TWO APPLICATIONS OF FINITE SIDE CONDITIONS AT ω_2

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Abstract. We present two applications of forcing with finite sequences of models as side conditions, adding objects of size ω_2. The first involves adding a $\square_{\omega_1}$ sequence and variants of such sequences. The second involves adding partial weak specializing functions for trees of height ω_2.

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1. Introduction

The method of using side conditions consisting of finite sequences of models to ensure properness goes back to the work of Todorcevic [23, 22], and has seen many applications since. Most relevant for us here is the development by Friedman [7] and Mitchell [13] of side conditions that enforce preservation of both ω_1 and a second cardinal that becomes ω_2. Friedman [7] used the new method to present a poset that adds a club in ω_2 using finite conditions, an analogue of a well known poset of Baumgartner [3, Section 3] that does this on ω_1. Mitchell [13] also used the method to add a club by finite conditions, in a cardinal θ that is collapsed to ω_2. The side conditions of Friedman and Mitchell were simplified in Neeman [16] by the addition of non-countable models. These side conditions, in the original or simplified version, have been applied to obtain several results, including in Friedman [8] to show that PFA does not imply that a model correct about ω_2 must contain all reals, in Mitchell [14] to show that the approachability ideal on ω_2 can be trivial, in Neeman [16] to obtain a finite support proof of the consistency of PFA, and in Velickovic-Venturi [27] to add thin very tall superatomic Boolean algebras and chains of length ω_2 in $(\omega_1^{\omega_1}, <_{\text{fin}})$ giving new proofs of results originally due to Baumgartner-Shelah [4] and Koszmider [10] respectively.

Here we use the framework of side conditions of Neeman [16] to add square sequences of length ω_2, and to add partial functions that satisfy weak specializing conditions introduced in Shelah [19] and Todorcevic [21, 24], for trees of height ω_2. Previous work on specializing for trees of height ω_2 has mostly been in the negative direction. The results we obtain on square principles show that some of this negative work applies also in our context, and places limitations on the type and domain of any specializing functions one can hope to obtain. We isolate additional limitations below. Our positive results here reach precisely to these limitations.

Recall that $\langle C_\alpha \mid \alpha < \kappa^+\rangle$ is a $\square_\kappa$ *sequence* if C_α is club in α of order type $\leq \kappa$, and $\beta \in \text{Limit}(C_\alpha) \to C_\beta = C_\alpha \cap \beta$. (Limit$(C_\alpha)$ here consists of the limit

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ordinals γ so that $X \cap \gamma$ is cofinal in γ.) More generally $\langle C_{\alpha,i} \mid \alpha < \kappa^+,\, i < n_\alpha \rangle$ is a $\square_{\kappa,<\delta}$ *sequence* if $C_{\alpha,i}$ is club in α of order type $\leq \kappa$, $n_\alpha < \delta$ for each α, and $\beta \in \mathrm{Limit}(C_{\alpha,i}) \rightarrow (\exists j) C_{\beta,j} = C_{\alpha,i} \cap \beta$. $\square_{\kappa,\delta}$ sequences are defined similarly with $n_\alpha = \delta$ for all α. $\square_{\kappa,\mathrm{fin}}$ denotes $\square_{\kappa,<\omega}$. These principles are standard. One new variant that comes up in the work here is that of a $\square^{\mathrm{ta}}_{\kappa,\delta}$ sequence (read $\square_{\kappa,\delta}$ with tail agreement). We call $\langle C_{\alpha,i} \mid \alpha < \kappa^+,\, i < \delta \rangle$ a $\square^{\mathrm{ta}}_{\kappa,\delta}$ *sequence* if it is a $\square_{\kappa,\delta}$ sequence and in addition, for every $\alpha < \kappa^+$ and every $i, j < \delta$, $C_{\alpha,i}$ and $C_{\alpha,j}$ agree on a tail, meaning that there is some $\beta < \alpha$ so that $C_{\alpha,i} - \beta = C_{\alpha,j} - \beta$.

These sequences are important for many reasons (we will see one below), and there are many ways to force their existence. In Section 3 we present a forcing that adds a $\square_{\omega_1}$ sequence using finite conditions. An earlier forcing to add a $\square_{\omega_1}$ sequence with finite conditions was given by Dolinar-Dzamonja [6], but in that forcing the individual clubs C_α are not added with finite conditions. A mathematical consequence of this is that the forcing in [6] is not strongly proper. The forcing we present in Section 3 adds the individual sets C_α using finite conditions, and is strongly proper. A poset doing this was also obtained by Krueger [11]. The posets here and in [11] were obtained independently around the same time. Krueger's poset collapses the continuum to ω_1 by Krueger-Mota [12]. In contrast the poset we give here is proper and ω_2-c.c. under assumptions which are compatible with arbitrarily large value of the continuum.

The poset we use to force $\square_{\omega_1}$ is easily modified to obtain variants of $\square_{\omega_1}$. We present two such modifications in Section 3. The first adds a $\square_{\omega_1,\mathrm{fin}}$ sequence, and the second adds a $\square^{\mathrm{ta}}_{\omega_1,\omega}$ sequence. Both principles are weakenings of $\square_{\omega_1}$, and the point of working on forcing them is that the posets used to force them enjoy additional properties, that are not satisfied by the poset forcing $\square_{\omega_1}$. These properties, which are given by Lemmas 3.15 and 3.23, can be used to show that the posets for $\square_{\omega_1,\mathrm{fin}}$ and $\square^{\mathrm{ta}}_{\omega_1,\omega}$ belong to certain classes of posets that can be iterated (with side conditions) while preserving ω_1 and ω_2. This includes some of the classes developed in Neeman's work on higher analogues of the proper forcing axiom. We refer the reader to [15] for more on the extent of square under these analogues. The posets also belong to a variant of the class $\aleph_{1.5}$-c.c. developed by Asperó-Mota [1]. In particular then, the corresponding forcing axiom implies both $\square_{\omega_1,\mathrm{fin}}$ and $\square^{\mathrm{ta}}_{\omega_1,\omega}$. We show this in Theorem 3.36.

The strengthening of the Asperó-Mota axiom that we assume in Theorem 3.36 involves requiring master conditions only for models N whose intersection with ω_2 belongs to a fixed set U which satisfies certain coherence conditions. We show in Section 3 that the work in Asperó-Mota [1] easily adapts to give the consistency of this strengthening. We also show, in Theorem 3.49, that the original axiom in Asperó-Mota [1] does *not* imply $\square_{\omega_1,\mathrm{fin}}$ and $\square^{\mathrm{ta}}_{\omega_1,\omega}$, or even for that matter $\square_{\omega_1,\omega}$.

Our second application, in Section 4, involves notions of specializing for trees. Let $\tau = \kappa^+$ be an infinite successor cardinal and let T be a tree of height τ. Recall that T is *Aronszajn* if it has no cofinal branches. T is *special* if there is a function $f \colon T \rightarrow \kappa$ which is injective on branches of T. The existence of such a function implies in particular that T is Aronszajn.

For the case of $\tau = \omega_1$ there are well known forcing notions that add specializing functions to Aronszajn trees, without collapsing ω_1. The simplest, adding the function with finite partial subfunctions, is c.c.c. This forcing notion has been used extensively in the context of forcing axioms.

Infinitary combinatorics imposes many more difficulties at the level of ω_2 than ω_1, and most of the work on specializing at ω_2 to date has been on negative results. Cummings [5], Shelah-Stanley [20], and Todorcevic [25], under various set theoretic hypotheses, all construct ω_2 Aronszajn trees that are not special, with the existence of such trees persisting to all generic extensions that preserve ω_1 and ω_2 in the case of [20] and [25]. This is in contrast to the situation on ω_1, where any tree can be specialized without collapsing any cardinals, and where the forcing to specialize trees can be iterated to a point where all trees are special. The hypothesis used by Shelah-Stanley [20] is the existence of a $\square_{\omega_1}$ sequence, and their construction adapts to use a $\square^{\mathrm{ta}}_{\omega_1,\omega}$ sequence instead (see Fact 3.25 and its proof sketch). Since we show in Section 3 that such sequences can be forced using posets that belong to iterable classes, it follows that non-special ω_2 Aronszajn trees exist provably under the corresponding forcing axioms. These results all suggest that at the level of ω_2 one should work with a weaker notion of specializing.

A function f defined on $X \subseteq T$ is a *weak specializing function* if $f(t) < \mathrm{height}(t)$ for every $t \in X$, and f is injective on chains of T. A tree T of height ω_2 is *weakly special on* $X \subseteq T$ if there is a weak specializing function on X. This is equivalent to a notion introduced in Todorcevic [21, 24]: if $\mathrm{height}(t)$ is sufficiently closed for all $t \in X$ then T is weakly special on X iff X is non-stationary in the sense of [21, 24]. (The equivalence uses the fact that the height of the tree T, in our case ω_2, is a successor cardinal.) It is also equivalent to a notion introduced in Shelah [19]: if $X = T|S$, by which we mean the set of nodes of T on levels $\alpha \in S$, and the ordinals in S are sufficiently closed, then being weakly special on X is exactly the same as being S-st-special in the sense of [19, Chapter IX]. If S is stationary then being weakly special on $T|S$ implies the nonexistence of cofinal branches, and hence for many purposes the weak notions are as good as the original notion.

Even with the weakened notion there is still an impediment for specializing at ω_2, that has no parallel at ω_1. We end the introduction with a description of this impediment. In Section 4 we show that this is the only impediment. The forcing notion we define there adds a weak specializing function whose domain consists of exactly the nodes of T where this impediment does not occur.

Definition 1.1. Let θ be some large regular cardinal, $A \subseteq H(\theta)^{<\omega}$, and $C \subseteq \omega_2$.

(1) A model M *overlaps* $t \in T$ if $t \notin M$ and there is a (non-cofinal) branch u of T with $t \in u$ and $u \in M$.
(2) $\beta \in C$ is an *extensive overlap point* for $t \in T$ (relative to θ, A, and C) if $\beta \leq \mathrm{height}(t)$, $\mathrm{cof}(\beta) = \omega_1$, and for every countable $a \subseteq \beta$ and countable $b \subseteq \omega_2 - \beta$, there is a countable $M \prec (H(\theta); A)$ so that $M \supseteq a$, $(M \cup \mathrm{Limit}(M)) \cap b = \emptyset$, and M overlaps t.
(3) $t \in T$ is *extensively overlapped* (relative to θ, A, and C) if $\mathrm{height}(t)$ or arbitrarily large $\beta < \mathrm{height}(t)$ are extensive overlap points for t.

When we omit reference to θ, A, and C, it is understood that θ is large enough, A codes all relevant objects, and C is the club of ordinals $\beta < \omega_2$ so that there are $M \prec (H(\theta); A)$ with $M \cap \omega_2 = \beta$.

Claim 1.2. *Suppose f is a partial weak specializing function for T, $t \in \mathrm{dom}(f)$, β is an extensive overlap point for t, and models M witnessing this can be found which are elementary relative to T and to f. Then $f(t) \geq \beta$.*

Proof. Suppose for contradiction that $f(t) < \beta$. Let $\alpha = f(t)$, $a = \{\alpha\}$, and $b = \emptyset$. Let M and u witness the extensive overlap for a and b, with M elementary relative to T and f. Since f is injective on chains, the set $\{s \in u \mid f(s) = \alpha\}$ has exactly one element, t. The set belongs to M since $u, \alpha \in M$. So $t \in M$, a contradiction. $\square$

The definition of an extensive overlap point β for t requires there to be many models $M \prec (H(\theta); A)$ witnessing many overlaps. All $\aleph_{1.5}$-c.c. posets preserve the elementarity of enough of these models that β continues to be an extensive overlap point for t in the generic extension, relative to θ, A', C where A' codes both A and the generic G. If G adds a partial weak specializing function $f = \dot{f}[G]$ on T, and A codes $\dot{f}$ and T, then it follows using Claim 1.2 that $f(t) \geq \beta$. Since $f(t) < \text{height}(t)$ this means in particular that $t \in \text{dom}(f)$ is impossible if t is extensively overlapped in V. The same is true with posets which are $\aleph_{1.5}$-c.c. relative to U assuming $\mathsf{MA}^{1.5}_{\omega_1}(U)$ (see Definitions 3.26 and 3.27), and with the ω_2-c.c. posets within the iterable classes developed in Neeman [15]. So partial weak specializing functions added by forcing within any of these classes, or for that matter any class of posets which preserve elementarity relative to the generic object sufficiently often to preserve extensive overlap points, cannot have in their domains nodes which are extensively overlapped in V. The largest domain one can expect is $\{t \in T \mid t \text{ is not extensively overlapped}\}$, as determined in V.

The poset we describe in Section 4 adds a partial weak specializing function with exactly this domain. We also discuss, at the end of the section, the fact that in some situations *all* posets that preserve ω_1 and ω_2 also preserve some extensive overlap points, and therefore none of them can add a total weak specializing function.

2. Preliminaries

This section includes a brief outline of relevant definitions and lemmas from Neeman [16], phrased specifically for the situations of interest to us here. The rest of the paper is mostly self-contained, granted the definitions and results in the current section.

The results from Neeman [16] that are outlined below are on a simplification of the side conditions developed by Friedman [7] and Mitchell [13]. The simplification involves explicitly adding uncountable models to the side conditions, and demanding closure under intersections with these models. The uncountable models may be viewed as implicitly hidden in the side conditions of the original posets of Friedman and Mitchell, generated there from Skolem hulls of the uncountable cofinality ordinals $\sup(M \cap \text{Ord} - N \cap \text{Ord})$, for countable models M, N in the side condition and in its closure under intersections with the generated hulls. Some of the work with the simplified poset, including for results in this section, then traces very directly to work by Friedman and Mitchell. We refer the reader to [16] for a better account of this tracing with more specific details.

We work throughout under the assumption that $\mathcal{S}$ and $\mathcal{T}$ satisfy conditions (ST1)–(ST5) below.

- (ST1) $K = \bigcup \mathcal{T}$ satisfies a large enough fragment of $\mathsf{ZFC}-\mathsf{Powerset}$, and $K \cap \text{Ord} = \omega_2$. For the exact consequences of ZFC needed in K see the first paragraph of [16, Section 2].
- (ST2) $\mathcal{T} \subseteq \{W \prec K \mid W \text{ is transitive}, W \in K, \text{ and } \omega_1 \in W\}$.
- (ST3) $\mathcal{S} \subseteq \{M \prec K \mid M \text{ is countable}, M \in K, \text{ and } \omega_1 \in M\}$.

(ST4) If $M \in \mathcal{S}$, $W \in \mathcal{T}$, and $W \in M$, then $M \cap W \in \mathcal{S}$ and $M \cap W \in W$.
(ST5) $\mathcal{S}$ and $\mathcal{T}$ are stationary in $\mathcal{P}(K)$.

Such sets can always be obtained, for example taking $K = H(\omega_2)$, $\mathcal{T} = \{W \prec K \mid W$ is transitive, $|W| = \omega_1$, and W is internal on a club$\}$, and $\mathcal{S} = \{M \prec K \mid M$ is countable$\}$. By *internal on a club* we mean that for a club of $P \subseteq W$, $P \in W$. Equivalently, $W = \bigcup_{\xi<\omega_1} P_\xi$ where $\langle P_\xi \mid \xi < \omega_1\rangle$ is increasing and continuous with $P_\xi \in W$ for all ξ. If W is internal on a club as witnessed by $\langle P_\xi \mid \xi < \alpha\rangle$, $K = H(\omega_2)$, and $M \prec K$ is countable with $W \in M$, then $M \cap W = P_\alpha$ where $\alpha = \sup(M \cap \omega_1)$, and in particular $M \cap W \in W$, as required for condition (ST4).

We refer to elements of $\mathcal{S}$ as *countable nodes*, and to elements of $\mathcal{T}$ as *transitive nodes*. $\mathcal{S}$ and $\mathcal{T}$ satisfying the above conditions are appropriate for ω, ω_1, and K in the sense of [16, Definition 2.2]. The next definition gives the resulting side condition poset of [16, Definition 2.4].

Definition 2.1. A *side condition* (relative to $\mathcal{S}$ and $\mathcal{T}$) is a finite sequence $s = \langle M_i \mid i < n\rangle$ so that:

1. Each M_i belongs to $\mathcal{S} \cup \mathcal{T}$.
2. s is increasing, meaning that $M_i \in M_{i+1}$ for all $i + 1 < n$.
3. s is closed under intersections, meaning that if M and W occur in s, M is countable, W is transitive, and $W \in M$, then $M \cap W$ occurs in s.

Side conditions are ordered by $t \leq s$ iff all nodes of s occur in t. The poset of side conditions is denoted $\mathbb{P}_{\text{side}}$ or $\mathbb{P}_{\text{side}}(\mathcal{S}, \mathcal{T})$.

Forcing with $\mathbb{P}_{\text{side}}$ adds a sequence of nodes of length ω_2, so that its restriction to transitive nodes is linearly ordered by $\in$, the countable nodes between any two successive transitive nodes are linearly ordered by $\in$, and the sequence is closed under intersections.

We often refer to $s = \langle M_i \mid i < n\rangle \in \mathbb{P}_{\text{side}}$ as a set $\{M_i \mid i < n\}$ rather than a sequence. There is no loss of information in this, since the ordering of the nodes in s is determined uniquely by condition (2). With this notation, $t \leq s$ iff $t \supseteq s$.

When working with side conditions we use interval notation in the natural way. For example $(M, W)_s$ consists of all nodes of s that occur strictly between M and W. We omit the subscript s when it is clear from the context.

For s satisfying conditions (1) and (2), closure under intersections is a consequence of the weaker condition that requires $M \cap W \in s$ only under the additional assumption that W is the largest transitive node of s below M. For a proof of this see [16, Claim 2.12].

Definition 2.2. Let $s \in \mathbb{P}_{\text{side}}$ and let $Q \in s$. Then $\text{res}_Q(s)$ is defined to be $s \cap Q$.

By [16, Lemma 2.18], $\text{res}_Q(s)$ is itself a side condition. The following result gives one of the most important properties of side conditions. It is a join of Lemmas 2.20 and 2.21 in [16].

Lemma 2.3. *Let $s \in \mathbb{P}_{\text{side}}$, let $Q \in s$, and let $t \in Q \cap \mathbb{P}_{\text{side}}$ extend* $\text{res}_Q(s)$*. Then s and t are compatible. Moreover, this is witnessed by $s \cup t$ if Q is transitive, and by the closure of $s \cup t$ under intersections if Q is countable.*

For countable Q the proof of the lemma involves an analysis of the way $\text{res}_Q(s)$ sits inside s. In this context, the *residue gaps* of s in Q are the intervals $[Q \cap W, W)$ for $W \in s \cap Q$ transitive. It is easy to check, and shown in [16, Section 2], that the residue gaps are disjoint from the residue, and the residue consists precisely of the nodes of s which occur below Q and do not belong to any residue gap. Under the assumptions of Lemma 2.3 (for Q countable) one can check that $s \cup t$ is $\in$-increasing when viewed as the sequence obtained from t by adding the nodes of each residue gap $[Q \cap W, W)$ right before W, and adding the nodes of s from Q upward above the largest node of t. $s \cup t$ need not be closed under intersection, and the main part of the proof of [16, Lemma 2.21] involves adding enough nodes to it to secure this closure. For $R \in t - s$ of transitive type, let $\bar{Q}_R$ be the bottom node of the first residue gap of s in Q above R if there is such, and let $\bar{Q}_R = Q$ if there are no residue gaps above R. Let E_R consist of the countable nodes of s from (and including) $\bar{Q}_R$ up, to (and not including) the first transitive node of s above $\bar{Q}_R$ if there is one. Let $F_R = \{M \cap R \mid M \in E_R\}$. We refer to F_R as the *tacked-on* sequence or interval associated to R. The side condition witnessing Lemma 2.3 in case Q is countable is the sequence obtained from $s \cup t$ by adding the nodes of each tacked-on interval F_R, in order and right before R. The proof of [16, Lemma 2.21] shows that this sequence is both increasing and closed under intersections. Closure under intersections is obtained by showing that for every countable M in the sequence which does not belong to t, and any transitive $R \in t - s$ with $R \in M$, $M \cap R$ is one of the nodes in the tacked-on interval F_R.

Let $\dot{G}$ be the canonical $\mathbb{P}_{\mathrm{side}}$ name for the generic filter. It follows from Lemma 2.3 that every $s \in \mathbb{P}_{\mathrm{side}}$ is a *strong master condition* for every $Q \in s$, meaning that s forces $\dot{G} \cap Q$ to be generic (over V) for $\mathbb{P}_{\mathrm{side}} \cap Q$. For more details on this see [16, Claim 4.1]. It also follows from the lemma that for any $t \in \mathbb{P}_{\mathrm{side}}$ and any $Q \in \mathcal{S} \cup \mathcal{T}$ with $t \in Q$, there is $r \leq t$ with $Q \in r$. (Use the lemma with $s = \{Q\}$.) This implies that $\mathbb{P}_{\mathrm{side}}$ is *strongly proper* for $\mathcal{S} \cup \mathcal{T}$, meaning that for all $Q \in \mathcal{S} \cup \mathcal{T}$, every $t \in Q \cap \mathbb{P}_{\mathrm{side}}$ extends to a strong master condition for Q. Since $\mathcal{S}$ and $\mathcal{T}$ are stationary in $\mathcal{P}_{<\omega_1}(K)$ and $\mathcal{P}_{<\omega_2}(K)$ respectively, strong properness for $\mathcal{S} \cup \mathcal{T}$ implies that $\mathbb{P}_{\mathrm{side}}$ preserves the cardinals ω_1 and ω_2. For more details on this see [16, Section 3].

Let G be generic for $\mathbb{P}_{\mathrm{side}}$. We say that Q *occurs* in G if $(\exists s \in G) Q \in s$. Abusing notation we write $Q \in G$ to mean that Q occurs in G. Let $Ot = \{\sup(W \cap \mathrm{Ord}) \mid W \in G \text{ and } W \text{ is transitive}\}$. Let $\dot{O}t$ name Ot.

Claim 2.4. *Ot is unbounded in ω_2, and ω_1 closed.*

Proof. That Ot is unbounded in ω_2 is clear using genericity of G and the stationarity of $\mathcal{T}$. Suppose $\alpha \in \mathrm{Limit}(Ot)$ (meaning that $Ot \cap \alpha$ is cofinal in α) and $\mathrm{cof}(\alpha) = \omega_1$. We prove that $\alpha \in Ot$.

Let Q be the least node in G with $\sup(Q \cap \mathrm{Ord}) \geq \alpha$. Q must be transitive, rather than countable. Otherwise $A = \{\sup(M \cap \mathrm{Ord}) \mid M \in Q\}$ is countable, hence bounded in α, and since (using condition (2) of Definition 2.1) A is cofinal in $\{\sup(M \cap \mathrm{Ord}) \mid M \text{ occurs in } G \text{ below } Q\}$, this contradicts the fact that α is a limit point of Ot.

If $\sup(Q \cap \mathrm{Ord}) = \alpha$ then $\alpha \in Ot$ and we are done. Suppose for contradiction $\sup(Q \cap \mathrm{Ord}) > \alpha$, and hence $\alpha \in Q$. Let $M \in G$ be a countable node with $\{\alpha, Q\} \subseteq M$. Such a node exists by genericity of G and the stationarity of $\mathcal{S}$. Using the closure under intersections given by condition (3) of Definition 2.1, it must be that $M \cap Q \in G$. But since $\alpha < \sup(M \cap Q \cap \mathrm{Ord}) < \sup(Q \cap \mathrm{Ord})$ this contradicts the minimality of Q. $\square$

The posets we give in Sections 3 and 4 can be defined using $\mathcal{S}$ and $\mathcal{T}$ satisfying conditions (ST1)–(ST5), and shown using these conditions to preserve ω_1, preserve ω_2, and be $|K|^+$-c.c. This allows for the possibility that the posets collapse cardinals in the interval $(\omega_2, |K|]$, and indeed they will collapse $|K|$ to ω_2 in case $|K| > \omega_2$.

With some additional assumptions on $\mathcal{S}$ and $\mathcal{T}$ the posets in Sections 3 and 4 can be shown to be ω_2-c.c. The additional assumptions which we use for this are conditions (ST6)–(ST9) below. These conditions abstract properties used in the proof of Claim 5.7 of Neeman [16] to show that in the poset of side conditions there, every side condition can be extended to incorporate any given transitive node, and hence *every* side condition is a strong master condition for *every* element of $\mathcal{T}$. Assuming conditions (ST6)–(ST9) we obtain a parallel result here, in Claim 2.5.

- (ST6) There is some relation $\Xi \subseteq K$ so that $W \in \mathcal{T}$ iff $W \prec (K; \in, \Xi)$, W is transitive, $\omega_1 \in W$, $|W| = \omega_1$, and $\operatorname{cof}(W \cap \omega_2) = \omega_1$.
- (ST7) Every $M \in \mathcal{S}$ is elementary in $(K; \in, \Xi)$.
- (ST8) Every $Q \prec_1 (K; \in, \Xi)$ is uniquely determined from $Q \cap \mathrm{Ord}$.
- (ST9) If $M \in \mathcal{S}$, $W \in \mathcal{T}$, and $M \cap \mathrm{Ord} \subseteq W$, then $M \in W$.

Condition (ST6) implies in particular that $\mathcal{T}$ is stationary in $\mathcal{P}_{<\omega_2}(K)$, making that part of condition (ST5) redundant. Condition (ST9) implies in particular that $M \cap W \in W$ for M and W as in condition (ST4), making the final part of condition (ST4) redundant. Conditions (ST6) and (ST8) imply that $\mathcal{T}$ is linearly ordered by $\in$ and ω_1-closed.

The existence of $\mathcal{S}$ and $\mathcal{T}$ satisfying the additional conditions is *not* provable in ZFC. It is equivalent to the existence of a thin stationary subset of $\mathcal{P}_{<\omega_1}(\omega_2)$, and can be obtained by forcing with $\mathbb{P}_{\mathrm{side}}(\mathcal{S}, \mathcal{T})$ for any $\mathcal{S}, \mathcal{T}$ satisfying the earlier conditions (ST1)–(ST5). We discuss both these facts next.

Recall that $\mathcal{R} \subseteq \mathcal{P}_{<\omega_1}(\omega_2)$ is *thin* in the sense of Friedman [7] if for every $\delta < \omega_2$ the set $\{x \cap \delta \mid x \in \mathcal{R}\}$ has size at most ω_1. The condition was used earlier, for example by Rubin-Shelah [17] where it appears as $(*)$ of Theorem 4.12. Thin stationary sets exist under many circumstances, but their existence is not provable in ZFC. For more on this we refer the reader to Friedman-Krueger [9].

We mentioned above that the existence of $\mathcal{S}$ and $\mathcal{T}$ satisfying conditions (ST1)–(ST9) is equivalent to the existence of a thin stationary set. In one direction, given $\mathcal{S}$ and $\mathcal{T}$, simply take $\mathcal{R} = \{Q \cap \omega_2 \mid Q \in \mathcal{S}\}$. To see that $\mathcal{R}$ is thin, note that for every $W \in \mathcal{T}$ and every $Q \in \mathcal{S}$, $Q \cap W \in W$. (This is clear if $Q \in W$. If $Q \notin W$, then by Remark 2.7 below there is a transitive $W^* \supseteq W$ with $W^* \in Q$. Taking a $\in$-minimal such W^* we have $Q \cap W = Q \cap W^* \in W$ by minimality.) It follows that for every $\delta < \omega_2$, and any transitive $W \supseteq \delta$, $\{x \cap \delta \mid x \in \mathcal{R}\}$ is contained in $\{M \cap \delta \mid M \in W\}$ and in particular has size at most ω_1.

In the other direction, given a thin set $\mathcal{R}$, let $f \colon \omega_2 \to \mathcal{P}_{<\omega_1}(\omega_2)$ enumerate all elements of $\{x \cap \delta \mid \delta < \omega_2 \wedge x \in \mathcal{R}\}$, ordered first by their supremum. Let $K = L_{\omega_2}[\Xi]$ where Ξ is the predicate $\{\langle \alpha, \xi \rangle \mid \xi \in f(\alpha)\}$. Let $\mathcal{T}$ consist of transitive $W \prec (K; \in, \Xi)$ of size ω_1 with $\omega_1 + 1 \subseteq W$ and $\operatorname{cof}(W \cap \omega_2) = \omega_1$. Let $\mathcal{S}$ consist of all models $Q \cap W$ where $Q \prec (K; \in, \Xi)$ is countable with $\omega_1 \in Q$, $Q \cap \omega_2 \in \mathcal{R}$, $W \in \mathcal{T}$, and $W \in Q$. This is similar to a more general construction of $\mathcal{S}$ and $\mathcal{T}$ from a thin stationary set in Subsection 5.2 of Neeman [16], restricting $\mathcal{S}$ to just the nodes that are used in the proof of [16, Claim 5.5]. (The move from Q to $Q \cap W$ is important there, to limit the set of supremums of countable nodes.

It is not so important here.) It is easy to check that $\mathcal{S}$ and $\mathcal{T}$ satisfy conditions (ST1)–(ST9), except possibly the requirement that $\mathcal{S}$ is stationary. The thinness of $\mathcal{R}$ is important in verifying condition (ST9). If $\mathcal{R}$ is stationary then $\mathcal{S}$ is stationary.

One can also obtain $\mathcal{S}'$ and $\mathcal{T}'$ satisfying conditions (ST1)–(ST9) by forcing with $\mathbb{P}_{\text{side}}(\mathcal{S},\mathcal{T})$ starting with $\mathcal{S},\mathcal{T}$ which satisfy only conditions (ST1)–(ST5). To see this, let G be generic for $\mathbb{P}_{\text{side}}(\mathcal{S},\mathcal{T})$, and let $\Xi \subseteq K$ be the set of nodes occurring in G. Then the sets $\mathcal{T}' = \{W \in \mathcal{T} \mid W \prec (K, \in, \Xi)\}$ and $\mathcal{S}' = \{M \in \mathcal{S} \mid M \in G \wedge M \prec (K; \in, \Xi)\}$ satisfy conditions (ST1)–(ST9), in $V[G]$. Indeed, most of the conditions are simply inherited from properties of $\mathcal{S}$ and $\mathcal{T}$, in some cases using the closure of G, genericity of G, and strong properness of $\mathbb{P}_{\text{side}}$. The use of G, and the closure given by Claim 2.4, allows us to obtain the right-to-left direction of the equivalence in condition (ST6). The use of G also allows us to avoid countable nodes that conflict with condition (ST9), and obtain condition (ST8). For the latter point note that if $Q \prec_1 (K; \in, \Xi)$ then Q is exactly equal to $\bigcup\{M \in \Xi \mid M$ is countable and $\sup(M \cap \text{Ord}) \in Q\}$.

This method for obtaining conditions (ST1)–(ST9) is important since it allows us to reduce our assumptions in forcing constructions, from $\mathcal{S}'$ and $\mathcal{T}'$ satisfying conditions (ST1)–(ST9), to $\mathcal{S}$ and $\mathcal{T}$ satisfying just conditions (ST1)–(ST5), simply by composing a poset which uses $\mathcal{S}'$ and $\mathcal{T}'$ with an initial application of $\mathbb{P}_{\text{side}}(\mathcal{S},\mathcal{T})$. Moreover, any properness we obtain for the original poset and nodes in $\mathcal{S}',\mathcal{T}'$ will typically continue to hold for the composition with $\mathbb{P}_{\text{side}}$ and nodes in $\mathcal{S},\mathcal{T}$, using the strong properness of $\mathbb{P}_{\text{side}}$.

In addition to $\mathbb{P}_{\text{side}}$ there are other finite conditions posets that can be used to obtain $\mathcal{S}'$ and $\mathcal{T}'$ which satisfy the full list of conditions (ST1)–(ST9) from assumptions which are satisfiable in ZFC, for example the two-type side condition posets with tower nodes in Neeman [15].

For the rest of the section we work with $\mathcal{S}$ and $\mathcal{T}$ satisfying all the conditions above, (ST1)–(ST9).

Claim 2.5. *(Assuming conditions (ST1)–(ST9).) Let $s \in \mathbb{P}_{\text{side}}$, and let W be a transitive node. Then there is $u \in \mathbb{P}_{\text{side}}$ extending s with $W \in u$.*

Proof. Similar to the proof of Claim 5.7 in [16]. Suppose $W \notin s$ as otherwise $u = s$ works. Suppose also that s has nodes outside W, since otherwise $u = s \cup \{W\}$ is a side condition and witnesses the claim.

Let Q be the first node of s outside W. Inductively we may assume that any s^* with a node outside W that has smaller ordinal height than Q, can be extended to u with $W \in u$.

By choice of Q, $\text{res}_s(Q) \subseteq W$. So $\text{res}_s(Q) \cup \{W\}$ is a side condition. If $W \in Q$ then $\text{res}_s(Q) \cup \{W\}$ belongs to Q, so by Lemma 2.3 it is compatible with s. Letting u witness the compatibility we have $W \in u$ as required.

