A FURTHER APPLICATIONS OF STERN ABSOLUTENESS

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Abstract. We prove, in ZFC, that certain $\mathbf{\Sigma}^1_2$ functions cannot injectively embed ω_1 into a Borel class of fixed countable rank. This had been proved under determinacy or large cardinals by Harrington and Hjorth for all $\mathbf{\Sigma}^1_2$ functions. Our contribution is to identify conditions under which the determinacy and large cardinal assumptions can be removed. These conditions are sufficient for a recent use of the in-existence of $\mathbf{\Sigma}^1_2$ injections of ω_1 into Borel classes by Day and Marks.

1. Introduction

Hjorth [3] discusses and applies an absoluteness principle due to Jacques Stern [4]. In this short article we provide another consequence of this principle, enhancing one of Hjorth's applications, to eliminate the large cardinal assumption in proving that certain $\mathbf{\Sigma}^1_2$ injections of ω_1 into levels of the Borel hierarchy cannot exist.

We give a quick survey of Stern's absoluteness principle in Section 2. Roughly quoting [3], the principle states that if a $\mathbf{\Pi}^0_\alpha$ set can be introduced into the universe by forcing, and membership in the set is sufficiently independent of the generic filter, then the set can be introduced by a small forcing notion.

In Section 3 we present the result on $\mathbf{\Sigma}^1_2$ injections of ω_1 into levels of the Borel hierarchy. This is a refinement of a well known theorem of Harrington [2, Theorem 4.5], that under the axiom of determinacy, there are no injections of ω_1 into the pointclass $\mathbf{\Pi}^0_\alpha$ for any $\alpha < \omega_1$. Hjorth [3, Theorem 3.1] re-proves Harrington's theorem using Stern's absoluteness principle. The first part of his proof uses determinacy to convert a counterexample to the theorem into a $\mathbf{\Sigma}^1_2$ function. His argument then proceeds to derive a contradiction using Stern absoluteness theorem and some additional mild consequences of determinacy. In a subsequent comment in [3], Hjorth notes that if the function were provably Δ^1_2, the contradiction could be derived in ZFC.

Our own contribution here is a slight improvement of this note, giving a ZFC arguments which applies to $\mathbf{\Sigma}^1_2$ functions whose domain is absolutely unbounded. This slight improvement is of some importance because it has a corollary, 3.8 below, which matches with the use of Harrington's theorem in the work of Day-Marks [1] on the decomposability conjecture, allowing their proof to go through in ZFC without any large cardinal assumptions.

2. Stern's absoluteness principle

Let $M \subseteq N$ be two wellfounded models of ZFC so that $\omega_1^N \subseteq \mathrm{Ord}^M$. Recall that $\mathbf{\Delta}^1_1$ sets in M have canonical extensions to N. There are several ways to reach

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these extensions. For example, given a $\mathbf{\Delta}^1_1$ subset $A \in M$ of Baire space, let T_{in} and T_{out} in M be trees on $\omega \times \omega$ which project to A and to $\mathbb{R} - A$ respectively, and then let A^{*N} be the projection of T_{in} taken in N. By absoluteness, A^{*N} is an extension of A, meaning that $A^{*N} \cap M = A$. By absoluteness and using the fact that $\omega_1^N \subseteq \mathrm{Ord}^M$, A^{*N} is the complement of the projection of T_{out} in N, and this implies that A^{*N} is independent of the choice of the tree T_{in}. The extension can be defined similarly for other Polish spaces.

Remark 2.1. Another way to obtain the extension A^{*N} is to take, in M, a real which codes the Borel definition of A, and then take A^{*N} to be the interpretation of the same definition in N. The precise way in which one sets up Borel codes is not important, so long as the process of obtaining a set from its code is Σ^1_1—as it will be with any of the standard coding approaches—so that the reinterpretation of the code in N is trivially equal to the extension A^{*N} defined above.

Remark 2.2. Let $M \subseteq N \subseteq P$ be three wellfounded models of ZFC so that $\omega_1^P \subseteq \mathrm{Ord}^M$. Let $A \in M$ be $\mathbf{\Delta}^1_1$. Then it is clear using absoluteness that $A^{*P} = (A^{*N})^{*P}$ and that $A^{*P} \cap N = A^{*N}$.

