$, $k_\alpha = k$ and $s_\alpha = s$ for all $\alpha \in I$. Let r' be obtained from r by extending r_{low} to s , let r'' be some k -step extension of r' , and for each $\alpha \in I$ let ξ_α be the value of $\dot{\xi}_\alpha$ determined by r'' .

Let r'' have length n . By construction $r'' \leq j_\mu^*(p)$, and for $\alpha < \beta$ both in I we have $r'' \Vdash_{j_\mu^*(\mathbb{P})}^N (j_\mu^*(\alpha), \xi_\alpha) <_{j_\mu^*(\dot{T})} (j_\mu^*(\beta), \xi_\beta)$. The desired conclusion follows by elementarity. $\square$

Lemma 10.13. *There is a forcing poset $\mathbb{R} \in V[L]$ such that, for all sufficiently large $\bar{n} < \omega$, there is a forcing poset $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3 \in V[L][R]$ such that:*

- $\lambda_{\bar{n}}^b$ is generically μ -supercompact in $V[L][A^{gg}][R]$ via $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$.
- $\mathbb{R}$ is countably closed and $< \mu$ -distributive in $V[L][A^{gg}]$.
- A^{gg} is $< \lambda_{\omega+2}^b$ -distributive in $V[L][R]$.
- $\mathbb{P}_1 \times \mathbb{P}_{2b} \times \mathbb{P}_3$ is $< \lambda_{\bar{n}-1}^b$ -distributive in $V[L][R][A^{gg}]$.
- In $V[L][R][A^{gg}][P_1 \times P_{2b} \times P_3]$, μ has cardinality and cofinality $\lambda_{\bar{n}-1}^b$, and $\mathbb{P}_{2a}$ has the $\lambda_{\bar{n}-1}^b$ -Knaster property.
- For $\lambda' = \lambda_{\bar{n}-3}^b$, there is a forcing poset in $V[L][R][A^{gg}]$ which adds λ' mutually generic filters for $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$, preserves the regularity of λ' , and forces that $\text{cf}(\mu) > \lambda'$.

Before starting the proof, we make a remark about how the generic embedding from the conclusion interacts with the supercompactness measures U_m for $17 \leq m < \bar{n}$. The key points are that $|P_\kappa \lambda_m^b| = \lambda_m^b$ and $2^{\lambda_m^b} = \lambda_{\omega+3}^b$, so that subsets of $P_\kappa \lambda_m^b$ are fixed, while the power set of $P_\kappa \lambda_m^b$ and the measure U_m are stretched.

Proof. We will use ideas from the proof of Lemma 4.5, in a context which is quite similar to that of Section 9.2. The analogy with Section 9.2 is slightly imperfect, because there we only needed to define the embedding on a submodel which contains all the subsets of the critical point. In the discussion below this means that we need to deal with the whole of the last component J^c in $L(\kappa)$, rather than just A_0^c .

Recall that $V[L] = V[L \restriction \kappa][L(\kappa)]$, where $L(\kappa) = L^b * I^b * (A_e \times J^c)$. Decomposing further:

- $I^b = A^b * U^b * S^b$, and is generic for an $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction with parameters $\mu_0 = \lambda_{17}^a$, $\mu_1 = \lambda_{\omega+1}^a$, $\mu_2 = \lambda_{\omega+2}^a$, $\mu_{n+3} = \lambda_n^b$.
- $\mathbb{A}_e = \text{Add}^{V^{I^b}(\kappa)}(\lambda_{17}^b, \lambda_{\omega+3}^b)$.
- $\mathbb{J}^c = \mathbb{J}_0^c * \mathbb{J}_1^c = (\mathbb{A}_0^c * \mathbb{U}_0^c * \mathbb{S}_0^c) * (\mathbb{A}_1^c * \mathbb{U}_1^c * \mathbb{S}_1^c)$, where the parameters are $\mu_0 = \lambda_{17}^b$, $\mu_1 = \lambda_{\omega+1}^b$, $\mu_2 = \lambda_{\omega+2}^b$, $\mu_3 = \lambda_{\omega+3}^b$.
- $\mathbb{A}_1^c * \mathbb{U}_1^c * \mathbb{S}_1^c$ is defined over $V^{I^{b_i}(\kappa)}[A_0^c * U_0^c * S_0^c]$.
- $\mathbb{A}_0^c = \text{Add}^{V^{I^b}(\kappa)}(\lambda_{17}^b, [\lambda_{\omega+1}^b, \lambda_{\omega+2}^b])$.
- $\mathbb{A}_1^c = \text{Add}^{V^{I^b}(\kappa)}(\lambda_{\omega+1}^b, [\lambda_{\omega+2}^b, \lambda_{\omega+3}^b])$.

The idea is now to construct a generic embedding as in the proof of Lemma 4.5, where:

- $V^{lb}(\kappa)$ plays the role of V_{def} (which coincides with V_{inn}).
- I^b plays the role of $A * U * S$.
- $\lambda_{\bar{n}}^b$ plays the role of μ_{n+2} .

Before we can do this we have to embed $A_e \times J^c$ into a generic object for a poset which meets the specifications of Lemma 4.5. This is the job of the auxiliary poset $\mathbb{R}$. In the terminology of Lemma 4.5 we will embed $\mathbb{A}_e \times \mathbb{J}^c$ into $\mathbb{D}^0 \times \mathbb{D}^2$, where $\mathbb{D}^0, \mathbb{D}^2 \in V^{lb}(\kappa)$. The poset $\mathbb{D}^0$ will be $\lambda_{\bar{n}-1}^b$ -Knaster in $V^{lb}(\kappa)$, and $\mathbb{D}^2$ will be $< \lambda_{\omega+1}^b$ -directed closed in this model.

We let $\mathbb{D}^0 = \mathbb{A}_e \times \mathbb{A}_0^c$. Before defining $\mathbb{D}^2$, we recall from the analysis in Section 3.2 that in $V^{lb}(\kappa)$ the poset $\mathbb{J}_0^c$ is a projection of $\mathbb{A}_0^c \times \mathbb{B}_0^c \times \mathbb{C}_0^c$, where $\mathbb{B}_0^c \times \mathbb{C}_0^c$ is $< \lambda_{\omega+1}^b$ -directed closed. Similarly in $V^{lb}(\kappa)[J_0^c]$ the poset $\mathbb{J}_1^c$ is a projection of $\mathbb{A}_1^c \times \mathbb{B}_1^c \times \mathbb{C}_1^c$, where $\mathbb{B}_1^c \times \mathbb{C}_1^c$ is $< \lambda_{\omega+2}^b$ -directed closed.

We let $\mathbb{D}^2 = \mathbb{A}_1^c \times \mathcal{A}^{V^{lb}(\kappa)}(\mathbb{I}^b, \mathbb{B}_0^c \times \mathbb{C}_0^c) \times \mathcal{A}^{V^{lb}(\kappa)}(\mathbb{I}^b * \mathbb{J}_0^c, \mathbb{B}_1^c \times \mathbb{C}_1^c)$.

Definition 10.14. Let $\mathbb{R} \in V[L]$ be the (iterated) quotient to term forcing to add generic objects for the second and third factors in $\mathbb{D}^2$ inducing generic filters $B_0^c \times C_0^c$ and $B_1^c \times C_1^c$, such that in turn $B_0^c \times C_0^c$ induces $U_0^c * S_0^c$ and $B_1^c \times C_1^c$ induces $U_1^c * S_1^c$.

Claim 10.15. Let $\mathbb{D}^2$, $\mathbb{R}$ and $\mathbb{A}^{gg}$ be as above.

- (1) $\mathbb{D}^2$ is $< \lambda_{\omega+1}^b$ -directed closed in $V^{lb}(\kappa)$.
- (2) $\mathbb{R}$ is countably closed and $< \mu$ -distributive in $V[L][A^{gg}]$.
- (3) $\mathbb{A}^{gg}$ is $< \lambda_{\omega+2}^b$ -distributive in $V[L][R]$.

Proof. We take each claim in turn.

- (1) $\mathbb{D}^2$ is the product of three factors. The first term $\mathbb{A}_1^c$ is clearly $< \lambda_{\omega+1}^b$ -directed closed in $V^{lb}(\kappa)$. The second factor is $< \lambda_{\omega+1}^b$ -directed closed by items 5 and 6 of Lemma 3.13 together with Lemma 2.33. and similarly the third factor is $< \lambda_{\omega+2}^b$ -directed closed.
- (2) $\mathbb{R}$ can be viewed as the product of two (iterated) quotient-to-term forcing posets, and each factor is countably closed by Lemma 2.44. Since λ_{ω}^b is singular, it will suffice for distributivity to show that $\mathbb{R}$ is λ_t^b -distributive in $V[L][A^{gg}]$ for all large enough $t < \omega$.

By the definition of $\mathbb{R}$, $V[L][A^{gg}][R] = V^{lb}(\kappa)[I^b \times D^0 \times D^2 \times A^{gg}]$. By item 7 of Lemma 3.13, in $V^{lb}(\kappa)$ we may view $\mathbb{I}^b$ as a projection of $\mathbb{I}_0^b \times \mathbb{I}_1^b$ where $\mathbb{I}_0^b$ is an initial segment of $\mathbb{I}^b$ with λ_{t+1}^b -cc and $\mathbb{I}_1^b$ is $< \lambda_{t+1}^b$ -closed. We extend $V[L][A^{gg}][R]$ to obtain $V^{lb}(\kappa)[I_0^b \times I_1^b \times D^0 \times D^2 \times A^{gg}]$.

Since $L \restriction \kappa * L^b * (I_0^b \times D^0)$ is generic for $\lambda_{\omega+2}^b$ -cc forcing, by Easton's lemma $\mathbb{A}^{gg}$ is $< \lambda_{\omega+2}^b$ -distributive in $V^{lb}(\kappa)[I_0^b \times D^0]$, so that $I_1^b \times D^2$ is λ_t^b -distributive in $V[L \restriction \kappa][L^b][I_0^b \times D^0 \times A^{gg}]$. It follows that every λ_t^b -

sequence of ordinals from $V[L][A^{gg}][R]$ lies in $V^{lb}(\kappa)[I_0^b \times D^0 \times A^{gg}]$, which is a submodel of $V[L][A^{gg}]$.

- (3) By items 2 and 3 of Lemma 3.13, $\mathbb{B}_0^c \times \mathbb{C}_0^c$ is $\lambda_{\omega+2}^b$ -cc in $V^{lb}(\kappa)$. Also $\lambda_{\omega+2}^b$ is supercompact in $V^{lb}(\kappa)$, and $|I^b| < \lambda_{\omega+2}^b$, so it follows by Lemma 2.35 that $\mathcal{A}^{V^{lb}(\kappa)}(\mathbb{I}^b, \mathbb{B}_0^c \times \mathbb{C}_0^c)$ is $\lambda_{\omega+2}^b$ -cc in $V^{lb}(\kappa)$. So $\mathbb{D}^2$ is the product of a $\lambda_{\omega+2}^b$ -cc poset $\mathbb{D}_{\text{low}}^2$ and a $< \lambda_{\omega+2}^b$ -closed poset $\mathbb{D}_{\text{high}}^2$ in the model $V^{lb}(\kappa)$.

Recall that $V[L][R][A^{gg}] = V^{lb}(\kappa)[I^b \times D^0 \times D_{\text{low}}^2 \times D_{\text{high}}^2 \times A^{gg}]$. Since $L \restriction \kappa * L^b$ is generic over V for $\lambda_{\omega+2}^b$ -cc forcing, by Easton's Lemma $\mathbb{A}^{gg}$ is $< \lambda_{\omega+2}^b$ -distributive in $V^{lb}(\kappa)$, so that $\mathbb{D}_{\text{high}}^2$ is $< \lambda_{\omega+2}^b$ -closed in $V^{lb}(\kappa)[A^{gg}]$. Since $L \restriction \kappa * L^b * (I^b \times D_0 \times D_{\text{low}}^2)$ is generic over V for $\lambda_{\omega+2}^b$ -cc forcing, by Easton's Lemma $\mathbb{L} \restriction \kappa * \mathbb{L}^b * (\mathbb{I}^b \times \mathbb{D}_0 \times \mathbb{D}_{\text{low}}^2)$ is $\lambda_{\omega+2}^b$ -cc in $V[A^{gg}]$, so that $\mathbb{I}^b \times \mathbb{D}_0 \times \mathbb{D}_{\text{low}}^2$ is $\lambda_{\omega+2}^b$ -cc in $V^{lb}(\kappa)[A^{gg}]$.

By Easton's Lemma $\mathbb{D}_{\text{high}}^2$ is $< \lambda_{\omega+2}^b$ -distributive in $V^{lb}(\kappa)[I^b \times D^0 \times D_{\text{low}}^2 \times A^{gg}]$ and $\mathbb{A}^{gg}$ is $< \lambda_{\omega+2}^b$ -distributive in $V^{lb}(\kappa)[I^b \times D_0 \times D_{\text{low}}^2]$. It follows that every $< \lambda_{\omega+2}^b$ -sequence of ordinals in $V^{lb}(\kappa)[I^b \times D^0 \times D_{\text{low}}^2 \times A^{gg} \times D_{\text{high}}^2]$ lies in $V^{lb}(\kappa)[I^b \times D^0 \times D_{\text{low}}^2]$, so that $\mathbb{A}^{gg}$ is $< \lambda_{\omega+2}^b$ -distributive in $V[L][R]$. □

For use later, we record some information about forcing with $\bar{\mathbb{P}}$ over $V[L][A^{gg}][R]$.

Claim 10.16. $\mu = \kappa^+$ in $V[L][A^{gg}][R][\bar{\mathbb{P}}]$.

Proof. By Lemma 10.15 the poset $\mathbb{R}$ is $< \mu$ -distributive and countably closed in $V[L][A^{gg}]$, and the proof of Lemma 8.7 easily shows that $\bar{\mathbb{P}}$ is μ -cc in $V[L][A^{gg}][R]$. The claim follows. □

By construction $V[L][R] = V^{lb}(\kappa)[D^0 \times D^2]$. Choosing $\bar{n}$ large enough, we may arrange that in the model $V^{lb}(\kappa)$, D^0 is generic for $\lambda_{\bar{n}-1}^b$ -cc forcing, and D^2 is generic for $< \lambda_{\bar{n}}^b$ -directed closed forcing. In fact since $\mathbb{D}^0$ is adding Cohen subsets to λ_{17}^b and $\mathbb{D}^2$ is $< \lambda_{\omega+1}^b$ -directed closed, any $\bar{n} \geq 19$ will work.

