Chang's Conjecture with $\square_{\omega_1,2}$ from an ω_1-Erdős Cardinal

Itay Neeman* and John Susice*

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Abstract

Answering a question of Sakai [7], we show that the existence of an ω_1-Erdős cardinal suffices to obtain the consistency of Chang's Conjecture with $\square_{\omega_1,2}$. By a result of Donder [3] this is best possible.

We also give an answer to another question of Sakai relating to the incompatibility of $\square_{\lambda,2}$ and $(\lambda^+,\lambda) \twoheadrightarrow (\kappa^+,\kappa)$ for uncountable κ.

1 Introduction

Chang's Conjecture is a model-theoretic principle asserting a strengthening of the Löwenheim-Skolem Theorem [1]. Chang's Conjecture was originally shown to be consistent assuming the existence of a Ramsey cardinal by Silver (see [6]) and this assumption was later weakened to the existence of an ω_1-Erdős cardinal [4]. This result is best possible, since Chang's Conjecture implies that ω_2 is ω_1-Erdős in the core model [3].

Chang's Conjecture is known to be incompatible with Jensen's square principle $\square_{\omega_1}$ (see [9]) but was recently shown to be consistent with Schimmerling's square principle $\square_{\omega_1,2}$ by Sakai [7], assuming the existence of a measurable cardinal. In light of this consistency upper bound, Sakai posed the following:

Question 1. *What is the consistency strength of the conjunction of Chang's Conjecture with $\square_{\omega_1,2}$?*

In Corollary 12 we show that the consistency of the given statement follows from the existence of an ω_1-Erdős cardinal, answering Sakai's question. Section 2.1 of the paper will cover some basic preliminaries, such as the definition of the relevant square principle and large cardinal. In Section 2.2 we describe our forcing poset. In Silver's consistency proof, he used what is now called a Silver forcing poset–a modification of the Levy Collapse forcing which allows larger supports [6]. Cummings and Schimmerling [2] have introduced another

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variant of the Levy Collapse forcing which collapses inaccessible κ to ω_2 while simultaneously adjoining a square sequence. Our forcing will be a hybrid of these two posets–in other words it will be a "Silverized" Cummings-Schimmerling poset.

Finally, in Section 2.3 we give the proof of our result, which is based on the methods of [7] and [4].

In Section 3 we investigate the relation between weak square principles and model theoretic transfer properties (i.e., generalizations of Chang's Conjecture) of the form $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ for $\kappa \geq \aleph_1$. Sakai proved the following:

Theorem 2 (Sakai, [7])**.** *Suppose that* $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$*, where* κ *is an uncountable cardinal and* λ *is a cardinal* $> \kappa$*. Moreover, suppose that either of the following holds:*

(I) $\lambda^{<\lambda} = \lambda$

(II) $\kappa < \aleph_{\omega_1}$*, and there are strictly more regular cardinals in the interval* $[\aleph_0, \kappa]$ *than in the interval* $(\kappa, \lambda]$*.*

Then $\square_{\lambda,\kappa}$ *fails.*

Although Theorem 2 imposes substantial constraints on the interaction of weak square principles and model theoretic transfer properties, there are many instances where it does not apply. For example, it does not answer the question of whether $(\aleph_4, \aleph_3) \twoheadrightarrow (\aleph_2, \aleph_1)$ is incompatible with $\square_{\omega_3,2}$ when $2^{\aleph_2} > \aleph_3$.

In light of these limitations, Sakai posed the following question:

Question 3 (Sakai, [7])**.** *Let* κ *be an uncountable cardinal and* λ *a cardinal* $> \kappa$*. Does* $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ *imply the failure of* $\square_{\lambda,2}$*?*

We answer this question in the affirmative in Corollary 20 (in fact we obtain the failure of $\square_{\lambda,\omega}$ under these hypotheses and more under slightly stronger hypotheses–see Corollaries 21 and 22). Taking $\kappa = \aleph_1$, $\lambda = \aleph_3$ in this theorem shows that indeed $(\aleph_4, \aleph_3) \twoheadrightarrow (\aleph_2, \aleph_1)$ is incompatible with $\square_{\omega_3,2}$, regardless of the value of $2^{\aleph_2}$.

2 The Consistency of Chang's Conjecture and $\square_{\omega_1,2}$ from an ω_1-Erdős Cardinal

2.1 Preliminaries

In the following, for any cardinal θ we denote by $H(\theta)$ the collection of all sets whose transitive closure has size $< \theta$. We frequently confuse a structure and its underlying set. I.e., if $\mathcal{M} = \langle M, \ldots \rangle$ is a structure and α is an ordinal, we write $\alpha \subseteq \mathcal{M}$ to mean $\alpha \subseteq M$. All structures we consider have at most countably many symbols in their signature.

