TODORCEVIC OPEN COLORING AXIOM FOR SPACES SMALLER THAN THE CONTINUUM

NEEMAN, ITAY

Abstract. We prove that the restriction of Todorcevic's open coloring axiom to spaces of size strictly less than the continuum is consistent with arbitrarily large values of the continuum.

§1. Introduction. Todorcevic's open coloring axiom ($\mathsf{OCA}_{\mathrm{Tod}}$) was introduced by Todorcevic in the early 1980s, see [11, Section 8]. Phrased as an axiom on graphs, it states that every open graph on a vertex set $X \subseteq \mathbb{R}$ is either countably chromatic, or has an uncountable clique. This is clearly a dichotomy. Here "open" is meant in the topology on $[X]^2 = \{\langle x, y\rangle \in X^2 \mid x \neq y\}$ inherited from $\mathbb{R}^2$. $\mathsf{OCA}_{\mathrm{Tod}}$ is a central consequence of the proper forcing axiom, and can also be forced directly assuming only ZFC. It negates the CH, and has several key consequences. To name just a few: Todorcevic [11] used it for an analysis of gaps, Farah [6] showed that it implies that all automorphisms of the Calkin algebra are inner, Velickovic [12] used $\mathsf{OCA}_{\mathrm{Tod}}$ in the context of MA to prove that all automorphisms of $\mathbb{P}(\omega)/\mathrm{Fin}$ are trivial, and de Bondt-Farah-Vignati [4] strengthened this result to remove the assumption of MA.

All models of $\mathsf{OCA}_{\mathrm{Tod}}$ known to date satisfy that the continuum is $\aleph_2$. It is a central open question whether the axiom determines the value of the continuum to be $\aleph_2$, or is consistent with higher values.

There have been only a few partial results on this question.

$\mathsf{OCA}_{\mathrm{Tod}}$ breaks into the conjunction of two statements: The first, which we denote $\mathsf{OCA}_{\mathrm{Tod}}(\aleph_1)$, is the restriction of $\mathsf{OCA}_{\mathrm{Tod}}$ to graphs on vertex sets X of size $\aleph_1$. The second, called *reflection*, is the statement that if an open graph G on a vertex set $X \subseteq \mathbb{R}$ is not countably chromatic, then there is $\bar{X} \subseteq X$ of size $\aleph_1$ so that the restriction of G to $\bar{X}$ is not countably chromatic.

Farah, in unpublished work [5], showed that $\mathsf{OCA}_{\mathrm{Tod}}(\aleph_1)$ is consistent with arbitrarily large values of the continuum. He also showed that the same is true adding $\mathsf{MA}(\aleph_1)$. Working on reflection, he noted that it holds in the forcing extension adding arbitrarily many Cohen reals to a model of CH.

Moore [8] considered the conjunction of $\mathsf{OCA}_{\mathrm{Tod}}$ with another open coloring axiom, that had been introduced prior to Todorcevic [11] by Abraham-Rubin-Shelah [1]. Their axiom $\mathsf{OCA}_{\mathrm{ARS}}$ states that if $X \subseteq \mathbb{R}$ has size $\aleph_1$, and $c\colon [X]^2 \to \{0, 1\}$ is open, then X can be partitioned into countably many c-homogeneous

This material is based upon work supported by the National Science Foundation under Grants No. DMS-1764029 and DMS-2246905

sets, meaning sets A_i so that c is constant on each $[A_i]^2$. Here "open" means that the c preimages of $\{0\}$ and $\{1\}$ are both open in the topology on $[X]^2$ inherited from $\mathbb{R}^2$.

$\mathsf{OCA}_{\text{Tod}}$ is often phrased as a statement about homogeneous sets for the characteristic function c_G of the graph G. Indeed this is closer to the original formulation in [11]. It then resembles $\mathsf{OCA}_{\text{ARS}}$, and in fact some of the methods leading to $\mathsf{OCA}_{\text{Tod}}$ trace back to Abraham-Rubin-Shelah [1], along with earlier work of Baumgartner [2, 3]. But there are some key differences between the two axioms: In $\mathsf{OCA}_{\text{Tod}}$ only one of the c_G-preimages of $\{0\}, \{1\}$ is assumed to be open, and the axiom only provides an actual partition of X on one side of its dichotomy, while $\mathsf{OCA}_{\text{ARS}}$ lacks the reflection component of $\mathsf{OCA}_{\text{Tod}}$.