Definition 2.3. Let $\mathbb{P}$ be a forcing notion, let τ be a $\mathbb{P}$-name for a $\mathbf{\Delta}^1_1$ set of reals, and let $p \in \mathbb{P}$. We say that $\langle \tau, p\rangle$ is *invariant* if for any generic $G_{\mathrm{l}} \times G_{\mathrm{r}}$ for $\mathbb{P} \times \mathbb{P}$ with $\langle p, p\rangle \in G_{\mathrm{l}} \times G_{\mathrm{r}}$, $\tau[G_{\mathrm{l}}]^{*V[G_{\mathrm{l}} \times G_{\mathrm{r}}]} = \tau[G_{\mathrm{r}}]^{*V[G_{\mathrm{l}} \times G_{\mathrm{r}}]}$. (The subscript "l" and "r" here stand for "left" and "right".) We say that a $\mathbf{\Delta}^1_1$ set of reals in an extension $V[G]$ by $\mathbb{P}$ is *invariant* if there is an invariant $\langle p, \tau\rangle$ so that $p \in G$ and $\tau[G] = A$.

Claim 2.4. *Let $\mathbb{P}$ be a forcing notion, let τ be a $\mathbb{P}$-name for a $\mathbf{\Delta}^1_1$ set of reals, let G be generic for $\mathbb{P}$ over V, and let $p \in G$. Suppose that for every $\mathbb{P}$-generic G' over $V[G]$ with $p \in G'$, the sets $\tau[G]^{*V[G \times G']}$ and $\tau[G']^{*V[G \times G']}$ are equal. Then $\tau[G]$ is invariant.*

Proof. Rephrasing the claim assumptions in the forcing language over $V[G]$ and using $\dot{G}'$ to name the generic, we have that p forces in $\mathbb{P}$ over $V[G]$ that $\tau[\dot{G}']^{*V[G][\dot{G}']} = \tau[G]^{*V[G][\dot{G}']}$. This is a statement about $V[G]$, so must be forced over V by a condition $q \in G$. We claim that $\langle \tau, q\rangle$ is invariant. Suppose $G_{\mathrm{l}} \times G_{\mathrm{r}}$ for $\mathbb{P} \times \mathbb{P}$ with $\langle q, q\rangle \in G_{\mathrm{l}} \times G_{\mathrm{r}}$. Let G' be generic for $\mathbb{P}$ over $V[G_{\mathrm{l}} \times G_{\mathrm{r}}]$ with $p \in G'$. Since $q \in G_{\mathrm{l}}$ and $p \in G'$ we have that $\tau[G_{\mathrm{l}}]^{*V[G_{\mathrm{l}} \times G']} = \tau[G']^{*V[G_{\mathrm{l}} \times G']}$. Using Remark 2.2 this implies that $\tau[G_{\mathrm{l}}]^{*V[G_{\mathrm{l}} \times G_{\mathrm{r}} \times G']} = \tau[G']^{*V[G_{\mathrm{l}} \times G_{\mathrm{r}} \times G']}$. Similarly $\tau[G_{\mathrm{r}}]^{*V[G_{\mathrm{l}} \times G_{\mathrm{r}} \times G']} = \tau[G']^{*V[G_{\mathrm{l}} G_{\mathrm{r}} \times \times G']}$. So $\tau[G_{\mathrm{l}}]^{*V[G_{\mathrm{l}} \times G_{\mathrm{r}} \times G']} = \tau[G_{\mathrm{r}}]^{*V[G_{\mathrm{l}} \times G_{\mathrm{r}} \times G']}$. Using Remark 2.2 again this implies that $\tau[G_{\mathrm{l}}]^{*V[G_{\mathrm{l}} \times G_{\mathrm{r}}]} = \tau[G_{\mathrm{r}}]^{*V[G_{\mathrm{l}} \times G_{\mathrm{r}}]}$. $\square$