Definition 4. Chang's Conjecture *is the assertion that for any structure* $\mathcal{N}$ *with* $\omega_2 \subseteq \mathcal{N}$*, there exists* $\mathcal{M} \preceq \mathcal{N}$ *such that* $|\mathcal{M}| = \aleph_1$ *and* $|\mathcal{M} \cap \omega_1| = \aleph_0$*.*

We observe that to verify Chang's Conjecture it suffices to verify it for models with underlying set $H(\omega_2)$:

Claim 5 (Folklore)**.** *Suppose that for all structures $\mathcal{H} = \langle H(\omega_2), \dots \rangle$ there exists $\mathcal{M} \preceq \mathcal{H}$ of cardinality $\aleph_1$ such that $|\mathcal{M} \cap \omega_1| = \aleph_0$. Then Chang's Conjecture holds.*

Proof. This is a standard model-theoretic argument. Suppose that $\mathcal{N} = \langle N, R_1, R_2, \dots \rangle$ is any structure with $\omega_2 \subseteq N$. We may assume without loss of generality that $|\mathcal{N}| = \aleph_2$. Let $\pi \colon N \to H(\omega_2)$ be any injection which is the identity on ω_2. Let $\mathcal{H} = \langle H(\omega_2), \tilde{N}, \tilde{R}_1, \tilde{R}_2, \dots \rangle$, where $\tilde{N}$ is a predicate representing membership in $\pi[N]$ and $\tilde{R}_i$ is a predicate representing R_i in the natural way. By our assumption there is $\mathcal{M} \preceq \mathcal{H}$ of cardinality $\aleph_1$ such that $|\mathcal{M} \cap \omega_1| = \aleph_0$. Pulling back via π, we get the desired submodel of $\mathcal{N}$. $\square$

Square properties are a family of "incompactness principles" regarding sequences of clubs.

Definition 6 ([8])**.** *Suppose that κ is an infinite cardinal and λ is a nonzero (but potentially finite) cardinal. A $\square_{\kappa,\lambda}$-sequence is a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \colon \alpha < \kappa^+ \rangle$ such that:*

1. *For all $\alpha < \kappa^+$, $1 \leq |\mathcal{C}_\alpha| \leq \lambda$.*
2. *For all $\alpha < \kappa^+$ and $C \in \mathcal{C}_\alpha$, C is a club subset of α and $\operatorname{otp} C \leq \kappa$.*
3. *(Coherence) For all $\alpha < \kappa^+$, every $C \in \mathcal{C}_\alpha$* threads *$\langle \mathcal{C}_\beta \colon \beta < \alpha \rangle$ in the sense that $C \cap \beta \in \mathcal{C}_\beta$ for all β which are limit points of C.*

We say that $\square_{\kappa,\lambda}$ holds if such a sequence exists.

In this section we will be concerned only with $\square_{\omega_1,2}$. In order to obtain our result, we will need to make use of a large cardinal hypothesis:

Definition 7. *A cardinal κ is said to be ω_1-*Erdős *if it is least such that for any partition $f \colon [\kappa]^{<\omega} \to 2$, there is $H \in [\kappa]^{\omega_1}$ which is homogeneous for f.*

Lemma 8 (Silver)**.** *If κ is ω_1-Erdős, then for any structure $\mathcal{M}$ with $\kappa \subseteq \mathcal{M}$, there is a set of indiscernibles $I \in [\kappa]^{\omega_1}$ for $\mathcal{M}$. Morever, if $\mathcal{M}$ has underlying set $H(\kappa)$ and includes among its predicates some $\lhd$ which is a well-ordering of its universe, we may assume I consists of inaccessible cardinals which are* remarkable *in the sense that for any $\gamma \in I$, $I \setminus \gamma$ is a set of indiscernibles for $\langle \mathcal{M}, (\delta)_{\delta<\gamma} \rangle$.*

Proof. See [5], [3]. $\square$

2.2 The Poset

Our poset $\mathbb{P}$ is a "Silverized" version of the one appearing in [2] in the sense that we modify their poset to allow conditions with ω_1-sized support. We define $\mathbb{P} = \mathbb{P}_\kappa$ as follows: set $p \in \mathbb{P}$ iff p is a function so that

(1) The domain of p is a closed $\leq \omega_1$-sized set of limit ordinals less than κ.

(2) If $\operatorname{cf} \alpha = \omega$ and $\alpha \in \operatorname{dom} p$ then $1 \leq |p(\alpha)| \leq 2$ and each set in $p(\alpha)$ is a club subset of α with countable order type.

(3) If $\operatorname{cf} \alpha = \omega_1$ and $\alpha \in \operatorname{dom} p$ then $p(\alpha) = \{C\}$ where C is a club subset of α with order type ω_1.

(4) If $\operatorname{cf} \alpha \geq \omega_2$ then $p(\alpha) = \{C\}$ where C is a closed bounded subset of α with countable order type such that $\max C = \sup(\operatorname{dom} p \cap \alpha)$.

(5) If $\alpha \in \operatorname{dom} p$, $C \in p(\alpha)$ and $\beta \in \lim(C)$, then $\beta \in \operatorname{dom} p$ and $C \cap \beta \in p(\beta)$.

(6) The supremum of $\operatorname{otp} C$ taken over all $C \in p(\alpha)$, $\operatorname{cf} \alpha \geq \omega_2$, is strictly below ω_1.

For two elements $p, q \in \mathbb{P}_\kappa$, we set $p \leq q$ iff:

1. $\operatorname{dom} q \subseteq \operatorname{dom} p$
2. For all $\alpha \in \operatorname{dom} q$:
   (a) If $\operatorname{cf} \alpha \in \{\omega, \omega_1\}$, then $p(\alpha) = q(\alpha)$.
   (b) If $\operatorname{cf} \alpha \geq \omega_2$, $p(\alpha) = \{C\}$ and $q(\alpha) = \{D\}$, then C is an end-extension of D in the sense that $D = C \cap (\max(D) + 1)$.