We will perform the construction from the proof of Lemma 4.5 to obtain a generic embedding with domain $V[L][R]$, and then derive a generic embedding with domain $V[L][A^{gg}][R]$. The construction in the proof involves a regular cardinal χ which in our case is $\max(\nu, |\mathbb{D}^0|, |\mathbb{D}^2|)^+$. We note for the record that $\chi > \mu$.

We summarize the key features of the construction from Lemma 4.5 in our current setting:

- Working in $V^{lb}(\kappa)[D^2 \times A_{[\bar{n}, \omega]}^b]$, we use the indestructible Laver function added by L^b to choose an embedding j which witnesses that $\lambda_{\bar{n}}^b$ is χ -supercompact and satisfies some technical conditions. We may assume that j is the ultrapower by some supercompactness measure W on $P_{\lambda_{\bar{n}}^b} \chi$, where we note that $(P_{\lambda_{\bar{n}}^b} \chi)^{V^{lb}(\kappa)[D^2 \times A_{[\bar{n}, \omega]}^b]} = (P_{\lambda_{\bar{n}}^b} \chi)^{V^{lb}(\kappa)}$.

- The lifting of j to $V[L][R]$ takes place in a generic extension $V[L][R][P_1 \times P_2 \times P_3]$.
- $\mathbb{P}_1 \times \mathbb{P}_3 \in V^{lb}(\kappa)[A^b * U^b * S^b \restriction [\lambda_{\bar{n}-1}^b, \lambda_\omega^b]][D^2]$ and is $< \lambda_{\bar{n}-1}^b$ -closed in this model.
- $\mathbb{P}_2 = \mathbb{P}_{2a} \times \mathbb{P}_{2b}$ where $\mathbb{P}_{2a} = j(\mathbb{A}_{\bar{n}-2}^b \times \mathbb{D}^0)/j[A_{\bar{n}-2}^b \times D^0]$ and $\mathbb{P}_{2b} = j(\mathbb{A}_{\bar{n}-1}^b)/j[A_{\bar{n}-1}^b]$.
- $\mathbb{P}_{2a} = \mathbb{P}_{2a}^{\text{low}} \times \mathbb{P}_{2a}^{\text{high}}$, where $\mathbb{P}_{2a}^{\text{high}}$ is a Cohen poset adding subsets to $\lambda_{\bar{n}-2}^b$, and $\mathbb{P}_{2a}^{\text{low}}$ is a Cohen poset adding subsets to λ_{17}^b , both defined in $V^{lb}(\kappa)$.
- $\mathbb{P}_{2b}$ is a Cohen poset adding subsets to $\lambda_{\bar{n}-1}^b$, again defined in $V^{lb}(\kappa)$.

Let $\lambda' = \lambda_{\bar{n}-3}^b$. With a view to using Fact 2.15 in the proof of Lemma 10.19, we construct an auxiliary forcing $\mathbb{P}_1^* \times (\mathbb{P}_{2a}^{\text{low}})^* \times (\mathbb{P}_{2a}^{\text{high}})^* \times \mathbb{P}_{2b}^* \times \mathbb{P}_3^* \in V[L][R]$ whose aim is to add λ' many mutually generic filters for $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$.

- $\mathbb{P}_1^*$ (resp $\mathbb{P}_3^*$) is the product of λ' copies of $\mathbb{P}_1$ (resp $\mathbb{P}_3$) computed with full support in $V^{lb}(\kappa)[A^b * U^b * S^b \restriction [\lambda_{\bar{n}-1}^b, \lambda_\omega^b]][D^2]$. As noted above $\mathbb{P}_1 \times \mathbb{P}_3$ is defined and $< \lambda_{\bar{n}-1}^b$ -closed in this model, so $\mathbb{P}_1^* \times \mathbb{P}_3^*$ is also $< \lambda_{\bar{n}-1}^b$ -closed in this model, and since A^{gg} is generic over this model for highly distributive forcing $\mathbb{P}_1^* \times \mathbb{P}_3^*$ is $< \lambda_{\bar{n}-1}^b$ -closed in $V^{lb}(\kappa)[A^b * U^b * S^b \restriction [\lambda_{\bar{n}-1}^b, \lambda_\omega^b]][D^2][A^{gg}]$.
- $(\mathbb{P}_{2a}^{\text{low}})^*$ is the product of λ' copies of $\mathbb{P}_{2a}^{\text{low}}$ computed with $< \lambda_{17}^b$ support in $V^{lb}(\kappa)$. Since $\mathbb{P}_{2a}^{\text{low}}$ is a Cohen poset adding subsets to λ_{17}^b defined in $V^{lb}(\kappa)$, $(\mathbb{P}_{2a}^{\text{low}})^*$ is a similar poset in this model.
- $(\mathbb{P}_{2a}^{\text{high}})^*$ (resp $\mathbb{P}_{2b}^*$) is the product of λ' copies of $\mathbb{P}_{2a}^{\text{high}}$ (resp $\mathbb{P}_{2b}$) computed with full support in $V^{lb}(\kappa)$. Since $\mathbb{P}_{2a}^{\text{high}}$ (resp $\mathbb{P}_{2b}$) is a Cohen poset adding subsets to $\lambda_{\bar{n}-2}^b$ (resp $\lambda_{\bar{n}-1}^b$) defined in $V^{lb}(\kappa)$, $(\mathbb{P}_{2a}^{\text{high}})^*$ (resp $\mathbb{P}_{2b}^*$) is a similar poset in this model.

Claim 10.17. *It is forced over $V[L][R][A^{gg}]$ by $\mathbb{P}_1^* \times \mathbb{P}_2^* \times \mathbb{P}_3^*$ that λ' remains regular and $\text{cf}(\mu) > \lambda'$.*

Proof. Recall that $V[L][A^{gg}][R] = V^{lb}(\kappa)[I^b \times D^0 \times D^2 \times A^{gg}]$, $\mathbb{D}^0 = \mathbb{A}_e \times \mathbb{A}_0^c$ is a Cohen poset adding subsets to λ_{17}^b defined in $V^{lb}(\kappa)$, while $\mathbb{D}^2$ is $< \lambda_{\omega+1}^b$ -directed closed forcing again defined in $V^{lb}(\kappa)$.

Since $\mathbb{P}_1^* \times \mathbb{P}_3^*$ is defined and $< \lambda_{\bar{n}-1}^b$ -closed in a generic extension of $V^{lb}(\kappa)$, we may force to extend the model $V^{lb}(\kappa)[I^b \times D^0 \times D^2 \times A^{gg}][P_1^* \times P_2^* \times P_3^*]$ to a model $V^{lb}(\kappa)[I^b \times D^0 \times D^2 \times A^{gg}][P_2^* \times T]$ where T is generic for a term forcing $\mathbb{T}$ which is defined and $< \lambda_{\bar{n}-1}^b$ -closed in $V^{lb}(\kappa)$. By more forcing we may use item 7 of Lemma 3.13 to extend to a model $V^{lb}(\kappa)[I_0^b \times I_1^b \times D^0 \times D^2 \times A^{gg}][P_2^* \times T]$ where $\mathbb{I}_0^b$ and $\mathbb{I}_1^b$ are defined in $V^{lb}(\kappa)$, and in that model $\mathbb{I}_0^b$ is $\lambda_{\bar{n}-3}^b$ -Knaster and $\mathbb{I}_1^b$ is $< \lambda_{\bar{n}-3}^b$ -closed.

We reorganize our expanded model as

$$V^{lb}(\kappa)[D^2 \times A^{gg}][I_0^b \times D^0 \times (P_{2a}^{\text{low}})^*][I_1^b \times (P_{2a}^{\text{high}})^* \times (P_{2b})^* \times T].$$

Note that $\mathbb{I}_0^b \times \mathbb{D}^0 \times (\mathbb{P}_{2a}^{\text{low}})^*$ is $\lambda_{\bar{n}-3}^b$ -cc in $V^{lb}(\kappa)$. Since $\mathbb{D}^2 \times \mathbb{A}^{gg}$ is highly distributive, $(\mathbb{P}_{2a}^{\text{high}})^* \times (\mathbb{P}_{2b})^* \times \mathbb{T}$ is $< \lambda_{\bar{n}-3}^b$ -closed in $V^{lb}(\kappa)[D^2 \times A^{gg}]$, and (since its definition

does not change) $\mathbb{I}_0^b \times \mathbb{D}^0 \times (\mathbb{P}_{2a}^{\text{low}})^*$ is $\lambda_{\bar{n}-3}^b$ -cc in $V^{lb}(\kappa)[D^2 \times A^{gg}]$.

It follows by Easton's Lemma that $\lambda_{\bar{n}-3}^b$ is still regular in $V^{lb}(\kappa)[I_0^b \times I_1^b \times D^0 \times D^2 \times A^{gg}][P_2^* \times T]$, and so *a fortiori* it is regular in $V^{lb}(\kappa)[I^b \times D^0 \times D^2 \times A^{gg}][P_1^* \times P_2^* \times P_3^*]$. To finish we note that by taking $\mathbb{I}^b$ as the projection of the product of a $\lambda_{\bar{n}-2}^b$ -Knaster poset and a $< \lambda_{\bar{n}-2}^b$ -closed poset, we may repeat the argument with $\lambda_{\bar{n}-2}^b$ in place of $\lambda_{\bar{n}-3}^b$. This allows us to conclude that $\lambda_{\bar{n}-2}^b$ is also still regular in $V^{lb}(\kappa)[I^b \times D^0 \times D^2 \times A^{gg}][P_1^* \times P_2^* \times P_3^*]$, and that $\text{cf}(\mu) \geq \lambda_{\bar{n}-2}^b$ in this model. $\square$

Claim 10.18. *The poset $\mathbb{P}_1 \times \mathbb{P}_{2b} \times \mathbb{P}_3$ is $< \lambda_{\bar{n}-1}^b$ -distributive in $V[L][R][A^{gg}]$.*

Proof. The argument is very similar to that for Claim 10.17 so we just sketch it. We force to extend $V^{lb}(\kappa)[I^b \times D^0 \times D^2 \times A^{gg}][P_1 \times P_{2b} \times P_3]$ to a model $V^{lb}(\kappa)[I^b \times D^0 \times D^2 \times A^{gg}][P_{2b} \times T]$ where T is generic for a term forcing $\mathbb{T}$ which is defined and $< \lambda_{\bar{n}-1}^b$ -closed in $V^{lb}(\kappa)$, and then to a model $V^{lb}(\kappa)[D^2 \times A^{gg}][I_0^b \times D^0][I_1^b \times P_{2b} \times T]$ where $\mathbb{I}_0^b$ and $\mathbb{I}_1^b$ are defined in $V^{lb}(\kappa)$, and in that model $\mathbb{I}_0^b$ is $\lambda_{\bar{n}-1}^b$ -Knaster and $\mathbb{I}_1^b$ is $< \lambda_{\bar{n}-1}^b$ -closed.

In the model $V^{lb}(\kappa)[D^2 \times A^{gg}]$ we appeal to Easton's lemma to see that all $< \lambda_{\bar{n}-1}^b$ -sequences of ordinals from $V[L][R][A^{gg}][P_1 \times P_{2b} \times P_3]$ lie in $V^{lb}(\kappa)[D^2 \times A^{gg}][I_0^b \times D^0]$. $\square$

It is now easy to see that μ has cardinality and cofinality $\lambda_{\bar{n}-1}^b$ in the model $V[L][R][A^{gg}][P_1 \times P_{2b} \times P_3]$, and that $\mathbb{P}_{2a}$ is $\lambda_{\bar{n}-1}^b$ -Knaster in this model.

At this point we have produced a generic χ -supercompactness embedding $j : V[L][R] \rightarrow N$ with critical point $\lambda_{\bar{n}}^b$ which exists in $V[L][R][P_{1,2,3}]$. By construction $N = \{j(F)(a) : F \in V[L][R], \text{dom}(F) = Z\}$ where $a = j[\chi]$ and $Z = (P_{\lambda_{\bar{n}}^b} \chi)^{V[L][\kappa][L_b]}$.

Now let $Z_0 = (P_{\lambda_{\bar{n}}^b} \mu)^{V[L][\kappa][L_b]}$. Factoring j in the standard way we obtain a generic μ -supercompactness embedding $j_0 : V[L][R] \rightarrow N_0$, such that $N_0 = \{j_0(F_0)(a_0) : F_0 \in V[L][R], \text{dom}(F_0) = Z_0\}$ where $a_0 = j_0[\mu]$. Since A^{gg} is generic for μ -distributive forcing in $V[L][R]$ and $|Z_0| = \mu$, it is easy to see that $j_0[A^{gg}]$ generates an N -generic filter and we may lift j_0 onto $V[L][R][A^{gg}]$. Note that the lifted j_0 exists in $V[L][A^{gg}][R][P_{1,2,3}]$. $\square$

Lemma 10.19. *There exist in $V[L][A^{gg}][R]$ a set J , a stem h and a sequence $\langle u_\alpha \mid \alpha \in J \rangle$ such that:*

- $J \subseteq I$ and J is unbounded.
- h has length n .
- u_α is a node of level α .
- For all $\alpha, \beta \in J$ with $\alpha < \beta$, $h \Vdash^* u_\alpha < u_\beta$.

Proof. We work for the moment in $V[L][A^{gg}]$. Let $\bar{n}$ be so large that an elementary embedding with critical point $\lambda_{\bar{n}}^b$ is guaranteed to fix each stem for a condition of length n , together with the set of all such stems. Let j_0 be an elementary embedding with critical point $\lambda_{\bar{n}}^b$ constructed as in Lemma 10.13.

Let S be the set of stems for conditions of length n in $\bar{\mathbb{P}}$. Define relations $(R_h)_{h \in S}$ on $I \times \kappa$ as follows: $(\alpha, \eta) R_h (\beta, \zeta)$ if and only if there is a condition $p \in \bar{\mathbb{P}}$ with length n and stem h such that $p \Vdash (\alpha, \eta) <_{\dot{T}} (\beta, \zeta)$. Since $\dot{T}$ names a tree and conditions with the same stem are compatible, it is easy to verify that this set of relations forms a system.