Lemma 2.5 (Stern's absoluteness principle)**.** *Let $\alpha < \omega_1$. Let $\mathbb{P}$ be a forcing notion and let G be generic for $\mathbb{P}$ over V. Suppose that G collapses $V_{\omega+\alpha}$ to ω and let $N \supseteq V$ be a submodel of $V[G]$ in which $V_{\omega+\alpha}$ is countable. Suppose that $A \in V[G]$ is an invariant $\mathbf{\Pi}^0_{1+\alpha}$ set. Then there is a $\mathbf{\Pi}^0_{1+\alpha}$ set $\bar{A} \in N$ so that $\bar{A}^{*V[G]} = A$.*

We refer the reader to Section 1 of Hjorth [3] for a proof of this lemma. Hjorth's presentation involves Borel codes, rather than Borel sets and their extensions. Using Remark 2.1 it is clear that Corollary 1.8 in [3] implies Lemma 2.5 as stated here.

3. Injections of ω_1 into $\mathbf{\Pi}^0_\alpha$

Our goal in this section is to prove in ZFC that there cannot be a $\mathbf{\Sigma}^1_2$ injection of ω_1 into a Borel class $\mathbf{\Pi}^0_\alpha$, whose domain is absolutely unbounded. We begin with the relevant definitions.

Recall that for $A \subseteq \mathbb{R} \times Z$, and for $w \in \mathbb{R}$, the *w-section* of A, denoted A_w, is the set $\{z \in Z \mid \langle w, z\rangle \in A\}$.

Remark 3.1. Let $M \subseteq N$ be two wellfounded models of ZFC so that $\omega_1^N \subseteq \mathrm{Ord}^M$. Let $A \in M$ be a $\mathbf{\Delta}^1_1$ set of pairs of reals and let $w \in \mathbb{R} \cap M$. Then $(A^{*N})_w = (A_w)^{*N}$. This is easy to check using a tree projecting to A_w which is induced from a tree projecting to A.

Fix $\alpha < \omega_1$, and fix a universal Π^0_α set U, meaning that $U \subseteq \mathbb{R} \times \mathbb{R}$ is Π^0_α and the sections U_w, $w \in \mathbb{R}$, produce *all* $\mathbf{\Pi}^0_\alpha$ subsets of $\mathbb{R}$.

Let LO be the Polish space of linear orders of ω. Let $WO \subseteq LO$ be the set of wellorders of ω. For $e \in WO$ we use $|e|$ to denote the ordertype of e.

Definition 3.2. A $\mathbf{\Sigma}^1_2$ *presentation* of a (partial) map from ω_1 into $\mathbf{\Pi}^0_\alpha$ is a partial function $f\colon WO \to \mathbb{R}$ so that that graph of f is $\mathbf{\Sigma}^1_2$ as a subset of $LO \times \mathbb{R}$, and so that $e_1, e_2 \in \mathrm{dom}(f) \wedge |e_1| = |e_2| \to U_{f(e_1)} = U_{f(e_2)}$. We view f as a $\mathbf{\Sigma}^1_2$ presentation of the function $|e| \mapsto U_{f(e)}$ for $e \in \mathrm{dom}(f)$, and we refer to this map as the *function presented by* f. A (partial) function from ω_1 into $\mathbf{\Pi}^0_\alpha$ is $\mathbf{\Sigma}^1_2$ if it has a $\mathbf{\Sigma}^1_2$ presentation.

For $B \subseteq Z \times \mathbb{R} \times \mathbb{R}$ we use $S(B)$ to denote the set $\{z \in Z \mid (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\langle z, x, y\rangle \in B\}$. Every $\mathbf{\Sigma}^1_2$ subset of Z is of the form $S(B)$ for a Borel set B.

Definition 3.3. Let f be a $\mathbf{\Sigma}^1_2$ subset of $WO \times \mathbb{R}$. We say that f has *absolutely unbounded domain* if there is a Borel $B \subseteq (LO \times \mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ so that:

(1) $f = S(B)$.
(2) For every forcing extension $V[G]$ of V, $\{|e| \mid (\exists w)\langle e, w\rangle \in S^{V[G]}(B^{*V[G]})\}$ is unbounded in $\omega_1^{V[G]}$.