Lemma 9. *Suppose that κ is inaccessible. Then $\mathbb{P} = \mathbb{P}_\kappa$ is κ-c.c. and countably closed, and collapses κ to $\aleph_2$ while adding a $\square_{\omega_1,2}$-sequence.*

Proof. The proof is very similar to that of the corresponding result in [2]. The fact that $\mathbb{P}$ is κ-c.c. follows from a standard Δ-system argument. If we can show that $\mathbb{P}$ is countably closed, then the second conclusion follows immediately. So suppose that $\langle p_n : n < \omega \rangle$ is a decreasing sequence of conditions.

Let X be the set of $\alpha \in \bigcup_{n<\omega} \operatorname{dom} p_n$ such that the value of $p_n(\alpha)$ does not eventually stabilize and let

$$Y = \{\sup_{n<\omega} \max p_n(\alpha) : \alpha \in X\}$$

Observe that $Y \cap \left(\bigcup_{n<\omega} \operatorname{dom} p_n\right) = \emptyset$, since if $\alpha \in X$ the fact that $\max(p_n(\alpha)) \geq \sup(\operatorname{dom} p_n \cap \alpha)$ for every n gives

$$\sup_{n<\omega} \max p_n(\alpha) \notin \bigcup_{n<\omega} \operatorname{dom} p_n$$

Let

$$Z = \overline{\left(\bigcup_{n<\omega} \operatorname{dom} p_n\right)} \cup Y$$

where the overline indicates closure in the ordinal topology. We claim that Z is closed. To show this it suffices to show that any limit point of Y lies in $\overline{\bigcup_{n<\omega} \operatorname{dom} p_n}$. Moreover, this will itself follow from the assertion that any element of Y lies in $\overline{\bigcup_{n<\omega} \operatorname{dom} p_n}$. But this is immediate by condition (4) in the definition of $\mathbb{P}$.

We will define a condition p_ω with domain Z which is a lower bound for $\langle p_n \colon n < \omega\rangle$. First, if $\alpha \in \bigcup_{n<\omega} \operatorname{dom} p_n \setminus X$, let $p_\omega(\alpha)$ be the eventual value of the sequence $\langle p_n(\alpha) : n < \omega\rangle$. If $\alpha \in X$, then set

$$p_\omega(\alpha) = \bigcup_{n<\omega} p_n(\alpha) \cup \{\sup_{n<\omega} \max p_n(\alpha)\}$$

Next, if $\alpha \in Y$ then $\alpha = \sup_{n<\omega} \max p_n(\beta)$ for a unique $\beta \in X$, and we set

$$p_\omega(\alpha) = \bigcup_{n<\omega} p_n(\beta) \cup \{\sup_{n<\omega} \max p_n(\beta)\}$$

for this β. Finally, suppose $\alpha \in \overline{\left(\bigcup_{n<\omega} \operatorname{dom} p_n\right)} \setminus \left(\bigcup_{n<\omega} \operatorname{dom} p_n\right)$ and $p_\omega(\alpha)$ is yet to be defined. Set

$$p_\omega(\alpha) = \{\max (\operatorname{dom} (p_n) \cap \alpha) \colon n < \omega\}$$

Clearly this set is unbounded in α. Moreover, this set has order-type ω, and therefore has no limit points below α (and is club in α). Therefore we are in no danger of violating coherence (condition (5) in the definition of $\mathbb{P}$) by defining $p_\omega(\alpha)$ as such.

We refer to the condition p_ω defined above as the *canonical lower bound* of $\langle p_n \colon n < \omega\rangle$. □

We also define a *threading poset* for a given $\square_{\omega_1,2}$-sequence. Supposing that $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \colon \alpha < \omega_2\rangle$ is such a sequence, we let $\mathbb{T} = \mathbb{T}_{\vec{\mathcal{C}}}$ be the poset of closed bounded subsets C of ω_2 of countable order type such that C threads $\langle \mathcal{C}_\alpha \colon \alpha \leq \max C\rangle$ in the sense that $C \cap \alpha \in \mathcal{C}_\alpha$ for all α which are limit points of C.

If $C, D \in \mathbb{T}$, then we set $C \leq D$ if and only if C is an end-extension of D.

Finally, suppose that $\mu < \kappa$ are two inaccessible cardinals. If G is the generic added by $\mathbb{P}_\mu$, then $\mathbb{Q} = \mathbb{Q}_{\mu,\kappa,G}$ is the poset in $V[G]$ defined by setting $q \in \mathbb{Q}$ iff $q \in V$ and:

(a) $\operatorname{dom} q$ is a closed $\leq \omega_1$-sized set of limit ordinals in the interval (μ, κ).

(b) If $\operatorname{cf} \alpha = \omega$ and $\alpha \in \operatorname{dom} q$, then $1 \leq |q(\alpha)| \leq 2$ and each element of $q(\alpha)$ is a club with countable order type.

(c) If $\operatorname{cf} \alpha = \omega_1$ and $\alpha \in \operatorname{dom} q$ then $q(\alpha) = \{C\}$ where C is a club subset of α with order type ω_1.

(d) If $\operatorname{cf} \alpha \geq \omega_2$, then $q(\alpha) = \{C\}$ where C is a closed bounded subset of α with countable order type such that $\max C = \sup (\operatorname{dom} q \cap \alpha)$.

(e) If $\alpha \in \operatorname{dom} q$, $C \in q(\alpha)$, and $\beta \in \lim C$, then:

  (A) If $\beta > \mu$, then $\beta \in \operatorname{dom} q$ and $C \cap \beta \in q(\beta)$.
  (B) If $\beta < \mu$, then $C \cap \beta \in \mathcal{C}_\beta$, where $\langle \mathcal{C}_\beta \colon \beta < \mu \rangle$ is $\bigcup G$.