Moore [8] showed that the *conjunction* $\mathsf{OCA}_{\text{Tod}} + \mathsf{OCA}_{\text{ARS}}$ implies that the continuum is exactly $\aleph_2$. Gilton-Neeman [7] later showed that $\mathsf{OCA}_{\text{ARS}}$ is consistent with arbitrarily large values of the continuum. These results underline the question of whether $\mathsf{OCA}_{\text{Tod}}$ determines the value of the continuum.

Moore [10] demonstrated some essential difficulties in obtaining $\mathsf{OCA}_{\text{Tod}}$ with large continuum, for example showing that a c.c.c. forcing adding reals over a model of $\mathsf{OCA}_{\text{Tod}}$ will destroy $\mathsf{OCA}_{\text{Tod}}$ unless it adds a dominating real. Since $\mathsf{OCA}_{\text{Tod}}$ implies that $\mathfrak{b} = \aleph_2$, adding too many dominating reals is itself a danger to $\mathsf{OCA}_{\text{Tod}}$.

As we noted above, the question of whether $\mathsf{OCA}_{\text{Tod}}$ is consistent with large continuum remains open. In this paper we provide the following partial result. Let $\mathsf{OCA}_{\text{Tod}}(< \kappa)$ denote the restriction of $\mathsf{OCA}_{\text{Tod}}$ to graph on vertex sets of size $< \kappa$. Define $\mathsf{OCA}_{\text{Tod}}(\kappa)$ similarly.

Theorem 1.1. *Let θ be a Mahlo cardinal, and let $\delta > \theta$ be a cardinal so that $\delta^{<\theta} = \delta$. Then there is a θ-c.c. forcing extension where $\theta = \aleph_2$, $\mathfrak{c} = \delta$, and $\mathsf{OCA}_{\text{Tod}}(< \mathfrak{c})$ holds. The extension preserves $\aleph_1$ if the CH holds in the ground model.*

In particular, assuming the consistency of a Mahlo cardinal, $\mathsf{OCA}_{\text{Tod}}(< \mathfrak{c})$ is consistent with arbitrarily large values of the continuum.

Let $\mathsf{MA}(\text{Knaster}, \aleph_1)$ denote the restriction of Marin's Axiom to meeting $\aleph_1$ dense sets in Knaster posets.

Theorem 1.2. *Under the assumptions of Theorem 1.1, one can in fact obtain a model of $\mathsf{OCA}_{\text{Tod}}(< \mathfrak{c}) + \mathsf{MA}(\text{Knaster}, \aleph_1)$.*

In particular, assuming the consistency of a Mahlo cardinal, $\mathsf{OCA}_{\text{Tod}}(< \mathfrak{c}) + \mathsf{MA}(\text{Knaster}, \aleph_1)$ is consistent with arbitrarily large values of the continuum.

Our models witnessing these theorems also satisfy that $\mathfrak{b} = \aleph_2$. Thus both theorems can be strengthened to include this in their conclusions.

Moore's proof that $\mathsf{OCA}_{\text{Tod}} + \mathsf{OCA}_{\text{ARS}}$ implies $\mathfrak{c} = \aleph_2$ does not use the full strength of $\mathsf{OCA}_{\text{Tod}}$, but only $\mathsf{OCA}_{\text{Tod}}(\aleph_2) + \mathfrak{b} = \aleph_2$, plus $\mathsf{OCA}_{\text{ARS}}$ of course. Our results, specifically the fact that $\mathsf{OCA}_{\text{Tod}}(\aleph_2) + \mathfrak{b} = \aleph_2$ is consistent with arbitrarily large continuum, therefore show that the techniques of Moore [8] would not work without assuming $\mathsf{OCA}_{\text{ARS}}$. This complements [10, Theorem 3].

The structure of our paper is as follows:

In Section 2 we specify a precise way of selecting which names for conditions are used in forcing iterations, that will help us with technical points later on.

In Section 3 we prepare for the countably closed part of our forcing construction. This part (working with δ and θ as in Theorems 1.1 and 1.2) involves adding δ subsets of ω_1, and then forcing with an iteration of length θ, where each iterand is a product of size δ of "stems and commitments" posets. Our work in Section 3 centers on maintaining a large degree of symmetry for these posets under permutations of δ, maintaining a large degree of compatibility between shifts of conditions under such permutations, and observing some consequences of the symmetry and compatibility. The purpose of the added subsets of ω_1 is not so much to increase $2^{\aleph_1}$ as in Moore [9], but to generate almost disjoint sets that reduce the interference between the clique adding factors of the large products to be used in the next section.