Suppose then that $W \notin Q$. Since $\mathcal{T}$ is linearly ordered by $\in$, it follows, using our earlier assumptions that $Q \notin W$ and $W \notin s$, that Q is countable rather than transitive. Let α be the least ordinal of Q at or above $\sup(W \cap \text{Ord})$. Such an ordinal must exist, since otherwise $Q \cap \text{Ord} \subseteq W$ and hence $Q \in W$ by condition (ST9). If $\alpha = \sup(W \cap \text{Ord})$ then by condition (ST8), W is the unique Σ_1 elementary substructure of $(K; \in, \Xi)$ with $W \cap \text{Ord} = \alpha$. Then W belongs to Q by elementarity, a contradiction. So $\alpha > \sup(W \cap \text{Ord})$.

By condition (ST8), for each $\nu < \omega_2$ there is at most one $Y \prec_1 (K; \in, \Xi)$ with $Y \cap \text{Ord} = \nu$. Let $h(\nu)$ denote this Y if it exists. Note that for every $\beta < \alpha$

and $n < \omega$, there is $\nu \in [\beta, \alpha)$ so that $h(\nu)$ is defined and is Σ_n elementary in $(K; \in, \Xi)$. Otherwise by elementarity of Q, a counterexample β exists in Q. Then $\beta < \sup(W \cap \mathrm{Ord})$ by minimality of α, and taking $\nu = \sup(W \cap \mathrm{Ord})$ gives a contradiction.

Letting $W^* = \bigcup_{\nu \in \alpha \cap \mathrm{dom}(h)} h(\nu)$ it follows that $W^* \cap \mathrm{Ord} = \alpha$ and that $W^* \prec (K; \in, \Xi)$. By condition (ST6) and since $\mathrm{cof}(\alpha) = \omega_1$, this implies that $W^* \in \mathcal{T}$. (To see $\mathrm{cof}(\alpha) = \omega_1$ note that otherwise Q is cofinal in α, and this contradicts the minimality of α.)

By elementarity of Q, $W^* \in Q$. Arguing as we did above in the case of $W \in Q$ it follows that there is a side condition s^* extending s, with $W^* \in s^*$. The first node of s^* outside W is either W^* or an earlier node, and in either case it has smaller ordinal height than Q. By our induction hypothesis there is therefore u extending s^* with $W \in u$. □

Remark 2.6. By the proof of Claim 2.5, a condition u witnessing the claim can be obtained from s through repeated applications of only the following consequences of Lemma 2.3: (a) For any condition $\hat{s} \in \mathbb{P}_{\mathrm{side}}$ and any transitive $\hat{W}$ with $\hat{s} \subseteq \hat{W}$, there is $\hat{r} \leq \hat{s}$ with $\hat{W} \in \hat{r}$; and (b) For any condition $\hat{s} \in \mathbb{P}_{\mathrm{side}}$, any $Q \in \hat{s}$, and any transitive $\hat{W} \in Q$ with $\hat{s} \cap Q \subseteq \hat{W}$, there is $\hat{r} \leq \hat{s}$ with $\hat{W} \in \hat{r}$.

Remark 2.7. The proof of Claim 2.5 shows that for any countable node Q, and any transitive node W with $Q \notin W$, there is a transitive node $W^* \supseteq W$ with $W^* \in Q$.

It follows from Claim 2.5 that, assuming conditions (ST1)–(ST9), every $s \in \mathbb{P}_{\mathrm{side}}$ is a strong master condition for every $W \in \mathcal{T}$. (No $s' \leq s$ can force the generic filter to avoid a dense subset of $\mathbb{P}_{\mathrm{side}} \cap W$, since s' extends to u with $W \in u$, and any such u is a strong master condition for W by Lemma 2.3.) In particular then for all large enough θ, and all sufficiently elementary $H \prec H(\theta)$ of cardinality ω_1 with $\omega_1 \subseteq H \wedge \mathrm{cof}(H \cap \omega_2) = \omega_1$, every condition in $\mathbb{P}_{\mathrm{side}}$ is a master condition for $H \cap K$, as $H \cap K \in \mathcal{T}$ by condition (ST6). Since strong properness implies properness in this context (see for example Neeman [16, Remarks 3.1,3.2]), every condition in $\mathbb{P}_{\mathrm{side}}$ is a master condition for H. Such a wealth of master conditions for models of size ω_1 implies that $\mathbb{P}_{\mathrm{side}}$ is ω_2-c.c., by analogues of the standard arguments one cardinal down.

3. Square

Work throughout with $\mathcal{S}$ and $\mathcal{T}$ satisfying conditions (ST1)–(ST9) in Section 2. We describe a strongly proper for $\mathcal{S} \cup \mathcal{T}$, ω_2-c.c., finite conditions poset which forces the existence of a $\square_{\omega_1}$ sequence, and variants which belong to iterable classes and force $\square_{\omega_1,\mathrm{fin}}$ and $\square^{\mathrm{ta}}_{\omega_1,\omega}$.

Conditions (ST6)–(ST9) are included in our assumptions for simplicity and to obtain the ω_2-chain condition. Posets forcing the same principles with finite conditions, which are strongly proper for $\mathcal{S} \cup \mathcal{T}$, can be obtained assuming just conditions (ST1)–(ST5), by prefixing the posets we define below with a preparatory forcing by $\mathbb{A} = \mathbb{P}_{\mathrm{side}}$, as described before Claim 2.5. Each of these combinations with $\mathbb{A}$ can be written as a single poset with finite conditions, but we leave the details to the reader. The results we obtain below under conditions (ST1)–(ST9) also hold for the combined posets under conditions (ST1)–(ST5), except for the claims related to the ω_2-chain condition.

Recall that $\dot{O}t$ names $\{\sup(W \cap \mathrm{Ord}) \mid W \in \mathcal{T}$ and W occurs in $\dot{G}\}$. By Claim 2.5, which is proved using conditions (ST6)–(ST9), $\dot{O}t$ is forced to name the ground model set $\{\sup(W \cap \mathrm{Ord}) \mid W \in \mathcal{T}\}$. In Definition 3.1 and later in the section we use Ot to denote $\{\sup(W \cap \mathrm{Ord}) \mid W \in \mathcal{T}\}$, and rely on the fact that this set is forced to be equal to $\dot{O}t$. When working with only conditions (ST1)–(ST5) one has to be more careful, and refer to $\dot{O}t$ and the $\mathbb{P}_{\mathrm{side}}$ forcing relation in the definitions.

Definition 3.1. $\mathbb{S}\mathrm{quare} = \mathbb{S}\mathrm{quare}(\mathcal{S}, \mathcal{T})$ is the poset of pairs $\langle s, c\rangle$ where:

(1) $s \in \mathbb{P}_{\mathrm{side}}(\mathcal{S}, \mathcal{T})$.
(2) c is a function on the nodes in s.
(3) (Linearity) For each $M \in \mathrm{dom}(c)$, $c(M)$ is an $\in$-linear set of countable nodes, contained in $s \cap M$.
(4) (Separation) If Ot is cofinal in $\sup(M \cap \mathrm{Ord})$, then Ot is cofinal in $\sup(\bar{M} \cap \mathrm{Ord})$ for every $\bar{M} \in c(M)$. If Ot is bounded in $\sup(M \cap \mathrm{Ord})$ then $c(M) = \emptyset$.
(5) (Coherence) If $\bar{M} \in c(M)$ then $c(\bar{M}) = c(M) \cap \bar{M}$.
(6) (Fullness) If $W \in M$ are nodes of s of transitive and countable type respectively, and $\sup(W \cap \mathrm{Ord})$ is a limit point of Ot, then $M \cap W \in c(W)$.

The ordering on $\mathbb{S}\mathrm{quare}$ is given by setting $\langle s^*, c^*\rangle \leq \langle s, c\rangle$ iff:

(i) $s^* \leq s$ in $\mathbb{P}_{\mathrm{side}}$.
(ii) $c^*(M) \cap s = c(M)$ for every $M \in s$.
(iii) (Novelty jump) For $M \in s$, $R \in c(M) \cup \{M\}$, and $P \in c^*(M) \cap R$, if $P \supseteq c(M) \cap R \neq \emptyset$ then $P \supseteq s \cap R$.

The fullness condition is equivalent to its restricted version that demands $M \cap W \in c(W)$ only in case that, in addition to the assumptions in the condition, there are no transitive nodes between W and M in s. The equivalence can be seen inductively by replacing a given M with $M \cap W^*$, for $W^* \in s$ largest transitive below M in case $W^* \neq W$.

The novelty jump condition is equivalent to its restricted version that applies only to $R = M$. We refer to this restricted version as the *end novelty jump* condition. It states that for any $P \in c^*(M)$, if $P \supseteq c(M) \neq \emptyset$ (or equivalently by linearity, P occurs in $c^*(M)$ above the largest node of $c(M)$) then $P \supseteq s \cap M$. To see that the end novelty jump condition implies the full novelty jump condition, note that by coherence of c and c^*, instances of the novelty jump condition for $c^*(M)$ with $R \in c(M)$ are the same as the end novelty jump condition for $c^*(R)$.

Remark 3.2. We could have dropped the requirement that $c(M) \cap R \neq \emptyset$ in the hypothesis of the novelty jump condition, and similarly the requirement $c(M) \neq \emptyset$ in the hypothesis of the end novelty jump condition. The results on $\mathbb{S}\mathrm{quare}$ would still go through, with the same proofs. The place where these requirements make a difference is in later results on $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}$. More specifically, the restriction to cases where $c(M) \cap R \neq \emptyset$ in the novelty jump condition is important for the first paragraph in the proof of Lemma 3.15. We could also have weakened the requirement in condition (ii) of Definition 3.1 to just $c^*(M) \supseteq c(M)$, and the results on $\mathbb{S}\mathrm{quare}$ would still go through, with the same constructions and in some cases slightly simpler arguments. The place where the requirement that $c^*(M) \cap s = c(M)$ becomes important is in later results on $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}$, specifically for Remark 3.10.

Claim 3.3. *The ordering on* $\mathbb{S}\mathrm{quare}$ *is transitive.*

Proof. The only nontrivial part involves the novelty jump condition. Suppose $\langle s^{**}, c^{**}\rangle \leq \langle s^*, c^*\rangle \leq \langle s, c\rangle$. It is enough to verify the end novelty jump condition for c^{**} relative to $\langle s, c\rangle$. Fix $M \in s$ and $P \in c^{**}(M)$. Suppose that $P \supseteq c(M)$. We prove that $P \supseteq s \cap M$.

If $P \supseteq c^*(M)$ then $P \supseteq s^* \cap M$ by the end novelty jump condition for c^{**} relative to $\langle s^*, c^*\rangle$, and therefore $P \supseteq s \cap M$.

Suppose $P \not\supseteq c^*(M)$ and let R be $\in$-minimal in $c^*(M) - P$. Then $c^*(M) \cap R \subseteq P$. If $R = P$ then $P \in c^*(M)$, so $P \supseteq s \cap M$ by the end novelty jump condition for c^* relative to $\langle s, c\rangle$. If $R \neq P$ then by linearity of $c^{**}(M)$ it must be that $P \in R$. By the novelty jump condition for c^{**} relative to $\langle s^*, c^*\rangle$ it follows that $P \supseteq s^* \cap R \supseteq s \cap R$. Since $P \in R$ we have $P \subseteq R$ and hence $R \supseteq c(M)$. By the end novelty jump condition for c^* relative to $\langle s, c\rangle$ (with the current R standing for P of the condition), $R \supseteq s \cap M$. So $P \supseteq s \cap R \supseteq s \cap M$. $\square$

For $\langle s, c\rangle \in \mathbb{S}\text{quare}$ and $Q \in s$ define $\mathrm{res}_Q(s, c)$ to be $\langle \mathrm{res}_Q(s), c'\rangle$ where $c'(M) = c(M) \cap Q$ for each $M \in \mathrm{res}_Q(M)$. If Q is of transitive type then $c(M) \subseteq M \subseteq Q$ for each $M \in \mathrm{res}_Q(s)$, so $c'(M) = c(M)$. If Q is countable, then by the same reasoning $c'(M) = c(M)$ for countable $M \in \mathrm{res}_Q(s)$. For transitive $W \in \mathrm{res}_Q(s)$ with $\sup(W \cap \mathrm{Ord})$ a limit point of Ot, $Q \cap W \in c(W)$ by fullness, and $c'(W)$ consists precisely of the nodes of $c(W)$ before $Q \cap W$, or equivalently by coherence, $c'(W) = c(Q \cap W)$. For transitive $W \in \mathrm{res}_Q(s)$ with Ot bounded in $\sup(W \cap \mathrm{Ord})$, $c(W) = \emptyset$ and hence $c'(W) = \emptyset$. Note in this case that by elementarity of Q and W, Ot is also bounded in $\sup(Q \cap W \cap \mathrm{Ord})$, so $c(Q \cap W) = \emptyset$, and hence again $c'(W) = c(Q \cap W)$.

It is easy using these observation to check that $\mathrm{res}_Q(s, c) \in \mathbb{S}\text{quare}$. It is also clear that $\langle s, c\rangle \leq \mathrm{res}_Q(s, c)$. The only non-trivial condition in verifying this is the end novelty jump condition at transitive W in $\mathrm{res}_Q(s)$ with Ot unbounded in $\sup(W \cap \mathrm{Ord})$. In this case the first node in $c(W) - c'(W) = c(W) - c(Q \cap W)$ is $Q \cap W$, and hence by linearity all nodes in $c(W) - c'(W)$ contain $Q \cap W \supseteq \mathrm{res}_Q(s) \cap W$.

Lemma 3.4. *Let $\langle s, c\rangle \in \mathbb{S}\text{quare}$, let $Q \in s$, and let $\langle t, d\rangle \in Q \cap \mathbb{S}\text{quare}$ extend $\mathrm{res}_Q(s, c)$. Then $\langle s, c\rangle$ and $\langle t, d\rangle$ are compatible. Moreover there is a condition $\langle r, b\rangle$ witnessing this so that $r = s \cup t$ if Q is transitive, and r is the closure of $s \cup t$ under intersections if Q is countable.*

Proof. By Lemma 2.3, s and t are compatible in $\mathbb{P}_{\text{side}}$, and moreover there is r witnessing this which is equal to $s \cup t$ if Q is transitive, and equal to the closure of $s \cup t$ under intersections if Q is countable. It remains to define b so that $\langle r, b\rangle \in \mathbb{S}\text{quare}$, $b(M) \cap s = c(M)$ for $M \in s$, $b(M) \cap t = d(M)$ for $M \in t$, and b satisfies the end novelty jump condition relative to $\langle s, c\rangle$ and $\langle t, d\rangle$.

If Q is transitive then the nodes of r below Q are exactly the nodes of t, and the nodes of r from Q upward are the ones from Q upward in s. Set in this case $b(M) = d(M)$ for M below Q. Note that then trivially $b(M) \cap t = d(M)$, and $b(M) \cap s = d(M) \cap s = c(M)$ since $\langle t, d\rangle \leq \mathrm{res}_Q(s, c)$. For M at or above Q, set $b(M) = c(M)$ if $c(M)$ has no nodes below Q, and otherwise set $b(M) = c(M) \cup b(P)$ where P is the largest node of $c(M)$ below Q. $b(M)$ in the latter case consists of $b(P)$ followed by the nodes of $c(M)$ from P onward, since $c(M) \cap P = c(P) \subseteq b(P)$ using coherence. This implies that $b(M)$ is $\in$-linear. We have $b(M) \cap s = c(M)$ since $b(P) \cap s = d(P) \cap s = c(P) = c(M) \cap P$, using the fact that $\langle t, d\rangle \leq \mathrm{res}_Q(s, c)$

and coherence. It is easy to check with these assignments that $\langle r, b\rangle$ satisfies the coherence required in Definition 3.1, and also the separation condition. Fullness holds for W from Q upward because all the relevant nodes then belong to s, and $b(W) \supseteq c(W)$. For W below Q, the restricted fullness condition, where there are no transitive nodes between W and M, holds because then both W and M are below Q, hence elements of t. The end novelty jump condition for b relative to $\langle t, d\rangle$ holds trivially since $b(M) - d(M) = \emptyset$ for $M \in t$. It holds trivially for b relative to $\langle s, c\rangle$ at nodes M from Q upward since the largest node of $b(M)$ belongs to $c(M)$, making the condition vacuous. For M below Q it is inherited from the end novelty jump condition for d relative to $\mathrm{res}_Q(s, c)$.

Suppose now that Q is countable. For $M \in t$ of countable type, set $b(M) = d(M)$. For each residue gap $[Q\cap W, W)$ of s in Q, set $b(Q\cap W) = d(W)$. Separation for $b(Q\cap W)$ is then inherited from separation for $d(W)$, since $\sup(Q\cap W\cap \mathrm{Ord})$ is a limit point of Ot iff $\sup(W\cap \mathrm{Ord})$ is a limit point of Ot, by the elementarity of Q and W. If Ot is bounded, in $\sup(W\cap \mathrm{Ord})$ and equivalently in $\sup(Q\cap W\cap \mathrm{Ord})$, then $b(Q \cap W) = d(W) = c(W) = c(Q \cap W) = \emptyset$. If Ot is unbounded then $c(Q\cap W) = c(W)\cap Q = d(W)\cap s\cap Q = d(W)\cap s = b(Q\cap W)\cap s$, with the first equality holding by coherence and fullness, the second holding since $\langle t, d\rangle \leq \mathrm{res}_Q(s, c)$, and the third since $d \in Q$. So either way, $b(Q \cap W) \cap s = c(Q \cap W)$.

Let U consist of the countable nodes in t, plus the bottom nodes of residue gap. So far we defined $b{\restriction}U$. For $M \in s - t$ other than the bottom nodes of residue gaps, set $b(M) = c(M) \cup b(P)$ where P is the largest node of $c(M)$ that belongs to U if there is one, and $b(M) = c(M)$ otherwise. Then as in paragraph dealing with transitive Q, b satisfies linearity, coherence, separation, and fullness on its domain. (One additional note needed here, in verifying fullness of b for transitive $W \in s - t$, is that there are no countable $M \in t$ with $W \in M$, since $M \subseteq Q$ for all countable $M \in t$. For verifying fullness this is used together with the fact that any countable $M \in r - t$ is either in s or of the form $M^* \cap W^*$ for countable $M^* \in s$, see the discussion following Lemma 2.3.) Again as in the argument for transitive Q, $b(M) \cap s = c(M)$. b also satisfies the end novelty jump condition at nodes in $U \cup (s - t)$ relative to $\langle s, c\rangle$ and $\langle t, d\rangle$. In the case of a bottom node $Q \cap W$ of a residue gap, the condition for $b(Q \cap W)$ relative to $\langle s, c\rangle$ is inherited from the condition for $d(W)$ relative to $\mathrm{res}_Q(s, c)$. The other cases are similar to the ones in the paragraph on transitive Q.

By the discussion following Lemma 2.3, r consists of the nodes of $s \cup t$, plus the nodes in tacked-on intervals F_W for transitive $W \in t - s$. The tacked-on intervals are pairwise disjoint, and disjoint from $s\cup t$. Moreover, for any transitive $W \in t - s$, and any $M \in r - t$ with $W \in M$, the intersection $M \cap W$ belongs to the tacked-on interval F_W.

To complete the definition of b, it remains to handle transitive nodes in t, and nodes in tacked-on sequences.

Suppose $W \in t$ is transitive and belongs to s. Then W is the top node of a residue gap of s in Q. By fullness, separation, coherence, and the facts that $\langle t, d\rangle$ belongs to Q and extends $\mathrm{res}_Q(s, c)$, $d(W) \cap s = c(W) \cap Q = c(Q \cap W)$, with $d(W) = c(W) = c(Q \cap W) = \emptyset$ in case Ot is bounded in $\sup(W \cap \mathrm{Ord})$, and $Q \cap W \in c(W)$ in case Ot is unbounded. Set $b(W) = d(W) \cup c(W)$. Then $b(W)$ is $\in$-linear, and indeed it consists of the nodes of $d(W)$ followed by the nodes of $c(W)$ from $Q \cap W$ upward. The equality $b(W) \cap t = d(W)$ holds because $c(W) - Q$ is disjoint from t and $c(W) \cap Q \subseteq d(W)$. The equality $b(W) \cap s = c(W)$ holds because $d(W) \cap s = c(W) \cap Q \subseteq c(W)$. Separation for $b(W)$ is inherited from separation for $c(W)$ and $d(W)$. Fullness of $c(W)$ and $d(W)$ in s and t respectively gives fullness of $b(W)$ in $s \cup t$, and hence also in r since all countable nodes of $r - (s \cup t)$ are of the form $M \cap R$ for countable $M \in s$. Our definition of b on countable nodes above, including in particular the case of bottom nodes of residue gaps, is such that $b(W)$ coheres with $b(M)$ for countable $M \in b(W)$. As for the end novelty jump condition, it holds vacuously for $b(W)$ relative to $\langle s, c\rangle$ since the largest node of $b(W)$ belongs to $c(W)$, and it holds relative to $\langle t, d\rangle$ since the first node in $b(W) - d(W)$ (in case $b(W) \neq \emptyset$) is $Q \cap W$, which contains $t \cap W$.

Suppose W is transitive and belongs to $t - s$. If Ot is bounded in $\sup(W \cap \mathrm{Ord})$, set $b(W) = \emptyset$. In this case Ot is also bounded in $\sup(P \cap \mathrm{Ord})$ for each P in the tacked-on interval F_W, since these nodes have the form $P^* \cap W$ for countable $P^* \in s$ with $W \in P$. Set $b(P) = \emptyset$ for all $P \in F_W$. These assignments trivially satisfy linearity, separation, coherence, fullness, and the novelty jump condition. In the case of coherence note that since F_W is disjoint from $s \cup t$, there is no interference with our previous definitions for b.

Suppose W is transitive and belongs to $t - s$, and Ot is cofinal in $\sup(W \cap \mathrm{Ord})$. Set $b(W) = d(W) \cup F_W$, where F_W is the tacked-on interval associated to W. The smallest node of the tacked-on interval is equal to $Q \cap W$. From this, the fact that F_W is $\in$-linear, the linearity of $d(W)$, and the fact that $d(W) \subseteq Q \cap W$, it follows that $b(W)$ is $\in$-linear, and that $b(W) \cap Q = d(W)$ hence in particular $b(W) \cap t = d(W)$. Using coherence of d and our earlier definitions it also follows that $b(W) \cap M = b(M)$ for every $M \in b(W)$ on which b has already been defined, meaning every $M \in b(W) - F_W$. For every countable $M^* \in r$ with $W \in M^*$, we have $M^* \cap W \in d(W) \subseteq b(W)$ if $M^* \in t$, and (by properties of the tacked-on intervals) $M^* \cap W \in F_W \subseteq b(W)$ if $M^* \in r - t$. So $b(W)$ is full. The end novelty jump condition holds for $b(W)$ (relative to $\langle t, d\rangle$) since each node in F_W contains the smallest node of F_W, which is equal to $Q \cap W$, and therefore contains $t \cap W$. Each $M \in F_W$ is of the form $M^* \cap W$ for a node $M^* \in s$ with $W \in M^*$, and by elementarity of W and M^* this implies that $\sup(M^* \cap W \cap \mathrm{Ord})$ is a limit point of Ot iff $\sup(W \cap \mathrm{Ord})$ is a limit point of Ot. Using this one can prove that the separation condition for $b(W)$ is inherited from the same condition for $d(W)$.

Finally, for each P that belongs to the tacked-on interval F_W, set $b(P) = d(W) \cup (F_W \cap P)$. Coherence for $b(P)$ is clear, and using the definition in the previous paragraph, so is coherence for $b(W)$ at $P \in F_W$. Linearity and separation hold as in the previous paragraph. There are no instances of the end novelty jump condition to check as $P \notin s \cup t$. This completes the proof of Lemma 3.4. □

Corollary 3.5. *The poset* $\mathbb{S}\mathrm{quare}(\mathcal{S}, \mathcal{T})$ *is strongly proper for* $\mathcal{S} \cup \mathcal{T}$*.*

Proof. Immediate from Lemma 3.4 with the usual argument: If $\langle s, c\rangle \in \mathbb{S}\mathrm{quare}$, $Q \in s$, and D is dense in $\mathbb{S}\mathrm{quare} \cap Q$, then by density there is $\langle t, d\rangle \in D$ extending $\mathrm{res}_Q(s, c)$, and by Lemma 3.4, $\langle t, d\rangle$ is compatible with $\langle s, c\rangle$. This implies that any $\langle s, c\rangle$ with $Q \in s$ is a strong master condition for Q. The lemma, when used with the initial condition $\langle\{Q\}, \{Q \mapsto \emptyset\}\rangle$, whose residue in Q is the empty condition, also shows that any condition in $\mathbb{S}\mathrm{quare} \cap Q$ extends to a condition that has Q as a node. □

Since $\mathcal{S}$ and $\mathcal{T}$ are both stationary, it follows from Corollary 3.5 that $\mathbb{S}$quare preserves ω_1 and ω_2.

Claim 3.6. *Let $\langle s, c\rangle \in \mathbb{S}\text{quare}$ and let $W \in \mathcal{T}$. Then there is $\langle s', c'\rangle \leq \langle s, c\rangle$ with $W \in s'$. Moreover such a condition can be found so that $c'(M) = c(M)$ for all $M \in s$.*

Proof. Similar to the proof of Claim 2.5, using Lemma 3.4 instead of Lemma 2.3. To show that the constructed c' has the property that $c'(M) = c(M)$ for all $M \in s$, it is enough, by Remark 2.6, to show that: (a) for any $\langle \hat{s}, \hat{c}\rangle \in \mathbb{S}\text{quare}$ and any transitive $\hat{W}$ with $\hat{s} \subseteq \hat{W}$, there is $\langle \hat{r}, \hat{b}\rangle \leq \langle \hat{s}, \hat{c}\rangle$ with $\hat{W} \in \hat{r}$ and $\hat{b}(M) = \hat{c}(M)$ for all $M \in \hat{s}$; and (b) for any $\langle \hat{s}, \hat{c}\rangle \in \mathbb{S}\text{quare}$, any $Q \in \hat{s}$, and any transitive $\hat{W} \in Q$ with $\hat{s} \cap Q \subseteq \hat{W}$, there is $\langle \hat{r}, \hat{b}\rangle \leq \langle \hat{s}, \hat{c}\rangle$ with $\hat{W} \in \hat{r}$ and the additional property that $\hat{b}(M) = \hat{c}(M)$ for every $M \in \hat{s}$. Part (a) holds trivially setting $\hat{r} = \hat{s} \cup \{\hat{W}\}$ and $\hat{b}(\hat{W}) = \emptyset$. Part (b) follows from an application of Lemma 3.4 to the condition $\langle \hat{t}, \hat{d}\rangle$ obtained from $\mathrm{res}_Q(\hat{s}, \hat{c})$ by adding the node $\hat{W}$ to $\hat{s} \cap Q$ and setting $\hat{d}(\hat{W}) = \emptyset$. A quick look through the proof of the lemma shows that the condition $\langle \hat{r}, \hat{b}\rangle$ it produces in this case has the additional property that $\hat{b}(M) = \hat{c}(M)$ for all $M \in \hat{s}$. Indeed, $\hat{b}$ is the function that extends $\hat{c}$ with just the following assignments: $\hat{b}(\hat{W}) = F_{\hat{W}}$ and $\hat{b}(P) = F_{\hat{W}} \cap P$ for $P \in F_{\hat{W}}$, where $F_{\hat{W}}$ is the tacked-on interval for $\hat{W}$, if Q is countable and Ot is unbounded in $\sup(\hat{W} \cap \mathrm{Ord})$; $\hat{b}(\hat{W}) = \hat{b}(P) = \emptyset$, where $P \in F_{\hat{W}}$, if Q is countable and Ot is bounded in $\sup(\hat{W} \cap \mathrm{Ord})$; and $\hat{b}(\hat{W}) = \emptyset$ if Q is transitive. □

Corollary 3.7. *The poset* $\mathbb{S}$quare *is ω_2-c.c.*

Proof. Suppose A is a maximal antichain in $\mathbb{S}$quare with $|A| = \omega_2$. Then for every $W \in \mathcal{T}$, every $\langle s, c\rangle \in A - W$ forces the generic filter to avoid $A \cap W$. Taking W sufficiently elementary that $A \cap W$ is a maximal antichain in $\mathbb{S}\text{quare} \cap W$ this gives a contradiction, since by Lemma 3.4 and Claim 3.6, every $\langle s, c\rangle \in \mathbb{S}\text{quare}$ extends to a strong master condition for W. □

Lemma 3.8. *Forcing with* $\mathbb{S}$quare *adds a $\square_{\omega_1}$ sequence.*

Proof. Let G be generic for $\mathbb{S}$quare. Let α_M denote $\sup(M \cap \mathrm{Ord})$. Let $X = \{\alpha_M \mid M \text{ occurs in } G\}$.

For $\alpha = \alpha_M \in X \cap \mathrm{Limit}(Ot)$ let $C_\alpha = \{\alpha_P \mid (\exists \langle s, c\rangle \in G)(M \in s \wedge P \in c(M))\}$. By the separation condition, $C_\alpha \subseteq \mathrm{Limit}(Ot)$. For $\alpha \in \mathrm{Limit}(Ot) - X$ set $C_\alpha = \emptyset$. We will show that $\langle C_\alpha \mid \alpha \in \mathrm{Limit}(Ot)\rangle$ is a $\square'_{\omega_1}$ sequence on $\mathrm{Limit}(Ot)$. Precisely this means that $C_\alpha \subseteq \mathrm{Limit}(Ot) \cap \alpha$ is closed in α and of ordertype $\leq \omega_1$; if $\mathrm{cof}(\alpha) > \omega$ and α is a limit point of $\mathrm{Limit}(Ot)$ then C_α is unbounded in α; and $C_\beta = C_\alpha \cap \beta$ whenever $\beta \in C_\alpha$.

Such a sequence can be turned into a $\square_{\omega_1}$ sequence by standard arguments: Let $\varphi\colon \omega_2 \to \mathrm{Limit}(Ot)$ be an order preserving bijection. Define D_α to be the preimage by φ of $C_{\varphi(\alpha)}$ in case $C_{\varphi(\alpha)}$ is unbounded in α. If $C_{\varphi(\alpha)}$ is bounded in α, note that $\mathrm{cof}(\alpha) \leq \omega$, and define D_α to be any cofinal subset of α of ordertype $\leq \omega$. It is easy to check that $\langle D_\alpha \mid \alpha < \omega_2\rangle$ is a $\square_{\omega_1}$ sequence.

It is clear using the coherence condition in Definition 3.1 that $\beta \in C_\alpha \to C_\beta = C_\alpha \cap \beta$. Using the linearity condition and since $c(M)$ consists only of countable nodes, it is clear that C_α is a subset of α with ordertype at most ω_1. By the fullness condition, Claim 3.6, and a genericity argument using Lemma 3.4 to show that $(\forall a \subseteq \omega_2 \text{ finite})(\exists M \text{ countable occurring in } G)(a \subseteq M)$, C_α is cofinal in α whenever $\alpha \in Ot \cap \text{Limit}(Ot)$. Since Ot is ω_1 closed this implies that C_α is cofinal in α for all α of cofinality ω_1 in Limit(Ot). The only remaining property to prove is that C_α is closed in α.

Fix $\alpha \in \text{Limit}(Ot)$, and suppose for contradiction that C_α is not closed in α. In particular $C_\alpha \neq \emptyset$, so $\alpha \in X$ and there is M occurring in G with $\alpha = \alpha_M$. Let $\beta < \alpha$ be a limit point C_α which does not belong to C_α. Fix $\langle s, c\rangle \in G$ forcing this. We may assume $M \in s$. Since β is a limit point of C_α we may also assume that there are nodes $U \in c(M)$ with $\alpha_U < \beta$. Let U be the largest such node. Let R be the first node of $c(M)$ above U if there is one, and otherwise set $R = M$. Since $\beta \notin C_\alpha$, we have $\alpha_R > \beta$.

By the separation condition in Definition 3.1, and since $\alpha = \alpha_M \in \text{Limit}(Ot)$, Ot is unbounded in $\sup(N \cap \text{Ord})$ for all $N \in c(M)$. Hence in particular Ot is unbounded in $\sup(R \cap \text{Ord})$. Let $\gamma \in Ot \cap \alpha_R$ be larger than β. Using Remark 2.7 we may, by increasing γ if necessary, assume that $\gamma = \sup(W \cap \text{Ord})$ for a transitive node $W \in R$. By Claim 3.6 there is $\langle s', c'\rangle \leq \langle s, c\rangle$ with $W \in s'$, and $c'(M) = c(M)$. But now by the novelty jump condition in Definition 3.1, for any $\langle s^*, c^*\rangle \leq \langle s', c'\rangle$, if $P \in c^*(M) \cap R$ is above U (which is the largest node of $c(M) \cap R = c'(M) \cap R$), then $P \supseteq s' \cap R$, and hence in particular $W \in P$. It follows that $\alpha_P > \gamma > \beta$. Hence $\langle s', c'\rangle$ forces that the first element of C_α above α_U, which is either of the form α_P for P as above, or else equal to α_R, is larger than β. This contradicts the fact that $\langle s, c\rangle$ forces β to be a limit point of C_α. $\square$

Though strongly proper, the poset $\mathbb{S}$quare need not belong to the iterable classes for which Neeman [15] obtains forcing axioms that allow meeting ω_2 maximal antichains, nor to the classes of Asperó-Mota [1, 2], or their modification in Definition 3.26. Indeed the forcing axioms in [15] and [1, 2] are compatible with failure of $\square_{\omega_1}$. We continue to describe two variants of $\mathbb{S}$quare that do belong to iterable classes. The first variant adds a $\square_{\omega_1,\text{fin}}$ sequence, the second adds a $\square^{\text{ta}}_{\omega_1,\omega}$ sequence. The additional properties they enjoy that allow showing that they belong to certain iterable classes are proved in Lemmas 3.15 and 3.23. After defining the posets and deriving the lemmas, we will discuss the way they connect to iterable classes.