(f) The supremum of $\operatorname{otp} C$ taken over all $C \in q(\alpha)$, $\operatorname{cf} \alpha \geq \omega_2$, is strictly below ω_1.

For two elements $p, q \in \mathbb{Q}_{\mu,\kappa}$, we set $p \leq q$ iff:

(1) $\operatorname{dom} q \subseteq \operatorname{dom} p$

(2) For all $\alpha \in \operatorname{dom} q$:

  (a) If $\operatorname{cf} \alpha \in \{\omega, \omega_1\}$, then $p(\alpha) = q(\alpha)$.
  (b) If $\operatorname{cf} \alpha \geq \omega_2$, $p(\alpha) = \{C\}$, $q(\alpha) = \{D\}$, then C is an end-extension of D.

Claim 10. *Suppose that μ, κ are inaccessible cardinals with $\mu < \kappa$, and $\dot{G}$ is the canonical name for the $\mathbb{P}_\mu$-generic. Then if we let $\dot{\mathbb{T}} = \check{\mathbb{T}}_{\bigcup \dot{G}}$, $\dot{\mathbb{Q}} = \check{\mathbb{Q}}_{\mu,\kappa,\dot{G}}$, there is an isomorphism between a dense subset of $\mathbb{P}_\kappa$ and a dense subset of $\mathbb{P}_\mu * \dot{\mathbb{T}} * \dot{\mathbb{Q}}$. In particular these two forcings are equivalent, so informally we may view them as being equal.*

Proof. As in [2]. $\square$

2.3 The Proof

Theorem 11. *Suppose that κ is an ω_1-Erdős cardinal. Let $\mathbb{P} = \mathbb{P}_\kappa$. Then for any $\mathbb{P}$-generic G, $V[G]$ satisfies Chang's Conjecture.*

Corollary 12. *The existence of an ω_1-Erdős cardinal is equiconsistent with "Chang's Conjecture plus $\square_{\omega_1,2}$."*

Proof of Corollary 12. By Theorem 11 and Lemma 9 an ω_1-Erdős cardinal suffices for the consistency of Chang's Conjecture plus $\square_{\omega_1,2}$. By [3], the consistency of Chang's Conjecture implies that of the existence of of an ω_1-Erdős cardinal. $\square$

Proof of Theorem 11. Suppose that G is a $\mathbb{P}$-generic over V. Then $\omega_2^{V[G]} = \kappa$ and $(H(\kappa))^{V[G]} = H(\kappa)[G]$. Let $\mathcal{H} = \langle H(\kappa), \in, \dot{R} \rangle$, which we view as a name for a structure $\mathcal{H}[G]$ with underlying set $H(\kappa)[G]$ and predicate $R = \dot{R}^G \subseteq H(\kappa)[G]$.

We seek a condition $p^* \in \mathbb{P}$ and a name $\dot{\mathcal{A}}$ for an elementary substructure $\mathcal{A}$ of $\mathcal{H}[G]$ such that p^* forces $|\dot{\mathcal{A}}| = \aleph_1$, $|\mathcal{A} \cap \omega_1| = \aleph_0$. With this in mind, let $I = \{\iota_\alpha \colon \alpha < \omega_1\}$ be a collection of remarkable indiscernibles for $\mathcal{H}$. For each $\alpha < \omega_1$, let $I_\alpha = \{\iota_\delta \colon \delta < \omega\alpha\}$ be the set of the first $\omega\alpha$ indiscernibles and let $\gamma_\alpha = \iota_{\omega\alpha}$. Let $\mathcal{M}_\alpha$ be the Skolem Hull of I_α in $\mathcal{H}$.

We construct a sequence $\langle p^*_\alpha \colon 1 \leq \alpha < \omega_1 \rangle$ by induction on α so that:

(a) If $1 \leq \alpha < \beta < \omega_1$ then $p^*_\beta \leq p^*_\alpha$.

(b) p^*_α is a master condition for $\mathbb{P}$ over $\mathcal{M}_\alpha$.

(c) p^*_α is an element of $\mathbb{P}_{\gamma_\alpha}$.

We begin with the base case $\alpha = 1$. Consider the set $\mathbb{P} \cap \mathcal{M}_1 = (\mathbb{P}_{\mathrm{ON}})^{\mathcal{M}_1}$, which is a proper class in $\mathcal{M}_1$. Observe that since $\mathcal{M}_1$ is elementary in $\mathcal{H}$, $\mathcal{M}_1$ satisfies "$\mathbb{P}$ has the <-ON chain condition." In other words, $\mathcal{M}_1$ believes that every antichain in $\mathbb{P}$ is a set. For each antichain A in $\mathcal{M}_1$, let $A^\downarrow = \{p \in \mathbb{P} \colon (\exists q \in A)\, p \leq q\}$ be the downwards closure of A. Let $\{A_i \colon i < \omega\}$ enumerate the collection of all maximal antichains which are elements of $\mathcal{M}_1$. By induction we may construct a descending sequence $\{r_i \colon i < \omega\}$ of elements of $\mathbb{P}$ such that $r_i \in A_i^\downarrow \cap \mathcal{M}_1$. Let $p^*_1 \in \mathbb{P}$ be the canonical lower bound for the sequence $\{r_i \colon i < \omega\}$. Then p^*_1 is a master condition for $\mathbb{P}$ over $\mathcal{M}_1$ and is an element of $\mathbb{P}_{\gamma_1}$, as desired.