Theorem 3.4. *Let $\alpha < \omega_1$. There are no $\mathbf{\Sigma}^1_2$ partial injections of ω_1 into $\mathbf{\Pi}^0_{1+\alpha}$ with absolutely unbounded domain.*

Proof. Suppose that f is a $\mathbf{\Sigma}^1_2$ presentation which provides a counterexample to the theorem. Let B witness that f has absolutely unbounded domain.

Let $\kappa = |V_{\omega+\alpha}|$ and let $\delta = 2^\kappa$. Let $\mathbb{P} = \mathrm{Col}(\omega, \delta)$ and let G be generic for $\mathbb{P}$ over V. Note that $\mathbb{P}$ subsumes the poset $\mathrm{Col}(\omega, \kappa)$, and let $H \in V[G]$ be generic for this poset over V. We have that $\aleph_1^{V[G]} = \delta^+$, and that $(2^{\aleph_0})^{V[H]} = \delta$.

We intend to extend the function presented by f to act in $V[G]$, show that the domain of the extended function is unbounded in $\aleph_1^{V[G]}$ and therefore has cardinality $\aleph_1^{V[G]}$, show that the extended function (like the original function) is injective, so that its range also has cardinality $\aleph_1^{V[G]}$, and show that all the Borel sets in its range are invariant. Using Lemma 2.5, namely the Stern absoluteness principle, it follows from the final item that each of these sets is the extension of a Borel set in $V[H]$. But this is a contradiction since there are only $(2^{\aleph_0})^{V[H]} = \delta < \delta^+ = \aleph_1^{V[G]}$ Borel sets in $V[H]$.

Before passing to work in $V[G]$, it is useful to note that the following statements, expressing the facts that $S(B)$ represents an injective function whose domain is contained in WO, are all $\mathbf{\Pi}^1_2$ and true in V:

(1) If $\langle e, w\rangle \in S(B)$ then $e \in WO$.
(2) If $\langle e_1, w_1\rangle, \langle e_2, w_2\rangle \in S(B)$ and $|e_1| = |e_2|$, then $U_{w_1} = U_{w_2}$.
(3) If $\langle e_1, w_1\rangle, \langle e_2, w_2\rangle \in S(B)$ and $|e_1| \neq |e_2|$, then $U_{w_1} \neq U_{w_2}$.

By Schoenfield absoluteness, these statements continue to be true in $V[G]$, with the Borel sets B and U replaced by their extensions $B^{*V[G]}$ and $U^{*V[G]}$. Letting $f^* = S^{V[G]}(B^{*V[G]})$ it follows that, in $V[G]$, f^* is a presentation of a $\mathbf{\Sigma}^1_2$ function, and that the function presented by f^* is injective. Let φ denote this function, namely $|e| \mapsto U^{*V[G]}_{f^*(e)}$.

Claim 3.5. *For every $\xi \in \mathrm{dom}(\varphi)$, $\varphi(\xi)$ is invariant.*

Proof. Fix ξ. Fix a $\mathbb{P}$-name $\dot{e} \in V$ so that $\dot{e}[G]$ is a wellorder of ω of ordertype ξ, and belongs to $\mathrm{dom}(f^*)$. Let $p \in G$ force that $|\dot{e}| = \check{\xi}$ and that there exists w so that $\langle \dot{e}, w\rangle \in S^{V[\dot{G}]}(B^{*V[\dot{G}]})$. Fix a name $\dot{w}$ so that $p \Vdash \langle \dot{e}, \dot{w}\rangle \in S^{V[\dot{G}]}(B^{*V[\dot{G}]})$.

Let τ name the set of z so that $\langle \dot{w}, z\rangle \in U^{*V[\dot{G}]}$. Note that $\tau[G] = \varphi(\xi)$. By Claim 2.4, it is enough to show that for every $\mathbb{P}$-generic G' over $V[G]$ with $p \in G'$, the sets $\tau[G]^{*V[G\times G']}$ and $\tau[G']^{*V[G\times G']}$ are equal. Fix such G'.