We will show that forcing with $\mathbb{R} * \mathbb{P}_{1,2,3}$ adds a system of branches $(b_{h,i})_{(h,i) \in S \times \kappa}$. Let $\gamma \in j_0(I)$ with $\sup j_0[\mu] < \gamma$. By the choice of $\bar{n}$ we have $|S| < \lambda_{\bar{n}}^b$ and $j_0(S) = S$. Let $\alpha \in \text{dom}(b_{h,i})$ if and only if there exist $\eta < \kappa$ and $p \in j_0(\bar{\mathbb{P}})$ with stem h such that $p \Vdash (j_0(\alpha), \eta) <_{j_0(\dot{T})} (\gamma, i)$, and in this case let $b_{h,i}(\alpha)$ be the unique η for which this holds. It is easy to see that this is a system of branches with $b_{h,i}$ forming a branch through R_h .

Using Claim 10.17 and appealing to Fact 2.15 in the model $V[L][A^{gg}][R]$, there exists $(h, i) \in S \times \kappa$ such that $b_{h,i} \in V[L][A^{gg}][R]$ and $\text{dom}(b_{h,i})$ is unbounded in μ . Now let $J = \text{dom}(b_{h,i})$, and for $\alpha \in J$ let $u_\alpha = (\alpha, b_{h,i}(\alpha))$. If $\alpha < \beta$ with $\alpha, \beta \in J$ there is a condition $p \in j_0(\bar{\mathbb{P}})$ with stem h such that $p \Vdash j_0(u_\alpha), j_0(u_\beta) <_{j_0(\dot{T})} (\gamma, i)$, so $p \Vdash j_0(u_\alpha) <_{j_0(\dot{T})} j_0(u_\beta)$. Since $j_0(h) = h$, by elementarity there is $p \in \bar{\mathbb{P}}$ with stem h such that $p \Vdash u_\alpha <_{\dot{T}} u_\beta$. $\square$

Suppose that a stem h' has the form $\langle q_{17}, x_{17}, \dots, q_{m-1}, x_{m-1}, [g]_{U_m} \rangle$, and that $(x, q) \in B_m$. We write $h' + (x, q)$ for the stem $\langle q_{17}, x_{17}, \dots, q_{m-1}, x_{m-1}, g(x), x, q \rangle$. This is technically illegal because it depends on the choice of g , but we will only use this notation in a context where the choice of g is explicit.

Global notation: $h' + (x, q)$

Fix J, h and $\langle u_\alpha : \alpha \in J \rangle$ as in the conclusion of Lemma 10.19.

Lemma 10.20. *Let h' be a stem of the form $\langle q_{17}, x_{17}, \dots, q_{m-1}, x_{m-1}, [g]_{U_m} \rangle$. Assume that there exists in $V[L][A^{gg}][R]$ an unbounded set $\bar{J} \subseteq J$, such that $h' \Vdash^* u_\alpha < u_\beta$ for all $\alpha, \beta \in \bar{J}$ with $\alpha < \beta$.*

Then there exist $\rho < \mu$ and a sequence $(E_\alpha)_{\alpha \in \bar{J} \setminus \rho}$ in $V[L][R][A^{gg}]$ such that:

- (1) $E_\alpha \subseteq B_m$ and $E_\alpha \in \mathcal{F}_m$.
- (2) For all $\alpha, \beta \in \bar{J} \setminus \rho$ with $\alpha < \beta$ and all $(x, q) \in E_\alpha \cap E_\beta$, $h' + (x, q) \Vdash^* u_\alpha < u_\beta$, where $h' + (x, q)$ is computed using the function g .

Before proving Lemma 10.20, we rewrite the conclusion in a way that is less concise but will be useful later. Refining E_α if necessary, we may fix $D_\alpha \in U_m$ and $F_\alpha \in C_{m+1}$ such that $E_\alpha = \{(x, q) \in B_m : x \in D_\alpha, q \leq [F_\alpha(x, -)]\}$. Now $([F_\alpha])_{\alpha \in \bar{J} \setminus \rho} \in M_1^*$ by the distributivity of A^{gg} and the closure of M_1^* , $[F_\alpha] \in K$ for all α , and K is generic over M_1^* . It follows that there is $F^* \in C_n$ such that $[F^*] \leq [F_\alpha]$ for all α , and shrinking D_α we may assume that $[F^*(x, -)] \leq [F_\alpha(x, -)]$ for all $x \in D_\alpha$. Refining E_α again we may assume that $E_\alpha = \{(x, q) \in B_m : x \in D_\alpha, q \leq [F^*(x, -)]\}$.

Then $(x, q) \in E_\alpha \cap E_\beta$ if and only if $x \in D_\alpha \cap D_\beta$ and $q \leq [F^*(x, -)]$, and the conclusion amounts to saying that if $x \in D_\alpha \cap D_\beta$ then $h' + (x, [F^*(x, -)]) \Vdash^* u_\alpha <$

u_β . Readers of [15] and [18] will notice that Lemma 10.20 is parallel to [15, Lemma 3.5] and [18, Lemma 16].

Proof. Choose n^* such that $m + 20 < n^* < \omega$, and let $j_1 : V[L][R][A^{gg}] \rightarrow N_1$ be a generic μ -supercompactness embedding with critical point $\lambda_{n^*}^b$ constructed as in Lemma 10.13. The embedding is added by a certain product $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$. We will work for the moment in $V[L][R][A^{gg}][P_1 \times P_2 \times P_3]$.

Let $\gamma \in j_1(\bar{J})$ be such that $\sup j_1[\mu] < \gamma$, and let $v = j_1(u)_\gamma$. By elementarity, for every $\alpha \in \bar{J}$ there is a condition $r_\alpha \in j_1(\mathbb{P})$ such that r_α has stem h' and $r_\alpha \Vdash j_1(u_\alpha) <_{j_1(T)} v$. Now $j_1(g) = g$, and $[g]_{U_m} = [j_1(g)]_{j_1(U_m)} = [f_m^{r_\alpha}]_{j_1(U_m)}$. Shrinking $A_m^{r_\alpha}$ if necessary, we may assume that $A_m^{r_\alpha} \subseteq \text{dom}(g)$ and $f_m^{r_\alpha} = g \upharpoonright A_m^{r_\alpha}$.

For each $y \in A_m^{r_\alpha}$, the minimal one-point extension of r_α by y forces $j_1(u_\alpha) <_{j_1(T)} v$. Since $g(y) = j_1(g)(y) = f_m^{r_\alpha}(y)$, the stem of the minimal one-point extension is $\langle q_{17}, x_{17}, \dots, q_{m-1}, x_{m-1}, g(y), y, r \rangle$ where $r = [F_{m+1}^{r_\alpha}(y, -)]_{U_{m+1}}$. We conclude that there is a $j_1(\mathcal{F}_m)$ -large set X_α such that $h' + (y, r) \Vdash_{j_1(\mathbb{P})}^* j_1(u_\alpha) <_{j_1(T)} v$ for all $(y, r) \in X_\alpha$. Membership of X_α in $j_1(\mathcal{F}_m)$ is witnessed by $A_m^{r_\alpha}$ and $F_{m+1}^{r_\alpha}$.

For $(x, q) \in B_m$, let $\bar{J}_{x,q} = \{\alpha \in \bar{J} : h' + (x, q) \Vdash_{j_1(\mathbb{P})}^* j_1(u_\alpha) < v\}$. It is easy to see that for $\beta \in \bar{J}_{x,q}$, we have that $\alpha \in \bar{J}_{x,q} \cap \beta$ if and only if $h' + (x, q) \Vdash_{\mathbb{P}}^* u_\alpha < u_\beta$, so that $\bar{J}_{x,q} \cap \beta \in V[L][A^{gg}]$.

Since μ has cardinality and cofinality $\lambda_{n^*-1}^b$ in $V[L][R][A^{gg}][P_1 \times P_{2b} \times P_3]$, and $\mathbb{P}_{2a}$ has the $\lambda_{n^*-1}^b$ -approximation property in this model, it follows that whenever $\bar{J}_{x,q}$ is unbounded in μ we have $\bar{J}_{x,q} \in V[L][R][A^{gg}][P_1 \times P_{2b} \times P_3]$. It is important to notice that even in this case the definition of j_1 (and hence $\bar{J}_{x,q}$) requires P_{2a} .

Working in $V[L][R][A^{gg}][P_1 \times P_{2b} \times P_3]$, let $\mathcal{J}_{x,q}$ be the set of unbounded subsets C of μ such that some condition in $\mathbb{P}_{2a}$ forces $\bar{J}_{x,q} = C$. It is easy to see that

- $|\mathcal{J}_{x,q}| < \lambda_{n^*-1}^b$.
- The function $(x, q) \mapsto \mathcal{J}_{x,q}$ is in $V[L][R][A^{gg}][P_1 \times P_{2b} \times P_3]$.
- If $C \in \mathcal{J}_{x,q}$ and $\beta \in C$, then $C \cap \beta$ is the set of $\alpha < \beta$ such that $h' + (x, q) \Vdash_{\mathbb{P}}^* u_\alpha < u_\beta$.
- If $C_1, C_2 \in \mathcal{J}_{x,q}$ with $C_1 \neq C_2$ then $C_1 \cap C_2$ is bounded in μ .

Let $\rho < \mu$ be such that $C_1 \cap C_2 \subseteq \rho$ for all $(x, q) \in B_m$ and all $C_1, C_2 \in \mathcal{J}_{x,q}$ with $C_1 \neq C_2$. For $(x, q) \in B_m$ and $\alpha \in \bar{J} \setminus \rho$, let $f(x, q, \alpha)$ be the unique $C \in \mathcal{J}_{x,q}$ such that $\alpha \in C$ if such a C exists, and let it be undefined otherwise.

Let $\alpha_0 = \min(\bar{J} \setminus \rho)$, and let A_α^* be the set of $(x, q) \in B_m$ such that:

- $f(x, q, \alpha)$ and $f(x, q, \alpha_0)$ are both defined.
- $f(x, q, \alpha) = f(x, q, \alpha_0)$.

Claim 10.21. $A_\alpha^* \in \mathcal{F}_m$.

Proof. Otherwise $B_m \setminus A_\alpha^* \in \mathcal{F}_m^+$, so that applying j_1 we have $B_m \setminus A_\alpha^* = j_1(B_m \setminus A_\alpha^*) \in j_1(\mathcal{F}_m)^+$. For each $\beta \in \bar{J} \setminus \rho$, choose $(x_\beta, q_\beta) \in X_{\alpha_0} \cap X_\alpha \cap X_\beta \cap (B_m \setminus A_\alpha^*)$.

Since $\text{cf}(\mu) > |B_m|$ in $V[L][A^{gg}][R][P_1 \times P_{2b} \times P_3]$, we may find $(x, q) \in B_m$ such that $(x_\beta, q_\beta) = (x, q)$ for unboundedly many β . For all such β we have by the

choice of X_β that $h' + (x, q) \Vdash_{j_1(\mathbb{P})}^* j_1(u_\beta) <_{j_1(T)} v$, so $\bar{J}_{x,q}$ is unbounded and hence $\bar{J}_{x,q} = C$ for some $C \in \mathcal{J}_{x,q}$.

Since $(x, q) \in A_\alpha$, $\alpha \in \bar{J}_{x,q} = C$ so that $f(x, q, \alpha)$ is defined and $f(x, q, \alpha) = C$. The same is true for α_0 , so $f(x, q, \alpha) = f(x, q, \alpha_0) = C$, and $(x, q) \in A_\alpha^*$. This is a contradiction since by construction $(x, q) \in B_m \setminus A_\alpha^*$. $\square$

Define relations $(R_{x,q})_{(x,q) \in B_m}$ on $(\bar{J} \setminus \rho) \times 1$ as follows: $(\alpha, 0) R_{x,q} (\beta, 0)$ if and only if $h' + (x, q) \Vdash^* u_\alpha <_T u_\beta$. It is easy to see that these relations form a system on $(\bar{J} \setminus \rho) \times 1$ in the sense of Definition 2.14: the main point is that if $\alpha, \beta \in \bar{J} \setminus \rho$ with $\alpha < \beta$ then by hypothesis $h' \Vdash^* u_\alpha < u_\beta$, and any minimal one-point extension of a suitable condition witnessing this will witness that $h' + (x, q) \Vdash^* u_\alpha < u_\beta$ for some $(x, q) \in B_m$.

For every $(x, q) \in B_m$, let $b_{x,q} = \{\alpha \in \bar{J} \setminus \rho : (x, q) \in A_\alpha^*\}$. If $\alpha, \beta \in b_{x,q}$ with $\alpha < \beta$, then $(x, q) \in A_\alpha^* \cap A_\beta^*$, so $f(x, q, \alpha) = f(x, q, \beta) = C \in \mathcal{J}_{x,q}$ and hence $h' + (x, q) \Vdash^* u_\alpha < u_\beta$, that is $(\alpha, 0) R_{x,q} (\beta, 0)$. Let $b_{x,q}^*$ be the function with domain $b_{x,q}$ and $b_{x,q}^*(\alpha) = (\alpha, 0)$ for all $\alpha \in b_{x,q}$.

Claim 10.22. $(b_{x,q}^*)$ is a system of branches through $(R_{x,q})$ in the sense of Definition 2.14.

Proof. Let $\beta \in b_{x,q}$ and let $\alpha < \beta$ be such that $h' + (x, q) \Vdash^* u_\alpha <_T u_\beta$. As $(x, q) \in A_\beta^*$, we have $f(x, q, \beta) = f(x, q, \alpha_0) = C$ where $C \in \mathcal{J}_{x,q}$. By the properties of $\mathcal{J}_{x,q}$ we have $\alpha \in C$, so that $f(x, q, \alpha) = C = f(x, q, \alpha_0)$ and $(x, q) \in A_\alpha^*$, hence $\alpha \in b_{x,q}$. Finally for every $\alpha \in \bar{J} \setminus \rho$ we have $\alpha \in b_{x,q}$ for any $(x, q) \in A_\alpha^*$. $\square$

Let E be the set of $(x, q) \in B_m$ such that $b_{x,q} \in V[L][A^{gg}][R]$ and $b_{x,q}$ is unbounded. By the distributivity of $\mathbb{P}_1 \times \mathbb{P}_{2b} \times \mathbb{P}_3$, we have $E \in V[L][A^{gg}][R]$. We now work below a condition in $\mathbb{P}_1 \times \mathbb{P}_{2b} \times \mathbb{P}_3$ that determines the value of E .

Claim 10.23. $E \in \mathcal{F}_m$.