In Section 4 we present one step of our intended iteration. This step handles a specific (name for) an open graph, but forces to simultaneously add cliques in all shifts of this name under permutations of δ. It breaks into a countably closed part, and a c.c.c. part. The countably closed part is a "stems and commitments" poset which can be folded into an iteration of large products as in Section 3. It adds a subset of the graph of size $\aleph_1$, so that forcing with finite subcliques of this subset is c.c.c. The second part is this c.c.c. clique-adding forcing. This kind of division is standard for forcing $\mathsf{OCA}_{\mathrm{Tod}}$. Crucially here we perform this step simultaneously, through a large symmetric product, on all shifts of the given graph name by permutations of δ. We use δ almost disjoint subsets of ω_1 to reduce the interference between the factors of this product, and to maintain the countable chain condition for the δ-sized product of the shifted clique-adding posets, see the proof of Lemma 4.11. To make this work we need a stronger assumption on the graph than not being countably chromatic. This is explained in Remark 4.1.

In Section 5 we put these components together to obtain a proof of Theorems 1.1 and 1.2. Even though we end with a continuum of size δ, our iteration is of length θ. In stage $\kappa < \theta$ we guess some specific (name for a) graph, and we simultaneously and symmetrically handle, in the manner of Section 4, this graph and all shifts of this graphs by permutations of δ. This is essential for obtaining a continuum of size δ. As mentioned above, doing this requires a stronger assumption on the graph than not being countably chromatic. It is in Section 5 that we must derive this stronger assumption for the graphs we are handling. One part of this derivation involves symmetrizing a countable coloring of the graph being handled. This is done in Claim 5.15, and relies on the use of finite supports when adding reals, in a manner similar to some of the key ideas of Farah [5], though the forcing construction here is very different. The second part of the proof involves internalizing the symmetrized coloring into an elementary substructure. This is done in the paragraphs that follow the proof of Claim 5.15. This is the part of the argument that relies on the vertex set of the graph being *smaller* than the continuum.

It is natural to ask whether some variant of our argument could be used to obtain the full $\mathsf{OCA}_{\mathrm{Tod}}$ with large continuum. Doing this would require a different internalization argument.

§2. A note on forcing compositions and iterations. There are differing standard approaches to specifying the precise set of conditions when defining forcing iterations. In some cases the choice of which approach to take can make a difference, at least on a technical level, to the argument. For example in closure arguments it can be important that the set of conditions is rich enough to be *full* at each coordinate. Fullness will be important for our argument, and more than this we will need the fullness to be witnessed by conditions which are frugal in their use of the first poset in our iteration. The following definitions and claims will be useful for this.

Definition 2.1. Let $\mathbb{P}$ be a poset (with largest element $\mathbb{1}$). A $\mathbb{P}$-name $\dot{x}$ is *efficiently below* $p \in \mathbb{P}$ if for all $\langle \dot{y}, q\rangle \in \dot{x}$:

1. $q \leq p$.
2. $\dot{y}$ is efficiently below q.

$\dot{x}$ is *efficient* if it is efficiently below $\mathbb{1}$.

Claim 2.2. *For every $\mathbb{P}$-name $\dot{x}$, and every $p \in \mathbb{P}$, there is a name $\dot{x}^*$ which is efficiently below p, so that $p \Vdash \dot{x}^* = \dot{x}$. In particular there is an efficient name $\dot{x}^*$ so that $\Vdash \dot{x}^* = \dot{x}$.*

Proof. By induction on the rank of $\dot{x}$. For each $\langle \dot{y}, q\rangle \in \dot{x}$ and $q^* \leq q$, find $\dot{y}^*_{\dot{y},q^*}$ which is efficiently below q^* and so that $q^* \Vdash \dot{y}^*_{\dot{y},q^*} = \dot{y}$, and fix any $A_q \subseteq \mathbb{P}$ which is maximal among conditions extending both q and p, in the sense that any such condition is compatible with a condition in A_q. Then set $\dot{x}^* = \{\langle \dot{y}^*_{\dot{y},q^*}, q^*\rangle \mid \langle \dot{y}, q\rangle \in \dot{x} \wedge q^* \in A_q\}$. ⊣

Say that a name $\dot{x}$ is of *forcing-hereditary size* at most κ, if $|\dot{x}| \leq \kappa$, and for every $\langle \dot{y}, p\rangle \in \dot{x}$, $\dot{y}$ is of forcing-hereditary size at most κ.