Definition 3.9. $\mathbb{S}\text{quare}_{\text{fin}} = \mathbb{S}\text{quare}_{\text{fin}}(\mathcal{S}, \mathcal{T})$ is the poset of pairs $\langle s, c\rangle$ where:

1. $s \in \mathbb{P}_{\text{side}}(\mathcal{S}, \mathcal{T})$.
2. c is a function with $\text{dom}(c) \subseteq \omega \times s$. We write $c_i(M)$ for $c(i, M)$. The sets $c_i(M)$ are distinct.
3. (Domain size) For each $M \in s$ there exists $n = n_c(M) \in [1, \omega)$ so that $\langle i, M\rangle \in \text{dom}(c)$ iff $i < n$. For transitive M, $n_c(M) = 1$.
4. (Linearity) $c_i(M)$ is an $\in$-linear set of countable nodes, contained in $s \cap M$.
5. (Separation) If Ot is cofinal in $\sup(M \cap \text{Ord})$, then Ot is cofinal in $\sup(\bar{M} \cap \text{Ord})$ for every $\bar{M} \in c_i(M)$. If Ot is bounded in $\sup(M \cap \text{Ord})$, then $c_i(M) = \emptyset$.
6. (Coherence) If $\bar{M} \in c_i(M)$ then there exists j so that $c_j(\bar{M}) = c_i(M) \cap \bar{M}$.
7. (Fullness) If $W \in M$ are nodes of s of transitive and countable type respectively, and $\sup(W \cap \text{Ord})$ is a limit point of Ot, then $M \cap W \in c_0(W)$.

The ordering on $\mathbb{S}\text{quare}_{\text{fin}}(\mathcal{C})$ is given by setting $\langle s^*, c^*\rangle \leq \langle s, c\rangle$ iff:

(i) $s^* \leq s$ in $\mathbb{P}_{\text{side}}$.

(ii) $n_{c^*}(M) = n_c(M)$ and $c^*_i(M) \cap s = c_i(M)$ for every $M \in s$ and each $i < n_c(M)$.

(iii) (Novelty jump) For $M \in s$, $i < n_c(M)$, $R \in c_i(M) \cup \{M\}$, and $P \in c^*_i(M) \cap R$, if $P \supseteq c_i(M) \cap R \neq \emptyset$ then $P \supseteq s \cap R$.

Note that the coherence witness j in condition (6) is necessarily unique, since by condition (2), the sets $c_j(\bar{M})$ are distinct.

Remark 3.10. Suppose $\langle s^*, c^* \rangle \leq \langle s, c \rangle \in \mathbb{S}\mathrm{quare}_{\mathrm{fin}}$. Let $M \in s$, $i < n_c(M)$, and $\bar{M} \in c_i(M)$. Let j witness the coherence condition for $c_i(M)$ at $\bar{M}$ in $\langle s, c \rangle$. Then the same j also witnesses the coherence condition for $c^*_i(M)$ at $\bar{M}$ in $\langle s^*, c^* \rangle$. To see this, note that for every $j' < n_c(\bar{M})$ other than j, $c_{j'}(\bar{M}) \neq c_j(\bar{M}) = c_i(M) \cap \bar{M}$, hence by condition (ii) of Definition 3.9, $c^*_{j'}(\bar{M}) \neq c^*_i(M) \cap \bar{M}$, and since $n_{c^*}(\bar{M}) = n_c(\bar{M})$ it follows that only j can witness coherence for $c^*_i(M)$ at $\bar{M}$ in $\langle s^*, c^* \rangle$.

Claim 3.11. *The ordering on* $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}$ *is transitive.*

Proof. Similar to the proof of Claim 3.3. We only note that here too the end novelty jump condition for $\langle s^{**}, c^{**} \rangle$ relative to $\langle s, c \rangle$ implies the full novelty jump condition, and that the proof of this uses the coherence preservation noted in Remark 3.10, to see that (in the notation of the novelty jump condition for $\langle s^{**}, c^{**} \rangle$ relative to $\langle s, c \rangle$) there is j so that both $c^{**}_i(M) \cap R = c^{**}_j(R)$ and $c_i(M) \cap R = c_j(R)$. $\square$

For $\langle s, c \rangle \in \mathbb{S}\mathrm{quare}_{\mathrm{fin}}$ and $Q \in s$ define $\mathrm{res}_Q(s, c)$ to be $\langle \mathrm{res}_Q(s), c' \rangle$, where $n_{c'}(M) = n_c(M)$ and $c'(i, M) = c(i, M) \cap Q$ for each $M \in \mathrm{res}_Q(s)$. Then $\mathrm{res}_Q(s, c)$ is a condition in $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}$, and $\langle s, c \rangle \leq \mathrm{res}_Q(s, c)$. The proof of this is similar to the corresponding proof for the poset $\mathbb{S}\mathrm{quare}$, noting in addition that for every $M \in \mathrm{res}_Q(s)$, the sets $c'(i, M)$ are distinct, because they are equal to the distinct sets $c(i, M)$ when M is countable, and because $n_{c'}(M) = 1$ when M is transitive.

Lemma 3.12. *Let* $\langle s, c \rangle \in \mathbb{S}\mathrm{quare}_{\mathrm{fin}}$*, let* $Q \in s$*, and let* $\langle t, d \rangle \in Q \cap \mathbb{S}\mathrm{quare}_{\mathrm{fin}}$ *extend* $\mathrm{res}_Q(s, c)$*. Then* $\langle s, c \rangle$ *and* $\langle t, d \rangle$ *are compatible. Moreover there is a condition* $\langle r, b \rangle$ *witnessing this so that* $r = s \cup t$ *if* Q *is transitive, and* r *is the closure of* $s \cup t$ *under intersections if* Q *is countable.*

Proof. Similar to the proof of Lemma 3.4. We only note the key new points.

When defining $b(Q \cap W)$ in the proof of Lemma 3.4, for $Q \cap W$ a bottom node of a residue gap of s in Q so that $\sup(Q \cap W \cap \mathrm{Ord}) \in \mathrm{Limit}(Ot)$, we set $b(Q \cap W) = d(W)$. Here we set $b_j(Q \cap W) = d_0(W)$ for the unique j so that $c_j(Q \cap W) = c_0(W) \cap Q$. Such j is given by the coherence and fullness conditions. For $i \neq j$ we define $b_i(Q \cap W)$ following the procedure in the next paragraph.

Several times in the proof of Lemma 3.4, when defining $b(M)$, we looked for the largest node $P \in c(M)$ which belongs to Q or is a bottom node of a residue gap of s in Q, and set $b(M) = c(M) \cup b(P)$. Here, when defining $b_i(M)$ in similar situations (and when defining $b_i(Q \cap W)$ for i not equal to the coherence witness used in the previous paragraph), we look for the largest node $P \in c_i(M)$ which belongs to Q or is a bottom node of a residue gap of s in Q, and set $b_i(M) = c_i(M) \cup b_l(P)$ for the unique l so that $c_l(P) = c_i(M) \cap P$. Such l exists by the coherence condition.

With these modifications, and with the obvious adaptations from the context of Definition 3.1 to the current context of Definition 3.9, the resulting pair $\langle r, b \rangle$ is an element of $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}$ extending both $\langle s, c \rangle$ and $\langle t, d \rangle$. The arguments involved in proving this are largely similar to the arguments in the proof of Lemma 3.4. The specific assignments mentioned in the previous paragraphs are used in verifying coherence. One new condition here that has no parallel in Lemma 3.4 is that for each $M \in r$ the sets $b_i(M)$, $i < n_b(M)$, must be distinct. This condition holds for $\langle r, b\rangle$ at transitive M and at M in tacked-on intervals since then $n_b(M) = 1$. For countable $M \in s$ it is inherited from the same condition for $\langle s, c\rangle$ since $b_i(M) \cap s = c_i(M)$. For countable $M \in t$ it is similarly inherited from the same condition for $\langle t, d\rangle$. □

Corollary 3.13. $\mathbb{S}\text{quare}_{\text{fin}}$ *is strongly proper for* $\mathcal{S} \cup \mathcal{T}$*, and hence preserves* ω_1 *and* ω_2*. Assuming conditions (ST6)–(ST9), for every* $\langle t, d\rangle \in \mathbb{S}\text{quare}_{\text{fin}}$ *and every* $W \in \mathcal{T}$ *there is* $\langle t^*, d^*\rangle \leq \langle t, d\rangle$ *with* $W \in t^*$*, and* $\mathbb{S}\text{quare}_{\text{fin}}$ *is* ω_2*-c.c.*

Proof. Similar to Corollary 3.5, Claim 3.6, and Corollary 3.7, but using Lemma 3.12. □

Lemma 3.14. *Forcing with* $\mathbb{S}\text{quare}_{\text{fin}}$ *adds a* $\square_{\omega_1,\text{fin}}$ *sequence.*

Proof. Similar to Lemma 3.8, but defining $C_{i,\alpha} = \{\alpha_P \mid (\exists\langle s, c\rangle \in G)(M \in s \wedge P \in c_i(M))\}$, for $\alpha = \alpha_M = \sup(M \cap \text{Ord})$ and $i < n_c(M)$. ($n_c(M)$ is the same for all $\langle s, c\rangle \in G$ with $M \in s$.) By Remark 3.10 the sequence defined this way satisfies coherence. It can be converted to a $\square_{\omega_1,\text{fin}}$ sequence as in the proof of Lemma 3.8. □

Lemma 3.15. *Let* $s \in \mathbb{P}_{\text{side}}$*, let* $Q \in s$*, and let* $\langle t, d\rangle \in Q \cap \mathbb{S}\text{quare}_{\text{fin}}$ *with* $t \supseteq \text{res}_Q(s)$*. Suppose* $\text{res}_Q(s)$ *has no countable nodes. Then there is* $\langle r, b\rangle \leq \langle t, d\rangle$ *with* $r \supseteq s$*.*

Proof. It is enough to show that there is c so that $\langle s, c\rangle \in \mathbb{S}\text{quare}_{\text{fin}}$, $n_c(M) = n_d(M)$, and $c_i(M) \cap Q = \emptyset$, for all $M \in \text{res}_Q(s)$. Then the facts that $t \supseteq \text{res}_Q(s)$ and $\text{res}_Q(s)$ has no countable nodes imply that $\langle t, d\rangle \leq \text{res}_Q(s, c)$ (note in particular that the novelty jump condition holds trivially since, letting $\langle s', c'\rangle = \text{res}_Q(s, c)$, there are no instances where $c'_i(M) \cap R \neq \emptyset$), and the existence of $\langle r, b\rangle \leq \langle t, d\rangle$ with $r \supseteq s$ follows by Lemma 3.12.

Since $\text{res}_Q(s)$ has no countable nodes, the requirements that $n_c(M) = n_d(M)$ and $c_i(M) \cap Q = \emptyset$ trivialize: $n_c(M) = n_d(M) = 1$ for all $M \in \text{res}_Q(s)$ by condition (3) of Definition 3.9 since all $M \in \text{res}_Q(s)$ are transitive, and all nodes in $c_i(M)$ are countable nodes of s, hence $c_i(M) \cap Q = \emptyset$. So it is enough to simply find some c so that $\langle s, c\rangle \in \mathbb{S}\text{quare}_{\text{fin}}$.

Let u consist of $M \in s$ so that $\sup(M \cap \text{Ord}) \in \text{Limit}(Ot)$. For $M \in s - u$ set $n_c(M) = 1$ and $c(M) = \emptyset$. For transitive $W \in u$, let $M_0, \ldots, M_l$ list all the countable nodes of s above W, up to and not including the next transitive node of s if there is one. Set $n_c(W) = 1$ and $c_0(W) = \{M_0 \cap W, \ldots, M_l \cap W\}$. The nodes $M_0 \cap W, \ldots, M_l \cap W$ all belong to u, and this ensures that separation holds at W. For each countable $M \in u$, let $n_c(M)$ be the number of $\in$-linear sets of countable nodes from $u \cap M$, and let $c_i(M)$, $i < n_c(M)$, enumerate these sets without repetitions. The domain size, linearity, separation, and fullness conditions of Definition 3.9 are clear for $\langle s, c\rangle$ with these assignments. Coherence holds because $c_i(M) \cap \bar{M}$ for $\bar{M} \in c_i(M)$ is an $\in$-linear set of countable nodes from $u \cap \bar{M}$, hence equal to one of the sets $c_j(\bar{M})$. □

Remark 3.16. Lemma 3.15 fails without an assumption limiting the nodes in $\text{res}_Q(s)$. For example one can arrange that there are countable nodes $M_1 \in M_2$ in $\text{res}_Q(s)$ so that in s there is transitive W with $\sup(W \cap \text{Ord}) \in \text{Limit}(Ot)$, and there are countable $M_1^* \in M_2^*$ with $W \in M_i^*$ and $M_i^* \cap W = M_i$, but there are no such W, M_1^*, M_2^* in $\mathrm{res}_Q(s)$. One can then construct $\langle t, d\rangle \in \mathbb{S}\mathrm{quare}_{\mathrm{fin}}$, in Q and with $t \supseteq \mathrm{res}_Q(s)$, so that for every j, $M_1 \notin d_j(M_2)$. But for any $\langle r, b\rangle \in \mathbb{S}\mathrm{quare}_{\mathrm{fin}}$ with $r \supseteq s$, by fullness at W and coherence for $b_0(W)$ at M_2, there must be j so that $M_1 \in b_j(M_2)$. So $\langle r, b\rangle$ cannot extend $\langle t, d\rangle$.

Remark 3.17. The parallel of Lemma 3.15 for the poset $\mathbb{S}\mathrm{quare}$, as opposed to $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}$, can fail. To see this, construct a side condition s that has transitive nodes N, W with $\sup(W \cap \mathrm{Ord}) \in \mathrm{Limit}(Ot)$, and countable nodes $M_0 \in M_1$ and $P_0 \in P_1$ with $N \in M_0$ and $W \in P_0$, so that $M_1 \cap N = P_1 \cap W$, and linearity fails for $M_0 \cap N$ and $P_0 \cap W$, meaning they are not equal, and neither is an element of the other. (Such side conditions can be constructed in some situations, see for example Remark 3.46. They cannot be constructed, at least not with arbitrary degree of elementarity, in models that satisfy $\square_{\omega_1}$, by an argument as in the proof of Claim 3.40.) Then there is no $\langle r, b\rangle \in \mathbb{S}\mathrm{quare}$ so that $r \supseteq s$, since coherence and fullness would combine to require that both $M_0 \cap N$ and $P_0 \cap W$ belong to $b(M_1 \cap N)$, contradicting linearity.

Definition 3.18. $\mathbb{S}\mathrm{quare}_{\mathrm{ta}} = \mathbb{S}\mathrm{quare}_{\mathrm{ta}}(\mathcal{S}, \mathcal{T})$ is the poset of pairs $\langle s, c\rangle$ where:

(1) $s \in \mathbb{P}_{\mathrm{side}}(\mathcal{S}, \mathcal{T})$.
(2) c is a finite function with $\mathrm{dom}(c) \subseteq \omega \times s$. We write $c_i(M)$ for $c(i, M)$. The sets $c_i(M)$ are distinct.
(3) (Domain size) For transitive $W \in s$, $\langle i, W\rangle \in \mathrm{dom}(c)$ iff $i = 0$.
(4) (Linearity) $c_i(M)$ is an $\in$-linear set of countable nodes, contained in $s \cap M$.
(5) (Separation) If Ot is cofinal in $\sup(M \cap \mathrm{Ord})$, then Ot is cofinal in $\sup(\bar{M} \cap \mathrm{Ord})$ for every $\bar{M} \in c_i(M)$. If Ot is bounded in $\sup(M \cap \mathrm{Ord})$, then $c_i(M) = \emptyset$.
(6) (Coherence) If $\bar{M} \in c_i(M)$ then there exists j so that $c_j(\bar{M}) = c_i(M) \cap \bar{M}$.
(7) (Fullness) If $W \in M$ are nodes of s of transitive and countable type respectively, and $\sup(W \cap \mathrm{Ord})$ is a limit point of Ot, then $M \cap W \in c_0(W)$.

The ordering on $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}$ is given by setting $\langle s^*, c^*\rangle \leq \langle s, c\rangle$ iff:

(i) $s^* \leq s$ in $\mathbb{P}_{\mathrm{side}}$.
(ii) $\langle i, M\rangle \in \mathrm{dom}(c^*)$ and $c_i^*(M) \cap s = c_i(M)$ for every $\langle i, M\rangle \in \mathrm{dom}(c)$.
(iii) (Novelty jump) For $\langle i, M\rangle \in \mathrm{dom}(c)$, $R \in c_i(M) \cup \{M\}$, and $P \in c_i^*(M) \cap R$, if $P \supseteq c_i(M) \cap R \neq \emptyset$ then $P \supseteq s \cap R$.
(iv) (Coherence preservation) If j witnesses the coherence condition for $c_i(M)$ at $\bar{M}$ in $\langle s, c\rangle$, then the same j also witnesses the coherence condition for $c_i^*(M)$ at $\bar{M}$ in $\langle s^*, c^*\rangle$.
(v) (Tail agreement) If M is countable, $\langle i, M\rangle, \langle j, M\rangle \in \mathrm{dom}(c)$, and $c_j(M) \neq \emptyset$, then every R in the tail-end of $c_j^*(M)$ above $c_j(M)$ belongs to the tail-end of $c_i^*(M)$ above $c_i(M)$, meaning precisely that if $R \in c_j^*(M) \wedge R \supseteq c_j(M)$ then $R \in c_i^*(M) \wedge R \supseteq c_i(M)$.

In contrast with Definition 3.9, for the ordering relation on $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}$ we allow the domain size to increase, meaning that we allow $\{i \mid \langle i, M\rangle \in \mathrm{dom}(c^*)\} \supsetneq \{i \mid \langle i, M\rangle \in \mathrm{dom}(c)\}$ for $M \in s$. One consequence of this is that the proof of coherence preservation for $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}$ in Remark 3.10 does not work for $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}$, and instead we added coherence preservation as an explicit condition. We also added the new condition of tail agreement. We require it only for countable M, but it holds also for transitive M, for the trivial reason that, by the domain size condition, $\langle i, M\rangle, \langle j, M\rangle \in \mathrm{dom}(c) \to i = j = 0$ for transitive M.

Claim 3.19. *The ordering on* $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}$ *is transitive.*

Proof. Let $\langle s^{**}, c^{**}\rangle \le \langle s^*, c^*\rangle \le \langle s, c\rangle$. That conditions (i)–(iii) of Definition 3.18 hold for $\langle s^{**}, c^{**}\rangle$ and $\langle s, c\rangle$ can be proved as in Claim 3.3. Condition (iv) is clearly transitive. We prove that condition (v) holds.

Fix $\langle i, M\rangle, \langle j, M\rangle \in \mathrm{dom}(c)$. Let $R \in c_j^{**}(M)$ with $R \supseteq c_j(M) \neq \emptyset$.

If $R \supseteq c_j^*(M)$ then by condition (v) for $\langle s^{**}, c^{**}\rangle$ and $\langle s^*, c^*\rangle$ we have $R \in c_i^{**}(M)$ and $R \supseteq c_i^*(M)$. The latter implies that $R \supseteq c_i(M)$.

If $R \in c_j^*(M)$ then by condition (v) for $\langle s^*, c^*\rangle$ and $\langle s, c\rangle$ we have $R \in c_i^*(M)$, which implies $R \in c_i^{**}(M)$, and $R \supseteq c_i(M)$.

Suppose $R \not\supseteq c_j^*(M)$ and $R \notin c_j^*(M)$. By linearity of $c_j^{**}(M)$ it follows that there is $U \in c_j^*(M)$ with $R \in U$. Fix the least such, so that $R \supseteq c_j^*(M) \cap U$. Since $R \supseteq c_j(M)$ we have $U \supseteq c_j(M)$. Hence by condition (v) for $\langle s^*, c^*\rangle$ and $\langle s, c\rangle$, $U \in c_i^*(M)$ and $U \supseteq c_i(M)$. By coherence there is l so that $c_l^*(U) = c_j^*(M) \cap U$, and k so that $c_k^*(U) = c_i^*(M) \cap U$. By the coherence preservation condition we have also $c_l^{**}(U) = c_j^{**}(M) \cap U$ and $c_k^{**}(U) = c_i^{**}(M) \cap U$. By condition (v) for $\langle s^{**}, c^{**}\rangle$ and $\langle s^*, c^*\rangle$ at U we have $R \in c_k^{**}(U)$ and $R \supseteq c_k^*(U)$. It follows that $R \in c_i^{**}(M)$ and $R \supseteq c_i^*(M) \cap U \supseteq c_i(M)$. ☐

For $\langle s, c\rangle \in \mathbb{S}\mathrm{quare}_{\mathrm{ta}}$ and $Q \in s$ define $\mathrm{res}_Q(s, c)$ to be $\langle \mathrm{res}_Q(s), c'\rangle$ where $\mathrm{dom}(c') = \mathrm{dom}(c) \cap Q$ and $c'(i, M) = c(i, M) \cap Q$. Then $\mathrm{res}_Q(s, c)$ is a condition in $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}$, and $\langle s, c\rangle \le \mathrm{res}_Q(s, c)$, using the fact that $c'(i, M) = c(i, M)$ for countable M to see that the tail agreement condition holds trivially.

Lemma 3.20. *Let* $\langle s, c\rangle \in \mathbb{S}\mathrm{quare}_{\mathrm{ta}}$*, let* $Q \in s$*, and let* $\langle t, d\rangle \in Q \cap \mathbb{S}\mathrm{quare}_{\mathrm{ta}}$ *extend* $\mathrm{res}_Q(s, c)$*. Then* $\langle s, c\rangle$ *and* $\langle t, d\rangle$ *are compatible. Moreover there is a condition* $\langle r, b\rangle$ *witnessing this so that* $r = s \cup t$ *if* Q *is transitive, and* r *is the closure of* $s \cup t$ *under intersections if* Q *is countable.*

Proof. For the most part this is similar to the proofs of Lemmas 3.4 and 3.12. The straightforward adaptation of the constructions there produces $\langle r, b\rangle \in \mathbb{S}\mathrm{quare}_{\mathrm{ta}}$ which satisfies conditions (i)–(iv) of Definition 3.18 relative to $\langle s, c\rangle$ and $\langle t, d\rangle$. The tail agreement condition (v) is also satisfied in all but one of the cases of the construction, for the trivial reasons that it is either vacuous—in many of the cases there are no new nodes of $b_i(M)$ above nodes of $c_i(M)$ or respectively $d_i(M)$—or directly inherited from the same condition for $\langle s, c\rangle$ and $\langle t, d\rangle$.

The one exception, and the one case where we have to change the construction from the previous lemmas, is the case of bottom nodes of residue gaps $[Q \cap W, W)$ of s in Q with $\sup(Q \cap W \cap \mathrm{Ord}) \in \mathrm{Limit}(Ot)$ and $c_0(W) \cap Q \neq \emptyset$. Let j be such that $c_j(Q \cap W) = c_0(W) \cap Q$. Such j exists by coherence and fullness, and it is unique since the sets $c_i(Q \cap W)$ are distinct. In the proof of Lemma 3.12 we set $b_j(Q \cap W) = d_0(W)$, and we do the same here.

For $i \neq j$ let $\bar{b}_i(Q \cap W)$ be the value defined for $b_i(Q \cap W)$ in the proof of Lemma 3.12. Precisely this is $c_i(Q \cap W) \cup d_l(M)$ where M is the largest node of $c_i(Q \cap W)$ and l is such that $c_l(M) = c_i(Q \cap W) \cap M$. (If $c_i(Q \cap W) = \emptyset$ we take $\bar{b}_i(Q \cap W) = \emptyset$ too.) We cannot set $b_i(Q \cap W)$ to the same value here, as this would violate the tail agreement condition at $Q \cap W$.

Let A list the elements of $b_j(Q \cap W) = d_0(W)$ above the largest element of $c_j(Q \cap W) = c_0(W) \cap Q$. By the novelty jump condition, $P \in A \to P \supseteq s \cap Q \cap W$. Since the largest node of $\bar{b}_i(Q \cap W)$ is a node of $c_i(Q \cap W)$, hence an element of $s \cap Q \cap W$, it follows that $\bar{b}_i \cup A$ is $\in$-linear for all i, with A forming its tail-end. Set $b_i(Q \cap W) = \bar{b}_i(Q \cap W) \cup A$ for $i \neq j$, instead of $b_i(Q \cap W) = \bar{b}_i(Q \cap W)$. It is clear that with this new assignment the tail agreement condition holds at $Q \cap W$.

For each $P \in A$ and each $i \neq j$ we can, by increasing the domain of b if necessary, arrange that there is some k so that $b_k(P) = b_i(Q \cap W) \cap P = \bar{b}_i(Q \cap W) \cup A \cap P$. We can do this while maintaining the fact that $b_m(P) = d_m(P)$ for m so that $\langle m, P\rangle \in \text{dom}(d)$. It is then easy to check that coherence holds for these assignments and the assignments in the previous paragraph. This completes the changes to the construction for the case of bottom nodes of residue gaps. The other aspects of the construction remain the same. $\square$

Remark 3.21. For the assignments in the final paragraph of the proof of Lemma 3.20 it is important that we are allowed to expand the domain of b, meaning that for $P \in t$ we are allowed to have m so that $\langle m, P\rangle \in \text{dom}(b)$ even though $\langle m, P\rangle \notin \text{dom}(d)$. It is for this reason that when requiring tail agreement we have to change from $\square_{\omega_1,\text{fin}}$ sequences to $\square_{\omega_1,\omega}$ sequences. The parallel of Lemma 3.20 would fail if we required $\{m \mid \langle m, P\rangle \in \text{dom}(b)\} = \{m \mid \langle m, P\rangle \in \text{dom}(d)\}$.

Corollary 3.22. *$\mathbb{S}\text{quare}_{\text{ta}}$ is strongly proper for $\mathcal{S} \cup \mathcal{T}$, and hence preserves ω_1 and ω_2. Assuming conditions (ST6)–(ST9), for every $\langle t, d\rangle \in \mathbb{S}\text{quare}_{\text{ta}}$ and every $W \in \mathcal{T}$ there is $\langle t^*, d^*\rangle \leq \langle t, d\rangle$ with $W \in t^*$, and $\mathbb{S}\text{quare}_{\text{ta}}$ is ω_2-c.c.*

Proof. Similar to the combined results given by Corollary 3.5, Claim 3.6, and Corollary 3.7, but using Lemma 3.20. $\square$

The next lemma establishes a condition similar to the one in Lemma 3.15, but stronger, dropping the requirement that $\text{res}_Q(s)$ has no countable nodes from the hypothesis.

Lemma 3.23. *Let $s \in \mathbb{P}_{\text{side}}$, let $Q \in s$, and let $\langle t, d\rangle \in Q \cap \mathbb{S}\text{quare}_{\text{ta}}$ with $t \supseteq \text{res}_Q(s)$. Then there is $\langle r, b\rangle \leq \langle t, d\rangle$ with $r \supseteq s$.*

Proof. Let r be the closure of $s \cup t$ under intersections, define $b(\langle i, M\rangle) = d(\langle i, M\rangle)$ for $\langle i, M\rangle \in \text{dom}(d)$ with M countable, define $b(\langle 0, W\rangle) = d(\langle 0, W\rangle) \cup F_W$ for transitive $W \in t$ with $\sup(W \cap \text{Ord}) \in \text{Limit}(Ot)$, where F_W consists of the nodes in $\{M^* \cap W \mid M^* \in r \wedge W \in M^*\}$ that do not belong to Q, define $b(\langle 0, W\rangle)$ so as to satisfy fullness for transitive $W \in s - t$ with $\sup(W \cap \text{Ord}) \in \text{Limit}(Ot)$, define $b_0(M) = \emptyset$ if Ot is bounded in $\sup(M \cap \text{Ord})$, and finally, expanding the domain of b as necessary, make sure that for every countable $M \in r$ with $\sup(M \cap \text{Ord}) \in \text{Limit}(Ot)$, every $\in$-linear subset of $M \cap \{P \in r \mid P$ is countable and $\sup(P \cap \text{Ord}) \in \text{Limit}(Ot)\}$ is equal to $b(\langle i, M\rangle)$ for a unique i. The last part uses the fact that $\{i \mid \langle i, M\rangle \in \text{dom}(b)\}$ is allowed to be larger than $\{i \mid \langle i, M\rangle \in \text{dom}(d)\}$ for countable M, and would not work with $\mathbb{S}\text{quare}_{\text{fin}}$. $\square$

Lemma 3.24. *Forcing with $\mathbb{S}\text{quare}_{\text{ta}}$ adds a $\square^{\text{ta}}_{\omega_1,\omega}$ sequence.*

Proof. Let G be generic for $\mathbb{S}\text{quare}_{\text{ta}}$ and let $X = \{\alpha_M \mid M$ occurs in $G\}$, where $\alpha_M = \sup(M \cap \text{Ord})$. For $\alpha = \alpha_M \in X \cap \text{Limit}(Ot)$, and $i < \omega$, set $C_{\alpha,i} = \{\alpha_P \mid (\exists\langle s, c\rangle \in G)(M \in s \wedge P \in c_i(M))\}$. For $\alpha \in \text{Limit}(Ot) - X$ set $C_{\alpha,i} = \emptyset$. By genericity, $C_{\alpha,i}$ is defined for all $i < \omega$ if $\text{cof}(\alpha) = \omega$, and for $i = 0$ if $\text{cof}(\alpha) = \omega_1$.

As in the proof of Lemma 3.8, the sequence $\langle C_{\alpha,i} \mid \alpha \in \mathrm{Limit}(Ot)\rangle$ is a coherent sequence of closed sets with the property that $C_{\alpha,0}$ is unbounded in α if α is a limit point of $\mathrm{Limit}(Ot)$ of cofinality ω_1.

In addition the current sequence has the following property of tail agreement: For every $\alpha \in \mathrm{Limit}(Ot)$ of countable cofinality, and every $i, j < \omega$ with $C_{\alpha,i}$ and $C_{\alpha,j}$ non-empty, there is $\beta < \alpha$ so that $C_{\alpha,i} - \beta = C_{\alpha,j} - \beta$. To see this, fix M so that $\alpha = \alpha_M$ (the case $\alpha \notin X$ is clear) and fix $\langle s, c\rangle \in G$ with $\langle i, M\rangle, \langle j, M\rangle \in \mathrm{dom}(c)$ and $c_i(M), c_j(M)$ non-empty. Let $\beta = \max\{\alpha_U \mid U \in c_i(M) \cup c_j(M)\} + 1$. Then by the tail agreement condition, for every $\langle t, d\rangle \leq \langle s, c\rangle$ and every P with $\sup(P \cap \mathrm{Ord}) \geq \beta$ we have $P \in d_i(M) \leftrightarrow P \in d_j(M)$. So $\sup(P \cap \mathrm{Ord}) \in C_{\alpha,i} \leftrightarrow \sup(P \cap \mathrm{Ord}) \in C_{\alpha,j}$.

The sequence $\langle C_{\alpha,i} \mid \alpha \in \mathrm{Limit}(Ot)\rangle$ can be converted to a $\Box^{\mathrm{ta}}_{\omega_1,\omega}$ sequence $\langle D_\alpha \mid \alpha < \omega_2\rangle$ using an argument as in the proof of Lemma 3.8, with slight additional care to maintain the tail agreement property. Let $\varphi \colon \omega_2 \to \mathrm{Limit}(Ot)$ be an order preserving bijection. For α and i so that $C_{\varphi(\alpha),i}$ is cofinal in $\varphi(\alpha)$, set $D_{\alpha,i}$ equal to the preimage of $C_{\varphi(\alpha),i}$ under φ. For α so that for all i, $C_{\varphi(\alpha),i}$ is bounded in $\varphi(\alpha)$, pick a cofinal subset of α of order type $\leq \omega$ and set $D_{\alpha,i}$, for all i, equal to this subset. These assignments may leave $D_{\alpha,l}$ undefined for some (but not all) l. These l may be ignored, or defined values of $D_{\alpha,i}$ can be duplicated to $D_{\alpha,l}$. □

The principle $\Box^{\mathrm{ta}}_{\omega_1,\omega}$ can serve as a replacement for the stronger $\Box_{\omega_1}$ in at least some applications. The following example of this is relevant for some results in Section 4. A sequence $\langle \bar{x}^\alpha \mid \alpha < \omega_2\rangle$ is an *ascent path* in a tree T of height ω_2 if for each α, $\bar{x}^\alpha$ is a sequence $\langle x^\alpha_n \mid n < \omega\rangle$ of distinct elements of the αth level of T, and for every $\alpha < \beta < \omega_2$, for all but finitely many n, $x^\alpha_n <_T x^\beta_n$. The notion is due to Laver. Shelah-Stanley [20] constructed an ω_2-Aronszajn tree with an ascent path, assuming $\Box_{\omega_1}$, but $\Box^{\mathrm{ta}}_{\omega_1,\omega}$ suffices:

Fact 3.25 (By Shelah-Stanley [20])**.** The existence of a $\Box^{\mathrm{ta}}_{\omega_1,\omega}$ sequence implies the existence of an ω_2 Aronszajn tree with an ascent path.