Proof. Suppose for a contradiction that $B_m \setminus E \in \mathcal{F}_m^+$. The set $\{b_{x,q}^* : (x, q) \in B_m \setminus E\}$ is still a system of branches through $(R_{x,q})$, since for every $\alpha \in \bar{J} \setminus \rho$ we may choose $(x, q) \in (B_m \setminus E) \cap A_\alpha^*$ to witness that $\alpha \in \text{dom}(b_{x,q}^*)$.

Now we appeal to Fact 2.15 with $\lambda = \lambda_{n^*-3}^b$, $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_{2b} \times \mathbb{P}_3$, and $\mathbb{Q} = \mathbb{P}_1^* \times \mathbb{P}_{2b}^* \times \mathbb{P}_3^*$ defined as in the discussion preceding Claim 10.17. It follows that there is $(x, q) \in B_m \setminus E$ such that $b_{x,q} \in V[L][A^{gg}][R]$ and $b_{x,q}$ is unbounded, an immediate contradiction. $\square$

By distributivity, $(b_{x,q})_{(x,q) \in E} \in V[L][A^{gg}][R]$. For every $\alpha \in \bar{J} \setminus \rho$, let $E_\alpha = \{(x, q) \in E : \alpha \in b_{x,q}\}$. Since $E_\alpha = E \cap A_\alpha^*$, $E_\alpha \in \mathcal{F}_m$. For all $\alpha, \beta \in \bar{J} \setminus \rho$ with $\alpha < \beta$ and all $(x, q) \in E_\alpha \cap E_\beta$, $\alpha, \beta \in b_{x,q}$ and hence $h' + (x, q) \Vdash^* u_\alpha < u_\beta$. $\square$

Lemma 10.24. *There exist $\rho < \mu$ and a sequence of conditions $(p_\alpha)_{\alpha \in \bar{J} \setminus \rho}$ in $V[L][A^{gg}][R]$ such that:*

- For all α the stem of p_α is h .

- For all α and β with $\alpha < \beta$, $p_\alpha \wedge p_\beta \Vdash u_\alpha < u_\beta$.

Proof. Let the stem h from the conclusion of Lemma 10.19 be $q_{17}, \dots, x_{n-1}, [g]$.

We will construct an increasing sequence $(\rho_m)_{n \leq m < \omega}$ of ordinals less than μ , together with sets $(A_m^\alpha)_{n \leq m, \alpha \in J \setminus \rho_{m+1}}$ and functions $(F_{m+1}^*)_{n \leq m}$ such that the following properties hold, along with another one to be stated below:

- $A_m^\alpha \in U_m$.
- $F_{m+1}^* \in C_{m+1}$.
- $A_m^\alpha \times^< A_{m+1}^{\alpha+1} \subseteq \text{dom}(F_{m+1}^*)$.

For $m \geq n$, $\alpha \in J \setminus \rho_m$ say that a lower part s is *good for α at m* if and only if:

- s has the the form $q'_{17}, \dots, x'_{m-1}$.
- $q'_k \leq q_k$ and $x'_k = x_k$ for $17 \leq k < n$.
- $x'_k \in A_k^\alpha$ for $n \leq k < m$.
- $q'_k \leq F_{k+1}^*(x'_k, x'_{k+1})$ for $n < k < m - 1$.
- $q'_n \leq g(x'_n)$ if $m > n$.

The final key property is that:

- For $\alpha, \beta \in J \setminus \rho_m$, if a lower part s of form $q'_{17}, \dots, x'_{m-1}$ is good for both α and β at m , then $s + [g] \Vdash^* u_\alpha < u_\beta$ if $m = n$, and $s + [F_m^*(x'_{m-1}, -)] \Vdash^* u_\alpha < u_\beta$ if $m > n$.

To initialize the construction we set $\rho_n = 0$, and verify that the key property holds for $m = n$. Suppose that $\alpha, \beta \in J$ with $\alpha < \beta$, and s is good for both α and β at n . That is to say s has the the form $q'_{17}, \dots, x_{n-1}$ where $q'_k \leq q_k$ for $17 \leq k < n$. Therefore $s + [g] \leq^* h$, and so by the conclusion of Lemma 10.19 $s + [g] \Vdash^* u_\alpha < u_\beta$.

Continuing the initialization apply Lemma 10.20 to the stem h , set J and sequence $(u_\alpha)_{\alpha \in J}$. Let ρ_{n+1} be the ordinal ρ from the conclusion of that lemma, and choose sets $(A_n^\alpha)_{\alpha \in J \setminus \rho_{n+1}}$ and a function $F_{n+1}^* \in C_{n+1}$ as in the discussion following the statement of Lemma 10.20. We verify that the key property holds for $m = n + 1$.

Suppose that $\alpha, \beta \in J \setminus \rho_{n+1}$ with $\alpha < \beta$, and s is good for both α and β at $n + 1$. That is to say s has the the form $q'_{17}, \dots, x_{n-1}, q, x$ where $q'_k \leq q_k$ for $17 \leq k < n$, $q \leq g(x)$, $x \in A_n^\alpha \cap A_n^\beta$. Let $t = q_{17}, \dots, x_{n-1}$, so that by construction $h + (x, [F_{n+1}^*(x, -)]) = t + g(x) + x + [F_{n+1}^*(x, -)] \Vdash^* u_\alpha < u_\beta$. Since $s \leq^* t + g(x) + x$, $s + [F_{n+1}^*(x, -)] \Vdash^* u_\alpha < u_\beta$ as required.

Now suppose that $m \geq n$ and we have constructed ρ_k for $k \leq m + 1$, $(A_k^\alpha)_{\alpha \in J \setminus \rho_{k+1}}$ for $k \leq m$ and F_k^* for $k \leq m + 1$. Let s be a lower part of form $q'_{17}, \dots, x'_m$ and let J^s be the set of $\alpha \in J \setminus \rho_{m+1}$ such that s is good for α at $m + 1$. By construction $s + [F_{m+1}^*(x'_m, -)] \Vdash^* u_\alpha < u_\beta$ for $\alpha, \beta \in J^s$ with $\alpha < \beta$.

For every lower part s such that J^s is bounded, let $\rho^s = \sup(J^s)$. For s such that J^s is unbounded, we apply Lemma 10.20 to the unbounded set J^s and the stem $s + [F_{m+1}^*(x'_m, -)]$. We obtain $\rho^s < \mu$ and sets $(E_\alpha^s)_{\alpha \in J^s \setminus \rho^s}$ in $\mathcal{F}_{m+1}$, such

that $s + F_{m+1}^*(x'_m, x) + (x, q) \Vdash^* u_\alpha < u_\beta$ for all $\alpha, \beta \in J^s \setminus \rho^s$ with $\alpha < \beta$ and all $(x, q) \in E_\alpha^s \cap E_\beta^s$.

Now let $\rho_{m+2} = \sup_s \rho^s$, and for $\alpha \in J \setminus \rho_{m+2}$ let $E_\alpha^{m+1} = \Delta_{s, \alpha \in J^s} E_\alpha^s$, that is $E_\alpha^{m+1} = \{x : \forall s \prec x \alpha \in J^s \implies x \in E_\alpha^s\}$. It follows that for $\alpha, \beta \in J \setminus \rho_{m+2}$ with $\alpha < \beta$, $(x, q) \in E_\alpha^{m+1} \cap E_\beta^{m+1}$, and s as above such that $s \prec x$ and s is good for both α and β at $m+1$ we have $s + F_{m+1}^*(x'_m, x) + (x, q) \Vdash^* u_\alpha < u_\beta$.

As in the discussion following Lemma 10.20, we now (shrinking E_α^{m+1} if necessary) choose $(A_{m+1}^\alpha)_{\alpha \in J \setminus \rho_{m+2}}$ and F_{m+2}^* such that $E_\alpha^{m+1} = \{(x, q) : x \in A_{m+1}^\alpha, q \leq [F_{m+2}^*(x, -)]\}$. To finish the construction we verify that we have maintained the key property. So let s be a lower part of form $q'_{17}, \dots, x'_m, q'_{m+1}, x'_{m+1}$ which is good for both α and β at $m+2$. Let t be the initial segment $q'_{17}, \dots, x'_m$ of s , so that $t \prec x'_{m+1}$ and t is good for α and β at $m+1$. By definition $q_{m+1} \leq F_{m+1}^*(x'_m, x'_{m+1})$ and $x'_{m+1} \in A_\alpha^{m+1} \cap A_\beta^{m+1}$, by construction $t + F_{m+1}^*(x'_m, x'_{m+1}) + x + [F_{m+2}^*(x, -)] \Vdash^* u_\alpha < u_\beta$, so $s + [F_{m+2}^*(x, -)] \Vdash^* u_\alpha < u_\beta$ as required.

Now let $\rho = \sup_{n \leq m < \omega} \rho_n$, and for $\alpha \in J \setminus \rho$ define p_α as follows:

- p_α has $q_{17}, \dots, x_{n-1}$ as an initial segment.
- $f_n^{p_\alpha} = g \upharpoonright A_n^\alpha$.
- $F_k^{p_\alpha} = F_k^* \upharpoonright A_{k-1}^\alpha \times \prec A_k^\alpha$ for $n < k < \omega$.

Let $\alpha, \beta \in J \setminus \rho$ with $\alpha < \beta$, and suppose for a contradiction that there is $q \leq p_\alpha, p_\beta$ such that $q \Vdash u_\alpha \not< u_\beta$. Let the lower part of q be $t = q'_{17}, \dots, x'_{m-1}$, where without loss of generality $m > n$. By definition t is good for both α and β at m , and $f_m^q \leq F^*(x'_{m-1}, -)$, so that a fortiori $[f_m^q] \leq [F^*(x'_{m-1}, -)]$. The stem of q is $h = t + [f_m^q]$, and so by construction $h \Vdash^* u_\alpha < u_\beta$ for an immediate contradiction. $\square$

Lemma 10.25. *The tree property at μ holds in the model $V[L][A^{gg}][R][\bar{P}]$.*

Proof. Let p be arbitrary. Our whole construction could have been done below p , so that the conditions p_α from the conclusion of Lemma 10.24 can be taken to be extensions of p . Since $\bar{\mathbb{P}}$ is μ -cc, by Lemma 2.1 there is α such that p_α forces $\{\beta : p_\beta \in G\}$ to be unbounded. But for $\beta < \gamma$ with $p_\beta, p_\gamma \in G$, we have $p_\beta \wedge p_\gamma \in G$, so that $u_\beta <_T u_\gamma$. It follows that p_α forces that the u_β such that $p_\beta \in G$ form a cofinal branch. $\square$

Lemma 10.26. *The tree property at μ holds in the model $V[L][A^{gg}][\bar{P}]$.*

Proof. We need to verify that we have satisfied the hypotheses of Lemma 2.21 as listed at the start of Section 2.5. $V[L][A^{gg}]$ plays the role of V , and $\bar{\mathbb{P}}$ plays the role of $\mathbb{P}$.

Hypothesis 1 is immediate. By Lemma 8.23 κ is a cardinal and $\mu = \kappa^+$ in $V[L][A^{gg}][\bar{P}]$, taking care of the first part of Hypothesis 2. Claim 10.16 takes care of the rest of Hypothesis 2. Hypothesis 3 follows from Lemma 10.15. The remaining hypotheses follow from Lemmas 8.6, 8.7 and 8.8.

The conclusion is now immediate from Lemmas 10.25 and 2.21. $\square$

10.2. The tree property at $\lambda = \aleph_{\omega^2+2}$

Recall from Section 10.1 that $\nu = \lambda_\omega^b$ and $\mu = \lambda_{\omega+1}^b$. Let $\lambda = \lambda_{\omega+2}^b$, so that in our final model the cardinal λ is destined to become $\aleph_{\omega^2+2}$. We will establish that the tree property holds at λ in $V[L][A^{gg}][\bar{P}]$. The argument is quite similar to that from [3, Theorem 3.1], but there are extra complications.

We start by constructing a suitable generic embedding with critical point λ whose domain is a generic extension of $V[L][A^{gg}]$. Now that the desired critical point is λ the poset $\mathbb{L} \restriction \kappa * \mathbb{L}^b * \mathbb{I}^b$ counts as small forcing, and the main obstacle is to deal with $\mathbb{J}_0^c * \mathbb{J}_1^c$. We note that the situation here is very like the $n = 0$ case in Section 9.1, in particular the proof of Claim 9.7 in that section.

We start by writing $V[L][A^{gg}] = V^{lbi}(\kappa)[J_0^c][J_1^c][A_e \times A^{gg}]$. Recall that $\mathbb{B}_1^c$ is defined and $< \lambda$ -closed in $V^{lbi}(\kappa)[J_0^c]$, and $\mathbb{U}_1^c = (\mathbb{B}_1^c)^{+A_1^c}$. Similarly $\mathbb{C}_1^c$ is defined and $< \lambda$ -closed in $V^{lbi}(\kappa)[J_0^c]$, $\mathbb{S}_1^c = (\mathbb{C}_1^c)^{+A_1^c}$.

Parallel to the proof of Claim 9.7, let $\text{TBC} = \mathcal{A}^{V^{lbi}(\kappa)[A_0^c * U_0^c]}(\mathbb{S}_0^c, \mathbb{B}_1^c \times \mathbb{C}_1^c)$, so that TBC is defined and $< \lambda$ -closed in $V^{lbi}(\kappa)[A_0^c * U_0^c]$. Let QTT be the two-step iteration of term forcing which adds a TBC -generic object inducing $U_1^c * S_1^c$. Exactly as in the proof of Claim 9.7, QTT is $< \mu$ -closed in $V^{lbi}(\kappa)[J_0^c * J_1^c]$. Forcing with QTT over $V[L][A^{gg}]$, we obtain a model $V[L][A^{gg}][QTT] = V^{lbi}(\kappa)[J_0^c][A_1^c \times \text{TBC}][A_e \times A^{gg}]$.

We will do the construction for Lemma 4.10 with appropriate parameter settings. Start by recalling that in the context of $\mathbb{A}_0^c * \mathbb{U}_0^c * \mathbb{S}_0^c \times \mathbb{A}_1^c$ we have $\mu_0 = \lambda_{17}^b$, $\mu_1 = \lambda_{\omega+1}^b = \mu$, $\mu_2 = \lambda_{\omega+2}^b = \lambda$, $\mu_3 = \lambda_{\omega+3}^b$. Accordingly we will set the parameters for Lemma 4.10 as follows:

- $n = 0$.
- $\eta = \mu_3 = \lambda_{\omega+3}^b$.
- V_{inn} is V .
- V_{def} is $V^{lbi}(\kappa)$, so that $V_{\text{def}}[A \restriction \mu_{n+2} * U \restriction \mu_{n+2}]$ is $V^{lbi}(\kappa)[A_0^c * U_0^c]$.
- V' is $V^{lbi}(\kappa)[A_0^c * U_0^c * S_0^c * A_1^c]$.
- $\mathbb{D}^2 = \mathbb{A}^{gg}$.
- $\mathbb{D}^3 = \text{TBC}$.
- $\mathbb{D}^0 = \mathbb{A}_e$.
- $\mathbb{D}^{\text{small}}$ is trivial forcing.
- $V'[D^{0,2,3}]$ is $V[L][A^{gg}][QTT]$.