Next suppose that α is limit. Choose a sequence $\langle \alpha_n \colon n < \omega \rangle$ cofinal in α, and let p^*_α be the canonical lower bound for $\langle p^*_{\alpha_n} \colon n < \omega \rangle$. It should be clear that properties (a)-(c) are satisfied, since $\mathbb{P} \cap \mathcal{M}_\alpha = \bigcup_{n<\omega} (\mathbb{P} \cap \mathcal{M}_{\alpha_n})$, and $\mathbb{P} \cap \mathcal{M}_\alpha = (\mathbb{P}_{\mathrm{ON}})^{\mathcal{M}_\alpha}$ has the <-ON chain condition in $\mathcal{M}_\alpha$.

Finally we consider the case where $\alpha = \bar{\alpha} + 1$ is a successor ordinal. We distinguish between the case where $\bar{\alpha}$ is a nonzero limit ordinal and where $\bar{\alpha}$ is itself a successor ordinal, considering first the latter. Since $p^*_{\bar{\alpha}}$ was chosen to be a master condition for $\mathbb{P}$ over $\mathcal{M}_{\bar{\alpha}}$, we have

$$p^*_{\bar{\alpha}} \Vdash \mathcal{M}_{\bar{\alpha}}[\dot{G}] \preceq \mathcal{H}[\dot{G}] \wedge \mathrm{ON} \cap \mathcal{M}_{\bar{\alpha}}[\dot{G}] = \mathrm{ON} \cap \mathcal{M}_{\bar{\alpha}}$$

Consider $\mathcal{M}_\alpha$. By remarkability of the indiscernibles which generate $\mathcal{M}_\alpha$, we have $H(\gamma_{\bar{\alpha}}) \cap \mathcal{M}_\alpha = \mathcal{M}_{\bar{\alpha}}$ and $\mathbb{P}_{\gamma_{\bar{\alpha}}} \cap \mathcal{M}_\alpha = \mathbb{P} \cap \mathcal{M}_{\bar{\alpha}}$. Moreover, $p^*_{\bar{\alpha}}$ is a master condition for the forcing $\mathbb{P}_{\gamma_{\bar{\alpha}}}$ over the model $\mathcal{M}_\alpha$, since $\mathbb{P}_{\gamma_{\bar{\alpha}}}$ has the $\gamma_{\bar{\alpha}}$-c.c. and therefore every antichain of $\mathbb{P}_{\gamma_{\bar{\alpha}}}$ in $\mathcal{M}_\alpha$ is an element of $H(\gamma_{\bar{\alpha}}) \cap \mathcal{M}_\alpha = \mathcal{M}_{\bar{\alpha}}$. So if we let $\dot{G}_{\gamma_{\bar{\alpha}}}$ be the canonical name for the $\mathbb{P}_{\gamma_{\bar{\alpha}}}$-generic, then

$$p^*_{\bar{\alpha}} \Vdash \mathcal{M}_\alpha[\dot{G}_{\gamma_{\bar{\alpha}}}] \preceq \mathcal{H}[\dot{G}_{\gamma_{\bar{\alpha}}}] \wedge \mathrm{ON} \cap \mathcal{M}_\alpha[\dot{G}_{\gamma_{\bar{\alpha}}}] = \mathrm{ON} \cap \mathcal{M}_\alpha$$

Working in V, let $\dot{\mathbb{T}} = \check{\mathbb{T}}_{\bigcup \dot{G}_{\gamma_{\bar{\alpha}}}}$ be the canonical name for the threading forcing associated to $G_{\gamma_{\bar{\alpha}}}$. Let $\{\dot{B}_i \colon i < \omega\}$ enumerate all names in $\mathcal{M}_\alpha$ which are forced by $p^*_{\bar{\alpha}}$ to be maximal antichains of $\dot{\mathbb{T}}$. By induction we may construct a descending sequence $\{\check{t}_i \colon i < \omega\}$ of "check-names" (by which we mean canonical names for elements of V) for elements of $\mathbb{T} = \dot{\mathbb{T}}^{G_{\gamma_{\bar{\alpha}}}}$ such that

$$p^*_{\bar{\alpha}} \Vdash \check{t}_i \in \dot{B}_i^\downarrow \cap \mathcal{M}_\alpha[\dot{G}_{\gamma_{\bar{\alpha}}}]$$

where $\dot{B}_i^\downarrow$ is a name for the downwards closure of $B_i = \dot{B}_i^{G_{\gamma_{\bar{\alpha}}}}$ in $\mathbb{T}$. Observe that we may take canonical names for elements of V $\check{t}_i$ rather than merely arbitrary names $\dot{t}_i$ since $p^*_{\bar{\alpha}}$ is a master condition for $\mathbb{P}_{\gamma_{\bar{\alpha}}}$ over $\mathcal{M}_\alpha$.