Sketch of proof. This is a modification of the proof in Section 1 of [20]. We briefly sketch the necessary changes. First note that a given $\Box^{\mathrm{ta}}_{\omega_1,\omega}$ sequence $\langle C_{\delta,k} \mid \delta < \omega_2,\, k < \omega\rangle$ can be adjusted to make sure that the sets $C'_{\delta,k} = \mathrm{Limit}(C_{\delta,k}) - \{\delta\}$ are all disjoint from a fixed non-reflecting stationary set S, by discarding from each $C_{\delta,k}$ all points γ so that some tail of a $C_{\delta,k} \cap \gamma$ has ordertype below ν, for some fixed large enough countable ν (and making standard adjustments to any $C_{\delta,k}$ which become bounded as a result). Then modify the argument in Section 1 of [20], to construct at each level $i < \omega_2$ not one candidate $\bar{x}^i = \langle x^i_n \mid n < \omega\rangle$ for the element of the ascent path on that level, but ω candidates $\bar{x}^{i,k} = \langle x^{i,k}_n \mid n < \omega\rangle$, with $\bar{x}^{i,k_1} = \bar{x}^{i,k_2}$ if $C_{i,k_1} = C_{i,k_2}$, and with tail agreement, meaning that for every i, k_1, k_2, for all sufficiently large n, $x^{i,k_1}_n = x^{i,k_2}_n$. Condition (1) of the construction of [20, Section 1] adapts naturally to this, modifying it to require that if $i < j$ then for all k, l, for all sufficiently large n, $x^{i,k}_n \lhd x^{j,l}_n \wedge \{f(z) \mid z \in (x^{i,k}_n, x^{j,l}_n]_\lhd\} \cap \lambda i = \emptyset$. (We are using the notation of [20].) Conditions (2)–(5) remain unaffected. Instead of one function h we now construct ω functions h^k. For each $j \notin S$ and each $k < \omega$, $h^k(x, j)$ is defined on all $x \in T_i$ for $i \in C'_{j,k}$, and assigns to each such x an element of T_j. For each j, the functions $h^k(x, j)$ for $k < \omega$ satisfy the following tail agreement, which matches the tail agreement of $C_{j,k}$: if i is large enough that $C_{j,k_1} - i = C_{j,k_2} - i$, then $h^{k_1}(x, j)$ and $h^{k_2}(x, j)$ agree on all $x \in T_i$. Condition (7) of the construction remains essentially as it is, applying individually to each $C'_{j,k}$. Conditions (6) and (8) are modified to reflect the coherence of the $\square^{\mathrm{ta}}_{\omega_1,\omega}$ sequence. In the case of condition (6) this means that the conclusion of the condition, for $i_1 < i_2$ both in $C'_{j,k}$ and $x \in T_{i_1}$, is that $h^k(x,j) = h^k(h^l(x,i_2),j)$ for all l so that $C_{i_2,l} = C_{j,k} \cap i_2$. (At least one such l exists by coherence.) In the case of condition (8) this means that the conclusion, for $i \in C'_{j,k}$, is changed to $h^k(x^{i,l}_n, j) = x^{j,k}_n$ for each l so that $C_{i,l} = C_{j,k} \cap i$. The construction itself proceeds largely as in [20], with some obvious adaptations to fit the changes above. The various tail agreement conditions above are important for the modified condition (1), which in turn is important in case 1 of the construction, namely the case that $\alpha \notin S$ and $C'_{\alpha,k}$ is bounded in α for some (equivalently for all) $k < \omega$. Finally, once the construction is over, each of the sequences $\langle \bar{x}^{\alpha,k_\alpha} \mid \alpha < \omega_2 \rangle$, for any choice of map $\alpha \mapsto k_\alpha$, gives an ascent path in the tree T. The construction ensures that T is Aronszajn, and in fact weakly special on $T|S$. $\square$

The property of $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}$ given by Lemma 3.23 can be viewed as a form of properness for models of two sizes. But rather than state that every condition in $\mathbb{S}\mathrm{quare}_{\mathrm{ta}} \cap Q$ extends to a master condition for Q (in fact a strong master condition), it strengthens the conclusion by obtaining a master condition for all nodes in a side condition s that has Q as an element, while in return requiring in the hypothesis that the side part of the starting condition in Q contains $\mathrm{res}_Q(s)$. This property allows showing that $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}$ belongs to the classes of posets that can be iterated using side conditions with nodes of three sizes, developed in Neeman [15]. Since the corresponding forcing axioms there allow meeting ω_2 dense sets, it follows that they imply $\square^{\mathrm{ta}}_{\omega_1,\omega}$. The property of $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}$ given by Lemma 3.15 is weaker, placing an additional constraint in the hypothesis, that $\mathrm{res}_Q(s)$ has no countable nodes. This is still sufficient to show that $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}$ belongs to some (though not all) of the classes developed in [15], and hence that $\square_{\omega_1,\mathrm{fin}}$ follows from the corresponding forcing axioms.

More precise details on this can be found in [15]. Here instead we will use the properties of $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}$ and $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}$ given by Lemmas 3.15 and 3.23 to show that the posets belong to a variant of the class of $\aleph_{1.5}$-.c.c. posets developed by Asperó-Mota [1]. The following definition gives a simple adaptation of their class; the special case where $U = \mathcal{P}_\omega(\omega_2)$, or equivalently the notion obtained by removing the requirement $N_i \cap \omega_2 \in U$ from the hypothesis, is precisely Definition 1.1 of [1].

Definition 3.26. Let $U \subseteq \mathcal{P}_\omega(\omega_2)$. A poset $\mathbb{P}$ has the $\aleph_{1.5}$*-chain condition relative to* U if for every regular θ large enough that $\mathbb{P} \in H(\theta)$, there is a club $D \subseteq \mathcal{P}_\omega(H(\theta))$, so that for every finite $\{N_i \mid i < n\} \subseteq D$ with $(\forall i) N_i \cap \omega_2 \in U$, for every $j < n$ with $\sup(N_j \cap \omega_1)$ minimal in the set $\{\sup(N_i \cap \omega_1) \mid i < n\}$, and for every $p \in N_j$, there exists $q \leq p$ which is a master condition for each of the models N_i.

Definition 3.27. The forcing axiom $\mathsf{MA}^{1.5}_{<\lambda}(U)$ is the statement that for every $\mathbb{P}$ with the $\aleph_{1.5}$-chain condition relative to U, and every collection $\mathcal{A}$ of fewer than λ dense subsets of $\mathbb{P}$, there is a filter on $\mathbb{P}$ which meets every set in $\mathcal{A}$. The forcing axiom $\mathsf{MA}^{1.5}_{\lambda}(U)$ is defined similarly, allowing λ dense sets. The special case when $U = \mathcal{P}_\omega(\omega_2)$ gives the axioms $\mathsf{MA}^{1.5}_{<\lambda}$ and $\mathsf{MA}^{1.5}_{\lambda}$ of Asperó-Mota [1, Definition 1.5].

Fact 3.28 (By Asperó-Mota [1])**.** Suppose $2^{\aleph_0} = \aleph_1$, $\kappa > \omega_2$ is regular, $(\forall \mu < \kappa)\mu^{\aleph_0} < \kappa$, and $\diamondsuit(\{\alpha < \kappa \mid \mathrm{cof}(\alpha) \geq \omega_1\})$ holds. Let $U \subseteq \mathcal{P}_\omega(\omega_2)$ be stationary.

Then there exists a poset $\mathbb{P}$, which is ω_2-c.c. and proper relative to the class of countable N so that $N \cap \omega_2 \in U$ (hence in particular cardinal preserving), and forces that $2^{\aleph_0} = \kappa$ and that $\mathsf{MA}^{1.5}_{<\kappa}(U)$ holds.

Sketch of proof. Adapt the proof in Section 2 of [1], restricting the countable models throughout the proof to ones whose intersection with ω_2 belongs to U. The proof involves fixing at the start a predicate T on $H(\theta)$ and working with models which are elementary relative to T. Strengthen T to make sure that it codes U. The various amalgamation lemmas then hold when restricted to models N with $N \cap \omega_2 \in U$, and the other proofs adapt easily to the same restriction. The adaption of Lemma 2.22 of [1] no longer establishes properness, but only properness relative to structures whose intersection with ω_2 belongs to U. Since U is stationary this is enough for ω_1 to be preserved. $\square$

For any $\lambda \geq \omega_1$, the axiom $\mathsf{MA}^{1.5}_{\lambda}(U)$ implies that for every $\alpha < \omega_2$, the set of restrictions of elements of U to α contains a club, meaning that there is a club in $\mathcal{P}_\omega(\alpha)$ which is contained in $\{x \cap \alpha \mid x \in U\}$. The reason is that the forcing to add such a club with finite conditions is $\aleph_{1.5}$-c.c. relative to U. In particular the axiom implies that U is stationary. But the axiom does not imply that U contains a club in $\mathcal{P}_\omega(\omega_2)$. For example, we will below obtain the consistency of the axiom with U of cardinality ω_2 in a model where the continuum is greater than ω_2. Since every club subset of $\mathcal{P}_\omega(\omega_2)$ has cardinality at least the continuum, this combination implies that U does not contain a club.

Definition 3.29. Let $\psi \colon \omega_2 \to \mathcal{P}_\omega(\omega_2)$. A set $U \subseteq \mathcal{P}_\omega(\omega_2)$ is *coherent* relative to ψ just in case that there is a club $C \subseteq \omega_2$, whose successor points all have cofinality ω_1, so that:

(1) For every $x, y \in U$, if $\sup(x) = \sup(y)$ then $x = y$.
(2) For every $x, y \in U$, if $\sup(x) < \sup(y)$ and $C \cap (\sup(x), \sup(y)) = \emptyset$, then $x \in \psi'' y$.
(3) For every $x \in U$, the set $\{\gamma \in C \cap \sup(x) \mid \mathrm{cof}(\gamma) = \omega_1 \text{ and } \gamma \in x\}$ is cofinal in the set $\{\gamma \in C \cap \sup(x) \mid \mathrm{cof}(\gamma) = \omega_1\}$.
(4) For every $x \in U$ and every $\alpha \in C$ of cofinality ω_1, $x \cap \alpha \in U$.

We say that U is coherent if it is coherent relative to some ψ. The club C is said to *witness* that U is coherent relative to ψ.

If U is stationary, then the conditions of Definition 3.29 imply in particular that every element of U belongs to the range of ψ. To see this, fix $x \in U$, let β be the least element of C above $\sup(x)$, and using the stationarity of U, find $y \in U$ so that $\sup(y \cap \beta) > \sup(x)$. By condition (4) of the definition, $y \cap \beta \in U$. Now by condition (2) of the definition, $x \in \psi''(y \cap \beta)$.

It is clear that forcing with the poset $\mathbb{P}_{\text{side}}$ of Section 2 adds a coherent set, namely the set of $Q \cap \omega_2$ for countable nodes Q belonging to the generic filter. Here we intend to work with a coherent U in the context of Fact 3.28, and since this requires a model where $(\forall \mu < \theta)\mu^{\aleph_0} < \theta$, we cannot use $\mathbb{P}_{\text{side}}$. Instead we use the countably closed poset given by the next lemma.

Lemma 3.30. *Suppose $\theta \geq \omega_2$ is regular and $(\forall \mu < \theta)\mu^{\aleph_0} < \theta$. In particular $\theta^{\aleph_0} = \theta$. Let $\psi \colon \theta \to \mathcal{P}_\omega(\theta)$ be a bijection. Then there is a countably closed θ-c.c. forcing extension which collapse all cardinals in the interval (ω_1, θ), turning θ to ω_2 of the extension, and adds a stationary coherent set relative to ψ.*

Proof. Let Z be the set of $\lambda < \theta$ so that $\mathrm{cof}(\lambda) \geq \omega_1$ and $\psi\restriction\lambda$ is a bijection from λ onto $\mathcal{P}_\omega(\lambda)$. By the lemma assumptions, Z is unbounded in θ. Let $C = Z \cup \mathrm{Limit}(Z)$. Since Z is ω_1-closed, it is equal to the set of $\lambda \in C$ so that $\mathrm{cof}(\lambda) \geq \omega_1$.

We use countable approximations to add U which is stationary and coherent relative to ψ with witness C. Define the poset $\mathbb{B}$ to consist of countable $u \subseteq \mathcal{P}_\omega(\theta)$ so that u and C satisfy conditions (1)–(4) of Definition 3.29, ordered by reverse inclusion, meaning that $u^* \leq u$ iff $u^* \supseteq u$. Since each of the conditions in Definition 3.29 involves only finitely many elements of u at a time, it is clear that $\mathbb{B}$ is countably closed. Indeed if u_n, $n < \omega$, is a descending sequence, then $\bigcup u_i$ is a condition in $\mathbb{B}$ extending each u_n.

Claim 3.31. *Suppose $M \prec (H(\theta); \psi, C)$ is countable, $u \in \mathbb{B}$, and $u \subseteq M$. Then there is $u^* \leq u$ so that $M \cap \theta \in u^*$. In particular there are non-empty conditions in $\mathbb{B}$.*

Proof. Let $u^* = u \cup \{M \cap \theta\} \cup \{M \cap \lambda \mid \lambda \in Z \cap M\}$. We verify the instances of the conditions of Definition 3.29 for $x, y \in u^*$ which do not both belong to u. The other instances are inherited from u.

No two elements of $u^* - u$ have the same supremum, nor does any of them have a supremum in M. This secures the instances of condition (1) of Definition 3.29 with $\{x, y\} \not\subseteq u$. For every $\alpha < \sup(M \cap \theta)$, $M \cap \alpha = M \cap \lambda$ for the least $\lambda \geq \alpha$ in $M \cap \theta$. If α belongs to C then by elementarity of M so does λ. If $\lambda > \alpha \in C$ then $\mathrm{cof}(\lambda) \geq \omega_1$, otherwise by elementarity M is cofinal in λ, contradicting the minimality of λ. It follows from this that u^* satisfies conditions (3) and (4) of Definition 3.29 at $x = M \cap \theta$, and similarly for all other $x \in u^* - u$.

For any two distinct $x, y \in u^* - u$, there is an element of Z between $\sup(x)$ and $\sup(y)$. The same is true for $x \in u^* - u$ and $y \in u \subseteq M$ with $\sup(y) > \sup(x)$. If $x = M \cap \theta$ and $y \in u$ then $y \in M$ and hence $y \in \psi''x$ by elementarity of M. Finally, if $x = M \cap \lambda$ for $\lambda \in Z \cap M$, $y \in u$, and $\sup(y) < \sup(x)$, then $y \subseteq \lambda$, hence by definition of Z there is $\nu < \lambda$ so that $y = \psi(\nu)$, and by elementarity of M such ν can be found in M, so $y \in \psi''x$. This establishes condition (2) of Definition 3.29. $\square$

Claim 3.32. *Let $\theta^* > \theta$, let $M^* \prec (H(\theta^*); \psi, C, \mathbb{B})$ be countable, and let $u \in \mathbb{B}$ belong to M^*. Then there is $u^* \leq u$ which is a master condition for M^* with $M^* \cap \theta \in u^*$.*

Proof. Let D_n, $n < \omega$, enumerate the dense subsets of $\mathbb{B}$ which belong to M^*. Let $u_0 = u$ and working inductively fix $u_{n+1} \leq u_n$ with $u_{n+1} \in M^* \cap D_n$. Let $u_\omega = \bigcup u_n$. Let u^* be the extension of u_ω given by Claim 3.31 applied to u_ω and $M = M^* \cap H(\theta)$. $\square$

For $\lambda \in Z$ and $u \in \mathbb{B}$, define $\mathrm{res}_\lambda(u) = u \cap \mathcal{P}_\omega(\lambda)$. It is clear that $\mathrm{res}_\lambda(u) \in \mathbb{B}$ and that $u \leq \mathrm{res}_\lambda(u)$.

Claim 3.33. *Suppose $\lambda \in Z$, $u, v \in \mathbb{B}$, $v \subseteq \mathcal{P}_\omega(\lambda)$, and $v \leq \mathrm{res}_\lambda(u)$. Then u, v are compatible, and indeed $u \cup v \in \mathbb{B}$.*

Proof. The only potential problems are with instances of conditions (1) and (2) of Definition 3.29 involving $x \in v - u$ and $y \in u - v$ or vice versa. The conditions hold in these instances since $x \in v - u$ implies $\sup(x) < \lambda$, $x \in u - v$ implies $\sup(x) > \lambda$ (using the assumption that $v \supseteq \mathrm{res}_\lambda(u)$), and $\lambda \in Z$. $\square$

Claim 3.34. *Every condition in $\mathbb{B}$ is a strong master condition for every $W \prec (H(\theta); \psi, C)$ which is transitive, countably closed, and bounded below θ. In particular $\mathbb{B}$ has the θ-chain condition.*

Proof. Let $\lambda = W \cap \theta$. Then $\mathrm{cof}(\lambda) \geq \omega_1$ by countable closure of M, and $\psi{\restriction}\lambda\colon \lambda \to \mathcal{P}_\omega(\lambda)$ is a bijection by the elementarity and countable closure of W, so $\lambda \in Z$. By the countable closure of W, $\mathbb{B}\cap W$ is equal to $\mathbb{B}\cap\mathcal{P}_\omega(\lambda)$. Hence by Claim 3.33, every condition $u \in \mathbb{B}$ is compatible with every $v \in \mathbb{B} \cap W$ which extends $\mathrm{res}_\lambda(u) \in W$. In particular for every dense $D \subseteq \mathbb{B} \cap W$ there is $v \in D$ compatible with u. This establishes the strong properness in the claim. By the cardinal assumptions of Lemma 3.30 there are stationarily many W as in the claim, and from this and the strong properness it follows as in the proof of Corollary 3.7 that $\mathbb{B}$ has no antichains of size θ. $\square$

Let G be generic for $\mathbb{B}$. Let $U = \bigcup\{u \mid u \in G\}$. U is stationary by Claim 3.32 and genericity. By the definition of $\mathbb{B}$, every countable subset of U satisfies the conditions of coherence relative to ψ with witness C, and it follows from this that U is coherent relative to ψ with witness C. The conditions of coherence imply that for any successive $\lambda, \lambda' \in C$, the set $Y = \{x \in U \mid \lambda < \sup(x) < \lambda'\}$ has ordertype at most ω_1 when ordered by $\sup(x)$. Hence $\bigcup Y$ has cardinality at most ω_1. Since U is closed under intersections with λ', and stationary, $\bigcup Y$ contains λ'. So all cardinals between ω_1 and θ are collapsed to ω_1. This completes the proof of Lemma 3.30. $\square$

Theorem 3.35. *Relative to the consistency of* ZFC*, it is consistent with arbitrarily large values of the continuum that $\mathsf{MA}^{1.5}_{<2^{\aleph_0}}(U)$ holds for a coherent U.*

Proof. Start with a model of the GCH, pass to a countably closed forcing extension adding a coherent stationary U using Lemma 3.30 applied at any regular $\theta \geq \omega_2$, force further to add square by initial segments at any $\kappa > \theta$, and then apply Fact 3.28 to force $\mathsf{MA}^{1.5}_{<2^{\aleph_0}}(U)$. The coherence of U is preserved since the definition of coherence involves only bounded quantifiers. $\square$

Theorem 3.36. *The statement that $\mathsf{MA}^{1.5}_{\omega_2}(U)$ holds for a coherent U implies both $\square_{\omega_1,\mathrm{fin}}$ and $\square^{\mathrm{ta}}_{\omega_1,\omega}$.*

Proof. Fix a coherent U, relative to some $\psi\colon \omega_2 \to \mathcal{P}_\omega(\omega_2)$, with witness C say, so that $\mathsf{MA}^{1.5}_{\omega_2}(U)$ holds. In particular U is stationary. We will find $\mathcal{S}$ and $\mathcal{T}$, satisfying conditions (ST1)–(ST9), so that $\{M\cap\omega_2 \mid M \in \mathcal{S}\}$ is equal to the restriction of U to a club, this restriction is coherent relative to ψ, and this is witnessed by the closure of $\{W \cap \omega_2 \mid W \in \mathcal{T}\}$. Using coherence we will then argue that every finite $u \subseteq \mathcal{S}$ extends to a side condition s with $\min\{N \cap \omega_1 \mid N \in s\} = \min\{N \cap \omega_1 \mid N \in u\}$. From this, using the properties of $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}(\mathcal{S},\mathcal{T})$ and $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}(\mathcal{S},\mathcal{T})$ given by Lemmas 3.15 and 3.23, we will show that both posets are $\aleph_{1.5}$-c.c. relative to U.

We begin with a lemma showing that coherence is robust under restrictions to clubs.

Lemma 3.37. *(Assuming $U \subseteq \mathcal{P}_\omega(\omega_2)$ is stationary and coherent relative to ψ with witness C.) Let $f\colon \omega_2^{<\omega} \to \omega_2$, and let $\hat{C} \subseteq C - \omega_1$ be club. Suppose every successor $\alpha \in \hat{C}$ is of cofinality ω_1, closed under f, and closed enough that for every $x \in U$ with $\sup(x) < \alpha$, there exists $\nu < \alpha$ so that $\psi(\nu) = x$. Then there is a* *club $E \subseteq \mathcal{P}_\omega(\omega_2)$ whose elements are all closed under f so that $U \cap E$ is coherent relative to ψ with witness $\hat{C}$.*

Proof. For each $\alpha \in \hat{C} \cup \{0\}$ fix a bijection $\phi_\alpha \colon \omega_1 \to \beta$ where β is the successor of α in $\hat{C} \cup \{0\}$. For $\alpha \in \omega_1 - \omega$ fix a bijection $\phi_\alpha \colon \omega \to \alpha$. Define $h \colon \omega_2^{<\omega} \to \omega_2$ through the following condition:

- $h(0, n) = n + 1$ for $n < \omega$.
- $h(1, \alpha)$ is the least ν so that $\psi(\nu) \in U$ and $\sup(\psi(\nu)) = \alpha$ if such ν exists.
- $h(2, \alpha, \xi) = \phi_\alpha(\xi)$ if $\phi_\alpha(\xi)$ is defined.
- $h(3, \mu)$ is the largest $\alpha \leq \mu$ in $\hat{C} \cup \{0\}$.
- $h(4, \alpha, \mu)$ is equal to the unique $\xi < \omega_1$ so that $\phi_\alpha(\xi) = \mu$ if such ξ exists.
- $h(5, \alpha, \xi)$ is the least ν so that $\psi(\nu) = \phi_\alpha'' \xi$, if such ν exists and $\phi_\alpha'' \xi \in U$.
- If $\alpha \in \hat{C}$ has cofinality ω then $\{h(6, \alpha, n) \mid n < \omega\}$ is cofinal in α and consists of successor points of $\hat{C}$.
- $h(\langle 7 \rangle ^\frown a) = f(a)$.
- In all other cases, h takes value 0.

Let E be the club of $x \in \mathcal{P}_\omega(\omega_2)$ which are closed under h. Let $\hat{U} = U \cap E$. We claim that $\hat{U}$ is coherent with witness $\hat{C}$.

Condition (1) of Definition 3.29 for $\hat{U}$ is immediate from the same condition for U. The same is true with condition (4) for $\hat{U}$ and $\hat{C}$, because every $\alpha \in \hat{C}$ is closed under h. Condition (3) holds for $\hat{C}$ at all x which are closed under h because from any $\mu \in x$, the values h takes on tuples starting with 3 and with 6 give points of $\hat{C}$ of cofinality ω_1 cofinal in the set of such points below $\mu + 1$.

For condition (2), fix successive $\alpha < \beta \in \hat{C} \cup \{0\}$, fix $x, y \in \hat{U}$, and suppose $\alpha \leq \sup(x) < \sup(y) \leq \beta$. Note that $\alpha \in y$ by closure of y and using the definition of h on tuples starting with 3.

If $\sup(x) = \alpha$ then since $x \in \text{range}(\psi)$ and x is the unique element of U with $\sup(x) = \alpha$, it follows that $h(1, \alpha)$ is defined and equal to the least ν so that $\psi(\nu) = x$. Then $\nu \in y$ by closure and hence $x \in \psi'' y$.

If $\sup(x) > \alpha$, then α belongs to both x and y. By closure of x and y, using the definition of h on tuples starting with 2 and 4, it follows that $x = \phi_\alpha'' \zeta$ and $y = \phi_\alpha'' \xi$ where $\zeta = x \cap \omega_1$ and $\xi = y \cap \omega_1$. Both ξ and ζ are ordinals, using the closure of x and y under the successor function on ω and under the functions ϕ_δ for $\delta \in \omega_1 - \omega$. Since $\psi_\alpha'' \zeta = x \in U$, $h(5, \alpha, \zeta)$ gives the least ν so that $x = \psi(\nu)$. ζ must be smaller than ξ, since otherwise $y \subseteq x$ contradicting the fact that $\sup(x) < \sup(y)$. So $\zeta \in y$. It follows that $h(5, \alpha, \zeta) \in y$, and hence $x \in \psi'' y$. □

Lemma 3.38. *(Assuming $U \subseteq \mathcal{P}_\omega(\omega_2)$ is stationary and coherent relative to ψ with witness C.) There is K, $\mathcal{S}$, $\mathcal{T}$, and a club $E \subseteq \mathcal{P}_\omega(\omega_2)$, so that the following conditions hold where $\hat{U} = U \cap E$:*

1. *K, $\mathcal{S}$, and $\mathcal{T}$ satisfy conditions (ST1)–(ST9).*
2. *$\{M \cap \omega_2 \mid M \in \mathcal{S}\}$ is exactly equal to $\hat{U}$.*
3. *$\hat{U}$ is coherent relative to ψ and this is witnessed by the closure of $\{W \cap \omega_2 \mid W \in \mathcal{T}\}$.*

Proof. Let Ξ be a relation on ω_2 which codes U, ψ, and C. Let $K = L_{\omega_2}[\Xi]$. Let $\mathcal{T}$ be defined as in condition (ST6). Let $\hat{C}$ be the closure of $\{W \cap \omega_2 \mid W \in \mathcal{T}\}$, namely the set of $\alpha < \omega_2$ of cofinality ω_1 so that $L_\alpha[\Xi]$ is elementary in $(K; \in, \Xi)$, plus the countable cofinality limits of such α. Let $\varphi\colon \omega_2^{<\omega} \to K$ be a Σ_1 Skolem function for K, and fix $f\colon \omega_2^{<\omega} \to \omega_2$ so that $x \subseteq \omega_2$ is closed under f iff $\omega_2 \cap \varphi''x = x$ and $\varphi''x \prec (K; \in, \Xi)$. Let E be given by Lemma 3.37. Define $\mathcal{S} = \{\varphi''x \mid x \in U \cap E\}$. It is easy to check that these assignments satisfy the requirements of the lemma. We only note that condition (ST4) for $\mathcal{S}$ and $\mathcal{T}$ uses the closure of $U \cap E$ under intersections with elements of $\hat{C}$ of cofinality ω_1, and that the stationarity of $\mathcal{S}$ required for condition (ST5) is a consequence of the stationarity of U. $\square$

Lemma 3.39. *Suppose K, $\mathcal{S}$, $\mathcal{T}$, and $\hat{U}$ are as in Lemma 3.38. Then for every finite $u \subseteq \mathcal{S}$ there exists $s \in \mathbb{P}_{\rm side}(\mathcal{S}, \mathcal{T})$ so that $s \supseteq u$. Moreover, for every transitive $W \in s$ there exists $Q \in u$ so that $W \in Q$, and for every countable $M \in s$ there exists $Q \in u$ and $W \in Q \cap \mathcal{T}$ so that $M = Q \cap W$.*

Proof. This is a direct consequence of the coherence of $\hat{U}$. Work by induction on $\max\{\sup(Q \cap \omega_2) \mid Q \in u\}$. Note that for every $Q, Q' \in u$, if $Q \neq Q'$ then $\sup(Q \cap \omega_2) \neq \sup(Q' \cap \omega_2)$. This is because Q is determined from $Q \cap \omega_2$ by condition (ST8) and $Q \cap \omega_2$ is determined from $\sup(Q \cap \omega_2)$ by condition (1) of Definition 3.29. Let Q_i, $i < n$, enumerate the elements of u, ordered so that $\sup(Q_i \cap \omega_2) < \sup(Q_{i+1} \cap \omega_2)$.

If $Q_i \in Q_{i+1}$ for every i, then u itself is a side condition and there is nothing further to prove. Suppose then that there exists i so that $Q_i \notin Q_{i+1}$, and let k be the largest such. Let Ξ be as in conditions (ST6)–(ST8). By elementarity of Q_{k+1}, and since Q_k is the Σ_1 hull of $Q_k \cap \omega_2$ in $(K; \in \Xi)$, it must be that $Q_k \cap \omega_2 \notin Q_{k+1}$, and hence using elementarity relative to ψ, $Q_k \cap \omega_2 \notin \psi''(Q_{k+1} \cap \omega_2)$. Since $Q_k \cap \omega_2$ and $Q_{k+1} \cap \omega_2$ both belong to $\hat{U}$ which is coherent with witness $\hat{C}$ equal to the closure of $\{W \cap \omega_2 \mid W \in \mathcal{T}\}$, it follows by condition (2) of Definition 3.29 that $\hat{C}$ has elements between $\sup(Q_k \cap \omega_2)$ and $\sup(Q_{k+1} \cap \omega_2)$. Let α be the least one. Then in particular α is a successor point of $\hat{C}$, and hence has cofinality ω_1. By condition (3) of Definition 3.29 there is $\gamma \geq \alpha$ with $\gamma \in \hat{C}$, $\operatorname{cof}(\gamma) = \omega_1$, and $\gamma \in Q_{k+1} \cap \omega_2$. Let W be the Σ_1 hull of γ in $(K; \in, \Xi)$. Then $W \in Q_{k+1}$ by elementarity, and $W \in \mathcal{T}$ by definition of $\hat{C}$ and condition (ST6). Since $Q_{k+1} \in Q_{k+2} \in \dots Q_{n-1}$ by maximality of k, $W \in Q_i$ for all $i \geq k+1$. By condition (ST4) then $Q_i \cap W \in \mathcal{S}$ and $Q_i \cap W \in W$, for $i \geq k+1$.

Let $\bar{u} = \{Q_i \mid i \leq k\} \cup \{Q_i \cap W \mid i \geq k+1\}$. Then $\bar{u}$ is a finite subset of $\mathcal{S}$, and $\bar{u} \subseteq W$, hence in particular $\max\{\sup(Q \cap \omega_2) \mid Q \in \bar{u}\} < \sup(Q_{k+1} \cap \omega_2) \leq \max\{\sup(Q \cap \omega_2) \mid Q \in u\}$. It follows by induction that there is a side condition $\bar{s} \supseteq \bar{u}$, satisfying the requirements of the lemma relative to $\bar{u}$. Since $\bar{u} \subseteq W$ these requirements imply in particular that $\bar{s} \subseteq W$. Using the fact that $W \in Q_{k+1} \in Q_{k+2} \in \dots \in Q_{n-1}$, and that $Q_i \cap W \in \bar{u} \subseteq \bar{s}$ for all $i \geq k+1$, it is easy now to check that $\bar{s} \cup \{W, Q_{k+1}, \dots, Q_{n-1}\}$ is a side condition, and satisfies the requirements of the lemma for u. $\square$

Using Lemmas 3.38 and 3.39 we can now complete the proof of Theorem 3.36. Let K, $\mathcal{S}$, $\mathcal{T}$, E be as in Lemma 3.38. Let Ξ witness conditions (ST6)–(ST8). It is enough to show that $\mathbb{S}\mathrm{quare}_{\rm fin}(\mathcal{S}, \mathcal{T})$ and $\mathbb{S}\mathrm{quare}_{\rm ta}(\mathcal{S}, \mathcal{T})$ have the $\aleph_{1.5}$-chain condition relative to U. Then noting that the proofs of Lemmas 3.14 and 3.24 only require a filter meeting a rich enough collection of ω_2 dense sets, it follows under $\mathsf{MA}^{1.5}_{\omega_2}(U)$ that there exist $\square_{\omega_1,\rm fin}$ and $\square^{\rm ta}_{\omega_1,\omega}$ sequences.