Following the argument for Lemma 4.10, we will start with a suitable embedding $j : V[A^{gg}] \rightarrow N$, defined in $V[A^{gg}]$ and witnessing that λ is χ -supercompact in $V[A^{gg}]$ for some large enough value of χ . Since $L \restriction \kappa * L^b * I^b$ is generic for small forcing, it is easy to lift j to an embedding $j : V^{lbi}(\kappa)[A^{gg}] \rightarrow N[L \restriction \kappa][L^b][I^b]$, defined in $V^{lbi}(\kappa)[A^{gg}]$ and witnessing that λ is χ -supercompact in this model. We note that the models V and $V[A^{gg}]$ agree on $< \lambda$ -sequences of ordinals, as do $V^{lbi}(\kappa)$ and $V^{lbi}(\kappa)[A^{gg}]$.

We will force over $V[L][A^{gg}][QTT]$ with a product $\mathbb{P}_2 \times \mathbb{P}_3$ defined as in the

proof of Lemma 4.10. To analyze $\mathbb{P}_2$, we note that

$$\begin{aligned}\mathbb{A}_0^c &= \text{Add}^{V^{lb}(\kappa)}(\lambda_{17}^b, [\lambda_{\omega+1}^b, \lambda_{\omega+2}^b]) \\ &= \text{Add}^{V[A^{gg}][L \restriction \kappa][L^b]}(\lambda_{17}^b, [\lambda_{\omega+1}^b, \lambda_{\omega+2}^b]).\end{aligned}$$

By elementarity,

$$j(\mathbb{A}_0^c) = \text{Add}^{N[L \restriction \kappa][L^b]}(\lambda_{17}^b, [j(\lambda_{\omega+1}^b), j(\lambda_{\omega+2}^b)]).$$

Since $V[A^{gg}][L \restriction \kappa][L^b] \models \text{``}N[L \restriction \kappa][L^b] \subseteq N[L \restriction \kappa][L^b]\text{"}$,

$$\begin{aligned}j(\mathbb{A}_0^c) &= \text{Add}^{V[A^{gg}][L \restriction \kappa][L^b]}(\lambda_{17}^b, [j(\lambda_{\omega+1}^b), j(\lambda_{\omega+2}^b)]) \\ &= \text{Add}^{V^{lb}(\kappa)}(\lambda_{17}^b, [j(\lambda_{\omega+1}^b), j(\lambda_{\omega+2}^b)]).\end{aligned}$$

By this and similar arguments for $\mathbb{A}_1^c$ and $\mathbb{A}_e$ we have:

- $j(\mathbb{A}_0^c)/\mathbb{A}_0^c = \text{Add}^{V^{lb}(\kappa)}(\lambda_{17}^b, j(\lambda_{\omega+2}^b) \setminus \lambda_{\omega+2}^b).$
- $j(\mathbb{A}_e)/\mathbb{A}_e = \text{Add}^{V^{lb}(\kappa)}(\lambda_{17}^b, j(\lambda_{\omega+3}^b) \setminus j[\lambda_{\omega+3}^b]).$
- $j(\mathbb{A}_1^c)/\mathbb{A}_1^c = \text{Add}^{V^{lb}(\kappa)}(\lambda_{\omega+1}^b, j(\lambda_{\omega+3}^b) \setminus j[\lambda_{\omega+3}^b]).$

In summary $\mathbb{P}_{2a}$ is a Cohen poset to add subsets of λ_{17}^b defined in $V^{lb}(\kappa)$, and $\mathbb{P}_{2b}$ is a Cohen poset to add subsets of $\lambda_{\omega+1}^b$ defined in the same model.

To analyze $\mathbb{P}_3$, it is useful to recall that $\mathbb{U}^c \restriction \mu_1$ and $\mathbb{S}^c \restriction \mu_1$ are both trivial. It follows that $\mathbb{P}_3$ is defined in $V^{lbi}(\kappa)[\mathbb{A}_0^c \times \mathbb{A}_1^c \times A^{gg}]$ and is $< \lambda_{\omega+1}^b$ -closed in $V^{lbi}(\kappa)[\mathbb{A}_0^c * \mathbb{U}_0^c * \mathbb{S}_0^c][\mathbb{A}_1^c \times \mathbb{B}_1^c \times \mathbb{C}_1^c \times A^{gg}]$. It is important that, as we noted in the proof of Lemma 4.5, $\mathbb{P}_3$ collapses λ to become an ordinal of cofinality μ and cardinality μ . We note for use later that *a fortiori* $\mathbb{P}_3$ is $< \mu$ -closed in $V^{lbi}(\kappa)[J_0^c][J_1^c \times A^{gg}]$.

As in Lemma 4.10 we lift j to obtain a generic embedding with critical point λ which has domain $V[L][A^{gg}][QTT]$ and exists in $V[L][A^{gg}][QTT][P_2 \times P_3]$. In the current setting we may restrict the domain to $V[L][A^{gg}]$, so we have a generic embedding with domain $V[L][A^{gg}]$ obtained by forcing over $V[L][A^{gg}]$ with $\mathbb{P}_2 \times (\mathbb{P}_3 \times QTT)$.

By Lemma 2.54, $\mathbb{P}_2 \times \mathbb{P}_2$ is λ -cc in $V[L][A^{gg}]$. It follows from Lemma 2.11 that $\mathbb{P}_2$ has the λ -approximation property in $V[L][A^{gg}]$. By another similar appeal to Lemmas 2.54 and 2.11, $\mathbb{P}_{2a}$ has the μ -approximation property in $V[L][A^{gg}][P_{2b} \times P_3 \times QTT]$.

By the preceding analysis QTT is $< \mu$ -closed in $V^{lbi}(\kappa)[J^c]$, and since $\mathbb{A}^{gg}$ is highly distributive the same is true in $V^{lbi}(\kappa)[J^c][A^{gg}]$, which is the submodel of $V[L][A^{gg}]$ missing only $\mathbb{A}_e$. As we noted above $\mathbb{P}_3$ is also $< \mu$ -closed in this model, so that $\mathbb{P}_3 \times QTT$ is $< \mu$ -closed in this model.

We claim that $\mathbb{P}_{2b}$ is $< \mu$ -distributive in $V^{lbi}(\kappa)[J^c][A^{gg}]$. To see this we note that we need to show that $\mathbb{P}_{2b}$ is λ_n^b -distributive for all n , and this will follow by the usual arguments using term forcing and Easton's Lemma.

It follows that $\mathbb{P}_3 \times \mathbb{QTT}$ is $< \mu$ -closed in $V^{lbi}(\kappa)[J^c][A^{gg}][P_{2b}]$. Therefore $\mathbb{P}_3 \times \mathbb{QTT}$ is formerly $< \mu$ -closed in the sense of Fact 2.12 in the model

$$V^{lbi}(\kappa)[J^c][A^{gg}][P_{2b}][P_{2a} \times A_e] = V[L][A^{gg}][P_2].$$

The key points are that A_0^c added λ subsets of λ_{17}^b , and that $\mathbb{P}_{2a} \times \mathbb{A}_e$ is μ -cc in $V^{lbi}(\kappa)[J^c][A^{gg}][P_{2b}]$.

Let $\dot{T} \in V[L][A^{gg}]$ be a $\mathbb{P}$ -name for a λ -tree. We assume that level α is a subset of $\{\alpha\} \times \mu$.

Lemma 10.27. *In $V[L][A^{gg}]$ there exist a stem h , an unbounded set $I \subseteq \lambda$ and $(u_\alpha)_{\alpha \in I}$ such that u_α is a node of level α for all $\alpha \in I$, and $h \Vdash^* u_\alpha <_{\dot{T}} u_\beta$ for all $\alpha, \beta \in I$ with $\alpha < \beta$.*

Proof. Let j be the generic embedding with domain $V[L][A^{gg}]$ and critical point λ , added by forcing over $V[L][A^{gg}]$ with $\mathbb{P}_2 \times (\mathbb{P}_3 \times \mathbb{QTT})$. Define a system on $\lambda \times \mu$ indexed by stems as follows: $uR_hv \iff h \Vdash^* u <_{\dot{T}} v$.

In $V[L][A^{gg}][P_2][P_3][QTT]$ define a branch b_h through R_h as follows: $\alpha \in \text{dom}(b_h)$ if and only if there is $\eta < \mu$ such that $h \Vdash_{j(\bar{P})}^* (\alpha, \eta) <_{j(\dot{T})} (\lambda, 0)$. It is routine to check that the branches b_h form a system of branches in the sense of Definition 2.14. It is also routine that if $\alpha \in \text{dom}(b_h)$ then $\beta \in \text{dom}(b_h) \cap \alpha$ if and only if there is ζ such that $h \Vdash_{\mathbb{P}}^* (\beta, \zeta) <_{\dot{T}} (\alpha, \eta)$, and in this case $b_h(\beta) = \zeta$ for the unique such ζ . In particular $b_h \restriction \alpha \in V[L][A^{gg}]$ for all $\alpha \in \text{dom}(b_h)$.

We now appeal to Lemma 2.16 with $\mathbb{P}_3 \times \mathbb{QTT}$ in place of $\mathbb{P}$, $\mathbb{P}_{2a} \times \mathbb{A}_e$ in place of $\mathbb{E}$, $V^{lbi}(\kappa)[J^c][A^{gg}][P_{2b}]$ in place of V , and λ_{17}^b in place of δ . It follows that there is h such that $b_h \in V[L][A^{gg}][P_2]$ and $\text{dom}(b_h)$ is unbounded in λ . Since $\mathbb{P}_2$ has the λ -approximation property in $V[L][A^{gg}]$, it follows that $b_h \in V[L][A^{gg}]$. Now we set $I = \text{dom}(b_h)$ and $u_\alpha = (\alpha, b_h(\alpha))$ to finish. $\square$

We can now describe the main idea of the proof that λ has the tree property. As in the proof of the tree property for μ in Section 10.1, we will construct conditions (p_α) for all sufficiently large α in I , such that $p_\alpha \wedge p_\beta \Vdash_{\mathbb{P}}^{V[L][A^{gg}]} u_\alpha < u_\beta$ for $\alpha < \beta$. This time we will construct this sequence of conditions in $V[L][A^{gg}][P_3 \times QTT]$: as in Section 10.1 this will give a branch in $V[L][A^{gg}][P_3 \times QTT][\bar{P}]$, and we will need to use a suitable branch lemma to find a branch in $V[L][A^{gg}][\bar{P}]$.

Let h be of the form $(\bar{s}, [g])$ for some lower part $\bar{s}$ and some 1-variable function g . We note that if $\beta \in I$ then $I \cap \beta = \{\alpha : h \Vdash^* u_\alpha < u_\beta\}$.

Let $\bar{q} = [g]$ and let the length of $h = (\bar{s}, \bar{q})$ be t . For all relevant q , fix g_q such that $q = [g_q]$. We take care to choose $g_{\bar{q}} = g$.

Remark 10.28. Let $F \in \mathbf{F}_n$, that is to say F is a potential value of F_n^p for some $p \in \mathbb{P}$. Then for all relevant x , $[F(x, -)] = [g_{[F(x, -)]}]$, that is to say $F(x, y) = g_{[F(x, -)]}(y)$ for many y . Taking a diagonal intersection, we may shrink the domain of F to arrange that $F(x, y) = g_{[F(x, -)]}(y)$ for all $(x, y) \in \text{dom}(F)$. In the sequel we will arrange that all 2-variable constraint functions have been treated in this way.

As in Section 10.1, we can use the functions g_q to prolong a stem (s, q) to stems $(s, q) + (x, r) = (s, g_q(x), x, r)$ for each (x, r) with $x \in \text{dom}(g_q)$. In the natural way we use this to define (recursively) a notion of *extension* for stems.

In the generic extension $V[L][A^{gg}][P_2][P_3][QTT]$ where j is defined, let $v = j(u)_{\bar{\lambda}}$ where $\bar{\lambda}$ is the λ^{th} point of $j(I)$. We work below some condition in $\mathbb{P}_2 \times \mathbb{P}_3 \times \mathbb{QTT}$ which fixes the values of $\bar{\lambda}$ and v . This condition forces “for all $\alpha \in I$, $(\bar{s}, \bar{q}) \Vdash^* u_\alpha < v$ ”. So in $V[L][A^{gg}][P_2][P_3][QTT]$ we may choose a sequence $(r_\alpha)_{\alpha \in I}$, such that $r_\alpha \in j(\mathbb{P})$ with stem $(\bar{s}, \bar{q})$ and $r_\alpha \Vdash u_\alpha < v$.

For all stems (s, q) extending $(\bar{s}, \bar{q})$, define

$$J_{s,q} = \{\alpha \in I \mid (s, q) \Vdash_{j(\bar{P})}^* u_\alpha <_{j(\dot{I})} v\}.$$

We note that in general the definition of $J_{s,q}$ involves the generic embedding j , so it takes place in $V[L][A^{gg}][P_2][P_3][QTT]$. However, it is clear that $J_{\bar{s}, \bar{q}} = I$. As usual if $\beta \in J_{s,q}$, then $J_{s,q} \cap \beta = \{\alpha < \beta : (s, q) \Vdash^* u_\alpha < u_\beta\}$, so in particular $J_{s,q} \cap \beta \in V[L][A^{gg}]$.

Recall that in $V[L][A^{gg}][P_3 \times QTT \times P_{2b}]$, λ has cardinality and cofinality μ , while $\mathbb{P}_{2a}$ is μ -cc and has the μ -approximation property. It follows that if $J_{s,q}$ is unbounded then $J_{s,q} \in V[L][A^{gg}][P_3 \times QTT \times P_{2b}]$.