Still working in V, we let

$$t = \bigcup_{i<\omega} t_i$$

$$p^{**}_{\bar{\alpha}} = p^*_{\bar{\alpha}} \cup \{(\sup(\mathcal{M}_{\bar{\alpha}} \cap \kappa), \{t\})\}$$

Then $p^*_{\bar{\alpha}} * \check{t}$ is a master condition for $\mathbb{P}_{\gamma_{\bar{\alpha}}} * \dot{\mathbb{T}}$ over $\mathcal{M}_\alpha$. Now observe that $\dot{\mathbb{Q}} = \check{\mathbb{Q}}_{\gamma_{\bar{\alpha}},\mathrm{ON},\dot{G}_{\gamma_{\bar{\alpha}}}}$ is definable over $\langle \mathcal{M}_\alpha[G_{\gamma_{\bar{\alpha}}}], \in, \mathcal{M}_\alpha\rangle$ (i.e. the structure $\mathcal{M}_\alpha[G_{\gamma_{\bar{\alpha}}}]$ with signature expanded to include a predicate for membership in $\mathcal{M}_\alpha$). So we may proceed as above to find $\dot{q} \in \dot{\mathbb{Q}}$ such that $p^*_{\bar{\alpha}} * \check{t} * \dot{q}$ is a master condition for $\mathbb{P}_{\gamma_{\bar{\alpha}}} * \dot{\mathbb{T}} * \dot{\mathbb{Q}}$ over $\mathcal{M}_\alpha$. Thus if we set $p^*_\alpha = p^{**}_{\bar{\alpha}} * \check{t} * \dot{q}$, we may view p^*_α as a master condition for $\mathbb{P}$ over $\mathcal{M}_\alpha$ which extends $p^*_{\bar{\alpha}}$. We note that $p^*_\alpha(\sup(M_{\bar{\alpha}} \cap \kappa)) = \{t\}$.

For nonzero limit $\bar{\alpha}$, the construction is exactly as above, except we modify $p^*_\alpha(\sup(M_{\bar{\alpha}} \cap \kappa))$ to be $\{t, F\}$, where t is a master condition for the threading poset associated to the generic for $\mathbb{P}_{\gamma_\alpha}$ (as above) and $F = \{\sup(\mathcal{M}_\delta \cap \kappa) : \delta < \bar{\alpha}\}$, rather than merely taking $p^*_\alpha(\sup(M_{\bar{\alpha}} \cap \kappa))$ to be $\{t\}$.

Observe that this is the only place in the proof where we use the allowed "two-ness" of the square sequence. Moreover, in adding F we preserve the coherence property since its initial segments of limit length were put on the square sequence at earlier successor of limit stages.

Finally, at the end of the construction we set

$$p^* = \bigcup_{\alpha<\omega_1} p^*_\alpha \cup \big\{(\sup((\bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha) \cap \kappa), F^*)\big\}$$

where $F^* = \{\sup(\mathcal{M}_\alpha \cap \kappa) : \alpha < \kappa\}$. The construction ensures that this is a condition in $\mathbb{P} = \mathbb{P}_\kappa$. In particular, successor of limit stages ensure that the initial segments of limit length of F^* appear on the square sequence, and so when adding F^* there is no danger of violating coherence. Moreover, p^* is a master condition for $\mathbb{P}$ over $\mathcal{M} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$. Thus p^* forces that $\mathcal{M}[G]$ is the desired elementary submodel of $\mathcal{H}[G]$. □

3 Higher Chang's Conjectures vs. Weak Squares

In this section we concern ourselves with generalizations of Chang's Conjecture to higher cardinals.

Definition 13. *Suppose that $\tau \leq \kappa < \lambda$ are cardinals. We write $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ if for every structure $\mathcal{N}$ with $\lambda^+ \subseteq \mathcal{N}$, there exists $\mathcal{M} \preceq \mathcal{N}$ such that $|\mathcal{M}| = \kappa^+$ and $|\mathcal{M} \cap \lambda| = \kappa$.*

Similarly, we write $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$ if for every structure $\mathcal{N}$ with $\lambda^+ \subseteq \mathcal{N}$, there exists $\mathcal{M} \preceq \mathcal{N}$ such that $|\mathcal{M}| = \kappa^+$, $|\mathcal{M} \cap \lambda| = \kappa$, and $\tau \subseteq \mathcal{M}$.

Observe that Chang's Conjecture is equivalent to $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ and that $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ is equivalent to $(\lambda^+, \lambda) \twoheadrightarrow_\omega (\kappa^+, \kappa)$ for any infinite cardinals $\kappa < \lambda$. Moreover, we also have:

Lemma 14. *Suppose that $\tau \leq \kappa < \lambda$ are infinite cardinals and there are at most τ many cardinals between κ and λ. Then $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$ implies $(\lambda^+, \lambda) \twoheadrightarrow_\kappa (\kappa^+, \kappa)$.*

Proof. The lemma is implicit in [7]. Specifically, the conclusion of the lemma holds by following the argument of Case (2) of Lemma 4.15 in [7]. $\square$

Lemma 15. *Suppose that $\tau \leq \kappa < \lambda$ are infinite cardinals such that $\lambda^\tau = \lambda$. Then $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ implies $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$.*

Proof. Take $B = \tau$ in Case (1) of Lemma 4.15 in [7]. $\square$

In the argument below we make use of the following claim without comment:

Claim 16. *Suppose that for all sufficiently large θ and all structures $\mathcal{H} = \langle H(\theta), \in, \dots \rangle$ there exists $\mathcal{M} \preceq \mathcal{H}$ such that $|\mathcal{M} \cap \lambda^+| = \kappa^+$, $|\mathcal{M} \cap \lambda| = \kappa$, and $\tau \subseteq \mathcal{M}$. Then $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$.*

The proof is entirely analogous to that of Claim 5.