Fix $\theta > \omega_2$. Let $D \subseteq \mathcal{P}_\omega(H(\theta))$ be a club so that every $N \in D$ is elementary in $(H(\theta); K, \Xi, E)$. Fix a finite $\{N_i \mid i < n\} \subseteq D$, satisfying $(\forall i < n) N_i \cap \omega_2 \in U$. Fix $j < n$ such that $N_j \cap \omega_1$ is minimal in $\{N_i \cap \omega_1 \mid i < n\}$. We have to prove that every condition in $N_j \cap \mathbb{S}\mathrm{quare}_{\mathrm{fin}}(\mathcal{S}, \mathcal{T})$ extends to a master condition for all N_i, and similarly with $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}(\mathcal{S}, \mathcal{T})$.

Let $Q_i = N_i \cap K$, and let $x_i = Q_i \cap \omega_2 = N_i \cap \omega_2$. Then $x_i \in E$ by the elementarity of N_i, and since $N_i \cap \omega_2 \in U$ it follows that $x_i \in \hat{U} = U \cap E$. By the conditions in Lemma 3.38 there is $M \in \mathcal{S}$ with $M \cap \omega_2 = x_i$. By condition (ST8) it must be that M is the Σ_1 hull of x_i in $(K; \in, \Xi)$, and hence by the elementarity of Q_i it must be that $M = Q_i$. So $Q_i \in \mathcal{S}$ for each i.

By Lemma 3.39 there is a side condition s containing $\{Q_i \mid i < n\}$, and moreover every countable node in s is the intersection of some Q_i with a transitive node. In particular, $\min\{Q \cap \omega_1 \mid Q \in s\} = \min\{N_i \cap \omega_1 \mid i < n\}$, and hence $Q_j \cap \omega_1$ is minimal in $\{Q \cap \omega_1 \mid Q \in s\}$. This in turn implies that $\mathrm{res}_{Q_j}(s)$ has no countable nodes.

Fix now any condition $\langle t, d\rangle \in N_j \cap \mathbb{S}\mathrm{quare}_{\mathrm{fin}}$. By Corollary 3.13 and since $\mathrm{res}_{Q_j}(s)$ has only transitive nodes, we may, extending $\langle t, d\rangle$ if necessary, but doing this inside the elementary N_j, assume that $t \supseteq \mathrm{res}_{Q_j}(s)$. Now by Lemma 3.15 there is $\langle r, b\rangle$ extending $\langle t, d\rangle$ with $r \supseteq s$. By Lemma 3.12 then $\langle r, b\rangle$ is a strong master condition for every $Q \in s$. Since $N_i \cap K \in s$ for each i, it follows in particular that $\langle r, b\rangle$ is a master condition, in fact a strong master condition, for each N_i.

The proof for $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}$ is similar, but using Lemma 3.23, which is stronger than Lemma 3.15. This completes the proof of Theorem 3.36. □

The hypothesis in Theorem 3.36 combines two elements. One is the axiom $\mathsf{MA}^{1.5}_{\omega_2}(U)$, the other is the coherence of U. We end this section by showing that neither of these elements by itself implies $\square_{\omega_1,\mathrm{fin}}$ and $\square^{\mathrm{ta}}_{\omega_1,\omega}$, or even for that matter $\square_{\omega_1,\omega}$. We will also comment on the question of whether Theorem 3.36 can be improved to give $\square_{\omega_1}$.

Claim 3.40. *Let $\tau \leq \kappa$ be regular cardinals, with $\kappa \geq \omega_1$. Suppose that for every club $E \subseteq \mathcal{P}_{<\kappa}(\kappa^+)$ there is an ordinal η, limit ordinals $\lambda^\xi < \kappa^+$ of cofinality κ, and sets $x^\xi_\iota \in E$, for $\xi < \tau$ and $\iota \leq \eta$, so that:*

1. *For every ξ and ι, $\lambda^\xi \in x^\xi_\iota$.*
2. *For every ξ and $\iota < \eta$, $\sup(x^\xi_\iota \cap \lambda^\xi) < \sup(x^\xi_\eta \cap \lambda^\xi)$.*
3. *For every ξ, ζ, and ι, $\sup(x^\xi_\iota \cap \kappa) = \sup(x^\zeta_\iota \cap \kappa)$.*
4. *For every ξ and ζ, $\sup(x^\xi_\eta \cap \lambda^\xi) = \sup(x^\zeta_\eta \cap \lambda^\zeta)$.*
5. *For every ξ and ζ, there exists $\iota < \eta$ so that $\sup(x^\xi_\iota \cap \lambda^\xi) \neq \sup(x^\zeta_\iota \cap \lambda^\zeta)$.*

Then $\square_{\kappa,<\tau}$ fails.

Proof. Suppose for contradiction that $\vec{C} = \langle C_{\alpha,i} \mid \alpha < \kappa^+, i < n_\alpha\rangle$, where $n_\alpha < \tau$ for each α, is a $\square_{\kappa,<\tau}$ sequence. Let E consist of the $x \in \mathcal{P}_{<\kappa}(\kappa^+)$ which are elementary in κ^+ relative to predicates for the successor function on κ and for the graph of the function sending $\alpha < \kappa^+$ and $\delta < \kappa$ to the δth element of $C_{\alpha,0}$ in ordertype. Let η, λ^ξ, and x^ξ_ι satisfy the conditions of the claim for the club E. Let α be the common value of $\sup(x^\xi_\eta \cap \lambda^\xi)$, given by condition (4). By elementarity of x^ξ_η and since $\lambda^\xi \in x^\xi_\eta$, $\alpha \in \mathrm{Limit}(C_{\lambda^\xi,0})$ for every ξ. By elementarity of x^ξ_ι, $\sup(x^\xi_\iota \cap \lambda^\xi)$ is the δ_ιth element of $C_{\lambda^\xi,0}$, where $\delta_\iota < \kappa$ is the common value of $\sup(x^\xi_\iota \cap \kappa)$ over $\xi < \tau$, given by condition (3). By coherence, for every ξ there exists some $j^\xi < n_\alpha$ so that $C_{\alpha,j^\xi} = C_{\lambda^\xi,0} \cap \alpha$. Since $\sup(x^\xi_\iota \cap \lambda^\xi) < \alpha$ by condition (2), it follows that for every $\iota < \eta$, $\sup(x^\xi_\iota \cap \lambda^\xi)$ is the δ_ιth element of C_{α,j^ξ}. Then by condition (5), the sets C_{α,j^ξ}, for $\xi < \tau$, are distinct. Hence $\xi \mapsto j^\xi$ injects τ into $n_\alpha < \tau$, a contradiction. □

It is easy to obtain objects satisfying the conditions in Claim 3.40 if κ^+ is replaced by a greater cardinal θ. If θ is Mahlo then one can do this in a way that persists to certain generic extensions where θ becomes κ^+. Claim 3.40 can then be used in the generic extensions to rule out $\square$ principles. We will follow this approach in the next theorem to show that the existence of a stationary coherent set is compatible with failure of $\square_{\omega_1,\omega}$, and in Remark 3.47 to show that the posets $\mathbb{S}\mathrm{quare}_{\mathrm{fin}}$ and $\mathbb{S}\mathrm{quare}_{\mathrm{ta}}$ need not add $\square_{\omega_1,n}$ sequences for any finite n.

Theorem 3.41. *(Assuming a Mahlo cardinal.) The existence of a coherent U does not imply $\square_{\omega_1,\omega}$.*

Proof. Let θ be Mahlo. Let $\psi\colon \theta \to \mathcal{P}_\omega(\theta)$ be a bijection. Let $\mathbb{B}$ be the poset used in the proof of Lemma 3.30. Let Z be as in the proof of the lemma, that is $Z = \{\lambda < \theta \mid \mathrm{cof}(\lambda) \geq \omega_1 \text{ and } \psi\restriction\lambda \text{ is a bijection onto } \mathcal{P}_\omega(\lambda)\}$. Let C be the set of cardinals τ so that $|H(\tau)| = \tau$.

In the extension by $\mathbb{B}$, $\omega_2 = \theta$ and there is a coherent set. It is enough then to show that $\square_{\omega_1,\omega}$ fails in the extension. For this in turn it is enough to establish the conditions in Claim 3.40 for $\tau = \kappa = \omega_1$. Fix a name $\dot{E}$ for a club in $\mathcal{P}_\omega(\theta)$. Fix a condition $u_0 \in \mathbb{B}$. We will find a generic G for $\mathbb{B}$ with $u_0 \in G$, so that in $V[G]$ the conditions in Claim 3.40 can be satisfied for $\dot{E}[G]$.

Fix a name $\dot{f}$ for a function from $\theta^{<\omega}$ into θ so that $\dot{E}$ is forced to be the set of countable x closed under $\dot{f}$. Let S be the set of countable $y \subseteq H(\theta)$ elementary relative to a wellordering of $H(\theta)$, the poset $\mathbb{B}$, the map ψ, and the relation $u \Vdash \dot{f}(a) = \mu$, with $u_0 \in y$. By elementarity and since $\mathbb{B}$ is θ-c.c., for every $y \in S$ and every $a \in y$, there is $D \in y$ which is a maximal antichain of conditions forcing values for $\dot{f}(a)$. Thus any master condition for $y \in S$ forces that $y \cap \theta$ belongs to $\dot{E}$.

Claim 3.42. *There is a regular $\lambda \in \mathrm{Limit}(Z \cap C)$, and $M \in S$ elementary relative to S, with $\lambda \in M$, so that for cofinally many $\lambda^* < \theta$, there is $M^* \in S$ and an isomorphism $\varphi_{\lambda,\lambda^*}\colon M_\lambda \to M_{\lambda^*}$ which respects S and $\mathbb{B}$, maps λ to λ^*, and is the identity on elements of $H(\lambda)$.*

Proof. For each regular $\lambda \in \mathrm{Limit}(Z \cap C)$ fix a countable $M_\lambda \subseteq H(\theta)$ satisfying the elementarity in the claim with $\lambda \in M$. Let D_λ be the elementary diagram of M_λ in a language with predicates for S and $\mathbb{B}$, a constant for λ, and additional constants for all elements of $M_\lambda \cap H(\lambda)$. D_λ and $M_\lambda \cap H(\lambda)$ can be coded by a countable subset of $H(\lambda)$, hence by an element of λ since $\lambda \in Z \cap C$. Since θ is Mahlo, M_λ is defined on stationarily many $\lambda < \theta$. By a pressing down argument there is a stationary set Y of regular $\lambda \in \mathrm{Limit}(Z \cap C)$ on which M_λ is fixed. Then for any $\lambda, \lambda^* \in Y$ there exists an isomorphism from M_λ to M_{λ^*} which respects $\mathbb{B}$ and S, maps λ to λ^*, and fixes elements of $H(\lambda)$. Since Y is unbounded in θ this proves the claim. □

Claim 3.43. *For every regular $\lambda \in \mathrm{Limit}(Z \cap C)$ and every countable u, there exist $y_l \in S$ and $\tau_l \in y_l \cap \lambda \cap Z \cap C$, for $l \in \{0, 1\}$, so that $u \subseteq y_0, y_1$, $\lambda, \tau_l \in y_l$, $\tau_1 > \sup(y_0 \cap \lambda)$, and $y_0 \cap H(\tau_0) = y_1 \cap H(\tau_1)$.*

Proof. Similar to the proof of Claim 3.42, and in fact one can further arrange that there is an isomorphism from y_0 onto y_1 which fixes elements of $H(\tau_0)$ and maps τ_0 to τ_1. Note that since the proof involves a pressing down argument with a map that assigns to each τ a countable subset of $H(\tau)$, it is important that λ is regular and belongs to $\mathrm{Limit}(Z \cap C)$. $\square$

Work with λ and M given by Claim 3.42. Let $y_{n,l}$ and $\tau_{n,l}$, for $n < \omega$ and $l \in \{0, 1\}$, be given by inductive applications of Claim 3.43, arranging that u_0 belongs to each of $y_{0,0}, y_{0,1}$, that $y_{n,0}, y_{n,1}$ both belong to each of $y_{n+1,0}, y_{n+1,1}$, and that the nth element in some fixed enumeration of M in ordertype ω belongs to each of $y_{n+1,0}, y_{n+1,1}$. Using the elementarity of M, choose $y_{n,l}$ which belong to M. Then $\bigcup_{n<\omega} y_{n,0} = \bigcup_{n<\omega} y_{n,1} = M$.

Claim 3.44. *There exists $u_s \in \mathbb{B} \cap M$, for $s \in 2^{<\omega}$, so that:*

(1) *u_s is a master condition for each of $y_{i,s(i)}$, $i < \mathrm{lh}(s)$.*
(2) *If $s \supseteq t$ then $u_s \leq u_t$. $u_\emptyset = u_0$.*
(3) *If $\mathrm{lh}(s) \leq n$ then u_s belongs to both $y_{n,0}$ and $y_{n,1}$.*
(4) *If $\mathrm{lh}(s) = \mathrm{lh}(t)$ then $\mathrm{res}_\lambda(u_s) = \mathrm{res}_\lambda(u_t)$.*

Proof. Work by induction on $\mathrm{lh}(s)$. For the base case, set $u_\emptyset = u_0$. Suppose inductively that u_s have been defined for s of length n. Let $\langle t_j, E_j \rangle$ enumerate all pairs $\langle t, E \rangle$ where $\mathrm{lh}(t) = n + 1$ and E is a dense subset of $\mathbb{B}$ which belongs to $y_{n,t(n)}$. For $t \in 2^{n+1}$, working by simultaneous induction on j, we construct conditions $v^t_j \in y_{n,t(n)}$, $w^0_j \in y_{n,0}$, and $w^1_j \in y_{n,1}$, maintaining inductively that:

(i) $\mathrm{res}_\lambda(v^t_j) = w^{t(n)}_j$.
(ii) $\mathrm{res}_{\tau_{n,0}}(w^0_j) = \mathrm{res}_{\tau_{n,1}}(w^1_j)$.

Set $v^t_0 = u_{t\restriction n}$ to begin. By condition (4) of the claim, $\mathrm{res}_\lambda(u_{t\restriction n})$ is independent of t. Set w^0_0 and w^1_0 equal to the constant value of $\mathrm{res}_\lambda(u_{t\restriction n})$ for $t \in 2^{n+1}$. By condition (3), $w^0_0 = w^1_0$ belongs to both $y_{n,0}$ and $y_{n,1}$. Since $y_{n,0} \cap y_{n,1} \cap H(\lambda) = y_{n,0} \cap H(\tau_{n,0}) = y_{n,1} \cap H(\tau_{n,1})$ it follows that $\mathrm{res}_{\tau_{n,0}}(w^0_0) = w^0_0 = w^1_0 = \mathrm{res}_{\tau_{n,1}}(w^1_0)$.

For the inductive step, suppose the conditions above hold for j. We define v^t_{j+1}, w^0_{j+1}, and w^1_{j+1}.

Suppose for definitiveness that $t_j(n) = 0$. The case that $t_j(n) = 1$ is similar. Let $v^{t_j}_{j+1}$ be some extension of $v^{t_j}_j$ in $y_{n,0}$ meeting E_j. Let $w^0_{j+1} = \mathrm{res}_\lambda(v^{t_j}_{j+1})$. For $t \in 2^{n+1}$ with $t(n) = 0$ and $t \neq t_j$, set $v^t_{j+1} = v^t_j \cup w^0_{j+1}$. This is a condition in $\mathbb{B}$ by Claim 3.33 applied at λ, since $w^0_{j+1} \leq w^0_j = \mathrm{res}_\lambda(v^t_j)$, and this same computation also implies $\mathrm{res}_\lambda(v^t_{j+1}) = w^0_{j+1}$.

Set $w^1_{j+1} = w^1_j \cup \mathrm{res}_{\tau_{n,0}}(w^0_{j+1})$. This is a condition in $\mathbb{B}$ by Claim 3.33 applied at $\tau_{n,1}$, since $\mathrm{res}_{\tau_{n,0}}(w^0_{j+1}) \leq \mathrm{res}_{\tau_{n,0}}(w^0_j) = \mathrm{res}_{\tau_{n,1}}(w^1_j)$ and since $\mathrm{res}_{\tau_{n,0}}(w^0_{j+1}) \in y_{n,0} \cap H(\tau_{n,0}) = y_{n,1} \cap H(\tau_{n,1}) \subseteq H(\tau_{n,1})$. These computations also show that $w^1_{j+1} \in y_{n,1}$ and that $\mathrm{res}_{\tau_{n,1}}(w^1_{j+1}) = \mathrm{res}_{\tau_{n,0}}(w^0_{j+1})$. For $t \in 2^{n+1}$ with $t(n) = 1$, set $v^t_{j+1} = v^t_j \cup w^1_{j+1}$. This is a condition in $\mathbb{B}$ by Claim 3.33 applied at λ, since $w^1_{j+1} \leq w^1_j = \mathrm{res}_\lambda(v^t_j)$, and this same computation also implies that $\mathrm{res}_\lambda(v^t_{j+1}) = w^1_{j+1}$. This completes the inductive construction of v^t_j, w^0_j, and w^1_j for all j.

Now for each $t \in 2^{n+1}$ set $v^t = \bigcup_{j<\omega} v^t_j$. Set $w^0 = \bigcup_{j<\omega} w^0_j$, and set $w^1 = \bigcup_{j<\omega} w^1_j$. Since $y_{n,0}$ and $y_{n,1}$ both belong to each of $y_{n+1,0}$ and $y_{n+1,1}$, we could have picked the enumeration $\langle t_j, E_j \rangle$ used above to be an element of both $y_{n+1,0}$ and $y_{n+1,1}$. Then by elementarity the entire construction of the previous paragraph can be done in the intersection of these sets, so that v^t, w^0, and w^1 all belong to both $y_{n+1,0}$ and $y_{n+1,1}$.

By construction v^t extends $u_{t\restriction n}$, $v^t \subseteq y_{n,t(n)}$, and v^t meets all dense subset of $\mathbb{B}$ in $y_{n,t(n)}$. Again by construction, $\mathrm{res}_\lambda(v^t) = w^{t(n)}$, so in particular $w^l \subseteq y_{n,l}$ for $l \in \{0,1\}$, and $\mathrm{res}_{\tau_{n,0}}(w^0) = \mathrm{res}_{\tau_{n,1}}(w^1)$.

Set $w = w^0 \cup w^1$. This is a condition in $\mathbb{B}$ by Claim 3.33 applied at $\tau_{n,1}$ with w^1 standing for u, since $w^0 \subseteq y_{n,0} \cap \lambda \subseteq H(\tau_{n,1})$ and $w^0 \leq \mathrm{res}_{\tau_{n,0}}(w^0) = \mathrm{res}_{\tau_{n,1}}(w^1)$.

Set finally $u_t = v^t \cup w$. This is a condition in $\mathbb{B}$ by Claim 3.33, with residue at λ equal to w (for all t of length $n+1$), since $w \leq w^{t(n)} = \mathrm{res}_\lambda(v^t)$. This completes the induction on length that constructs the conditions u_s witnessing Claim 3.44. $\square$

Let w_n be the common value of $\mathrm{res}_\lambda(u_s)$ over $s \in 2^{<\omega}$ of length n. Let $w = \bigcup_{n<\omega} w_n$. Let G be generic for $\mathbb{B}$ with $w \in G$, and hence in particular $u_0 \in G$. It remains to prove that in $V[G]$ we can find objects satisfying the conditions in Claim 3.40 for $\dot{E}[G]$.

Claim 3.45. *For every $r \in 2^\omega$ there exists $\lambda^r < \theta$ of cofinality ω_1 in $V[G]$, and $x_i^r \in \dot{E}[G]$ for $i < \omega$, so that $\lambda^r \in x_i^r$ and $x_i^r \cap \lambda^r = y_{i,r(i)} \cap \lambda$.*

Proof. Fix $p \in G$ extending w. We will find $p^* \leq p$ forcing the existence of λ^r and x_i^r. This is enough to prove the claim.

Let λ^*, M^*, and $\varphi_{\lambda,\lambda^*}$ be given by Claim 3.42, with λ^* large enough that $p \in H(\lambda^*)$. Let $q = \bigcup_{n<\omega} u_{r\restriction n}$. Let $q^* = \varphi''_{\lambda,\lambda^*} q = \bigcup_{n<\omega} \varphi_{\lambda,\lambda^*}(u_{r\restriction n})$. Since $\varphi_{\lambda,\lambda^*}$ sends λ to λ^* and is the identity on elements of $H(\lambda)$, $\mathrm{res}_{\lambda^*}(q^*) = \mathrm{res}_\lambda(q) = \bigcup_{n<\omega} \mathrm{res}_\lambda(u_{r\restriction n}) = \bigcup_{n<\omega} w_n = w$. Since $p \leq w$ and $p \in H(\lambda^*)$ it follows by Claim 3.33 that p and q^* are compatible. Let p^* witness this.

By elementarity of $\varphi_{\lambda,\lambda^*}$, and since q is a master condition for each of the models $y_{i,r(i)}$, q^* is a master condition for each of the models $\varphi_{\lambda,\lambda^*}(y_{i,r(i)})$. Setting $x_i^r = \varphi_{\lambda,\lambda^*}(y_{i,r(i)}) \cap \theta$ and using the fact that $\varphi_{\lambda,\lambda^*}(y_{i,r(i)}) \in S$ it follows that $q^* \Vdash x_i^r \in \dot{E}$. Since $\varphi_{\lambda,\lambda^*}$ maps λ to λ^* and fixes elements of $H(\lambda)$, and since $\lambda \in y_{i,r(i)}$, we have further that $x_i^r \cap \lambda^* = y_{i,r(i)} \cap \lambda$ and $\lambda^* \in x_i^r$. $\square$

Now let r^ξ, $\xi < \omega_1$, list distinct elements of 2^ω. Let $\lambda^\xi = \lambda^{r^\xi}$, $x_i^\xi = x_i^{r^\xi}$ for $i < \omega$, and $x_\omega^\xi = \bigcup_{i<\omega} x_i^\xi$. Using the fact that $x_i^r \cap \lambda^r = y_{i,r(i)} \cap \lambda$ it is then easy to check λ^ξ and x_i^ξ satisfy the conditions in Claim 3.40 in $V[G]$ with $\tau = \kappa = \omega_1$. The key points are that $\sup(x_\omega^\xi \cap \lambda^\xi)$ is equal to $\sup(M \cap \lambda)$ independently of ξ, that $\sup(x_i^\xi \cap \kappa)$ is equal to either $\sup(y_{i,0} \cap \kappa)$ or $\sup(y_{i,1} \cap \kappa)$ but the two are the same since $\kappa < \tau_0, \tau_1$, and that $\sup(x_i^\xi \cap \lambda^\xi)$ is equal to $\sup(y_{i,r^\xi(i)} \cap \lambda)$. Since $\sup(y_{i,0} \cap \lambda) < \tau_{i,1} < \sup(y_{i,1} \cap \lambda)$, the last point implies that for $\xi \neq \zeta$ there exists i so that $\sup(x_i^\xi \cap \lambda^\xi) \neq \sup(x_i^\zeta \cap \lambda^\zeta)$, specifically any i so that $r^\xi(i) \neq r^\zeta(i)$. Through a use of Claim 3.40, this completes the proof of Theorem 3.41 $\square$

Remark 3.46. Let θ be Mahlo, let $\mathbb{P} = \mathbb{P}_{\mathrm{side}}(\mathcal{S}, \mathcal{T})$ where $\mathcal{T}$ consists of all countably closed $H(\lambda) \prec H(\theta)$ and $\mathcal{S}$ consists of all countable $x \prec H(\theta)$, and let G be generic for $\mathbb{P}$. Let $\mathcal{S}'$ and $\mathcal{T}'$ be defined from G as indicated in Section 2 so that $\mathcal{S}'$ and $\mathcal{T}'$ satisfy conditions (ST1)–(ST9). For any club $C \subseteq \theta$ and any club $Y \subseteq \mathcal{P}_\omega(H(\theta))$, using Claim 3.42 and 3.43, one can find $M, M^* \in Y$, $y_0, y_1 \in Y \cap M$, $\lambda \in C \cap y_0 \cap y_1$, $\lambda^* \in M^* \cap C$, $\tau_0 \in y_0 \cap C \cap \lambda$, $\tau_1 \in y_1 \cap C \cap \lambda$, and $\varphi\colon M \to M^*$, so that $\tau_1 > \sup(y_0 \cap \lambda)$, $y_0 \cap H(\tau_0) = y_1 \cap H(\tau_1)$, τ_0, τ_1 have uncountable cofinality, λ, λ^* are inaccessible, $\lambda^* > \sup(M \cap \theta)$, and φ is an isomorphism sending λ to λ^* and fixing all elements of $H(\lambda)$. Choosing C and Y so that $(\forall \rho \in C) H(\rho) \prec H(\theta)$ and $(\forall x \in Y) x \prec H(\theta)$, it is easy to check that $s = \{y_0 \cap y_1 \cap H(\lambda), H(\tau_0), y_0 \cap H(\lambda), H(\tau_1), y_1 \cap H(\lambda), H(\lambda), y_0, M, H(\lambda^*), \varphi(y_1), M^*\}$ is then a condition in $\mathbb{P}$, with its nodes arranged in the order listed. By Lemma 2.3, s is compatible with any u which belongs to $y_0 \cap y_1 \cap H(\lambda) = y_0 \cap H(\tau_0) = y_1 \cap H(\tau_1)$. It follows by genericity that objects as above can be found so that $s \in G$. One can arrange, through choice of the clubs C and Y and strong properness, that the nodes of s are sufficiently closed in $V[G]$ that the countable nodes in s belong to $\mathcal{S}'$, and the transitive nodes in s are limits of transitive nodes in $\mathcal{T}'$. The nodes of s then give rise to the configuration mentioned in Remark 3.17, when setting $M_0 = y_0$, $M_1 = M$, $P_0 = \varphi(y_1)$, $P_1 = M^*$, $N = H(\lambda)$, and $W = H(\lambda^*)$. It follows by the remark that the parallel of Lemma 3.15 fails for the poset $\mathbb{S}\text{quare}(\mathcal{S}', \mathcal{T}')$ in $V[G]$.

Remark 3.47. Continuing with the objects in Remark 3.46, note that by Lemma 3.15, every condition $\langle t, d\rangle \in \mathbb{S}\text{quare}_{\text{fin}}(\mathcal{S}', \mathcal{T}') \cap y_0 \cap y_1 \cap H(\lambda)$ extends to a condition $\langle r, b\rangle \in \mathbb{S}\text{quare}_{\text{fin}}(\mathcal{S}', \mathcal{T}')$ with $r \supseteq s$. Letting H be generic for $\mathbb{S}\text{quare}_{\text{fin}}(\mathcal{S}', \mathcal{T}')$ over $V[G]$ it follows using a density argument that s as in Remark 3.46 can be found with all its nodes belonging to H. Using strong properness and restricting the clubs C and Y in Remark 3.46 it follows further that for any club $E \subseteq \mathcal{P}_\omega(H(\theta))^{V[G*H]}$, one can find such s with all its countable nodes in E. By Claim 3.40, with $\kappa = \omega_1$ and $\tau = 2$, this implies that $\square_{\omega_1}$ fails in $V[G*H]$. A similar argument with more nodes (but still finitely many, as conditions in $\mathbb{P}_{\text{side}}(\mathcal{S}, \mathcal{T})$ and $\mathbb{S}\text{quare}_{\text{fin}}(\mathcal{S}', \mathcal{T}')$ are required to be finite) can be used to show that for every $n < \omega$, $\square_{\omega_1,n}$ fails in $V[G*H]$. The same arguments work with $\mathbb{S}\text{quare}_{\text{ta}}(\mathcal{S}', \mathcal{T}')$, using Lemma 3.23. Thus, with $V[G]$, $\mathcal{S}'$, and $\mathcal{T}'$ as in Remark 3.46, for every $n < \omega$, the posets $\mathbb{S}\text{quare}_{\text{fin}}(\mathcal{S}', \mathcal{T}')$ and $\mathbb{S}\text{quare}_{\text{ta}}(\mathcal{S}', \mathcal{T}')$ force the negation of $\square_{\omega_1,n}$.

Abstractly the only consequence of Lemmas 3.15 and 3.23 used here for the posets $\mathbb{S}\text{quare}_{\text{fin}}$ and $\mathbb{S}\text{quare}_{\text{ta}}$ in $V[G]$ is that for every large enough θ^* there is a club $D \subseteq \mathcal{P}_\omega(H(\theta^*))^{V[G]}$ so that for every finite $s^* \subseteq D$, if $\{U \cap H(\theta) \mid U \in s^*\}$ extends to $s \in \mathbb{P}_{\text{side}}(\mathcal{S}', \mathcal{T}')$ through only the addition of transitive nodes and closure under intersections, then every condition $p \in \bigcap s^*$ extends to a master condition for all $U \in s^*$. Any poset $\mathbb{Q}$ with this property forces the failure of $\square_{\omega_1,n}$, for all $n < \omega$, over $V[G]$.

Claim 3.48. *Suppose that for every club $E \subseteq \mathcal{P}_\omega(\omega_2)$ there exists $\lambda < \omega_2$ of uncountable cofinality, and $x, z \in E$, so that $\lambda \in z$, $\sup(z \cap \lambda) \in x$, and $x \cap z$ is bounded below $\sup(z \cap \lambda)$. Then $\square_{\omega_1,\omega}$ fails.*

Proof. Suppose for contradiction that $\vec{C} = \langle C_{\alpha,i} \mid \alpha < \omega_2, i < \omega\rangle$ is a $\square_{\omega_1,\omega}$ sequence. For each α of cofinality ω, each of the sets $C_{\alpha,i}$ has countable ordertype, and hence $\bigcup_{i<\omega} C_{\alpha,i}$ is countable. Let $h\colon \omega_2 \times \omega \to \omega_2$ be a function so that for every α of countable cofinality $\{h(\alpha, i) \mid i < \omega\} \supseteq \bigcup_{i<\omega} C_{\alpha,i}$. Let E consist of the countable $x \subseteq \omega_2$ which contain ω, and are closed under h, under the function which maps α, i, ξ to the ξth element of $C_{\alpha,i}$, and under the function which maps α, i, γ to the least ξ so that the ξth element of $C_{\alpha,i}$ is greater than γ. Let λ, x, and z be as in the claim, for the club E. Let $\alpha = \sup(z \cap \lambda) < \lambda$. By the closure of z and since $\lambda \in z$, α is a limit point of $C_{\lambda,0}$ and z includes a cofinal subset of $C_{\lambda,0} \cap \alpha$. By the coherence of $\vec{C}$ there is $n < \omega$ so that $C_{\lambda,0} \cap \alpha = C_{\alpha,n}$. Since $\alpha \in x$ it follows by the closure of x under h that $C_{\lambda,0} \cap \alpha \subseteq x$. But then since z contains a cofinal subset of $C_{\lambda,0} \cap \alpha$ it follows that $z \cap x$ is cofinal in α, contradicting the conditions in the claim. $\square$

Theorem 3.49. *(Assuming a measurable cardinal.) The axiom* $\mathsf{MA}^{1.5}_{<2^{\aleph_0}}$*, with arbitrarily large value for* $2^{\aleph_0}$*, does not imply* $\square_{\omega_1,\omega}$.

Proof. Let (†) denote the following strengthening of Chang's conjecture, taken from condition (2) in Theorem 2.5 of Shelah [19, Chapter XII]: for every large enough regular θ, for every wellordering $<$ of $H(\theta)$, for every countable $M \prec (H(\theta); \in, <)$, and for every $\alpha < \omega_2$, there exists a countable $M^* \prec (H(\theta); \in, <)$ so that $M^* \supseteq M$, $M^* \cap \omega_1 = M \cap \omega_1$, and $M^* \cap \omega_2 \not\subseteq \alpha$. We will use (†) to obtain models as in the hypothesis of Claim 3.48, and then use a strengthening of properness for the poset of Asperó-Mota [1] to ensure the models remain sufficiently closed when forcing to obtain $\mathsf{MA}^{1.5}_{<2^{\aleph_0}}$. The next two facts give (†) and the necessary strengthening of properness.

Fact 3.50 (By Shelah [19])**.** Let τ be measurable. Let G be generic for $\mathrm{Col}(\omega_1, <\tau)$. Then (†) holds in $V[G]$.

Sketch of proof. This is Conclusion 2.8 of [19, Chapter XII]. The proof in [19] involves winning strategies in a certain game, and this proof has other applications, for example in Todorcevic [26, Section 2]. But for us only the end principle is relevant, and it can be obtained directly through the following argument. The direct proof has its own applications, for example as a starting point for the variant in Sakai [18].