Working in $V[L][A^{gg}][P_3 \times QTT \times P_{2b}]$, let $\mathcal{J}_{s,q}$ be the set of all possible unbounded values for $J_{s,q}$. Since $\mathbb{P}_{2a}$ is μ -cc, $|\mathcal{J}_{s,q}| < \mu$, and for any name $\dot{C}$ for a bounded subset of λ there is $\beta < \lambda$ such that $\Vdash_{\mathbb{P}_{2a}} \dot{C} \subseteq \beta$. We use these facts to choose $\rho < \lambda$ so large that:

- It is forced by $\mathbb{P}_{2a}$ that for all (s, q) , if $J_{s,q}$ is bounded in λ then $J_{s,q} \subseteq \rho$.
- For all (s, q) and all distinct $C, D \in \mathcal{J}_{s,q}$, $C \cap D \subseteq \rho$.

For $\alpha \in I \setminus \rho$, define a partial function f , by setting $f((s, q), \alpha)$ equal to the unique $C \in \mathcal{J}_{s,q}$ such that $\alpha \in C$. We note that $\mathcal{J}_{\bar{s}, \bar{q}} = \{I\}$, so that $f((\bar{s}, \bar{q}), \alpha) = I$ for all $\alpha \in I \setminus \rho$.

Let $\alpha_0 = \min(I \setminus \rho)$. Fix a length $k \geq t$, we will consider stems (s, q) of this length extending $(\bar{s}, \bar{q})$. Let $\alpha \in I \setminus \rho$, and define $B_{k,\alpha}$ as the set of pairs (s, q) such that (s, q) extends $(\bar{s}, \bar{q})$, (s, q) has length k , and $f((s, q), \alpha) = f((s, q), \alpha_0)$.

Given $(s, q) \in B_{k,\alpha}$, define

$$F_\alpha^{s,q} = \{(x, r) : (s, q) + (x, r) \in B_{k+1,\alpha}\}$$

We note that for all $\alpha \in I \setminus \rho$:

- Since $f((\bar{s}, \bar{q}), \alpha) = f((\bar{s}, \bar{q}), \alpha_0) = I$, $(\bar{s}, \bar{q}) \in B_{t,\alpha}$.
- By the distributivity of $\mathbb{P}_3 \times \mathbb{QTT} \times \mathbb{P}_{2b}$, $(B_{k,\alpha})_{t \leq k < \omega} \in V[L][A^{gg}]$ and $(F_\alpha^{s,q})_{(s,q) \text{ extends } (\bar{s}, \bar{q})} \in V[L][A^{gg}]$.

Remark 10.29. The following Claim is an assertion in $V[L][A^{gg}][P_3 \times QTT \times P_{2b}]$ about sets which all lie in $V[L][A^{gg}]$, but are defined in terms of the function f which only exists in $V[L][A^{gg}][P_3 \times QTT \times P_{2b}]$, and in turn is defined using the embedding j which only exists in $V[L][A^{gg}][P_3 \times QTT \times P_2]$. We will prove it (as

one would expect) by a forcing argument involving both $\mathbb{P}_3 \times \mathbb{QTT} \times \mathbb{P}_{2b}$ and $\mathbb{P}_{2a}$. Similar remarks apply to Claim 10.31 below. Throughout we will only discuss f and j in appropriate generic extensions, or in formulae which are being forced to hold in such extensions.

Claim 10.30. *For all $\alpha \in I \setminus \rho$ and all $(s, q) \in B_{k,\alpha}$, $F_\alpha^{s,q} \in \mathcal{F}_{k+1}$.*

Proof. If not then let $p \in \mathbb{P}_3 \times \mathbb{QTT} \times \mathbb{P}_{2b}$ be such that

$$p \Vdash_{\mathbb{P}_3 \times \mathbb{QTT} \times \mathbb{P}_{2b}}^{V[L][A^{gg}]} "(s, q) \in \dot{B}_{k,\alpha} \text{ and } \dot{F}_\alpha^{s,q} \notin \mathcal{F}_{k+1}."$$

Since j fixes sets of rank below λ ,

$$(p, 1_{\mathbb{P}_{2a}}) \Vdash_{\mathbb{P}_3 \times \mathbb{QTT} \times \mathbb{P}_2}^{V[L][A^{gg}]} "B_{k+1} \setminus \dot{F}_\alpha^{s,q} \in j(\mathcal{F}_{k+1})^+."$$

Forcing below p we obtain $P_3 \times \mathbb{QTT} \times P_{2b}$, such that $(s, q) \in B_{k,\alpha}$ and $F_\alpha^{s,q} \notin \mathcal{F}_{k+1}$ in $V[L][A^{gg}][P_3 \times \mathbb{QTT} \times P_{2b}]$.

Let $f((s, q), \alpha) = f((s, q), \alpha_0) = C$. Since $C \in \mathcal{J}_{s,q}$, C is a possible value for $J_{s,q}$, and so we may choose $\bar{p} \in \mathbb{P}_{2a}$ such that

$$\bar{p} \Vdash_{\mathbb{P}_{2a}}^{V[L][A^{gg}][P_3 \times \mathbb{QTT} \times P_{2b}]} "C = \dot{J}_{s,q}."$$

Forcing below $\bar{p}$ we obtain P_{2a} , such that $C = J_{s,q}$ in $V[L][A^{gg}][P_3 \times \mathbb{QTT} \times P_2]$.

So $\alpha_0, \alpha \in J_{s,q}$, that is to say $(s, q) \Vdash_{j(\bar{\mathbb{P}})}^* u_{\alpha_0}, u_\alpha < v$. We choose $p' \in j(\bar{\mathbb{P}})$ with stem (s, q) such that $p' \Vdash_{j(\bar{\mathbb{P}})} u_{\alpha_0}, u_\alpha < v$.

Take a minimal one-step extension p'' of p' , arranging that the stem of p'' is $(s, q) + (x, r)$ and $(x, r) \in B_{k+1} \setminus F_\alpha^{s,q}$. $p'' \Vdash_{j(\bar{\mathbb{P}})} u_{\alpha_0}, u_\alpha < v$, so that $(s, q) + (x, r) \Vdash_{j(\bar{\mathbb{P}})}^* u_{\alpha_0}, u_\alpha < v$.

We have $\alpha_0, \alpha \in J_{(s,q)+(x,r)}$. Since $\alpha, \alpha_0 > \rho$ we see that $J_{(s,q)+(x,r)}$ is unbounded. So $J_{(s,q)+(x,r)} \in \mathcal{J}_{(s,q)+(x,r)}$, say it is D .

Returning to the model $V[L][A^{gg}][P_3 \times \mathbb{QTT} \times P_{2b}]$, we have $\alpha_0, \alpha \in D$, so $f((s, q) + (x, r), \alpha) = D = f((s, q) + (x, r), \alpha_0)$, that is to say $(s, q) + (x, r) \in B_{k+1,\alpha}$. Therefore $(x, r) \in F_\alpha^{s,q}$ by definition, contradicting our choice of (x, r) as an element of $B_{k+1} \setminus F_\alpha^{s,q}$. $\square$

The following claim will ultimately be used to create a branch using the nodes u_α .

Claim 10.31. *Let $\alpha, \beta \in I \setminus \rho$ and let (s, q) have length k with $(s, q) \in B_{k,\alpha} \cap B_{k,\beta}$. Then $(s, q) \Vdash^* u_\alpha < u_\beta$.*

Proof. We work in $V[L][A^{gg}][P_3 \times \mathbb{QTT} \times P_{2b}]$. By the definitions of $B_{k,\alpha}$ and $B_{k,\beta}$, $f((s, q), \alpha) = f((s, q), \alpha_0) = f((s, q), \beta) = C$ say. There is $p \in \mathbb{P}_{2a}$ forcing that $C = J_{s,q}$: if we force below p then in the extension $\alpha, \beta \in J_{s,q}$ and we may choose $r \in j(\bar{\mathbb{P}})$ with stem (s, q) such that $r \Vdash_{j(\bar{\mathbb{P}})} u_\alpha, u_\beta < v$, from which it follows that $r \Vdash_{j(\bar{\mathbb{P}})} u_\alpha < u_\beta$. By elementarity there is $r_0 \in \bar{\mathbb{P}}$ with stem (s, q) such that $r_0 \Vdash_{\bar{\mathbb{P}}} u_\alpha < u_\beta$, so $(s, q) \Vdash^* u_\alpha < u_\beta$. $\square$

Claim 10.32 exposes a “monotonicity” property of the sets $B_{k,\alpha}$ which will be crucial in the proof of Lemma 10.34 below.

Let $(s, q) \in B_{k,\alpha}$ and (s', q') be a direct extension of (s, q) , then $(s', q') \in B_{k,\alpha}$.

Proof. Let $f((s, q), \alpha) = f((s, q), \alpha_0) = C$ and let $p \in \mathbb{P}_{2a}$ force that $J_{s,q} = C$. Force below p , choose $r \in j(\bar{\mathbb{P}})$ with stem (s, q) such that $r \Vdash u_{\alpha_0}, u_\alpha < v$, and refine r to a condition r' with stem (s', q') , so that $r' \Vdash u_{\alpha_0}, u_\alpha < v$ and hence $(s', q') \Vdash^* u_{\alpha_0}, u_\alpha < v$. So $\alpha_0, \alpha \in J_{s',q'}$, since $\rho < \alpha_0 < \alpha$ we see that $J_{s',q'}$ is unbounded, say $J_{s',q'} = D \in \mathcal{J}_{s',q'}$. Then $f((s', q'), \alpha) = f((s', q'), \alpha_0) = D$, so that $(s', q') \in B_{k,\alpha}$. $\square$

At this point we are ready to construct the conditions p_α for $\alpha \in I \setminus \rho$. We will perform the construction of the entries in p_α in $V[L][A^{gg}][P_3 \times QTT \times P_{2b}]$, and it will follow by distributivity that $p_\alpha \in V[L][A^{gg}]$ (so that $p_\alpha \in \bar{\mathbb{P}}$) for each α . However the sequence $(p_\alpha)_{\alpha \in I \setminus \rho}$ only exists in $V[L][A^{gg}][P_3 \times QTT \times P_{2b}]$.

Define p_α for $\alpha \in I \setminus \rho$, where

$$p_\alpha = \langle \bar{s}, g \restriction A_t^\alpha, A_t^\alpha, F_{t+1}^\alpha, A_{t+1}^\alpha, F_{t+2}^\alpha, \dots \rangle.$$

To start the construction of p_α , recall that $(\bar{s}, \bar{q}) \in B_{t,\alpha}$, so that $F_{\alpha}^{\bar{s}, \bar{q}} \in \mathcal{F}_{t+1}$ by Claim 10.30. We will begin by choosing A_t^α and $\bar{F}_{t+1}^\alpha$ so that:

- A_t^α and $\bar{F}_{t+1}^\alpha$ witness that $F_{\alpha}^{\bar{s}, \bar{q}} \in \mathcal{F}_{t+1}$, that is to say $(x, [\bar{F}_{t+1}^\alpha(x, -)]) \in F_{\alpha}^{\bar{s}, \bar{q}}$ for all $x \in A_t^\alpha$.
- $A_t^\alpha \subseteq \text{dom}(g)$.
- $x \in A_t^\alpha$ and $g_{[\bar{F}_{t+1}^\alpha(x, -)]}(y) = \bar{F}_{t+1}^\alpha(x, y)$ for all $(x, y) \in \text{dom}(\bar{F}_{t+1}^\alpha)$.

We note for the record that for all $x \in A_t^\alpha$:

- $(\bar{s}, \bar{q}) + (x, [\bar{F}_{t+1}^\alpha(x, -)]) \in B_{t+1,\alpha}$.
- Since $g = g_{\bar{q}}$, the stem of the minimal extension of p_α by adding x will be $(\bar{s}, \bar{q}) + (x, [\bar{F}_{t+1}^\alpha(x, -)])$.

We will complete the choice of F_{t+1}^α once we have chosen A_{t+1}^α , by defining $F_{t+1}^\alpha = \bar{F}_{t+1}^\alpha \restriction A_t^\alpha \times_{\prec} A_{t+1}^\alpha$. Note that F_{t+1}^α retains all the properties listed above for $\bar{F}_{t+1}^\alpha$.

Now assume that $k \geq t$ and we have defined sets (A_i^α) for $t \leq i \leq k$, together with functions F_i^α for $t < i \leq k$ and a function $\bar{F}_{k+1}^\alpha$, satisfying:

- $\text{dom}(F_i^\alpha) = A_{i-1}^\alpha \times_{\prec} A_i^\alpha$ for $t \leq i < k$,
- $x \in A_k^\alpha$ for all $(x, y) \in \text{dom}(\bar{F}_{k+1}^\alpha)$.
- $g_{[\bar{F}_{k+1}^\alpha(x, -)]}(y) = \bar{F}_{k+1}^\alpha(x, y)$ for all $(x, y) \in \text{dom}(\bar{F}_{k+1}^\alpha)$.
- $g_{[\bar{F}_i^\alpha(x, -)]}(y) = \bar{F}_i^\alpha(x, y)$ for all $(x, y) \in A_{i-1}^\alpha \times_{\prec} A_i^\alpha$.
- For all $\prec$ -increasing sequences $\vec{x} = (x_i)_{t \leq i \leq k}$ with $x_i \in A_i^\alpha$, let $h'(\vec{x})$ be the stem

$$(\bar{s}, \bar{q}) + (x_t, [F_{t+1}^\alpha(x_t, -)]) + \dots + (x_{k-1}, [F_k^\alpha(x_{k-1}, -)]) + (x_k, [\bar{F}_{k+1}^\alpha(x_k, -)]),$$

then:

- $h'(\vec{x}) \in B_{k+1,\alpha}$.
- $h'(\vec{x})$ is the stem of the minimal extension of p_α by $(x_i)_{t \leq i \leq k}$.