Lemma 17 (Folklore)**.** *Suppose that $\kappa < \lambda$ are infinite cardinals and θ is a sufficiently large regular cardinal. Let M be an elementary substructure of $\langle H(\theta), \in \rangle$ such that $|M \cap \lambda^+| = \kappa^+$ and $|M \cap \lambda| = \kappa$. Then the order type of $M \cap \lambda^+$ is κ^+.*

Proof. Suppose otherwise for a contradiction. Since $|M \cap \lambda^+| = \kappa^+$, the order type of $M \cap \lambda^+$ must be strictly greater than κ^+. Let α be the κ^+ element of $M \cap \lambda^+$. Observe that $\alpha \geq \lambda$ (since there are only κ many elements of M below λ) and hence by elementarity $\lambda = |\alpha|$ is an element of M. Applying elementarity again, there is $f \in M$ which is a bijection from α to λ. In particular,

$$f``(M \cap \alpha) \subseteq M \cap \lambda$$

which is a contradiction since the left hand side has cardinality κ^+ (since f is a bijection) whereas the right hand side has cardinality κ. $\square$

Lemma 18 (Folklore)**.** *Suppose that M is an elementary substructure of $\langle H(\theta), \in \rangle$ for some sufficiently large θ and $\alpha \in M$. Letting $\mu = \mathrm{cf}\, \alpha$, if $f \in M$ is an increasing function from μ into α whose range is cofinal in α, then*

$$\sup (f``(M \cap \mu)) = \sup (M \cap \alpha)$$

Proof. Clearly $\sup (f``(M \cap \mu)) \leq \sup (M \cap \alpha)$, since $f \in M$ and $f \colon \mu \to \alpha$. For equality, suppose for a contradiction that

$$\sup (f``(M \cap \mu)) < \sup (M \cap \alpha)$$

and choose $\beta \in M \cap \alpha$ such that $\beta > \sup (f``(M \cap \alpha))$. By elementarity

$$M \models (\exists \xi \in \mu)\, (f(\xi) > \beta)$$

and so choosing $\xi_0 \in M \cap \mu$ to witness the existential statement above we have:

$$\sup\left(f``(M \cap \mu)\right) < \beta < f(\xi_0)$$

an obvious contradiction. $\square$

Theorem 19. *Suppose that $\kappa < \lambda$ are uncountable cardinals and $\tau \leq \kappa$ is infinite. Suppose moreover that $\square_{\lambda,\tau}$ holds. Then $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$ fails.*

Corollary 20. *Suppose that $\kappa < \lambda$ are uncountable cardinals and $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ holds. Then $\square_{\lambda,\omega}$ fails.*

Proof. Immediate from the theorem and the fact that $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ is equivalent to $(\lambda^+, \lambda) \twoheadrightarrow_\omega (\kappa^+, \kappa)$. $\square$

Corollary 21. *Suppose that $\kappa < \lambda$ are uncountable cardinals and there are at most countably many cardinals between κ and λ. Then $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ implies the failure of $\square_{\lambda,\kappa}$.*

Proof. This follows immediately from Theorem 19 and Lemma 14 by taking $\tau = \omega$. $\square$

Observe that the same argument shows that if there are at most τ many cardinals between κ and λ then $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$ implies the failure of $\square_{\lambda,\kappa}$.

Corollary 22. *Suppose that $\kappa < \lambda$ are uncountable cardinals and $\tau \leq \kappa$ is some infinite cardinal with $\lambda^\tau = \lambda$. Then $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ implies the failure of $\square_{\lambda,\tau}$.*

Proof. Immediate from Theorem 19 and Lemma 15. $\square$

Proof of Theorem 19. Suppose for a contradiction that $\square_{\lambda,\tau}$ held in conjunction with $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$, and let $\vec{\mathcal{C}} = \langle \mathcal{C}_\xi : \xi < \lambda^+ \rangle$ be a $\square_{\lambda,\tau}$ sequence. Choose M an elementary substructure of $\langle H(\theta), \in, \vec{\mathcal{C}} \rangle$ (for sufficiently large θ) such that $|M \cap \lambda^+| = \kappa^+$, $|M \cap \lambda| = \kappa$, and $\tau \subseteq M$.

Fix a club $C^* \in \mathcal{C}_{\sup(M \cap \lambda^+)}$. By Lemma 17, we may choose a club D in $\sup(M \cap \lambda^+)$ of ordertype κ^+. We assume moreover that D consists only of limits of ordinals in M.

Claim 23. *For all sufficiently large $\alpha \in C^*$, the ordertype of $C^* \cap \alpha$ is not an element of M.*

Proof. These ordertypes are distinct elements of λ, and since $|M \cap \lambda| = \kappa$, at most κ of them can belong to M. Since the cofinality of $\sup C^* = \sup(M \cap \lambda^+)$ is κ^+, the result follows immediately. $\square$

Claim 24. *For all sufficiently large $\alpha \in \operatorname{Lim} C^*$, $\alpha \notin M$.*

Proof. Choose $\alpha \in \operatorname{Lim} C^*$ and note that $C^* \cap \alpha \in \mathcal{C}_\alpha$. If $\alpha \in M$, then $\mathcal{C}_\alpha \subseteq M$ (since $|\mathcal{C}_\alpha| \leq \tau$ and $\tau \subseteq M$) and so in particular $C^* \cap \alpha \in M$, giving $\operatorname{otp}(C^* \cap \alpha) \in M$. By Claim 23, this may happen for only boundedly many $\alpha \in C^*$. $\square$

For each α below $\sup(M \cap \lambda^+)$, let $\alpha^\uparrow$ denote the least element of M which is $\geq \alpha$.