Let $\theta > 2^{2^\tau}$, $<$, M, and α be as in the hypothesis of (†), interpreted in $V[G]$. Then $\tau = \omega_2^{V[G]} \in M$, and $\alpha < \tau$. Let K be generic for $\mathrm{Col}(\omega_1, \tau^+)$ over $V[G]$, chosen using countable closure to contain a master condition for M. Then $M[K]$ is elementary in $(H(\theta)^{V[G*K]}; \in, <)$ and $M[K] \cap \omega_1 = M \cap \omega_1$. In $V[G * K]$ there is a measure over $\mathcal{P}^{V[G]}(\tau)$, and by elementarity there exists such a measure μ in $M[K]$. Let $\pi\colon V[G] \to V^*[G^*]$ be the ultrapower embedding by μ. By the elementarity of $M[K]$ and the definability of π from μ, the Skolem hull of $\{\tau\} \cup \pi'' M$ in $\pi((H(\theta)^{V[G]}; \in, <))$ is contained in $M[K]$. This hull then witnesses the conclusion of (†) for $\pi(M)$ and $\alpha = \pi(\alpha)$. By elementarity of π a witness for the conclusion of (†) for M and α then exists in $V[G]$. $\square$

When we talk about Skolem hulls in $(H(\theta); \in, <)$, in the proof of Fact 3.50 and further below, we mean the closure under functions which select $<$-minimal witnesses to true existential formulas in $(H(\theta); \in, <)$.

Fact 3.51 (By Asperó-Mota [1])**.** Suppose $2^{\aleph_0} = \aleph_1$, $\kappa > \omega_2$ is regular, $(\forall \mu < \kappa)\mu^{\aleph_0} < \kappa$, and $\Diamond(\{\alpha < \kappa \mid \mathrm{cof}(\alpha) \geq \omega_1\})$ holds. Let $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ be the sequence of posets defined in Section 2.2 of Asperó-Mota [1] (or as modified for Fact 3.28), so that in the extension by $\mathbb{P} = \mathbb{P}_\kappa$, the continuum is κ and $\mathsf{MA}^{1.5}_{<2^{\aleph_0}}$ holds (or $\mathsf{MA}^{1.5}_{<2^{\aleph_0}}(U)$ holds, if working as in Fact 3.28). Then there exists $T \subseteq H(\kappa)$ so that for every large enough regular θ, for every finite set $\{N_i^* \mid i < m\}$ of countable subsets of $H(\theta)$, and for every $p \in \mathbb{P} \cap \bigcap_{i<m} N_i^*$, if $\{N_i^* \cap H(\kappa) \mid i < m\}$ is a partial T-symmetric system in the sense of [1, Definition 2.2], then there is $q \leq p$ which is a master condition for all the models N_i^*, $i < m$.

Sketch of proof. This follows from the proof of Lemma 2.2 of [1], starting with a finite set of models $\{N_i^* \mid i < m\}$ instead of a single model N^*. The assumed symmetry of $\{N_i^* \cap H(\kappa) \mid i < m\}$ (relative to the predicate T used in the definition of $\mathbb{P}$) implies that $\{N_i^* \cap H(\kappa) \mid i < m\}$ can be amalgamated with any system of models Δ occurring in a condition of $\mathbb{P}$ which belongs to $\bigcap_{i<m} N_i^*$. This allows modifying condition $(1)_\alpha$ of the lemma to give q' so that all the pairs $(N_i^* \cap H(\kappa), \alpha)$, or $(N_i^* \cap H(\kappa), \sup(N \cap \kappa))$ in case $\alpha = \kappa$, belong to $\Delta_{q'}$. From this and condition $(2)_\alpha$ of the lemma it follows that q' is a master condition for N_i^* for all $i < m$. $\square$

Given a measurable cardinal τ we can, by passing to an inner model, reach a class model where τ is still measurable, and $\Diamond$ holds everywhere. The forcing extension of this class model by $\mathrm{Col}(\omega_1, \tau)$ then satisfies the CH, $\Diamond$ everywhere above ω_2, and, by Fact 3.50, (†). To prove Theorem 3.49 it is therefore enough to work over such a model, and force further to obtain $\mathsf{MA}^{1.5}_{<2^{\aleph_0}}$, an arbitrarily large value for the continuum, and failure of $\square_{\omega_1,\omega}$.

Work then assuming (†), the CH, and $\Diamond$ above ω_2. Let $\kappa > \omega_2$ be regular with $(\forall \mu < \kappa)\mu^{\aleph_0} < \kappa$. Let $\mathbb{P}$ be the poset of Asperó-Mota [1, Section 2.2]. (This uses $\Diamond$ and the CH.) It is forced in $\mathbb{P}$ that the continuum is κ and $\mathsf{MA}^{1.5}_{<2^{\aleph_0}}$ holds. We will be done if we can establish that the assumption of Claim 3.48 is forced in $\mathbb{P}$, as by the claim this gives the failure of $\square_{\omega_1,\omega}$ in the extension by $\mathbb{P}$.

Fix $\dot{E}$ which names a club subset of $\mathcal{P}_\omega(\omega_2)$ in the extension by $\mathbb{P}$. Fix $p \in \mathbb{P}$. We will find $\lambda < \kappa$ of uncountable cofinality, x and z as in Claim 3.48, and $q \leq p$ which forces that x and z belong to $\dot{E}$.

Let $\theta_1 < \theta_2 < \theta_3$ be regular, with θ_1 large enough for (†) and for Fact 3.51, and large enough that $\dot{E}, H(\kappa) \in H(\theta_1)$. Let $T \subseteq H(\kappa)^{V[G]}$ be given by Fact 3.51. Let $<$ be a wellordering of $H(\theta_1)$.

Let $N \prec H(\theta_3)$ be countable with $\{\dot{E}, T, \mathbb{P}, p, <, \kappa, \theta_1, \theta_2\} \subseteq N$. Let $\langle Q_i \mid i < \omega\rangle \in N$ be a sequence of models so that $\omega_1 \cup \{\dot{E}, T, \mathbb{P}, p, <, \kappa, \theta_1\} \subseteq Q_i$, $Q_0 \prec Q_1 \prec \cdots \prec H(\theta_2)$, each Q_i is countably closed, and $|Q_i| = \omega_1$. That such a sequence exists follows from the CH. The sequence can be picked in N by elementarity. Let $\alpha_i = Q_i \cap \omega_2$.

Let D be the elementary diagram of $N \cap H(\theta_1)$ in a language with the relation T and constants for the elements of $N \cap Q_0 \cap H(\theta_1)$. By the countable closure of Q_0, $D \in Q_0$. By elementarity of Q_0, an elementary substructure of $H(\theta_1)$ with the same diagram D can be found inside Q_0. Fix such a structure $\bar{N}$. Then $\dot{E}, T, \mathbb{P}, p \in \bar{N}$, $\bar{N} \cap H(\kappa)$ is elementary in $(H(\kappa), \in, T)$, and there is an isomorphism $\varphi_{\bar{N},N} : \bar{N} \cap H(\kappa) \to N \cap H(\kappa)$ which respects T and is the identity on $\bar{N} \cap N \cap H(\kappa) = N \cap Q_0 \cap H(\kappa)$. We will use this later to argue for partial T-symmetry.

Let $R \prec H(\theta_2)$ be countable with $\{\dot{E}, T, \mathbb{P}, p, <, \kappa, \theta_1, Q_0, \bar{N}\} \subseteq R$. Let $M_0 = R \cap Q_0 \cap H(\theta_1)$. Let Y be the set of $\eta < \omega_2$ so that there is $R' \supseteq M_0$, elementary in $(H(\theta_1); \in, <)$, with $R' \cap \omega_1 = M_0 \cap \omega_1$ and $\min(R' \cap \omega_2 - M_0) = \eta$. By the countable closure of Q_0, $M_0 \in Q_0$, and hence by elementarity $Y \in Q_0$. If Y is bounded in ω_2, then by elementarity of Q_0 a bound must exist in Q_0, meaning that the bound is smaller than α_0. But this is impossible, since α_0 itself belongs to Y as witnessed by $R' = R \cap H(\theta_1)$. So it must be that Y is unbounded in ω_2. Using this inside Q_1, fix $\eta_0 \in Y \cap Q_1$ with $\eta_0 > \alpha_0$. Let $M_1 \in Q_1$ be the Skolem hull of $M_0 \cup \{\eta_0\}$ in $(H(\theta_1); \in, <)$. Then M_1 is a subset of every R' witnessing that $\eta_0 \in Y$, hence in particular $M_1 \cap \omega_1 = M_0 \cap \omega_1$, and $M_1 \cap \eta_0 \subseteq M_0 \cap \omega_2 \subseteq Q_0 \cap \omega_2 = \alpha_0$, so $\alpha_0 \notin M_1$.

Now by induction on $n \in [1, \omega)$ define $M_{n+1} \in Q_{n+1}$ as follows. Using (†) inside Q_{n+1} find some countable $M' \prec (H(\theta_1); \in, <)$, in Q_{n+1}, so that $M' \supseteq M_n$,

$M' \cap \omega_1 = M_n \cap \omega_1$, and $M' \cap \omega_2 \not\subseteq \alpha_n$. Fix $\eta_n \in M' \cap \omega_2 - \alpha_n$. Let M_{n+1} be the Skolem hull of $M_n \cup \{\eta_n\}$ in $(H(\theta_1); \in, <)$.

Let $M_\omega = \bigcup_{n<\omega} M_n$. Through another application of (†), find $\eta_\omega > \sup_{n<\omega} \alpha_n$ so that the Skolem hull of $M_\omega \cup \{\eta_\omega\}$ in $(H(\theta_1); \in, <)$ has the same intersection with ω_1 as M_ω. Let $M = M_{\omega+1}$ be this Skolem hull.

To summarize some of the properties of the construction, M_n for $1 \leq n \leq \omega+1$ is the Skolem hull of $M_0 \cup \{\eta_i \mid i < n\}$ in $(H(\theta); \in, <)$, $M_n \cap \omega_1 = M_0 \cap \omega_1 = R \cap \omega_1$, $\alpha_n \leq \eta_n < \alpha_{n+1}$ for $n < \omega$, and $\alpha_0 \notin M_1$.

Claim 3.52. *$M \cap N \subseteq Q_0$.*

Proof. Let $\vec{f} = \langle f_\xi \mid \xi < \omega \rangle$ be the $<$-least enumeration of all functions on $\omega_2^{<\omega}$ in $Q_0 \cap H(\theta_1)$. By elementarity the enumeration belongs to both N and R. Let $\delta_N = N \cap \omega_1$ and let $\delta_R = R \cap \omega_1$. Since $\bar{N} \in R$, and since $\bar{N} \cap \omega_1 = N \cap \omega_1$, $\delta_R > \delta_N$. By elementarity of R, the functions on $\omega_2^{<\omega}$ which belong to $M_0 = R \cap Q_0 \cap H(\theta_1)$ are precisely the functions f_ξ for $\xi < \delta_R$. By the elementarity of M_0, every element in the Skolem hull of $M_0 \cup \{\eta_i \mid i \leq \omega\}$ can be obtained by applying these functions to finite tuples contained in $\{\eta_i \mid i \leq \omega\}$.

Suppose for contradiction that $M \cap N \not\subseteq Q_0$ and fix $x \in M \cap N - Q_0$ witnessing this. By the conclusion of the previous paragraph, there is some $\xi < \delta_R$ and some finite $a \subseteq \{\eta_i \mid i \leq \omega\}$ so that $f_\xi(a) = x$. By the elementarity of N, and since both $\vec{f}$ and x belong to N, there must then exist $\zeta < \omega_1$ and a finite $b \subseteq \omega_2^{<\omega}$, both in N, so that $x = f_\zeta(b)$. Note that $\zeta \in R$, since $N \cap \omega_1 = \delta_N < \delta_R$. Hence $f_\zeta \in R \cap Q_0 \cap H(\theta_1) = M_0 \subseteq M$. Letting $c \in \omega_2^{<\omega}$ be lexicographically least so that $x = f_\zeta(c)$ it follows by the elementarity of M and N that $c \in M \cap N$.

Since $x \notin Q_0$, and $f_\zeta \in Q_0$, it must be that $c \notin Q_0$. Since $c \in M \cap N$, and c is a finite tuple of ordinals below ω_2, it follows that there is $\beta \in \omega_2 \cap M \cap N$ with $\beta \notin \omega_2 \cap Q_0 = \alpha_0$. This implies that in fact $\alpha_0 \in M$. Certainly this is the case if $\beta = \alpha_0$. If $\beta > \alpha_0$, then letting g be the $<$-least bijection of ω_1 onto β we have by elementarity of M and N that $\beta \cap N = g''\delta_N \subseteq g''\delta_R = g''(M \cap \omega_1) = \beta \cap M$, and since $\alpha_0 \in N$ it follows again that $\alpha_0 \in M$.

But letting $h \in M_1$ be a bijection of ω_1 onto $\eta_0 > \alpha_0$ we have that $\eta_0 \cap M_1 = h''(M_1 \cap \omega_1) = h''(M \cap \omega_1) = \eta_0 \cap M$. So $\alpha_0 \in M$ iff $\alpha_0 \in M_1$. Since we picked M_1 in such a way that $\alpha_0 \notin M_1$, this is a contradiction. $\square$

Claim 3.53. *$\{\bar{N} \cap H(\kappa), N \cap H(\kappa), M \cap H(\kappa)\}$ is partial T-symmetric in the sense of Definition 2.2 of* [1].

Proof. This is clear from the definition, using the isomorphism $\varphi_{\bar{N},N} \colon \bar{N} \cap H(\kappa) \to N \cap H(\kappa)$ that we obtained above, noting that $M \cap \omega_1 > N \cap \omega_1 = \bar{N} \cap \omega_1$, and noting further that, using Claim 3.52, $M \cap N \cap H(\kappa) = \bar{N} \cap N \cap H(\kappa)$. $\square$

Using Fact 3.51 we can now fix $q \leq p$ which is a master condition for both M and N (and in fact also for $\bar{N}$, but we have no use for this model beyond its use in obtaining partial T-symmetry above). Letting $x = N \cap \omega_2$ and $z = M \cap \omega_2$ it follows that q forces both x and z to belong to $\dot{E}$.

Let $\alpha = \sup_{i<\omega} \alpha_i = \sup_{i<\omega} \eta_i$. Let $\lambda = \min(M \cap \omega_2 - \alpha)$. Then $\alpha \in N \cap \omega_2 = x$ since $\langle Q_i \mid i < \omega \rangle \in N$. Since $M \cap N \subseteq Q_0$ and $\alpha > \alpha_0 = Q_0 \cap \omega_2$ it follows that $\alpha \notin M$, and hence $\lambda > \alpha$. This in turn implies that $\mathrm{cof}(\lambda) \geq \omega_1$, otherwise M is cofinal in λ contradicting the fact that $M \cap [\alpha, \lambda) = \emptyset$. By minimality of λ, and since $\{\eta_i \mid i < \omega\}$ is a subset of M cofinal in α, $\sup(z \cap \lambda) = \sup(M \cap \lambda) = \alpha$. By

Claim 3.52, $x \cap z = N \cap M \cap \omega_2 \subseteq Q_0 \cap \omega_2 = \alpha_0 < \alpha$. Thus x and z satisfy the conditions in the hypothesis of Claim 3.48. This completes the proof of Theorem 3.49. $\square$

The results above do not address the question of whether $\mathsf{MA}^{1.5}_{\omega_2}(U)$ for a coherent U implies $\square_{\omega_1,n}$ for any $n < \omega$. It seems conceivable that, working over a model $V[G]$ where G is generic for the poset $\mathbb{B}$ of Lemma 3.30 for adding a coherent U, constructing master conditions for models as in Remark 3.46 using the methods in the proof of Theorem 3.41, and then arguing as in Remark 3.47 using a variant of Fact 3.51 to give enough of the needed abstraction of the consequences of Lemmas 3.15 and 3.23, one could reach an extension of $V[G]$ where $\mathsf{MA}^{1.5}_{\omega_2}(U)$ holds and $\square_{\omega_1,n}$ fails for every $n < \omega$. But Fact 3.51 itself is not sufficient, because of its restriction to symmetric systems. Indeed this scenario would not work if forcing $\mathsf{MA}^{1.5}_{\omega_2}(U)$ over $V[G]$ using the poset indicated in the proof of Fact 3.28. Using the symmetry of the side conditions in that poset, the fact that they involve only models whose intersection with ω_2 belongs to U, and the coherence of U, one can check that the poset adds a $\square_{\omega_1}$ sequence.

4. Specializing

Let T be a tree of height ω_2. Let $\theta > \omega_2$ with $T \in H(\theta)$, $A \subseteq H(\theta)^{<\omega}$, and $C \subseteq \omega_2$ club. Let E be the set of nodes of T that are extensively overlapped (relative to θ, A, and C). We describe a poset which weakly specializes the restriction of T to nodes outside E on a club of levels, while preserving ω_1 and ω_2.

We allow trees T which have cofinal branches. But the next claim and corollary show that for any cofinal branch h, for a club of levels α, the node of h on level α is extensively overlapped. These nodes are therefore left out of the domain of the weak specializing function that we force to add.

Claim 4.1. *Let $P \subseteq H(\theta)^{<\omega}$ and let D be the set of $\beta < \omega_2$ so that $\mathrm{cof}(\beta) = \omega_1$ and for every countable $a \subseteq \beta$, and every $\gamma < \omega_2$, there is a countable $M \prec (H(\theta); A, P)$ with $a \subseteq M$, $M \cap \omega_2 \not\subseteq \beta$, and $\min(M - \beta) \geq \gamma$. Then D is stationary.*

Proof. Suppose not. Then $Z = \{X \prec (H(\theta); A, P) \mid X$ is internal on a club and $\sup(X \cap \omega_2) \notin D\}$ is stationary. For each $X \in Z$ fix a_X and γ_X witnessing that $\sup(X \cap \omega_2) \notin D$. Increasing a_X if needed, using the fact that X is internal on a club, we may assume $a_X \in X$. Then the function $X \mapsto a_X$ is pressing down, and thinning Z if necessary we may assume it takes a constant value, call it a.

Let $B = \{\beta \mid (\exists X \in Z) \sup(X \cap \omega_2) = \beta\}$, and for $\beta \in B$ let $\gamma_\beta = \gamma_X$ for some $X \in Z$ with $\sup(X \cap \omega_2) = \beta$. Let $\delta < \omega_2$ in $\mathrm{Limit}(B)$ be a closure point, of cofinality ω_1, of the map $\beta \mapsto \gamma_\beta$. Let $M \prec (H(\theta); A, P)$ be countable with $\delta, a \in M$. Fix $\beta \in B$ between $\sup(M \cap \delta)$ and δ. Let $X \in Z$ be such that $\sup(X \cap \omega_2) = \beta$ and $\gamma_\beta = \gamma_X$. Then $M \supseteq a = a_X$ and $\min(M - \beta) = \delta > \gamma_X$, contradicting the fact that a_X and γ_X witness $\sup(X \cap \omega_2) \notin D$. $\square$

The proof of Claim 4.1 in fact shows that there is a club of $X \prec H(\theta)$ so that $\sup(X \cap \omega_2) \in D$ if X is internal on a club.

Corollary 4.2. *Suppose h is a cofinal branch of T. Then for a club of $\alpha < \omega_2$, the node of h on level α is extensively overlapped.*

Proof. Recall that extensive overlaps are relative to fixed θ, A, and C. Let D be the set in Claim 4.1 for $P = h$. Let $\alpha < \omega_2$, and let $\beta \leq \alpha$ belong to $D \cap C$. We prove that β is an extensive overlap point for the unique node of b on level α. If $\alpha \in \text{Limit}(D \cap C)$ this implies that the node is extensively overlapped.

Fix $a \subseteq \beta$ and $b \subseteq \omega_2 - \beta$ both countable. Let $\gamma < \omega_2$ be above $\max\{\sup(b), \alpha\}$. By Claim 4.1, there is countable $M \prec (H(\theta); A, h)$ with $a \subseteq M$, $M \cap \omega_2 \not\subseteq \beta$, and $\min(M - \beta) \geq \gamma$. The latter implies in particular that $(M \cup \text{Limit}(M)) \cap b = \emptyset$. Let μ be the least ordinal in M above β. Let u be the restriction of b to levels up to μ. Then $u \in M$ by elementarity of M relative to b. Since $\mu > \alpha$ and $\alpha \notin M$, u witnesses that M overlaps the node of b on level α. $\square$

We work throughout with $\mathcal{S}$ and $\mathcal{T}$ satisfying conditions (ST1)–(ST9) in Section 2. As in Section 3, conditions (ST6)–(ST9) are included for simplicity and to obtain the ω_2-chain condition. We could manage with just conditions (ST1)–(ST5), by prefixing our poset with a preparatory forcing by $\mathbb{A} = \mathbb{P}_{\text{side}}$, as described before Claim 2.5. The combination can be written as a single poset with finite conditions, and the results we obtain below hold for the combined poset, except for the claims related to the ω_2-chain condition. As in Section 3 we use Ot to denote $\{\sup(W \cap \text{Ord}) \mid W \in \mathcal{T}\}$, and rely on the fact that in $\mathbb{P}_{\text{side}}$, this set is forced to be equal to $\{\sup(W \cap \text{Ord}) \mid W \text{ occurs in } \dot{G}\}$. This fact uses conditions (ST6)–(ST9). For a combined poset under conditions (ST1)–(ST5) one has to replace the use of Ot in Definition 4.3 with a name $\dot{O}t$ for the set consisting of $\sup(W \cap \text{Ord})$ for W which occur in the side conditions part of the poset.

For simplicity we may assume that $|T| = \omega_2$; if not then the preliminary poset $\mathbb{A}$ can be arranged to collapse $|T|$ to ω_2, for example using a poset as in Neeman [16, Subsection 5.3]. Passing to an isomorphic copy of T we may assume $T \subseteq K$. Restricting $\mathcal{S}$ and $\mathcal{T}$ if necessary we may assume that $\sup(W \cap \text{Ord}) \in C$ for all $W \in \mathcal{T}$, and that each $Q \in \mathcal{S} \cup \mathcal{T}$ expands to $Q^* \prec (H(\theta); A, T)$ with $Q^* \cap K = Q$. Then for every $x \in T$ and $W \in \mathcal{T}$ so that $\sup(W \cap \text{Ord}) \leq \text{height}(x)$ and $\sup(W \cap \text{Ord})$ is not an extensive overlap point for x, there exists countable $a \subseteq \sup(W \cap \text{Ord})$ and countable $b \subseteq \omega_2 - W$, so that for every $M \in \mathcal{S}$, if $M \supseteq a$ and $(M \cup \text{Limit}(M)) \cap b = \emptyset$ then M does not overlap x.

If $x \in T - E$ then there is some $\beta < \text{height}(x)$ so that no ordinal in $[\beta, \text{height}(x)]$ is an extensive overlap point for x. Let β_x denote the least such. Restricting the class $\mathcal{S}$ we may assume that every $M \in \mathcal{S}$ is elementary relative to $T - E$ and closed under the function $x \mapsto \beta_x$.

We use $\text{Level}(\alpha)$ to denote level α of T, namely the set $\{x \in T \mid \text{height}(x) = \alpha\}$. For $x \in \text{Level}(\alpha)$ and $\beta < \alpha$, we use $\text{Proj}(x, \beta)$ to denote the $<_T$ predecessor of x on level β of T. More generally, if u is a (non cofinal) branch of T height α, and $\beta < \alpha$, then we use $\text{Proj}(u, \beta)$ to denote the unique node in $u \cap \text{Level}(\beta)$. If u can be capped by some node $x \in T$ then this is the same as $\text{Proj}(x, \beta)$, but we use the notation also in cases where u is not capped in T.

Recall from Definition 1.1 that $M \in \mathcal{S}$ *overlaps* $x \in T$ if there is a (non cofinal) branch u of T so that $u \in M$, $x \in u$, and $x \notin M$. In this case in fact $\text{height}(x) \notin M$, since otherwise x, which is definable from u and $\text{height}(x)$, would belong to M. There are ordinals in M above $\text{height}(x)$, since the height of u is such an ordinal. For α the least ordinal of M above $\text{height}(x)$, there is a unique branch z of T of height α that belongs to M with $x \in z$. The reason is that any two distinct branches of height α, that both belong to M, must diverge below $\sup(M \cap \alpha) < \text{Level}(x)$, and hence cannot both extend x. Define $\text{Lift}(x, M)$ to denote this unique z.

If M overlaps x, and is elementary relative to a weakly specializing function φ, then $\varphi(x)$ must be outside M. (If $\xi = \varphi(x) \in M$ then x would belong to M by elementarity, being the only element of u on which φ takes value ξ.) This motivates condition (8) in Definition 4.3. We noted already in Section 1 that $\varphi(x)$ cannot drop below an extensive overlap point for x, and this motivates condition (3).

Definition 4.3. $\mathbb{S}\text{pecialize} = \mathbb{S}\text{pecialize}(T, \mathcal{S}, \mathcal{T})$ consists of pairs $\langle s, \varphi \rangle$ where:

(1) $s \in \mathbb{P}_{\text{side}}$.
(2) (Domain) φ is a finite partial function on $T - E$. For every $x \in \text{dom}(\varphi)$, $\text{height}(x) \in \text{Limit}(Ot)$, there is $M \in s$ with $\text{height}(x) \leq \sup(M \cap \text{Ord})$, and for the least such M, either $\text{height}(x) = \sup(M \cap \text{Ord})$, or else M is countable, $\text{height}(x) \in M$, and $(\text{height}(x), \sup(M \cap \text{Ord})) \cap Ot = \emptyset$.
(3) (Lower bound) For all $x \in \text{dom}(\varphi)$, $\varphi(x) \geq \beta_x$.
(4) (Pressing down) For all $x \in \text{dom}(\varphi)$, $\varphi(x) < \text{height}(x)$.
(5) (Chain injectivity) If $x, y \in \text{dom}(\varphi)$ are comparable in T then $\varphi(x) \neq \varphi(y)$.
(6) (Interior) If $M \in s$ and $x \in \text{dom}(\varphi) \cap M$, then $\varphi(x) \in M$.
(7) (Space) Let $M \in s$ be countable, and let $U \in s$ with $\sup(U \cap \omega_1) > \sup(M \cap \omega_1)$. Then there exists countable $e \in U$ so that $e \supseteq U \cap \{\text{Proj}(z, \alpha) \mid z \in M$ and $\alpha \in \text{Limit}(M)\}$.
(8) (Overlap) Suppose $x \in \text{dom}(\varphi)$ is overlapped by $M \in s$. Then $\varphi(x) \notin M$.

The ordering on $\mathbb{S}\text{pecialize}$ is given by $\langle s^*, \varphi^* \rangle \leq \langle s, \varphi \rangle$ iff $s^* \supseteq s$ and $\varphi^* \supseteq \varphi$.

Claim 4.4. *For transitive nodes, the interior condition (6) of Definition 4.3 follows from the pressing down condition (4).*

Proof. If $x \in \text{dom}(\varphi) \cap W$ for transitive W, then by elementarity $\text{height}(x) \in W$, hence by condition (4), $\varphi(x) < \sup(W \cap \text{Ord})$, and by transitivity of W, $\varphi(x) \in W$. □

Claim 4.5. *The space condition of Definition 4.3 follows from its restriction to countable U which occur before M.*

Proof. Note first that the condition holds automatically for transitive U, since $U \cap \{\text{Proj}(z, \alpha) \mid z \in M \text{ and } \alpha \in \text{Limit}(M)\}$ is a countable subset of U, and U is internal on a club, meaning in particular that every countable subset of U is contained in a countable element of U.

If U is countable and $M \in U$, then $\{\text{Proj}(z, \alpha) \mid z \in M \text{ and } \alpha \in \text{Limit}(M)\} \in U$ and the space condition is clear.

If U is countable, U occurs above M in s, and $M \notin U$, then there must be some transitive $W \in U$ occurring in s above M. Since $U \cap W$ occurs in s before U, we may by induction assume that the space condition holds for $U \cap W$ and M. Let $e \in U \cap W$ witness this. Since $\{\text{Proj}(z, \alpha) \mid z \in M \text{ and } \alpha \in \text{Limit}(M)\} \subseteq W$, we have $U \cap \{\text{Proj}(z, \alpha) \mid z \in M \text{ and } \alpha \in \text{Limit}(M)\} \subseteq U \cap W$, and hence $U \cap \{\text{Proj}(z, \alpha) \mid z \in M \text{ and } \alpha \in \text{Limit}(M)\} \subseteq e$. So e also witnesses the space condition for U and M. □

The *universe* of a condition $\langle s, \varphi \rangle$, denoted $v(s, \varphi)$, is the smallest set that contains $s \cup \text{dom}(\varphi) \cup \text{range}(\varphi)$ and is closed under the following operations: $M \mapsto \sup(M \cap \text{Ord})$; $x \mapsto \text{height}(x)$ (for $x \in T$); $z, \beta \mapsto \text{Proj}(z, \beta)$; $x, Q \mapsto \text{Lift}(x, Q)$. It is easy to check that $v(s,\varphi)$ is finite, using the facts that s and φ are finite and that closure under the first two operators by itself produces all the ordinals in $v(s,\varphi)$.

We say that two conditions $\langle s,\varphi\rangle$ and $\langle t,\chi\rangle$ are *isomorphic* if there is a bijection $i\colon v(s,\varphi)\to v(t,\chi)$ which preserves the truth values of each of the following statements (meaning that truth value is preserved when applying i to the variables of the statement and replacing s and φ by t and χ): $M\in s$; M occurs before N in s; $M\cap W=N$; $M\supseteq N$; $x\in M$; $\alpha\in M$; $\alpha=\sup(M\cap\mathrm{Ord})$; $\alpha<\beta$; $x<_T y$; $\mathrm{height}(x)=\alpha$; $\mathrm{Proj}(z,\beta)=y$; M overlaps x; $\mathrm{Lift}(x,M)=u$; $\varphi(x)=\xi$.

$\langle s,\varphi\rangle$ and $\langle t,\chi\rangle$ are *isomorphic at* $\langle Q,\bar{Q}\rangle$ if in addition $i(Q)=\bar{Q}$ and $i\restriction Q$ and $i^{-1}\restriction\bar{Q}$ are both identity. i is then called an *isomorphism at* $\langle Q,\bar{Q}\rangle$. Since $\bar{Q}$ is determined uniquely from the other objects, we sometimes omit it from the notation and say that $\langle t,\chi\rangle$ is *isomorphic to* $\langle s,\varphi\rangle$ *at* Q. For $a\subseteq Q$ we say that $\langle t,\chi\rangle$ is *isomorphic to* $\langle s,\varphi\rangle$ *at* Q *above* a if in addition $\bar{Q}\supseteq a$.

The *type* of a condition $\langle s,\varphi\rangle$ is its isomorphism class. Since $v(s,\varphi)$ is finite and only finitely many formulas have to be preserved by isomorphisms, the type is completely determined by a finite truth table, and can therefore be coded by a natural number. The *type at* Q of a condition $\langle s,\varphi\rangle$ with $Q\in s$ is the class of $\langle t,\chi\rangle$ which are isomorphic to $\langle s,\varphi\rangle$ at Q. The type of $\langle s,\varphi\rangle$ at Q is completely determined by a finite subset of Q and a finite truth table, and can therefore be coded by an element of Q.

Lemma 4.6. *Let $\langle s,\varphi\rangle\in\mathbb{S}\mathrm{pecialize}$ and let $Q\in s$. Then there is a finite $a\subseteq Q$ so that for every $\langle t,\chi\rangle\in Q\cap\mathbb{S}\mathrm{pecialize}$ which is isomorphic to $\langle s,\varphi\rangle$ at Q above a, either $\langle t,\chi\rangle$ is compatible with $\langle s,\varphi\rangle$, or else there is $y\in\mathrm{dom}(\chi)-\mathrm{dom}(\varphi)$ and $x\in v(s,\varphi)$ so that $y<_T x$ and $\chi(y)\in v(s,\varphi)$.*

Proof. Let $a\subseteq Q$ be a finite set containing witnesses e for all instances of the space condition of Definition 4.3 for $\langle s,\varphi\rangle$ with $U=Q$.

Let i witness that $\langle s,\varphi\rangle$ and $\langle t,\chi\rangle$ are isomorphic at Q above a. Since $i\restriction Q=\mathrm{id}$, $\mathrm{res}_Q(s)\subseteq t$. It follows by Lemma 2.3 that s and t are compatible, and in fact there is r witnessing this which is equal to $s\cup t$ if Q is transitive, and to the closure of $s\cup t$ under intersections if Q is countable. Since $i\restriction Q$ is the identity, $x\in Q\to\varphi(x)\in Q$, and $\langle t,\chi\rangle\in Q$, the maps φ and χ must agree on their common domain. So $\upsilon=\varphi\cup\chi$ is a function. υ inherits the lower bound and pressing down conditions, (3) and (4), of Definition 4.3 from χ and φ. We work toward establishing the remaining conditions, and in cases where an argument for one of these conditions fails, show that there are y,x as in the lemma.

Claim 4.7. *The domain condition (2) of Definition 4.3 holds for $\langle r,\upsilon\rangle$.*

Proof. The only cases of the condition which do not automatically transfer from $\langle s,\varphi\rangle$ and $\langle t,\chi\rangle$ to $\langle r,\upsilon\rangle$ are ones where $\mathrm{height}(x)\in M$ (as opposed to $\mathrm{height}(x)=\sup(M\cap\mathrm{Ord})$). We work only on these cases.