Remark 10.33. By the choice of the functions F_i^α ,

$$h'(\vec{x}) = (\bar{s}, g(x_t), F_{t+1}^\alpha(x_t, x_{t+1}), \dots, x_{k-1}, F_k^\alpha(x_{k-1}, x_k), x_k, [\bar{F}_{k+1}^\alpha(x_k, -)]).$$

In this round of the construction we will choose A_{k+1}^α and $\bar{F}_{k+2}^\alpha$, and will then define F_{k+1}^α as $\bar{F}_{k+1}^\alpha \upharpoonright A_k^\alpha \times_{\prec} A_{k+1}^\alpha$. For each $\prec$ -increasing sequence $\vec{x} = (x_i)_{t \leq i \leq k}$ with $x_i \in A_i^\alpha$, $h'(\vec{x}) \in B_{k+1,\alpha}$ by our induction hypothesis, and thus $F_\alpha^{h'(\vec{x})} \in \mathcal{F}_{k+2}$ by Claim 10.30. By Lemma 10.9 we have $\Delta_{\vec{x}} F_\alpha^{h'(\vec{x})} \in \mathcal{F}_{k+2}$, where

$$\Delta_{\vec{x}} F_\alpha^{h'(\vec{x})} = \{(x, r) : \forall \vec{x} \ x_k \prec x \implies (x, r) \in F_\alpha^{h'(\vec{x})}\}.$$

We choose A_{k+1}^α and $\bar{F}_{k+2}^\alpha$ so that:

- A_{k+1}^α and $\bar{F}_{k+2}^\alpha$ witness that $\Delta_{\vec{x}} F_\alpha^{h'(\vec{x})} \in \mathcal{F}_{k+2}$, or to be more explicit $(x, [\bar{F}_{k+2}^\alpha(x, -)]) \in \Delta_{\vec{x}} F_\alpha^{h'(\vec{x})} \in \mathcal{F}_{k+2}$ for all $x \in A_{k+1}^\alpha$.
- $x \in A_{k+1}^\alpha$ and $g_{[\bar{F}_{k+2}^\alpha(x, -)]}(y) = \bar{F}_{k+2}^\alpha(x, y)$ for all $(x, y) \in \text{dom}(\bar{F}_{k+2}^\alpha)$.

Then we define F_{k+1}^α as $\bar{F}_{k+1}^\alpha \upharpoonright A_k^\alpha \times_{\prec} A_{k+1}^\alpha$.

It remains to check that we have propagated our induction hypotheses. Only the last clause requires any work. Let $(x_i)_{t \leq i \leq k+1}$ with $x_i \in A_i^\alpha$, and let h' be the stem

$$(\bar{s}, \bar{q}) + (x_t, [F_{t+1}^\alpha(x_t, -)]) + \dots + (x_k, [F_{k+1}^\alpha(x_k, -)]) + (x_{k+1}, [\bar{F}_{k+2}^\alpha(x_{k+1}, -)]),$$

where we note that

$$h' = (\bar{s}, g(x_t), x_t, F_{t+1}^\alpha(x_t, x_{t+1}), \dots, x_k, F_{k+1}^\alpha(x_k, x_{k+1}), x_{k+1}, [\bar{F}_{k+2}^\alpha(x_{k+1}, -)]).$$

Let $\vec{x} = (x_i)_{t \leq i \leq k}$, and note that $[\bar{F}_{k+1}^\alpha(x_k, -)] = [F_{k+1}^\alpha(x_k, -)]$, so that $h'(\vec{x})$ as defined above is the stem obtained from h' by deleting the last entry. Because $x_k \prec x_{k+1}$ and $(x_{k+1}, [\bar{F}_{k+2}^\alpha(x_{k+1}, -)]) \in \Delta_{\vec{y}} F_\alpha^{h'(\vec{y})}$, $(x_{k+1}, [\bar{F}_{k+2}^\alpha(x_{k+1}, -)]) \in F_\alpha^{h'(\vec{x})}$, so that $h' \in B_{k+2,\alpha}$.

To finish we consider the stem of the minimal extension of p_α by the sequence $(x_i)_{t \leq i \leq k+1}$, recalling that by our induction hypothesis $h'(\vec{x})$ is the stem of the minimal extension of p_α by $(x_i)_{t \leq i \leq k}$. Recalling that the one-variable function in the minimal extension by $(x_i)_{t \leq i \leq k}$ is $F_{k+1}^\alpha(x_k, -)$, and that by our induction hypothesis $g_{[\bar{F}_{k+1}^\alpha(x_k, -)]}(x_{k+1}) = F_{k+1}^\alpha(x_k, x_{k+1})$, it is clear that h' is the stem of the minimal extension of p_α by $(x_i)_{t \leq i \leq k+1}$ as required.

We have now constructed in $V[L][A^{gg}][P_3 \times QTT \times P_{2b}]$ a sequence $(p_\alpha)_{\alpha \in I}$ such that

$$p_\alpha = \langle \bar{s}, g \upharpoonright A_t^\alpha, A_t^\alpha, F_{t+1}^\alpha, A_{t+1}^\alpha, F_{t+2}^\alpha, \dots \rangle.$$

and for every minimal extension q of p_α by a sequence $(x_i)_{t \leq i \leq k}$, the stem of q is in $B_{k+1,\alpha}$.

The following Lemma is analogous to Lemma 10.24.

Lemma 10.34. *For $\alpha, \beta \in I \setminus \rho$ with $\alpha < \beta$, $p_\alpha \wedge p_\beta \Vdash u_\alpha < u_\beta$.*

Proof. Suppose for a contradiction that $p \leq p_\alpha, p_\beta$ and $p \Vdash u_\alpha \not< u_\beta$. Assume that p is an s -step extension of p_α and p_β for some $s \geq 2$. Let

$$p = \langle s', c_t, x_t \dots c_{t+s-1}, x_{t+s-1}, f_{t+s}, A_{t+s}, F_{t+s+1}, A_{t+s+1}, \dots \rangle.$$

Let $\vec{x} = (x_i)_{t \leq i < t+s}$. The stem of p directly extends the stem of the minimal extension of p_α by $\vec{x}$. By construction the stem of this minimal extension lies in $B_{t+s, \alpha}$, and so by Claim 10.32 $\text{stem}(p) \in B_{t+s, \alpha}$. Similarly $\text{stem}(p) \in B_{t+s, \beta}$. By Claim 10.31 $\text{stem}(p) \Vdash^* u_\alpha < u_\beta$, contradicting $p \Vdash u_\alpha \not< u_\beta$. $\square$

Lemma 10.35. *The tree T has a cofinal branch in $V[L][A^{gg}][P_3 \times QTT \times P_{2b}][\bar{P}]$.*

Proof. The proof is essentially the same as the proof of Lemma 10.25. The main difference is that λ is no longer a cardinal in $V[L][A^{gg}][P_3 \times QTT \times P_{2b}]$, in fact it has become an ordinal of cofinality μ . Since $\bar{\mathbb{P}}$ has only ν stems and conditions with the same stem are compatible, $\bar{\mathbb{P}}$ still enjoys the μ -cc in $V[L][A^{gg}][P_3 \times QTT \times P_{2b}]$, and the argument goes through. $\square$

Lemma 10.36. *The tree T has a cofinal branch in $V[L][A^{gg}][\bar{P}]$.*

Proof. We start by claiming that $\mathbb{P}_3 \times \mathbb{QTT}$ is formerly $< \mu$ -closed in $V[L][A^{gg}][P_{2b}][\bar{P}]$, with a view to using Fact 2.12. This is easy: because of the robust μ -cc of $\bar{\mathbb{P}}$, $\mathbb{A}_e \times \bar{\mathbb{P}}$ is μ -cc in $V^{lbi}(\kappa)[J^c][A^{gg}][P_{2b}]$, and we can argue as in the discussion preceding Lemma 10.27. it follows that T has a cofinal branch in $V[L][A^{gg}][P_{2b}][\bar{P}]$.

Now we claim that $\mathbb{P}_{2b}$ has the λ -approximation property in $V[L][A^{gg}][\bar{P}]$. Again this is easy, because $\bar{\mathbb{P}} \times \mathbb{P}_{2b}$ is λ -Knaster in $V[L][A^{gg}]$. It follows that T has a cofinal branch in $V[L][A^{gg}][\bar{P}]$ as required. $\square$

10.3. The tree property at $\aleph_{\omega^2+3}$

The proof that the tree property holds at $\aleph_{\omega^2+3}$ in our final model is very similar to that for $\aleph_{\omega^2+2}$, so we only sketch it. The main point is to get a suitable generic embedding with domain $V[L][A^{gg}]$ and critical point $\lambda_{\omega+3}^b$. This is much more straightforward than it was for $\lambda_{\omega+2}^b$ in Section 10.2, mostly because $\lambda_{\omega+3}^b$ is supercompact in $V^{lbi}(\kappa)[J_0^c]$, so that we only need to account for J_1^c , A_e and A^{gg} . Moreover A_e and A^{gg} are both adding Cohen subsets to cardinals below the critical point.

We recall that J_1^c is a single round of the $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction defined in $V^{lbi}(\kappa)[J_0^c]$. The parameters are $\lambda_{\omega+1}^b$, $\lambda_{\omega+2}^b$, and $\lambda_{\omega+3}^b$.

- $\mathbb{A}_1^c = \text{Add}^{V^{lb}(\kappa)}(\lambda_{\omega+1}^b, [\lambda_{\omega+2}^b, \lambda_{\omega+3}^b])$.
- $\mathbb{A}^{gg} = \text{Add}^V(\lambda_{\omega+2}^b, \lambda^*)$, where $\lambda^* = j_{01}(\lambda_0^a)$.
- $\mathbb{A}_e = \text{Add}^{V^{lb}(\kappa)}(\lambda_{17}^b, \lambda_{\omega+3}^b)$.

We fix an embedding j witnessing that $\lambda_{\omega+3}^b$ is highly supercompact in V , and lift it trivially to $V^{lbi}(\kappa)[J_0^c]$. We will lift j onto $V[L][A^{gg}]$ much as we would in Lemma 4.5, if our goal was to prove that $\lambda_{\omega+3}^b$ has the tree property in $V[L][A^{gg}]$.

We will step through the construction from the proof of Lemma 4.5, adapted to a situation where the $\mathbb{A} * \mathbb{U} * \mathbb{S}$ construction only runs for one round. In our current context the parameters are set as follows:

- $n = 0$.
- $\mu_0 = \lambda_{\omega+1}^b$, $\mu_1 = \lambda_{\omega+2}^b$, $\mu_2 = \lambda_{\omega+3}^b$.
- $\mathbb{A}_0 * \mathbb{U}_0 * \mathbb{S}_0 = \mathbb{J}_1^c$.
- $V_{\text{def}} = V^{lbi}(\kappa)[J_0^c]$. There is no need for V_{inn} .
- $\mathbb{D}^1 = A^{gg}$.
- $\mathbb{D}^0 = \mathbb{A}_e$.

The generic embedding is added to $V[L][A^{gg}]$ by a product forcing $\mathbb{P}_2 \times \mathbb{P}_3 = \mathbb{P}_{2a} \times \mathbb{P}_{2b} \times \mathbb{P}_3$, where:

- $\mathbb{P}_{2b} = j(\mathbb{A}^{gg})/A^{gg}$, so $\mathbb{P}_{2b} = \text{Add}^V(\lambda_{\omega+2}^b, j(\lambda^*) \setminus j[\lambda^*])$.
- $\mathbb{P}_{2a} = j(\mathbb{A}_e \times \mathbb{A}_1^c)/A_e \times A_1^c$, so $\mathbb{P}_{2a} = \text{Add}^{V^{lb}(\kappa)}(\lambda_{17}^b, j(\lambda_{\omega+3}^b) \setminus \lambda_{\omega+3}^b) \times \text{Add}^{V^{lb}(\kappa)}(\lambda_{\omega+1}^b, j(\lambda_{\omega+3}^b) \setminus \lambda_{\omega+3}^b)$.
- $\mathbb{P}_3$ is defined in and $< \lambda_{\omega+2}^b$ -closed in $V^{lbi}(\kappa)[J^c]$, and retains this closure in $V^{lbi}(\kappa)[J^c][A^{gg}]$.

As in the proof of Lemma 4.5, $\mathbb{P}_3$ is still $< \lambda_{\omega+2}^b$ -closed in

$$V^{lbi}(\kappa)[J^c][A^{gg}][P_{2b}] = V^{lbi}(\kappa)[J^c][\hat{A}^{gg}],$$

where $\hat{A}^{gg}$ is the $j(\mathbb{A}^{gg})$ -generic object obtained by combining A^{gg} and P_{2b} . In this model $\mathbb{A}_e \times \mathbb{P}_{2a}$ is $\lambda_{\omega+2}^b$ -Knaster and $2^{\lambda_{\omega+1}^b} = \lambda_{\omega+3}^b$.

To summarize the key points:

- $\mathbb{P}_2$ is $\lambda_{\omega+3}^b$ -Knaster in $V[L][A^{gg}]$.
- $\mathbb{P}_3$ is formerly $< \lambda_{\omega+2}^b$ -closed forcing in $V[L][A^{gg}][P_2]$.

With this information in hand, we may finish the proof exactly as in Section 10.2.

Acknowledgements

The results in this paper were conceived during a series of visits to the American Institute of Mathematics, as part of the Institute's SQuaRE collaborative research program. The authors are deeply grateful to all the staff at AIM for providing an ideal working environment.

Cummings was partially supported by NSF grant DMS-2054532. Hayut was partially supported by Israel Science Fund grant 1967/21. Neeman was partially supported by NSF grants DMS-1764029 and DMS-2246905. Sinapova was partially supported by NSF grant DMS-2246781. Unger was partially supported by an NSERC discovery grant.

Appendix A. A lifting argument

As promised, we give here the details of the generic supercompactness for θ in the statement of Lemma 5.2.

Let $\mathbb{Q} \in V$ be $< \theta$ -directed closed and let H be $\mathbb{Q}$ -generic over V . Decompose V as $V_0[A^0 * U^0 \restriction \theta][L^0]$ where L^0 is generic over $V_0[A^0 * U^0 \restriction \theta]$ for the Laver preparation $\mathbb{L}^0$. Let $\dot{R}$ be an $\text{Add}(\omega, \theta) * \dot{\mathbb{U}}^0 \restriction \theta$ -name for the two-step iteration $\mathbb{R} = \mathbb{L}^0 * \dot{\mathbb{Q}}$. Appealing to the properties of θ and ϕ_0 in V_0 we fix $i : V_0 \rightarrow N_0$ such that for an appropriate $\gamma > \delta$ (which may be chosen arbitrarily large):

- i witnesses that θ is γ -supercompact in V_0 .
- γ^{++} is a fixed point of i .
- $i(\psi_0)(\theta) = \dot{R}$.
- The first point of $\text{dom}(i(\phi_0))$ past θ is greater than γ .

Let A' be $\text{Add}(\omega, i(\theta) \restriction \theta)$ -generic over $V[H]$. Our goal is to find a lifting of i onto $V[H][A']$ defined in $V[H][A']$.