Claim 25. *For all sufficiently large $\alpha \in \mathrm{Lim}(C^* \cap D)$, $\alpha^\uparrow$ is strictly greater than α.*

Proof. Immediate from Claim 24. □

Now define:

$$Z = \{\mu \leq \lambda \colon \mu = \mathrm{cf}(\alpha^\uparrow) \text{ for unboundedly many } \alpha \text{ in } \mathrm{Lim}(C^* \cap D)\}$$

Claim 26. $|Z| \leq \kappa$.

Proof. For each $\alpha \in \mathrm{Lim}(C^* \cap D)$, $\alpha^\uparrow$ is an element of M below λ^+, and therefore its cofinality is an element of $M \cap (\lambda + 1)$, which has cardinality κ. □

Claim 27. *There is $\mu \in Z$ with $\mu \geq \kappa^+$.*

Proof. By Claim 26, it is enough to find unboundedly many $\alpha \in \mathrm{Lim}(C^* \cap D)$ such that $\mathrm{cf}(\alpha^\uparrow) \geq \kappa^+$.

Fix any $\alpha \in \mathrm{Lim}(C^* \cap D)$ large enough for Claim 25, with $\mathrm{cf}(\alpha) = \kappa$. Observe that there are unboundedly many such α since the ordertype of $C^* \cap D$ is κ^+. By choice of α, $\sup(M \cap \alpha^\uparrow) = \alpha < \alpha^\uparrow$. Then:

$$\kappa = \mathrm{cf}\,\alpha < \mathrm{cf}\,\alpha^\uparrow$$

by Lemma 18. □

Claim 28. $|Z| \geq 2$.

Proof. By Claim 26, it suffices to find disjoint $A_1, A_2 \subseteq \mathrm{Lim}(C^* \cap D)$ such that A_1, A_2 are unbounded and for any $\alpha_1 \in A_1$, $\alpha_2 \in A_2$, we have $\mathrm{cf}(\alpha_1^\uparrow) \neq \mathrm{cf}(\alpha_2^\uparrow)$.

To do so, choose distinct regular $\eta_1, \eta_2 \leq \kappa$. Observe that this is possible since κ is uncountable. Now let

$$A_1 = \{\alpha \in \mathrm{Lim}(C^* \cap D) \colon \mathrm{cf}\,\alpha = \eta_1\}$$
$$A_2 = \{\alpha \in \mathrm{Lim}(C^* \cap D) \colon \mathrm{cf}\,\alpha = \eta_2\}$$

Clearly A_1, A_2 are disjoint and unbounded. Moreover, for any $\alpha_1 \in A_1$, $\alpha_2 \in A_2$, we have $\mathrm{cf}(\alpha_1^\uparrow) \neq \mathrm{cf}(\alpha_2^\uparrow)$ by Lemma 18. □

Now to prove the theorem:

Fix distinct $\mu_1, \mu_2 \in Z$ with $\mu_1 > \mu_2$ and $\mu_1 \geq \kappa^+$. Fix $\alpha_1, \alpha_2 \in \mathrm{Lim}(C^* \cap D)$, large enough for Claims 23 and 24 and with $\alpha_1 < \alpha_2$, so that $\mathrm{cf}(\alpha_1^\uparrow) = \mu_1$ and $\mathrm{cf}(\alpha_2^\uparrow) = \mu_2$. Fix $E \in M$ cofinal in $\alpha_2^\uparrow$ of ordertype μ_2.

Let

$$U = \Big\{ \sup(C \cap \alpha_1^\uparrow) + 1 \colon C \in \bigcup_{\xi \in E} \mathcal{C}_\xi \text{ with } \sup(C \cap \alpha_1^\uparrow) < \alpha_1^\uparrow \Big\}$$

Claim 29. $\sup(C^* \cap \alpha_1^\uparrow) + 1 \in U$.

Proof. Note first that $C^* \cap \alpha_1^\uparrow$ is bounded in $\alpha_1^\uparrow$, since otherwise we would have $\alpha_1^\uparrow \in \operatorname{Lim} C^*$, and as $\alpha_1^\uparrow > \alpha_1$ is an element of M this would contradict choice of α_1. Now since E is club in $\alpha_2^\uparrow$ and belongs to M, we have $\alpha_2 = \sup(M \cap \alpha_2^\uparrow) \in E$, where the equality follows from Lemma 18. Since $C^* \cap \alpha_2 \in \mathcal{C}_{\alpha_2}$, and since $C^* \cap \alpha_2 \cap \alpha_1^\uparrow = C^* \cap \alpha_1^\uparrow$ is bounded in $\alpha_1^\uparrow$, it follows by definition that $\sup(C^* \cap \alpha_1^\uparrow) + 1$ is in U. $\square$

We have $U \in M$ by elementarity and since the parameters used are in M. U has cardinality $< \mu_1$ by definition and since $\mu_1 > \max(\mu_2, \kappa)$. Since $\operatorname{cf}(\alpha_1^\uparrow) = \mu_1$, it follows that U is bounded in $\alpha_1^\uparrow$.

Moreover, since $U \in M$ we have $\sup U \in M$, and since $\sup U < \alpha_1^\uparrow$ it follows that $\sup U \leq \alpha_1$. But this contradicts Claim 29, since $\alpha_1 \in C^*$ and therefore

$$\sup(C^* \cap \alpha_1^\uparrow) + 1 \geq \alpha_1 + 1 > \alpha_1$$

$\square$

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