Let $x\in\mathrm{dom}(\chi)$, let M witness condition (2) of Definition 4.3 for x in $\langle t,\chi\rangle$, and suppose $\mathrm{height}(x)\in M$. To see that M continues to witness the condition in $\langle r,\upsilon\rangle$, it is enough to prove that there is no node $\bar{M}\in r$ before M with $\mathrm{height}(x)\leq\sup(\bar{M}\cap\mathrm{Ord})$. But if such a node exists, then the predecessor of M in r is such a node, and this is impossible since the predecessor of M in r is an element of $M\subseteq Q$, hence itself a node in t.

Let $x\in\mathrm{dom}(\varphi)-\mathrm{dom}(\chi)$. If $\mathrm{height}(x)\in Q$, then $\mathrm{height}(i(x))=\mathrm{height}(x)$, and since by the previous paragraph condition (2) holds for $i(x)$ in $\langle r,\upsilon\rangle$, it holds also for x. Suppose $\mathrm{height}(x) \notin Q$. Let M witness condition (2) for x in s, and suppose $\mathrm{height}(x) \in M$. Then $M \notin Q$. If the predecessor of M in r is the same as its predecessor in s, then as in the previous paragraph M witnesses condition (2) for x in $\langle r, \upsilon\rangle$. Since $M \notin Q$ the only other possibility is that M is either Q or the bottom node $Q \cap W$ of a residue gap of s in Q. But in each of these cases $\mathrm{height}(x) \in M \to \mathrm{height}(x) \in Q$, contradicting an earlier assumption. □

Claim 4.8. *If the chain injectivity condition (5) of Definition 4.3 fails for $\langle r, \upsilon\rangle$, then there is $y \in \mathrm{dom}(\chi) - \mathrm{dom}(\varphi)$ and $x \in \upsilon(s, \varphi)$ so that $y <_T x$ and $\chi(y) \in \upsilon(s, \varphi)$.*

Proof. Suppose $y <_T x$ are both in $\mathrm{dom}(\upsilon)$ and $\upsilon(y) = \upsilon(x)$. x and y cannot both belong to $\mathrm{dom}(\varphi)$, and cannot both belong to $\mathrm{dom}(\chi)$, since the chain injectivity condition holds for each of $\langle s, \varphi\rangle$ and $\langle t, \chi\rangle$. If $x \in \mathrm{dom}(\varphi) - \mathrm{dom}(\chi)$ and $y \in \mathrm{dom}(\chi) - \mathrm{dom}(\varphi)$, then x and $\chi(y) = \varphi(x)$ both belong to $\upsilon(s, \varphi)$, and we are done. Suppose instead that $x \in \mathrm{dom}(\chi) - \mathrm{dom}(\varphi)$ and $y \in \mathrm{dom}(\varphi) - \mathrm{dom}(\chi)$. Then in particular $x \in Q$ and $y \notin Q$ (as $y \in \mathrm{dom}(\varphi) \cap Q \to y = i(y) \in \mathrm{dom}(\chi)$). Since $y <_T x$ it follows that Q overlaps y. By the overlap condition (8) of Definition 4.3 for $\langle s, \varphi\rangle$ it follows that $\varphi(y) \notin Q$. But then since $\chi(x) \in Q$ it follows in particular that $\upsilon(y) = \varphi(y) \neq \chi(x) = \upsilon(x)$. □

Claim 4.9. *The interior condition (6) of Definition 4.3 holds for $\langle r, \upsilon\rangle$.*

Proof. It is enough to verify the condition for nodes in $s \cup t$, since the interior condition for any two nodes implies the same condition for their intersection. By Claim 4.4 it is enough to verify the condition for countable $M \in s \cup t$. Fix M, and let $x \in \mathrm{dom}(\upsilon) \cap M$. If $M \in s \wedge x \in \mathrm{dom}(\varphi)$ or $M \in t \wedge x \in \mathrm{dom}(\chi)$ then the interior condition for M and x is directly inherited from the same condition in $\langle s, \varphi\rangle$ and $\langle t, \chi\rangle$. The case that $M \in t - s$ and $x \in \mathrm{dom}(\varphi) - \mathrm{dom}(\chi)$ is impossible, since $x \in M \in t \to x \in Q$, and $x = i(x) \in \mathrm{dom}(\chi)$ for any $x \in \mathrm{dom}(\varphi) \cap Q$.

The only remaining case is that $M \in s - t$ and $x \in \mathrm{dom}(\chi) - \mathrm{dom}(\varphi)$. We work on this case, first with transitive Q, and then with countable Q.

If Q is transitive, then M occurs above Q. Let W be the largest transitive node of s at or above Q. Then $x \in \mathrm{dom}(\chi) \subseteq Q \subseteq W$. Hence $x \in M \cap W$. The interior condition for M and x therefore reduces to the same condition for $M \cap W$ and x, which we may assume holds by induction.

Suppose Q is countable, and M occurs at or above Q. If there are no transitive nodes of s between Q and M, then $Q \subseteq M$, hence $\chi(x) \in M$, as required. If there are transitive nodes of s between Q and M, then letting W be the largest one, we have that $M \cap W$ is a node of s occurring before M, and as in the previous paragraph, the interior condition for M and x reduces to the same condition for $M \cap W$ and x, since $x \in Q \subseteq W$. The case that Q is countable and M occurs in a residue gap $[Q \cap R, R)$ of s in Q is similar, but using $Q \cap R$ instead of Q. The only additional observations needed for this case are that $x \in Q \cap R$ (since $x \in M \subseteq R$) and $\chi(x) \in Q \cap R$ (since $\chi(x) < \mathrm{height}(x) < \sup(R \cap \mathrm{Ord})$). □

Claim 4.10. *The space condition (7) of Definition 4.3 holds for r.*

Proof. It is enough to establish the weak space condition in Claim 4.5. Suppose first that Q is transitive. We prove the condition in Claim 4.5 by induction on U. The only instances which are not directly inherited from the space condition for s and t are instances where $M \in s - t$ and $U \in t - s$. M must then occur above Q, and U must occur above $i(Q)$. The latter in particular implies that there is a transitive node $W \in U$ that occurs at or above $i(Q)$. By our definition of a above, there is a witness e for the space condition for M and Q (standing for U) in a. This witness then belongs to $i(Q)$. So $U \cap \{\mathrm{Proj}(z,\alpha) \mid z \in M$ and $\alpha \in \mathrm{Limit}(M)\} \subseteq Q \cap \{\mathrm{Proj}(z,\alpha) \mid z \in M$ and $\alpha \in \mathrm{Limit}(M)\} \subseteq e \subseteq i(Q) \subseteq W$. It follows that the space condition for M and U reduces to the space condition for M and $U \cap W$, which holds by induction.

Suppose next that Q is countable. The countable nodes of r are the ones in s, in t, and in tacked-on sequences. The latter have the form $P \cap W$ for countable $P \in s - t$ and transitive $W \in t - s$. The space condition for $M = P \cap W$ and any U is an immediate consequence of the same condition for $M' = P$ and U, since $\{\mathrm{Proj}(z,\alpha) \mid z \in P \cap W$ and $\alpha \in \mathrm{Limit}(P \cap W)\} \subseteq \{\mathrm{Proj}(z,\alpha) \mid z \in P$ and $\alpha \in \mathrm{Limit}(P)\}$. The space condition for M and $U = P \cap W$ is a consequence of the same condition for M and $U' = P$, since given $e' \in P$ witnessing the latter, by elementarity of P and the fact that W is internal on a club, one can find countable $e \in P \cap W$ with $e \supseteq e' \cap W$. It is therefore enough to verify instances of the space condition for nodes in $s \cup t$. We do this by induction on U. We consider only the cases that $M \in s - t \wedge U \in t - s$ or $M \in t - s \wedge U \in s - t$. For other instances the condition is directly inherited from the space condition for s and t.

Suppose $M \in s - t$ and $U \in t - s$. Let $\hat{U} = i^{-1}(U)$. Then $\hat{U} \in s - Q$. Suppose for definitiveness that $\hat{U}$ occurs in a residue gap $[Q \cap W, W)$ of s in Q. The case that $\hat{U}$ occurs above Q is similar. By assumption of the space condition, $\sup(U \cap \omega_1) > \sup(M \cap \omega_1)$. Since $U \in Q$ this implies $\sup(Q \cap \omega_1) > \sup(M \cap \omega_1)$. By the space condition for M and Q in s, there is countable $e \in Q$ containing $Q \cap \{\mathrm{Proj}(z,\alpha) \mid z \in M$ and $\alpha \in \mathrm{Limit}(M)\}$. By our choice of a, such e can be found in a, and hence in $i(Q)$. By elementarity of $i(Q)$ and since W is internal on a club, there exists some countable $e' \in i(Q) \cap W$ with $e' \supseteq e \cap W$. Note $e' \supseteq U \cap \{\mathrm{Proj}(z,\alpha) \mid z \in M$ and $\alpha \in \mathrm{Limit}(M)\}$ since $U \subseteq Q \cap W$. If there are no transitive nodes in s between $Q \cap W$ and $\hat{U}$, then $\hat{U} \supseteq Q \cap W$, and since i is an isomorphism, $U \supseteq i(Q) \cap i(W) = i(Q) \cap W$. So e' belongs to U, and witnesses the space condition for M and U. If there are transitive nodes in s between $Q \cap W$ and $\hat{U}$, then since i is an isomorphism, there are transitive nodes in t between $i(Q) \cap W$ and U. Let R be the largest one. Then $R \in U$, and $e' \subseteq i(Q) \subseteq R$. The latter implies that $U \cap \{\mathrm{Proj}(z,\alpha) \mid z \in M$ and $\alpha \in \mathrm{Limit}(M)\} \subseteq U \cap R$. Hence the space condition for M and U reduces to the space condition for M and $U \cap R$, which holds by induction.

Suppose $M \in t - s$ and $U \in s - t$. Since U occurs before M, it must occur before Q. So U belongs to a residue gap of s in Q, say $[Q \cap W, W)$. Let $e = W \cap \{\mathrm{Proj}(z,\alpha) \mid z \in M$ and $\alpha \in \mathrm{Limit}(M)\}$. By elementarity of Q and since $W, M \in Q$, we have $e \in Q$. Since W is internal on a club, and again using the elementarity of Q, there is countable $e' \in Q \cap W$ so that $e' \supseteq e$. If there are no transitive nodes in s between $Q \cap W$ and U, then e' belongs to U, and witnesses the space condition for M and U since $U \subseteq W$. If there are transitive nodes in s between $Q \cap W$ and U, let R be the largest one. Then $e' \subseteq R$. It follows that $U \cap \{\mathrm{Proj}(z,\alpha) \mid z \in M$ and $\alpha \in \mathrm{Limit}(M)\} \subseteq R$, so the space condition for M and U reduces to the space condition for M and $U \cap R$, which holds by induction. □

Claim 4.11. *The overlap condition (8) of Definition 4.3 holds for $\langle r, v \rangle$ in the case that $x \in \mathrm{dom}(\varphi)$ and $M \in t$.*

Proof. Let $u \in M$ witness that M overlaps of x. Note $u \in Q$ since $M \subseteq Q$.

If $x \in Q$ then $x = i(x) \in \mathrm{dom}(\chi)$, and condition (8) for x and M is inherited from $\langle t, \chi \rangle$. Suppose $x \notin Q$. Since $u \in Q$ this implies that Q overlaps x. (Q must be countable, since otherwise $u \subseteq Q$ and $x \in u \to x \in Q$.) Hence $\varphi(x) \notin Q$ by condition (8) of Definition 4.3 applied to $\langle s, \varphi \rangle$. In particular $\varphi(x) \notin M$. □

Claim 4.12. *The overlap condition (8) of Definition 4.3 holds $\langle r, \upsilon \rangle$ in the case that $x \in \mathrm{dom}(\chi)$, $M \in s$, and $\sup(M \cap \omega_1) \geq \sup(Q \cap \omega_1)$.*

Proof. Since $\sup(M \cap \omega_1) \geq \sup(Q \cap \omega_1)$, Q is countable. The fact that $\sup(M \cap \omega_1) \geq \sup(Q \cap \omega_1)$ also implies that either $M \supseteq Q$ or there is some transitive node $W \in Q$ so that $M \supseteq Q \cap W$ and $(M \cup \mathrm{Limit}(M)) \cap (Q - W) = \emptyset$. To see this, suppose otherwise, and let M be a minimal counterexample. If M occurs at or above Q, then since $M \not\supseteq Q$ there is transitive $R \in M$ occurring above Q. But in this case $M \cap R$ gives a smaller counterexample. If M occurs in a residue gap $[Q \cap W, W)$ of s in Q, and there is a transitive $R \in M$ above $Q \cap W$, then again $M \cap R$ gives a smaller counterexample. If there is no such R, then $M \supseteq Q \cap W$, and $(M \cup \mathrm{Limit}(M)) \cap (Q - W) = \emptyset$.

$M \supseteq Q$ is impossible here since $x \in Q$ is overlapped by M and therefore $x \notin M$. So there is a transitive $W \in Q$ so that $M \supseteq Q \cap W$, and $(M \cup \mathrm{Limit}(M)) \cap (Q - W) = \emptyset$. Let $\beta = \sup(W \cap \mathrm{Ord})$. Note $\beta \leq \mathrm{height}(x)$, since otherwise $\mathrm{height}(x) \in Q \cap W \subseteq M$, contradicting the fact that M overlaps x.

We show below that β is an extensive overlap point for x. This is enough to establish the current claim, since it implies that $\beta_x > \beta$, hence $\chi(x) > \beta$ by condition (3) of Definition 4.3, and in particular $\chi(x) \notin M$, since $\chi(x) \in Q$ and $M \cap (Q - W) = \emptyset$.

Suppose for contradiction that β is not an extensive overlap point for x, and let $a \subseteq \beta$ and $b \subseteq \omega_2 - \beta$ witness this. By elementarity of Q, we may pick $a, b \in Q$. Then $a, b \subseteq Q$. So $M \supseteq Q \cap W \supseteq Q \cap \beta \supseteq a$, and $(M \cup \mathrm{Limit}(M)) \cap b \subseteq (M \cup \mathrm{Limit}(M)) \cap (Q - W) = \emptyset$ using the properties of M obtained above. Since M overlaps x this contradicts the fact that a and b witness that β is not an extensive overlap point. □

Claim 4.13. *If the overlap condition (8) of Definition 4.3 fails for $\langle r, \upsilon \rangle$, then there exists $y \in \mathrm{dom}(\chi) - \mathrm{dom}(\varphi)$ and $x \in \upsilon(s, \varphi)$ so that $y <_T x$ and $\chi(y) \in \upsilon(s, \varphi)$.*

Proof. Let $y \in \mathrm{dom}(\upsilon)$ and let $M \in r$ overlap y. Let $u = \mathrm{Lift}(y, M) \in M$ witness the overlap. Without loss of generality we may assume that $M \in s \cup r$. (The only other possibility is that M belongs to a tacked-on sequence. Then M has the form $M^* \cap W$ for some $M^* \in s$ and transitive $W \in t$, and we may replace M by M^*.) Suppose y and M witness the failure of the overlap condition for $\langle r, \upsilon \rangle$. Then $y \in \mathrm{dom}(\chi) - \mathrm{dom}(\varphi)$, $M \in s - t$, and $\sup(M \cap \omega_1) < \sup(Q \cap \omega_1)$, since in the other configurations, the overlap condition is either inherited from the same condition for $\langle s, \varphi \rangle$ and $\langle t, \chi \rangle$, or given by Claims 4.11 and 4.12.

We work to prove that $u \in \upsilon(s, \varphi)$. From u we will then derive $x >_T y$ in $\upsilon(s, \varphi)$, and the structure we obtain for the proof will allow us show that $\chi(y) \in \upsilon(s, \varphi)$.

Let R_0 be the least transitive node of s with $R_0 \in M$ and $\mathrm{height}(y) < \sup(R_0 \cap \mathrm{Ord})$. Such a node must exist: otherwise all nodes P of s before M with $\sup(P \cap \mathrm{Ord}) > \mathrm{height}(y)$ are countable, hence they all belong to M. Then since $M \notin Q$, $\sup(Q \cap \omega_1) > \sup(M \cap \omega_1)$, and $\sup(Q \cap \mathrm{Ord}) > \mathrm{height}(y)$, it must be that M occurs inside a residue gap $[Q \cap W, W)$ of s in Q for some W above M, and that

$\sup(Q \cap W \cap \mathrm{Ord}) \leq \mathrm{height}(y)$. But this is impossible since $\mathrm{height}(y) \in Q$ and $\mathrm{height}(y) < \sup(M \cap \mathrm{Ord}) < \sup(W \cap \mathrm{Ord})$.

Suppose, for contradiction, that $\sup(M \cap R_0 \cap \mathrm{Ord}) > \mathrm{height}(y)$. Then it must be that $M \cap R_0 \in Q$, since otherwise an argument as in the previous paragraph produces a transitive $R \in M \cap R_0$ with $\mathrm{height}(y) < \sup(R \cap \mathrm{Ord})$, contradicting the minimality of R_0. In particular $i(M \cap R_0) = M \cap R_0$, $M \cap R_0 \in t$, and $M \cap R_0 \subseteq i(Q)$. Let $P \in t$ witness condition (2) of Definition 4.3 for $\mathrm{height}(y)$. Since $\mathrm{height}(y) \notin M \cap R_0$ it must be that P occurs in a residue gap of t in $M \cap R_0$, say $[M \cap R', R')$ for $R' \in M \cap R_0$ a node of t. Since i^{-1} is identity on $i(Q)$ we have $i^{-1}(R') = R'$ and therefore R' is a node of s. But this contradicts the minimality of R_0.

By the previous paragraph, $\sup(M \cap R_0 \cap \mathrm{Ord}) \leq \mathrm{height}(y)$. Working inductively define a descending sequence of transitive nodes of s in M, starting from R_0, as follows. If $M \cap R_n \in Q$, end the definition. Otherwise, $M \cap R_n$ must belong to a residue gap of s in Q, say $[Q \cap W_n, W_n)$. Since $\sup(M \cap R_n \cap \omega_1) < \sup(Q \cap W_n \cap \omega_1)$ there must be a transitive node of s in $M \cap R_n$ above $Q \cap W_n$. Let R_{n+1} be the least such.

Let k be largest so that R_k is defined. Then $M \cap R_k \in Q$. Let $\gamma_n = \sup(M \cap R_n \cap \mathrm{Ord})$ and $\gamma_n^* = \sup(R_n \cap \mathrm{Ord})$ for $n \leq k$. Let $\beta_n = \sup(Q \cap W_n \cap \mathrm{Ord})$ and $\beta_n^* = \sup(W_n \cap \mathrm{Ord})$ for $n < k$. Then $\beta_{n+1}^* < \beta_n < \gamma_{n+1}^* < \gamma_n < \beta_n^*$.

Note that $\mathrm{height}(u) = \gamma_0^*$ as γ_0^* is the first ordinal of M above $\mathrm{height}(y)$. Note further that $\beta_0^* \leq \mathrm{height}(y)$, since $\mathrm{height}(y)$, being an element of Q, cannot belong to the interval $[\beta_0, \beta_0^*)$. By closure of $v(s, \varphi)$ under lifting, and using the facts that $y \in Q$ and $u \in M$ to see that the relevant lifts are defined, we have $\mathrm{Proj}(u, \gamma_n) \in v(s, \varphi) \to \mathrm{Proj}(u, \gamma_n^*) = \mathrm{Lift}(\mathrm{Proj}(u, \gamma_n), M) \in v(s, \varphi)$ and $\mathrm{Proj}(u, \beta_n) \in v(s, \varphi) \to \mathrm{Proj}(u, \beta_n^*) = \mathrm{Lift}(\mathrm{Proj}(u, \beta_n), Q) \in v(s, \varphi)$. We also have $\mathrm{Proj}(u, \gamma_{n+1}^*) \in v(s, \varphi) \to \mathrm{Proj}(u, \beta_n) \in v(s, \varphi)$ and $\mathrm{Proj}(u, \beta_n^*) \in v(s, \varphi) \to \mathrm{Proj}(u, \gamma_n) \in v(s, \varphi)$, by closure of $v(s, \varphi)$ under projections. Putting these together, it follows that $\mathrm{Proj}(u, \gamma_k) \in v(s, \varphi) \to u = \mathrm{Proj}(u, \gamma_0^*) \in v(s, \varphi)$. We continue to show that $\mathrm{Proj}(u, \gamma_k) \in v(s, \varphi)$.

By the space condition (7) of Definition 4.3, there is a countable $e \in Q$ so that $e \supseteq Q \cap \{\mathrm{Proj}(z, \alpha) \mid z \in M \text{ and } \alpha \in \mathrm{Limit}(M)\}$. By our choice of a at the start of the proof of Lemma 4.6, $e \in a \subseteq i(Q)$. Since γ_k and y both belong to Q we have $\mathrm{Proj}(y, \gamma_k) \in Q$. As $\mathrm{Proj}(y, \gamma_k) = \mathrm{Proj}(u, \gamma_k)$ it follows that $\mathrm{Proj}(y, \gamma_k) \in e$, and therefore $\mathrm{Proj}(y, \gamma_k) \in i(Q)$.

$M \cap R_k$ is a node of s that belongs to Q, so $i(M \cap R_k) = M \cap R_k$ and hence $M \cap R_k \in t$. Since $y \in \mathrm{dom}(\chi)$ it follows that $\mathrm{Proj}(y, \gamma_k) \in v(t, \chi)$. Combining this with the conclusion of the previous paragraph and the fact that i^{-1} is identity on $i(Q)$, we get $\mathrm{Proj}(y, \gamma_k) = i^{-1}(\mathrm{Proj}(y, \gamma_k)) \in v(s, \varphi)$.

We have so far established that $u = \mathrm{Lift}(y, M)$ belongs to $v(s, \varphi)$. We also saw that $\mathrm{height}(u) = \gamma_0^* = \sup(R_0 \cap \mathrm{Ord})$. If R_0 belongs to a residue gap of s in Q, then let P be the bottom node of the gap, and let $\alpha = \sup(P \cap \mathrm{Ord})$. If R_0 occurs above Q in s, then let $\alpha = \sup(Q \cap \mathrm{Ord})$. If $R_0 \in Q$, then let P be the largest node of s below R_0 and let $\alpha = \sup(P \cap \mathrm{Ord})$. In each of these cases we have $\alpha \in v(s, \varphi)$ and $\mathrm{height}(y) \leq \alpha < \gamma_0^*$. So letting $x = \mathrm{Proj}(u, \alpha)$ we have $x \in v(s, \varphi)$ and $y <_T x$.

By construction, $M \cap \gamma_n^* \subseteq \gamma_n$ and $Q \cap \beta_n^* \subseteq \beta_n$. It follows using the inequalities on these ordinals established above that $Q \cap M \cap \gamma_0^* \subseteq \gamma_k$. Since $\chi(y) < \mathrm{height}(y) < \gamma_0^*$ belongs to both Q and M (the latter because y and M witness failure of the overlap condition for $\langle r, v\rangle$) we therefore have $\chi(y) < \gamma_k$, and this implies that $\chi(y) \in M \cap R_k$. Since $M \cap R_k$ is a node of s that belongs to Q we have $M \cap R_k = i(M \cap R_k) \subseteq i(Q)$. So $\chi(y) \in i(Q)$. This implies that $\chi(y) = i^{-1}(\chi(y))$ and hence in particular $\chi(y) \in v(s, \varphi)$. □

Claim 4.13 completes the proof of Lemma 4.6. □

Remark 4.14. The proof of Lemma 4.6 gives more information on the obstructions for compatibility than is stated in the lemma. For $\langle t, \chi\rangle$ as in the lemma the proof establishes one of the following conditions:

(1) $\langle s, f\rangle$ and $\langle t, \chi\rangle$ are compatible, and moreover there is a condition $\langle r, v\rangle$ witnessing this so that $r = s \cup t$ if Q is transitive, and r is equal to the closure of $s \cup t$ under intersections if Q is countable.
(2) There is $x \in \mathrm{dom}(\varphi) - \mathrm{dom}(\chi)$ and $y \in \mathrm{dom}(\chi) - \mathrm{dom}(\varphi)$ so that $y <_T x$ and $\chi(y) = \varphi(x)$.
(3) There is $y \in \mathrm{dom}(\chi) - \mathrm{dom}(\varphi)$ and $M \in s$ overlapping y, so that $\sup(M \cap \omega_1) < \sup(Q \cap \omega_1)$, $\mathrm{Lift}(y, M) \in v(s, \varphi)$, and $\chi(y) \in M$.

Conditions (2) and (3) were obtained in the proofs of Claims 4.8 and 4.13, from failure of one of the conditions of Definition 4.3 for $\langle r, v\rangle$.

Claim 4.15. *Let $\langle s, \varphi\rangle \in \mathbb{S}\mathrm{pecialize}$, let W be a transitive node, and suppose that either $s \subseteq W$ or there exists $Q \in s$ so that $\mathrm{res}_Q(s) \subseteq W \in Q$. Then there is $\langle r, \varphi\rangle \leq \langle s, \varphi\rangle$ with $W \in r$.*

Proof. If $s \subseteq W$, then $\langle s \cup \{W\}, \varphi\rangle$ is easily seen to be a condition in $\mathbb{S}\mathrm{pecialize}$. Suppose $\mathrm{res}_Q(s) \subseteq W \in Q$ for Q in s. By Lemma 2.3, $\mathrm{res}_Q(s) \cup \{W\}$ and s are compatible. Let r be a side condition witnessing this. If Q is transitive, we can take $r = s \cup \{W\}$ and it is easy to check that $\langle r, \varphi\rangle \in \mathbb{S}\mathrm{pecialize}$. Suppose Q is countable. Then we can take r to be the closure of $s \cup \{W\}$ under intersections, and by the discussion following Lemma 2.3, r is obtained from $s \cup \{W\}$ by adding one tacked-on sequence, the sequence F_W associated to W, right before W. The lowest element of this sequence is $Q \cap W$.

The space condition (7) of Definition 4.3 holds for r by an argument similar to parts of the proof of Claim 4.10. The interior condition transfers from countable M to intersections $M \cap W$, and hence transfers from the countable nodes of $\langle s, \varphi\rangle$ to all countable nodes of $\langle r, \varphi\rangle$. By Claim 4.4 it then holds for $\langle r, \varphi\rangle$. The overlap condition similarly transfers from countable $M \in s$ to $M \cap W$, and hence holds for $\langle r, \varphi\rangle$. The lower bound, pressing down, and chain injectivity conditions do not involve any nodes, and hold for φ. Finally, all instances of the domain condition trivially transfer from $\langle s, \varphi\rangle$ to $\langle r, \varphi\rangle$ except when $M = Q$, since for $M \neq Q$, the predecessor of M in r is the same as its predecessor in s. In case $M = Q$ the condition again transfers trivially if $\mathrm{height}(x) = \sup(M \cap \mathrm{Ord})$. If $\mathrm{height}(x) < \sup(M \cap \mathrm{Ord})$ then by the domain condition, $(\mathrm{height}(x), \sup(M \cap \mathrm{Ord})) \cap Ot = \emptyset$, so it must be that $\mathrm{height}(x) \geq \sup(W \cap \mathrm{Ord})$. In the case of strict inequality M continues to witness the condition in $\langle r, \varphi\rangle$, and in the case of equality the condition is witnessed by W. This establishes that $\langle r, \varphi\rangle \in \mathbb{S}\mathrm{pecialize}$. □

Corollary 4.16. *Let $\langle s, \varphi\rangle \in \mathbb{S}\mathrm{pecialize}$ and let W be a transitive node. Then there is $\langle r, \varphi\rangle \leq \langle s, \varphi\rangle$ with $W \in s$.*

Proof. Immediate by an argument similar to the proof of Claim 2.5. The proof relies on the two properties stated in Remark 2.6. Both these properties hold for $\mathbb{S}$pecialize, by Claim 4.15. $\square$

Claim 4.17. *Let $\langle s, \varphi\rangle \in \mathbb{S}\text{pecialize}$, let Q be a countable node, and suppose $\langle s, \varphi\rangle \in Q$. Then there is $\langle r, \varphi\rangle \leq \langle s, \varphi\rangle$ with $Q \in r$.*

Proof. By Lemma 2.3, and since $s \in Q$ and $s \leq \text{res}_Q(\{Q\})$, there is $r \leq s$ with $Q \in r$. Moreover we can take r to be the closure of $s \cup \{Q\}$ under intersections. By the discussion following Lemma 2.3, r is generated from $s \cup \{Q\}$ by adding, for each transitive $W \in s$, the node $Q \cap W$ right before W. (The tacked-on sequence associated to W consists of just this one node.) The lower bound, pressing down, and chain injectivity conditions of Definition 4.3 transfer from $\langle s, \varphi\rangle$ to $\langle r, \varphi\rangle$ since they are phrased with no reference to nodes. Instances of the overlap condition with $M \in s$ transfer trivially to $\langle r, \varphi\rangle$, and the remaining instances hold vacuously in $\langle r, \varphi\rangle$ because all $x \in \text{dom}(\varphi)$ belong to Q and hence are not overlapped by Q or by any of the nodes $Q \cap W$. Instances of the interior condition with $M \in s$ similarly transfer from $\langle s, \varphi\rangle$ to $\langle r, \varphi\rangle$, the instance with $M = Q$ holds because $\varphi(x) \in Q$ for all $x \in \text{dom}(\varphi)$, and instances with $M = Q \cap W$ hold because for every $x \in \text{dom}(\varphi) \cap W$, $\varphi(x) \in Q \cap \text{height}(x) \subseteq Q \cap W$. The domain condition transfers from $\langle s, \varphi\rangle$ to $\langle r, \varphi\rangle$ since for every countable node $M \in s$, the predecessor of M in r is its predecessor in s. This implies that for every $x \in \text{dom}(\varphi)$, the node M witnessing the domain condition for x in $\langle s, \varphi\rangle$ continues to witness it in $\langle r, \varphi\rangle$.

It remains to consider the space condition. Fix U and M as in the condition. By Claim 4.5 we need only consider instances where U is countable. In all these instances $\sup(M \cap \omega_1) < \sup(U \cap \omega_1) \leq \sup(Q \cap \omega_1)$, and in particular then $M \in s$. Instances with $U \in s$ transfer trivially to $\langle r, \varphi\rangle$ from $\langle s, \varphi\rangle$. Instances with $U = Q$ hold because the set $\{\text{Proj}(z, \alpha) \mid z \in M \text{ and } \alpha \in \text{Limit}(M)\}$ belongs to Q by elementarity. For instances with $U = Q \cap W$, apply the space condition for M and W in $\langle s, \varphi\rangle$, and note that by elementarity of Q, a set e witnessing the condition can be found in Q. The same e then witnesses the condition for M and $Q \cap W$. $\square$

With Lemma 4.6 at hand we can proceed to establish properness of $\mathbb{S}$pecialize for models, countable and of size ω_1, which restrict to elements of $\mathcal{S} \cup \mathcal{T}$. Work below with some fixed regular θ^* so that T, $\mathcal{S}$, $\mathcal{T}$, and $\mathbb{S}$pecialize all belong to $H(\theta^*)$.

Lemma 4.18. *Let $Q^* \prec H(\theta^*)$ with $T, \mathcal{S}, \mathcal{T}, \mathbb{S}\text{pecialize} \in Q^*$. Let $Q = Q^* \cap K$ and suppose $Q \in \mathcal{T}$. Let $\langle s, \varphi\rangle \in \mathbb{S}\text{pecialize}$. Let $D \in Q^*$ be a subset of $\mathbb{S}$pecialize so that $\langle s, \varphi\rangle \in D$. Suppose $Q \in s$. Then there is $\langle t, \chi\rangle \in D \cap Q$ which is compatible with $\langle s, \varphi\rangle$.*

Proof. Let $\mathfrak{tp}$ be the type of $\langle s, \varphi\rangle$ at Q. Let X be the set of $W \in \mathcal{T}$ so that there exists $\langle s_W, \varphi_W\rangle \in D$ with $W \in s_W$ and the type of $\langle s_W, \varphi_W\rangle$ at W equal to $\mathfrak{tp}$.

By elementarity of Q^* and since $\mathfrak{tp}$ can be coded by an element of Q, we have $X \in Q^*$ and may pick the map $W \mapsto \langle s_W, \varphi_W\rangle$ in Q^*.

By the lemma assumptions, $Q \in X$. It follows that for every $b \in K \cap Q^* = Q$, there is $W \in X$ with $b \in W$. By elementarity of Q^*, the same is true for every $b \in K$. The sets $\{W \in X \mid b \in W\}$, for $b \in K$, generate a filter. Let $\mathcal{U}$ be an extension of this filter to an ultrafilter. By elementarity of Q^* we can pick $\mathcal{U} \in Q^*$. Below we write $(\forall^*_{\mathcal{U}} W)$ to mean $(\exists Z \in \mathcal{U})(\forall W \in Z)$.

Fix some $\bar{W} \in X \cap Q$, and let $\langle \bar{s}, \bar{\varphi} \rangle = \langle s_{\bar{W}}, \varphi_{\bar{W}} \rangle$. For each $W \in X$ let $i_W : v(s_W, \varphi_W) \to v(\bar{s}, \bar{\varphi})$ be an isomorphism at $\langle W, \bar{W} \rangle$. By elementarity we can arrange that $W \mapsto i_W$ belongs to Q^*. Let $\hat{i} : v(s, \varphi) \to v(\bar{s}, \bar{\varphi})$ be an isomorphism at $\langle Q, \bar{W} \rangle$. Such isomorphisms exist since the types of $\langle s_W, \varphi_W \rangle$ at W, $\langle s, \varphi \rangle$ at Q, and $\langle \bar{s}, \bar{\varphi} \rangle$ at $\bar{W}$ are all the same.