The main point is that $i(\mathbb{B}^0)$ agrees with $\mathbb{B}^0$ up to θ , uses the name $\dot{R}$ at stage θ , and then has nothing in its support until past γ . By a straightforward adaptation of the argument of [16, Claim 4.7], and using the gap in the support which we just mentioned, $i(\mathbb{B}^0) \restriction (\theta, i(\theta))$ is γ -closed in N_0 . An easy counting argument shows that the set of maximal antichains of $i(\mathbb{B}^0) \restriction (\theta, i(\theta))$ which lie in N_0 has cardinality at most γ^+ in V_0 , so we may build $B \in V_0$ which is $i(\mathbb{B}^0) \restriction (\theta, i(\theta))$ -generic over N_0 .

Let $\hat{\mathbb{R}}$ be the term forcing $\mathcal{A}^{V_0}(\text{Add}(\omega, \theta) * \dot{\mathbb{U}}^0 \restriction \theta, \dot{R})$, so that $\hat{\mathbb{R}}$ is $< \theta$ -closed in V_0 , and hence $i(\hat{\mathbb{R}})$ is $< i(\theta)$ -closed in N_0 . Since $\gamma < i(\theta)$ the poset $i(\hat{\mathbb{R}})$ is γ -closed in V_0 . By choosing γ large enough we may assume that the set of maximal antichains of $i(\hat{\mathbb{R}})$ which lie in N_0 has size γ^+ in V_0 , and we may build $R^* \in V_0$ which is $i(\hat{\mathbb{R}})$ -generic over N_0 . We will eventually make sure that R^* contains a term for a master condition but we defer the description of this term.

Let A^* be obtained by combining A^0 and A' in the natural way, so that A^* is $\text{Add}(\omega, i(\theta))$ -generic over V_0 and $i[A^0] \subseteq A^*$. Keep in mind that A' was obtained by forcing over $V[H] = V_0[A^0 * U^0 * H]$, so it is mutually generic with $U^0 * H$ over $V_0[A^0]$. We note for use later that by this analysis:

- $A^* * U^0 \restriction \theta$ is generic over N_0 for $\text{Add}(\omega, i(\theta)) * i(\mathbb{U}^0) \restriction \theta$.
- A' is mutually generic with $L^0 * H$ over $V_0[A^0 * U^0 \restriction \theta]$.

Recall that we built $B \in V_0$ to be generic over N_0 for the forcing $i(\mathbb{B}^0) \restriction (\theta, i(\theta))$, which is γ -closed in N_0 . It is easy to see that $\text{Add}(\omega, i(\theta)) * i(\mathbb{U}) \restriction \theta$ is θ -cc in N_0 , and so by Easton's Lemma B is $i(\mathbb{B}^0) \restriction (\theta, i(\theta))$ -generic over $N_0[A^* * U \restriction \theta]$.

Now we recall that $L^0 * H$ is $\mathbb{L}^0 * \mathbb{Q}$ -generic over $V_0[A^0 * U^0 \restriction \theta]$, so *a fortiori* it is $\mathbb{L}^0 * \mathbb{Q}$ -generic over $N_0[A^0 * U^0 \restriction \theta]$. As we noted above A' is mutually generic with $L^0 * H$ over $V_0[A^0 * U^0 \restriction \theta]$, so these objects are mutually generic over $N_0[A^0 * U^0 \restriction \theta]$ and hence $L^0 * H$ is $\mathbb{L}^0 * \mathbb{Q}$ -generic over $N_0[A^* * U^0 \restriction \theta]$. Since $\dot{R} = i(\psi_0)(\theta)$ and it names $\mathbb{L}^0 * \mathbb{Q}$, we see that $A^* * (U^0 \restriction \theta * L^0 * H)$ is $\text{Add}(\omega, i(\theta)) * i(\mathbb{U}^0) \restriction \theta + 1$ -generic

over N_0 .

Choosing γ large enough we can arrange that $L^0 * H$ is generic for γ^+ -cc forcing, so that by Easton's Lemma again B is $i(\mathbb{B}^0)(\theta, i(\theta))$ -generic over $N_0[A^* * (U^0 \restriction \theta * L^0 * H)]$. As in Fact 3.3 it follows that the upwards closure of B in $i(\mathbb{B}^0)^{A^* * (U \restriction \theta * L^0 * H)}(\theta, i(\theta))$ is generic for this forcing over $N_0[A^* * (U^0 \restriction \theta * L^0 * H)]$: combining the upwards closure of B with $A^* * (U^0 \restriction \theta * L^0 * H)$ we obtain $A^* * U_{i(\theta)}^*$ which is $\text{Add}(\omega, i(\theta)) * i(\mathbb{U}^0 \restriction \theta)$ -generic over N_0 . Note that we can rearrange $A^* * (U^0 \restriction \theta * L^0 * H)$ as $A^0 * U^0 * H * A'$, and that $U_{i(\theta)}^* \in V_0[A^0 * U^0 * H * A'] = V[H][A']$. By standard arguments we can lift i to obtain a generic embedding $i : V_0[A^0 * (U^0 \restriction \theta)] \rightarrow N_0[A^* * U_{i(\theta)}^*]$.

Recall that $\hat{\mathbb{R}}$ is defined in V_0 as the set of $\mathbb{A}^0 * \mathbb{U}^0 \restriction \theta$ -names for elements of $\hat{\mathbb{R}}$. Choosing γ large enough we may arrange that $i \restriction \hat{\mathbb{R}} \in N_0$, and it follows readily that $i \restriction \mathbb{R} \in N_0[A^* * U_{i(\theta)}^*]$. By the construction of $U_{i(\theta)}^*$ we have that $L^0 * H \in N_0[A^* * U_{i(\theta)}^*]$, so that $i[L^0 * H] \in N_0[A^* * U_{i(\theta)}^*]$. Since $i(\mathbb{R})$ is $< i(\theta)$ -directed closed, $i[L^0 * H]$ has a lower bound in $i(\mathbb{R})$ and we claim that we can choose a term $\dot{r} \in i(\hat{\mathbb{R}})$ which is forced to denote a lower bound: this is easy because $\dot{r}$ has a simple definition in terms of $i \restriction \hat{\mathbb{R}}$ and the $i(\mathbb{A}^0 * \mathbb{U}^0 \restriction \theta)$ -generic object. At this point we return to the choice of R^* , an object which has not been used up to now, and make sure that $\dot{r} \in \mathbb{R}^*$.

Now we can realize the set of names R^* , and obtain a filter $R^+ \subseteq i(\mathbb{R})$ such that $i[L^0 * H] \subseteq R^+$. In order to complete the lifting and obtain $i : V[H] \rightarrow N_0[A^* * U_{i(\theta)}^*][R^+]$, it only remains to verify that R^+ is generic over $N_0[A^* * U_{i(\theta)}^*]$. Recall that we chose $R^* \in V_0$ to be generic over N_0 . Since R^* is generic over N_0 for $< i(\theta)$ -closed forcing and $A^* * U_{i(\theta)}^*$ is generic over N_0 for $i(\theta)$ -cc forcing, by Easton's Lemma R^* is generic over $N_0[A^* * U_{i(\theta)}^*]$. It follows that R^+ is generic over $N_0[A^* * U_{i(\theta)}^*]$.

References

1. U. Abraham, Aronszajn trees on $\aleph_2$ and $\aleph_3$, *Annals of Pure and Applied Logic* **24** (1983) 213–230.
2. J. Cummings and M. Foreman, The tree property, *Advances in Mathematics* **133** (1998) 1–32.
3. J. Cummings, Y. Hayut, M. Magidor, I. Neeman, D. Sinapova and S. Unger, The tree property at the two immediate successors of a singular cardinal, *Journal of Symbolic Logic* **86**(2) (2021) 600–608.
4. M. Foreman, M. Magidor and R.-D. Schindler, The consistency strength of successive cardinals with the tree property, *Journal of Symbolic Logic* **66** (2001) 1837–1847.
5. M. Gitik and A. Sharon, On SCH and the approachability property, *Proceedings of the American Mathematical Society* **136**(1) (2008) 311–320 (electronic).
6. J. D. Hamkins, Small forcing makes any cardinal superdestructible, *The Journal of Symbolic Logic* **63**(1) (1998) 51–58.

130 REFERENCES

7. R. Jensen and J. Steel, K without the measurable, *The Journal of Symbolic Logic* **78** (2013) 708–734.
8. J. Keisler and A. Tarski, From accessible to inaccessible cardinals. Results holding for all accessible cardinal numbers and the problem of their extension to inaccessible ones, *Fundamenta Mathematicae* **53** (1963/1964) 225–308.
9. D. König, Über eine Schlussweise aus dem Endlichen ins Unendliche, *Acta Scientiarum Mathematicarum (Szeged)* **3** (1927) 121–130.
10. R. Laver, Making the supercompactness of κ indestructible under κ -directed closed forcing, *Israel Journal of Mathematics* **29** (1978) 385–388.
11. M. Magidor, On the existence of nonregular ultrafilters and the cardinality of ultrapowers, *Transactions of the American Mathematical Society* **249** (1979) 97–111.
12. M. Magidor and S. Shelah, The tree property at successors of singular cardinals, *Archive for Mathematical Logic* **35**(5-6) (1996) 385–404.
13. W. Mitchell, Aronszajn trees and the failure of the transfer property, *Annals of Mathematical Logic* **5** (1972) 21–46.
14. D. Monk and D. Scott, Additions to some results of Erdős and Tarski, *Polska Akademia Nauk. Fundamenta Mathematicae* **53** (1964) 335–343.
15. I. Neeman, Aronszajn trees and failure of the singular cardinal hypothesis, *Journal of Mathematical Logic* **9** (2009) 139–157.
16. I. Neeman, The tree property up to $\aleph_{\omega+1}$, *Journal of Symbolic Logic* **79**(2) (2014) 429–459.
17. E. Schimmerling and M. Zeman, Characterization of $\square_\kappa$ in core models, *Journal of Mathematical Logic* **4** (2004) 1–72.
18. D. Sinapova, The tree property and the failure of the singular cardinal hypothesis at $\aleph_{\omega^2}$, *Journal of Symbolic Logic* **77**(3) (2012) 934–946.
19. D. Sinapova, The tree property at $\aleph_{\omega+1}$, *Journal of Symbolic Logic* **77** (2012) 279–290.
20. D. Sinapova and S. Unger, The tree property at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$, *Journal of Symbolic Logic* **83** (2018) 669–682.
21. R. M. Solovay, Strongly compact cardinals and the GCH, in *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971)* (Published for the Association for Symbolic Logic by the American Mathematical Society, Providence, RI, 1974), volume Vol. XXV of *Proc. Sympos. Pure Math.*, pp. 365–372.
22. E. Specker, Sur un problème de Sikorski, *Colloquium Mathematicae* **2** (1949) 9–12.
23. S. Unger, Fragility and indestructibility of the tree property, *Archive for Mathematical Logic* **51**(5-6) (2012) 635–645.
24. S. Unger, Aronszajn trees and the successors of a singular cardinal, *Archive for Mathematical Logic* **52**(5-6) (2013) 483–496.
25. S. Unger, The tree property below $\aleph_{\omega^2}$, *Annals of Pure and Applied Logic* **167**(3) (2016) 247–261.

26. S. Unger, Successive failures of approachability, *Israel Journal of Mathematics* **242** (2021) 663–695.

Index of Notation

A^b , 56	$\mathbb{B}^0$, 50	δ , 49
$A^e(\tau)$, 58	$\mathbb{B}^b$, 56	$\Vdash^*$, 107
E , 67	$\mathbb{C}^b$, 56	κ , 49
$E^{(k)}$, 77	$\mathbb{L}$, 56	κ_α , 49
E_0 , 49	$\mathbb{L}^0$, 51	λ^a , 55
$F(x, -)$, 77	$\mathbb{L}^b$, 56	λ^b , 55
I^b , 56	$\mathbb{I}^b$, 56	λ_i , 53
$I^b(\tau)$, 58	$\mathbb{J}^c$, 56	$\mathbb{L}_{\text{aux}}(\rho, \lambda)$, 53
$J^c(\tau)$, 58	$\Lambda^a(\tau)$, 55	$\text{lh}(p)$, 74
K , 70	$\Lambda_i^a(\tau)$, 55	Index, 53
$L(\tau)$, 58	$\Lambda^b(\tau)$, 55	ϕ , 50
$L^b(\tau)$, 58	$\Lambda_i^b(\tau)$, 55	ϕ_0 , 49
L_n , 78	$\mathbb{Q}(\alpha, \kappa)$, 73	ψ , 56
M , 50	$\mathbb{Q}(\tau, \tau^*)$, 59	$\mathbb{R}_{\text{aux}}(\lambda)$, 53
M_2^* , 70	$\mathbb{Q}(x, y)$, 74	ρ , 55
M_0 , 49, 67	$\mathbb{Q}^*(\tau)$, 67	$\text{stem}(p)$, 74
M_1 , 67	$\mathbb{Q}^*(x)$, 74	τ_i , 87
M_1^* , 70	$\mathbb{Q}_0^*(\tau)$, 67	θ , 49
M_2 , 67	$\mathbb{Q}_1^*(\tau)$, 67	$\times^<$, 72
N_n , 70	$\mathbb{Q}_2^*(\tau)$, 67	$h' + (x, q)$, 114
N_n^+ , 77	$\mathbb{Q}_0(\tau, \tau^*)$, 60	i_n , 71
N_n^A , 77	$\mathbb{Q}_1(\tau, \tau^*)$, 61	i_n^A , 77
N_n^{A+} , 77	$\mathbb{Q}_2(\tau, \tau^*)$, 62	j , 50
S^b , 56	$\mathbb{Q}_\infty$, 70	j_{01}^* , 70
U^b , 56	$\mathbb{S}^b$, 56	j_{02}^* , 70
U_n , 70	$\mathbb{T}(\tau, \tau^*)$, 62	j_0 , 49
V , 50	$\mathbb{U}^0$, 50	j_E , 67
V_0 , 49	$V^l(\tau)$, 59	j_n , 70
$W(\rho)$, 53	$V^{lb}(\tau)$, 59	j_n^A , 77
W_η , 67	$V^{lbi}(\tau)$, 59	j_{01} , 67
Y , 56	α^* , 49	j_{02} , 67
$\mathbb{A}^0$, 50	$\mathbb{P}$, 74	j_{12} , 67
$\mathbb{A}^b$, 56	$\bar{\theta}$, 55	k_n , 70
$\mathbb{A}_e$, 56	$\bar{\theta}(\rho)$, 53	$p \smallfrown \vec{x}$, 76
A^{gg} , 70	$\bar{W}$, 55	$\mathbf{F}_n$, 77
		$\mathcal{F}_n$, 107