Let $e = \dot{e}[G]$, $w = \dot{w}[G]$, $e' = \dot{e}[G']$, and $w' = \dot{w}[G']$. Since $p \in G, G'$ we have that $|e| = \xi = |e'|$, that $\langle e, w\rangle \in S^{V[G]}(B^{*V[G]})$, and that $\langle e', w'\rangle \in S^{V[G']}(B^{*V[G']})$. By Remark 2.2 and Π^1_1 absoluteness it follows from the last two items that $\langle e, w\rangle$ and $\langle e', w'\rangle$ both belong to $S^{V[G\times G']}(B^{*V[G\times G']})$. Using condition (2) above, and transferring the condition to apply to $B^{*V[G\times G']}$ and $U^{*V[G\times G']}$ in $V[G\times G']$ using Schoenfield absoluteness, it follows that $(U^{*V[G\times G']})_w = (U^{*V[G\times G']})_{w'}$. By Remarks 2.2 and 3.1 the set on the left-hand-side is exactly $\tau[G]^{*V[G\times G']}$, and the set on right-hand-side is exactly $\tau[G']^{*V[G\times G']}$, so these two sets are equal. $\square$

By Lemma 2.5 it now follows that $\varphi(\xi)$ is the extension of a Borel set in $V[H]$. Indeed by Remark 2.2 that Borel set is exactly $\varphi(\xi) \cap V[H]$. Since two distinct sets cannot be the extensions of two identical sets, and since φ is injective, it must be that the map $\xi \mapsto \varphi(\xi) \cap V[H]$ is an injection of $\mathrm{dom}(\varphi)$ into the set of Borel sets in $V[H]$. But this is a contradiction, since the set of Borel sets in $V[H]$ is equinumerous with δ, which is countable in $V[G]$, while the the assumption that f represents a function with absolutely unbounded domain, together with our choice of B to witness this, imply that $\mathrm{dom}(\varphi)$ is unbounded in $\aleph_1^{V[G]}$. $\square$

The criterion of having absolutely unbounded domain is perhaps somewhat artificial, tailored specifically to the proof of Theorem 3.4. The following criterion is typically more natural:

Definition 3.6. Let f be a $\mathbf{\Sigma}^1_2$ subset of $WO \times \mathbb{R}$. We say that f has *provably unbounded domain via a* Π^1_2 *condition* if there is a (lightface) Π^1_2 set $C \subseteq \mathbb{R}$, a lightface Borel set $B \subseteq R \times LO \times \mathbb{R}^3$, and a real r, so that:

(1) ZFC proves that for every $t \in C$, $\mathrm{dom}(S(B_t))$ is unbounded in ω_1.
(2) $r \in C$ and $f = S(B_r)$.

Notice that the assumption that C and B are lightface definable is implicitly used in condition (1) above, as the references to these sets in the condition are really references to their definitions.

Claim 3.7. *Suppose that f has provably unbounded domain via a Π^1_2 condition. Then f has absolutely unbounded domain.*

Proof. Clear, using the facts that membership in C reflects from V to $V[G]$ by Schoenfield absoluteness, that any statement provable in ZFC must hold in $V[G]$, Remark 3.1, and that the extension of a lightface Borel set from V to $V[G]$ is the interpretation of the set's lightface Borel code in $V[G]$. $\square$

Corollary 3.8. *Let $\alpha < \omega_1$. There are no partial $\boldsymbol{\Sigma}^1_2$ injections of ω_1 into $\boldsymbol{\Pi}^0_{1+\alpha}$ which have provably unbounded domain via a Π^1_2 condition.*

Proof. Clear from Theorem 3.4 and Claim 3.7. $\square$

Remark 3.9. The results above are for for functions in the class $\boldsymbol{\Sigma}^1_2$, and conditions which are Π^1_2. The factor in the proofs which limits us to this level of definability is our reliance on Schoenfield absoluteness. If absoluteness holds for higher classes of descriptive set theoretic complexity, as is the case for example under large cardinal assumptions, then the results trivially adapt to apply to these higher classes.

References

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