Let $k = |v(\bar{s}, \bar{\varphi})|^2 + 1$. Let $a \subseteq Q$ be the finite set witnessing Lemma 4.6 for $\langle s, \varphi \rangle$. Working in Q^*, fix a sequence W_n, $n < k$, of elements of X, so that the following holds:

(i) $a \in W_n$ and $(\forall m < n) v(s_{W_m}, \varphi_{W_m}) \subseteq W_n$

(ii) $(\forall m < n)(\forall \bar{y} \in v(\bar{s}, \bar{\varphi}))$ W_n belongs to a set in $\mathcal{U}$ that witnesses either $(\forall^*_{\mathcal{U}} W)(i^{-1}_{W_m}(\bar{y}) <_T i^{-1}_W(\bar{y}))$ or $(\forall^*_{\mathcal{U}} W)(i^{-1}_{W_m}(\bar{y}) \not<_T i^{-1}_W(\bar{y}))$, depending on which of the two holds.

(iii) $(\forall \bar{y} \in v(\bar{s}, \bar{\varphi}))$ W_n belongs to a set in $\mathcal{U}$ that witnesses (the first quantifier in) either $(\forall^*_{\mathcal{U}} V)(\forall^*_{\mathcal{U}} W)(i^{-1}_V(\bar{y}) <_T i^{-1}_W(\bar{y}))$ or $(\forall^*_{\mathcal{U}} V)(\forall^*_{\mathcal{U}} W)(i^{-1}_V(\bar{y}) \not<_T i^{-1}_W(\bar{y}))$, depending on which of the two holds.

(iv) $(\forall m < n)(\forall \bar{y} \in v(\bar{s}, \bar{\varphi}))$ if $(\forall^*_{\mathcal{U}} W)(\beta_{i^{-1}_W(\bar{y})} \geq \sup(W_m \cap \mathrm{Ord}))$ then W_n belongs to a set in $\mathcal{U}$ that witnesses this. (Recall that, for each $x \in T$ which is not extensively overlapped, β_x is an ordinal witnessing this.)

This can be done using the finite completeness of $\mathcal{U}$, and the fact that the sets $\{W \in X \mid b \in W\}$ for $b \in K$ all belong to $\mathcal{U}$. Let $\langle s_n, \varphi_n \rangle = \langle s_{W_n}, \varphi_{W_n} \rangle$ and let $i_n = i_{W_n}$. Note that $\langle s_n, \varphi_n \rangle \in Q$ since the construction above is done inside Q^*.

If there is some $n < k$ so that $\langle s_n, \varphi_n \rangle$ is compatible with $\langle s, \varphi \rangle$ then Lemma 4.18 holds with $\langle t, \chi \rangle = \langle s_n, \varphi_n \rangle$ and we are done. Suppose no such n exists. We will derive a contradiction.

By Lemma 4.6 there is, for each $n < k$, is some $y_n \in \mathrm{dom}(\varphi_n) - \mathrm{dom}(\varphi)$ and $x_n \in v(s, \varphi)$ so that $y_n <_T x_n$ and $\varphi_n(y_n) \in v(s, \varphi)$.

If $y_n \in W_n$ then $y_n = i_n(y_n) \in \bar{W}$ as $i_n \restriction W_n$ is identity, and therefore $y_n = \hat{i}^{-1} \circ i_n(y_n)$ as $\hat{i}^{-1} \restriction \bar{W}$ is identity. But then $y_n \in \mathrm{dom}(\varphi_n) \to y_n \in \mathrm{dom}(\varphi)$, contradicting the fact that $y_n \in \mathrm{dom}(\varphi_n) - \mathrm{dom}(\varphi)$. So it must be that $y_n \notin W_n$.

Let $\bar{y}_n = i_n^{-1}(y_n)$. By choice of k, there are $m < n < k$ so that $\bar{y}_m = \bar{y}_n$ and $x_m = x_n$. Let $\bar{y}$ and x denote the common values. y_m and y_n are then comparable in $<_T$, as both are $<_T x$. Since $y_n \notin W_n$ and $y_m \in W_n$, it must be that height$(y_n) >$ height(y_m). So $y_m <_T y_n$. It follows by condition (ii) that $(\forall^*_{\mathcal{U}} W)(i_m^{-1}(\bar{y}) <_T i_W^{-1}(\bar{y}))$. It then follows using condition (iii) that $(\forall^*_{\mathcal{U}} V)(\forall^*_{\mathcal{U}} W)(i_V^{-1}(\bar{y}) <_T i_W^{-1}(\bar{y}))$. Let $Z \in \mathcal{U}$ witness the first quantifier in this statement.

We finish the proof of the lemma by showing that the nodes $i_V^{-1}(\bar{y})$, $V \in Z$, form a cofinal branch of T, then using the extensive overlaps that result from this by Corollary 4.2 to obtain a contradiction to condition (iv) above and the lower bound condition (3) in Definition 4.3.

The proof that the nodes $\{i_V^{-1}(\bar{y}) \mid V \in Z\}$ form a cofinal branch of T is standard, similar to the final part of the ultrafilters proof that the standard poset for specializing trees of height ω_1 has the countable chain condition. Note first that height$(i_V^{-1}(\bar{y})) \notin V$. (Otherwise using the fact that i_V is an isomorphism at $\langle V, \bar{W} \rangle$, we have height$(\bar{y}) \in \bar{W}$. This in turn implies that height$(y_m) =$ height$(y_n) =$ height$(\bar{y})$ since i_m and i_n are isomorphism at $\langle W_m, \bar{W} \rangle$ and $\langle W_n, \bar{W} \rangle$ respectively. But then $y_m = y_n$, contradicting the fact that $y_m \in W_n$ and $y_n \notin W_n$.) Since the sets $\{W \in X \mid \xi \in W\}$, $\xi < \omega_2$, all belongs to $\mathcal{U}$, and since $\xi \in W \to \xi \subseteq W$, it follows that $\{\mathrm{height}(i_V^{-1}(\bar{y})) \mid V \in Z\}$ is cofinal in ω_2. It remains to show that every two elements of $\{i_V^{-1}(\bar{y}) \mid V \in Z\}$ are comparable in T. Fix $V_1, V_2 \in Z$. Then the sets $\{W \in X \mid i_{V_1}^{-1}(\bar{y}) <_T i_W^{-1}(\bar{y})\}$ and $\{W \in X \mid i_{V_2}^{-1}(\bar{y}) <_T i_W^{-1}(\bar{y})\}$ both belong to $\mathcal{U}$ by definition of Z. Their intersection is therefore non-empty. Fixing any W that belongs to both we have $i_{V_1}^{-1}(\bar{y}) <_T i_W^{-1}(\bar{y})$ and $i_{V_2}^{-1}(\bar{y}) <_T i_W^{-1}(\bar{y})$. So $i_{V_1}^{-1}(\bar{y})$ and $i_{V_2}^{-1}(\bar{y})$ are comparable.

Let h be the cofinal branch of T generated by $\{i_V^{-1}(\bar{y}) \mid V \in Z\}$. The proof of Corollary 4.2 shows that for unboundedly many $\beta < \omega_2$, for all $\alpha \geq \beta$, the node of h on level α has an extensive overlap point at β. In particular for all $V \in Z$ with $\sup(V \cap \mathrm{Ord})$ sufficiently large, $i_V^{-1}(\bar{y})$ has an extensive overlap point above $\sup(W_m \cap \mathrm{Ord})$, and hence $\beta_{i_V^{-1}(\bar{y})} > \sup(W_m \cap \mathrm{Ord})$. By condition (iv) above it follows that $\beta_{y_n} = \beta_{i_{W_n}^{-1}(\bar{y})} > \sup(W_m \cap \mathrm{Ord})$. By condition (3) of Definition 4.3 it then follows that $\varphi_n(y_n) > \sup(W_m \cap \mathrm{Ord})$.

But on the other hand, our use of Lemma 4.6 above to obtain y_n and x_n gave $\varphi_n(y_n) \in v(s, \varphi)$. This means that $\varphi_n(y_n) \in \mathrm{dom}(i_n^{-1} \circ \hat{i})$. Since $i_n^{-1} \circ \hat{i}$ is an isomorphism at $\langle Q, W_n \rangle$, and since $\varphi_n(y_n) \in Q$, it follows that $\varphi_n(y_n) = (i_n^{-1} \circ \hat{i})(\varphi_n(y_n)) \in W_n$. Since $i_m^{-1} \circ i_n$ is an isomorphism at $\langle W_n, W_m \rangle$ this implies that $\varphi_n(y_n) \in W_m$, and hence $\varphi_n(y_n) < \sup(W_m \cap \mathrm{Ord})$, a contradiction. □

Corollary 4.19. *Let $Q^* \prec H(\theta^*)$ with $T, \mathcal{S}, \mathcal{T}, \mathbb{S}\mathrm{pecialize} \in Q^*$ and $Q^* \cap K \in \mathcal{T}$. Then any $\langle s, \varphi \rangle \in \mathbb{S}\mathrm{pecialize}$ with $Q^* \cap K \in s$ is a master condition for Q^*.*

Hence $\mathbb{S}\mathrm{pecialize}$ is proper on $\mathcal{T}$, meaning that for a club of $Q^* \prec H(\theta^*)$, if $Q^* \cap K \in \mathcal{T}$, then every condition in $\mathbb{S}\mathrm{pecialize} \cap Q^*$ extends to a master condition for Q^* in $\mathbb{S}\mathrm{pecialize}$.

Proof. The second part is immediate from the first using Claim 4.15. Suppose the first part fails. Extending $\langle s, \varphi \rangle$ we may fix a specific dense open $D \in Q^*$ so that $\langle s, \varphi \rangle$ is not compatible with any element of $D \cap Q^*$. Extending further, we may assume $\langle s, \varphi \rangle \in D$. But then by Lemma 4.18, $\langle s, \varphi \rangle$ *is* compatible with an element of $D \cap Q^*$. □

Since $\mathcal{T}$ is stationary, Corollary 4.19 implies that $\mathbb{S}\mathrm{pecialize}$ does not collapse ω_2. Lemma 4.18 can be used further, to show that in fact $\mathbb{S}\mathrm{pecialize}$ is ω_2-c.c.

Corollary 4.20. $\mathbb{S}\mathrm{pecialize}$ *is ω_2-c.c.*

Proof. Suppose not, and let $\langle s_\xi, \varphi_\xi \rangle$, $\xi < \omega_2$, be an antichain. Let $Q^* \prec H(\theta^*)$ be as in Lemma 4.18 with $\langle s_\xi, \varphi_\xi \mid \xi < \omega_2 \rangle \in Q^*$. Such Q^* can be found since $\mathcal{T}$ is stationary. Let $\alpha = \sup(Q^* \cap \omega_2)$. By Corollary 4.16, there is $\langle s, \varphi \rangle \leq \langle s_\alpha, \varphi_\alpha \rangle$ with $Q^* \cap K \in s$. By Lemma 4.18 there is $\langle r, \chi \rangle \leq \langle s_\xi, \varphi_\xi \rangle$, for some $\xi \in Q$, which is compatible with $\langle s, \varphi \rangle$. But then $\langle s_\xi, \varphi_\xi \rangle$ is compatible with $\langle s_\alpha, \varphi_\alpha \rangle$, and since $\xi < \alpha$ this is a contradiction. □

Lemma 4.21. *Let $Q^* \prec H(\theta^*)$ with $T, \mathcal{S}, \mathcal{T}, \mathbb{S}\mathrm{pecialize} \in Q^*$ and $Q^* \cap K \in \mathcal{S}$. Suppose further that for every $a \in Q^*$, there is $W^* \in Q^*$ so that $T, \mathcal{S}, \mathcal{T}, \mathbb{S}\mathrm{pecialize} \in W^* \prec H(\theta^*)$ and $W^* \cap K \in \mathcal{T}$. Let $Q = Q^* \cap K$. Let $\langle s, \varphi \rangle \in \mathbb{S}\mathrm{pecialize}$. Let $D \in Q^*$ be a subset of $\mathbb{S}\mathrm{pecialize}$ so that $\langle s, \varphi \rangle \in D$. Suppose $Q \in s$. Then there is $\langle t, \chi \rangle \in D \cap Q$ which is compatible with $\langle s, \varphi \rangle$.*

Proof. Fix $W^* \in Q^*$ as in the assumption of the lemma, with $D \in W^*$. Let $W = W^* \cap K$. Without loss of generality we may assume that D is open in $\mathbb{S}\mathrm{pecialize}$. Extending s if necessary we may by Corollary 4.16 assume that $W \in s$. By closure of s under intersections we also have $Q \cap W \in s$.

Let $\hat{D}$ be the set of $\langle \hat{s}, \hat{\varphi}\rangle \in D$ so that $Q\cap W \in \hat{s}$ and there is no $\langle t, \chi\rangle \in D\cap Q\cap W$ which is compatible with $\langle \hat{s}, \hat{\varphi}\rangle$. If $\langle s, \varphi\rangle \notin \hat{D}$ then any $\langle t, \chi\rangle$ witnessing this also witnesses the lemma, and there is nothing further to prove. Suppose that $\langle s, \varphi\rangle \in \hat{D}$. We will derive a contradiction.

Note that $\hat{D} \in W^*$, since it is defined using the parameters $Q\cap W$ and D, which belong to W^*. By Lemma 4.18 and using the assumption that $\langle s, \varphi\rangle \in \hat{D}$, there is $\langle \hat{s}, \hat{\varphi}\rangle \in \hat{D}\cap W$ which is compatible with $\langle s, \varphi\rangle$. Then:

(i) $\langle \hat{s}, \hat{\varphi}\rangle \in D$.
(ii) There is no $\langle t, \chi\rangle \in D\cap Q\cap W$ which is compatible with $\langle \hat{s}, \hat{\varphi}\rangle$.
(iii) $\langle \hat{s}, \hat{\varphi}\rangle \in W$.

Since $\hat{s} \subseteq W$ and $Q\cap W \in \hat{s}$, $\hat{s}\cup\{W, Q\}$ is a side condition. For any condition $\langle r, v\rangle$ witnessing that $\langle \hat{s}, \hat{\varphi}\rangle$ is compatible with $\langle s, \varphi\rangle$, we have $r \supseteq \hat{s}\cup\{W, Q\}$ and $v \supseteq \hat{\varphi}$. It follows that $\langle \hat{s}\cup\{W, Q\}, \hat{\varphi}\rangle$ is a condition in $\mathbb{S}$pecialize.

For every residue gap $[Q\cap R, R)$ of $\hat{s}\cup\{W, Q\}$ in Q, and every $x \in T\cap v(\hat{s}, \hat{\varphi})$ with $\text{height}(x) \geq \sup(Q\cap R\cap \text{Ord})$, if there is a branch u through T, of height $\sup(R\cap\text{Ord})$, with $u \in Q$ and $\text{Proj}(u, \gamma) = \text{Proj}(x, \gamma)$ for all $\gamma < \sup(Q\cap R\cap\text{Ord})$, then there is a unique such branch. (The reason is that by elementarity of Q, any disagreement between branches $u_1, u_2 \in Q$ of height $\sup(R\cap\text{Ord})$ must occur at a level below $\sup(Q\cap R\cap\text{Ord})$.) Let $u(x, R)$ denote the unique such u, when it exists.

Let $a \subseteq Q$ be a finite set large enough to witness Lemma 4.6 for $\langle \hat{s}\cup\{W, Q\}, \hat{\varphi}\rangle$, and large enough that $u(x, R) \in a$ for all $x, R \in v(\hat{s}, \hat{\varphi})$ so that $u(x, R)$ is defined.

Since W is internal on a club, we may write $W = \bigcup_{\xi<\omega_1} P_\xi$ where $\langle P_\xi \mid \xi < \omega_1\rangle$ is an increasing continuous sequence of countable models. By elementarity of Q^* we can pick $\langle P_\xi \mid \xi < \omega_1\rangle \in Q^*$. Let $\alpha = \sup(Q^*\cap\omega_1)$. Then $Q^*\cap W = Q\cap W$ is exactly equal to P_α.

Let $\mathfrak{tp}$ be the type of $\langle \hat{s}\cup\{W, Q\}, \hat{\varphi}\rangle$ at Q. Let X be the set of $\xi < \omega_1$ so that there is $\langle s_\xi\cup\{W, Q_\xi\}, \varphi_\xi\rangle \in \mathbb{S}$pecialize with $Q_\xi\cap W = P_\xi$, the type of $\langle s_\xi\cup\{W, Q_\xi\}, \varphi_\xi\rangle$ at Q_ξ equal to $\mathfrak{tp}$, $Q_\xi \supseteq a$, and $\langle s_\xi, \varphi_\xi\rangle \in D$. Note that in particular this implies $\langle s_\xi, \varphi_\xi\rangle \in W$, since $\langle \hat{s}, \hat{\varphi}\rangle \in W$. By elementarity of Q^* we have $X \in Q^*$, and can also arrange that the map $\xi \mapsto \langle s_\xi, \varphi_\xi, Q_\xi\rangle$ belongs to Q^*. Let $\bar{\xi}$ be the least element of X and let $\bar{Q} = Q_{\bar{\xi}}$, $\bar{s} = s_{\bar{\xi}}$, and $\bar{\varphi} = \varphi_{\bar{\xi}}$. For each $\xi \in X$ let $i_\xi \colon v(s_\xi\cup\{W, Q_\xi\}, \varphi_\xi) \to v(\bar{s}\cup\{W, \bar{Q}\}, \bar{\varphi})$ be an isomorphism at $\langle Q_\xi, \bar{Q}\rangle$. By elementarity we can arrange that $\xi \mapsto i_\xi$ belongs to Q^*. Let $\hat{i}\colon v(\hat{s}\cup\{W, Q\}, \hat{\varphi}) \to v(\bar{s}\cup\{W, \bar{Q}\}, \bar{\varphi})$ be an isomorphism at $\langle Q, \bar{Q}\rangle$.

Since $\langle \hat{s}, \hat{\varphi}\rangle \in D$ and $Q\cap W = P_\alpha$, we have $\alpha \in X$. Using the elementarity of Q^* it follows that X is unbounded in ω_1. Let $\mathcal{U}$ be an ultrafilter on X, with the sets $X - \zeta$ in $\mathcal{U}$ for all $\zeta < \omega_1$. By elementarity of Q^* we can pick $\mathcal{U} \in Q^*$.

Let $k = |v(s, \varphi)|^2 + 1$. Working inductively construct $\xi_n \in Q$, for $n < k$, so that:

(v) $(\forall m < n) v(s_{\xi_m}, \varphi_{\xi_m}) \subseteq Q_{\xi_n}\cap W$.
(vi) $(\forall m < n)(\forall \bar{y} \in v(\bar{s}, \bar{\varphi}))$ ξ belongs to a set in $\mathcal{U}$ that witnesses either $(\forall^*_{\mathcal{U}}\xi)(i^{-1}_{\xi_m}(\bar{y}) <_T i^{-1}_\xi(\bar{y}))$ or $(\forall^*_{\mathcal{U}}\xi)(i^{-1}_{\xi_m}(\bar{y}) \not<_T i^{-1}_\xi(\bar{y}))$, depending on which of the two holds.
(vii) $(\forall \bar{y} \in v(\bar{s}, \bar{\varphi}))$ ξ_n belongs to a set in $\mathcal{U}$ that witnesses (the first quantifier in) either $(\forall^*_{\mathcal{U}}\zeta)(\forall^*_{\mathcal{U}}\xi)(i^{-1}_\zeta(\bar{y}) <_T i^{-1}_\xi(\bar{y}))$ or $(\forall^*_{\mathcal{U}}\zeta)(\forall^*_{\mathcal{U}}\xi)(i^{-1}_\zeta(\bar{y}) \not<_T i^{-1}_\xi(\bar{y}))$, depending on which of the two holds.

(viii) $(\forall x \in v(s,\varphi))(\forall \bar{y} \in v(\bar{s},\bar{\varphi}))$ if there is a set $Z \in \mathcal{U} \cap Q^*$ so that $(\forall \xi \in Z \cap Q)(i_\xi^{-1}(\bar{y}) \not<_T x)$ then ξ_n belongs to such a set.

In contrast with the situation in Lemma 4.18 where we had $\bigcup_{W \in X \cap Q} = Q$, here we do not have $\bigcup_{\xi \in X \cap Q}(Q_\xi) = Q$. But we do have $\bigcup_{\xi \in X \cap Q} Q_\xi \cap W = Q \cap W$, and this is all we need to ensure that any sufficiently large $\xi_n \in X \cap Q = X \cap \alpha$ satisfies condition (v). Conditions (vi)–(viii) require ξ_n to belong to each set in some finite list of sets in $\mathcal{U} \cap Q^*$, and can be secured using the finite completeness of $\mathcal{U}$. That the elements of $\mathcal{U}$ involved can all be picked in Q^* is clear from the statement in the case of condition (viii), and uses the elementarity of Q^* in the case of conditions (vi) and (vii).

Let $Q_n = Q_{\xi_n}$, $s_n = s_{\xi_n}$, $\varphi_n = \varphi_{\xi_n}$, and $i_n = i_{\xi_n}$. We have $\langle s_n, \varphi_n \rangle \in D \cap Q \cap W$ by construction. It follows by condition (ii) that $\langle s_n, \varphi_n \rangle$ is not compatible with $\langle \hat{s}, \hat{\varphi} \rangle$. Hence by Lemma 4.6 there must exist $y_n \in \operatorname{dom}(\varphi_n) - \operatorname{dom}(\hat{\varphi})$ and $x_n \in v(\hat{s},\hat{\varphi})$ so that $y_n <_T x_n$ and $\varphi_n(y_n) \in v(\hat{s},\hat{\varphi})$.

Let $\bar{y}_n = i_n(y_n)$. By choice of k, there exist $m < n < k$ so that $\bar{y}_m = \bar{y}_n$ and $x_m = x_n$. Let $\bar{y}$ and x denote these common values. Let R be the least transitive node of $\operatorname{res}_Q(\hat{s} \cup \{W, Q\})$ above $\hat{i}^{-1}(\operatorname{height}(\bar{y}))$. (Such a node exists since $\hat{\varphi} \subseteq W$. R may be W itself, or a smaller transitive node of $\hat{s} \cap Q$.) Adapting the corresponding arguments in the proof of Lemma 4.18 one can check that $\operatorname{height}(y_m) < \sup(Q_n \cap R \cap \mathrm{Ord}) \leq \operatorname{height}(y_n)$, that consequently $y_m <_T y_n$, and that, using conditions (vi) and (vii), there is $Z \in \mathcal{U}$ so that $(\forall \zeta \in Z)(\forall^*_{\mathcal{U}} \xi)(i_\zeta^{-1}(\bar{y}) <_T i_\xi^{-1}(\bar{y}))$. Continuing to adapt the arguments in the proof of Lemma 4.18 it then follows that the nodes $\{i_\zeta^{-1}(\bar{y}) \mid \zeta \in Z\}$ form a branch through T, call it u, that $\sup(Q_\zeta \cap R \cap \mathrm{Ord}) \leq \operatorname{height}(i_\zeta^{-1}(\bar{y})) < \sup(R \cap \mathrm{Ord})$, and that consequently the height of the branch u is exactly $\sup(R \cap \mathrm{Ord})$.

By elementarity of Q^*, the set Z above can be picked inside Q^*. It then follows that $u \in Q^*$. By condition (viii) and since the sets $Z - \beta$ for $\beta < \alpha$ all belong to $\mathcal{U} \cap Q$, it must be that $(\forall \xi \in Z \cap Q)(i_\xi^{-1}(\bar{y}) <_T x)$. (Otherwise, letting β be the least counterexample, we have that $(\forall \xi \in Z - \beta)(i_\xi^{-1}(\bar{y}) \not<_T x)$. Then by condition (viii) it follows that $i_n^{-1}(\bar{y}) \not<_T x$. But this is a contradiction since $y_n <_T x$.) Since $\{\operatorname{height}(i_\xi^{-1}(\bar{y})) \mid \xi \in Z \cap Q\}$ is cofinal in $\sup(Q \cap R \cap \mathrm{Ord})$ it follows that $\operatorname{Proj}(u, \gamma) <_T x$ for all $\gamma < \sup(Q \cap R \cap \mathrm{Ord})$. This implies that u is equal to $u(x, R)$. In particular then $u \in a$, and hence $u \in Q_n$ (in fact $u \in Q_\xi$ for all ξ).

But this means that y_n is overlapped by Q_n in $\langle s_n \cup \{W, Q_n\}, \varphi_n \rangle$. Hence by the overlap condition (8) of Definition 4.3 for $\langle s_n \cup \{W, Q_n\}, \varphi_n \rangle$, $\varphi_n(y_n) \notin Q_n$. On the other hand, by our choice of y_n (through an application of Lemma 4.6) we have $\varphi_n(y_n) \in v(\hat{s} \cup \{W, Q\}, \hat{\varphi})$. Since $i_n^{-1} \circ \hat{i} \colon v(\hat{s} \cup \{W, Q\}, \hat{\varphi}) \to V(s_n \cup \{W, Q_n\}, \varphi_n)$ is an isomorphism at $\langle Q, Q_n \rangle$, and since $\varphi_n(y_n) \in Q$, this implies that $\varphi_n(y_n) = i_n^{-1} \circ \hat{i}(\varphi_n(y_n)) \in Q_n$, a contradiction. □

Corollary 4.22. $\mathbb{S}$pecialize *is proper on $\mathcal{S}$. Moreover, for a club of $Q^* \prec H(\theta^*)$, if $Q^* \cap K \in \mathcal{S}$ then every $\langle s, \varphi \rangle \in \mathbb{S}$pecialize with $Q^* \cap K \in s$ is a master condition for Q^*.*

Proof. Fix any $Q^* \prec H(\theta^*)$ which is sufficiently closed to satisfy the initial assumptions of Lemma 4.21. Suppose $Q^* \cap K \in \mathcal{S}$. Let $\langle s, \varphi \rangle \in Q^* \cap \mathbb{S}$pecialize. By Claim 4.17, there is $\langle s', \varphi' \rangle \leq \langle s, \varphi \rangle$ with $Q^* \cap K \in s'$. By Lemma 4.21, and using an argument as in the proof of Corollary 4.19, $\langle s', \varphi' \rangle$ is a master condition for Q^*. □

Claim 4.23. *Forcing with* $\mathbb{S}$pecialize *adds a weak specializing function on the restriction of T to nodes outside E on a club of levels. (Recall that E consists of the extensively overlapped nodes of T.)*

Proof. Immediate by the definition of $\mathbb{S}$pecialize and genericity. The function is $\bigcup\{\varphi \mid (\exists s)\langle s, \varphi\rangle \in G\}$, where G is generic for $\mathbb{S}$pecialize, and the club of levels is $\text{Limit}(Ot)$. We only note that for every $\alpha \in \text{Limit}(Ot)$ there is some M occurring in G satisfying condition (2) of Definition 4.3. The proof of this uses Corollary 4.16, unboundedness of the countable nodes of G in K, and an argument as in the proof of Claim 2.5 to show that for every countable M and every $\alpha \in \text{Limit}(Ot)$ below $\sup(M \cap \text{Ord})$, if $\alpha \notin M$ then the first ordinal of M above α belongs to Ot. □

We end the section with a brief example showing that in some cases there are ω_2 Aronszajn trees for which *no* forcing that preserves ω_1 and ω_2 can add a total weak specializing function.

Recall from Fact 3.25 that $\square^{\text{ta}}_{\omega_1,\omega}$ implies the existence of an ω_2 Aronszajn tree T with an ascent path, namely a sequence $\langle \bar{x}^\alpha \mid \alpha < \omega_2\rangle$ where $\bar{x}^\alpha = \langle x^\alpha_n \mid n < \omega\rangle$ with x^α_n for $n < \omega$ distinct elements of $\text{Level}(\alpha)$, so that for every $\alpha < \beta < \omega_2$, for all but finitely many n, $x^\alpha_n <_T x^\beta_n$. The existence of an ascent path persists to all generic extensions. In light of this and Claim 1.2, the following lemma establishes that it is impossible to add a total weakly specializing function on a tree with an ascent path without collapsing one of ω_1 and ω_2. More precisely the lemma shows that, without collapsing one of ω_1 and ω_2, it is impossible to add a weak specializing function f so that for stationarily many $\delta \in \omega_2$ of cofinality ω_1, for stationarily many α in δ, for infinitely many n, $x^\alpha_n \in \text{dom}(f)$.

Lemma 4.24. *Let T be a tree of height ω_2 with an ascent path, given by $\bar{x}^\alpha = \langle x^\alpha_n \mid n < \omega\rangle$ for $\alpha < \omega_2$. Then (relative to any θ, A, and C as in Definition 1.1) for a club of $\delta < \omega_2$ of cofinality ω_1, for a club of $\alpha < \delta$, for all sufficiently large n, x^α_n is extensively overlapped in T.*

Proof. Fix θ, A, and C. Let D be the set in Claim 4.1 for P that codes the ascent path. Let $\delta \in \text{Limit}(D \cap C)$ have cofinality ω_1. Let S be the set of $\alpha < \delta$ so that for infinitely many n, x^α_n is not extensively overlapped in T. Suppose for contradiction that S is stationary in δ. Shrinking S if needed we may assume $S \subseteq \text{Limit}(D \cap C)$. We will be done if we can show that for some $\alpha \in S$, for all sufficiently large n, for arbitrarily large $\beta < \alpha$ in $D \cap C$, β is an extensive overlap point for x^α_n.

For each $\beta \in D \cap \delta$, each countable $a \subseteq \beta$, and each $\gamma < \omega_2$, let $M_{\beta,a,\gamma} \prec H(\theta, A, P)$ be countable with $M_{\beta,a,\gamma} \supseteq a$, $M_{\beta,a,\gamma} \cap \omega_2 \not\subseteq \beta$, and $\min(M_{\beta,a,\gamma} - \beta) \geq \max\{\gamma, \delta\}$. Such a model exists by the definition of D in Claim 4.1. Let $\nu(\beta, a, \gamma) = \min(M_{\beta,a,\gamma} - \beta)$. Let $k_{\beta,a,\gamma} < \omega$ be such that for all $n \geq k_{\beta,a,\gamma}$, $x^\delta_n <_T x^{\nu(\beta,a,\gamma)}_n$. Such k exists by the properties of an ascent path.

Claim 4.25. *For each $\beta \in D \cap \delta$ there is k_β so that for cofinally many countable $a \subseteq \beta$, and cofinally many $\gamma < \omega_2$, $k_{\beta,a,\gamma} = k_\beta$.*

Proof. Suppose not. Then for every $k < \omega$ there is a_k and γ_k so that for all countable $a \subseteq \beta$ with $a \supseteq a_k$, and all $\gamma < \omega_2$ with $\gamma \geq \gamma_k$, $k_{\beta,a,\gamma} \neq k$. But then letting $a^* = \bigcup_{k<\omega} a_k$ and $\gamma^* = \sup_{k<\omega} \gamma_k$ we have that $(\forall k < \omega) k_{\beta,a^*,\gamma^*} \neq k$, contradiction. □

Fix a specific $k < \omega$ so that the set $X = \{\beta \in D \cap C \cap \delta \mid k_\beta = k\}$ is unbounded in δ. Such k exists since $\delta \in \text{Limit}(D \cap C)$ and $\text{cof}(\delta) = \omega_1$.

Now take any $\alpha \in S \cap \mathrm{Limit}(X)$. Let $l < \omega$ be such that for all $n \geq l$, $x_n^\alpha <_T x_n^\delta$. We claim that for every $\beta \in X \cap \alpha$, and any $n \geq \max\{l, k\}$, β is an extensive overlap point for x_n^α. This will complete the proof of the lemma.

Fix $\beta \in X \cap \alpha$ and $n \geq \max\{l, k\}$. Let $a \subseteq \beta$ and $b \subseteq \omega_2 - \beta$ be countable. Let $\gamma < \omega_2$ be greater than $\sup(b)$. Let $M = M_{\beta,a,\gamma}$. We have $a \subseteq M$, and as in the proof of Corollary 4.2, $(M \cup \mathrm{Limit}(M)) \cap b = \emptyset$. Increasing a and γ if necessary, we may by Claim 4.25 assume that $k_{\beta,a,\gamma} = k_\beta$, and hence $k_{\beta,a,\gamma} = k$ since $\beta \in X$. Since $n \geq k$ we have $x_n^\delta <_T x_n^{\nu(\beta,a,\gamma)}$. Since $n \geq l$ we also have $x_n^\alpha <_T x_n^\delta$. So $x_n^\alpha <_T x_n^{\nu(\beta,a,\gamma)}$. Since M is elementary relative to the ascent path, and since $\nu(\beta, a, \gamma) \in M$, we have $x_n^{\nu(\beta,a,\gamma)} \in M$. So M overlaps x_n^α. $\square$

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