Claim 2.3. *Suppose $\mathbb{P}$ is κ^+-c.c. Then given $\dot{x}$ of forcing-hereditary size at most κ, the conclusion of Claim 2.2 holds with $\dot{x}^*$ which is also of forcing hereditary size at most κ.*

Proof. Follow the proof of Claim 2.2, pick A_q to be antichains, so that they have size at most κ, and inductively pick $\dot{y}^*_{\dot{y},q^*}$ to be of forcing-hereditary size at most κ. Then the resulting $\dot{x}^*$ is of forcing-hereditary size at most κ. ⊣

By the von Neumann rank of a set u we mean the least α so that $u \in V_{\alpha+1}$.

Claim 2.4. *Suppose $\mathbb{P} \in V_\kappa$, $p \in \mathbb{P}$, $\dot{x}$ is efficiently below p, and p forces that the von Neumann rank of $\dot{x}$ is α. Then the name $\dot{x}$ belongs to $V_{\kappa+3\cdot(\alpha+1)}$.*

Proof. By induction on α. Note that if $\langle \dot{y}, q\rangle \in \dot{x}$, then $q \leq p$, $\dot{y}$ is efficiently below q, and $q \Vdash \dot{y} \in \dot{x}$, so that in particular q forces the von Neumann rank of $\dot{y}$ to be strictly smaller than α. It follows inductively that $\dot{y} \in V_{\kappa+3\cdot\alpha}$. Since $\mathbb{P} \in V_\kappa$, this implies that the ordered pair $\langle \dot{y}, q\rangle$ belongs to $V_{\kappa+3\cdot\alpha+2}$. So $\dot{x}$ is a subset of $V_{\kappa+3\cdot\alpha+2}$, i.e., an element of $V_{\kappa+3\cdot(\alpha+1)}$. ⊣

Corollary 2.5. *For each $\alpha \in$ Ord, and for each $p \in \mathbb{P}$, the collection of names $\dot{x}$ which are efficiently below p and forced by p to have rank $\leq \alpha$, is a set (not a proper class). In particular for any name $\dot{\mathbb{Q}}$, the collection of efficient names for elements of $\dot{\mathbb{Q}}$ is a set.*

PROOF. Clear from Claim 2.4. ⊣

We take $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \delta, \beta < \delta \rangle$ to be a *countable (respectively finite) support forcing iteration* of length δ if:

1. $\mathbb{P}_0$ is the trivial poset with $\emptyset$ as the sole condition.
2. $\dot{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for a poset.
3. Conditions in $\mathbb{P}_\alpha$ are function with countable (respectively finite) domain contained in α.
4. For $p \in \mathbb{P}_\alpha$ and $\mu \in \mathrm{dom}(p)$, $p(\mu)$ is an efficient $\mathbb{P}_\mu$-name forced to be an element of $\dot{\mathbb{Q}}_\mu$.
5. $p^* \leq_{\mathbb{P}_\alpha} p$ iff for all $\mu \in \mathrm{dom}(p)$, $p^* \restriction \mu \Vdash_{\mathbb{P}_\mu} p^*(\mu) \leq_{\dot{\mathbb{Q}}_\mu} p(\mu)$.

This definition falls into the standard template for forcing iterations with countable (respectively finite) support. The only part where we are being more specific is the use of efficient names in condition (4). Note that our iteration posets $\mathbb{P}_\alpha$ are all sets (not proper classes), by Claim 2.5. Note also that the collection of $p(\mu)$ allowed by condition (4) is *full*, meaning that for every antichain A in $\mathbb{P}_\mu$, and for every list of names $\dot{u}_a$ for $a \in A$ so that $a \Vdash \dot{u}_a \in \dot{\mathbb{Q}}_\mu$, there is a name $\dot{u}^*$ allowed as $p(\mu)$ by condition (4), so that $a \Vdash \dot{u}^* = \dot{u}_a$ for each $a \in A$. To see this, use the standard merging argument to get a name $\dot{v}$ so that $\Vdash \dot{v} \in \dot{\mathbb{Q}}_\mu$ and $a \Vdash \dot{v} = \dot{u}_a$, and then apply Claim 2.2 to get an efficient $\dot{u}^*$ which is forced to be equal to $\dot{v}$. The fullness implies that all the standard results about forcing iterations, including in particular the ones constructing lower bounds in descending chains with countable support, hold for the definition above.

For consistency, we define forcing compositions with the same restriction to efficient names: conditions in $\mathbb{P} * \dot{\mathbb{Q}}$ are pairs $\langle p, \dot{q} \rangle$ where $p \in \mathbb{P}$, and $\dot{q}$ is an efficient $\mathbb{P}$-name forced to be an element of $\dot{\mathbb{Q}}$.

§3. Preparing for the countably closed preparation. Fix a cardinal δ of cofinality at least $\aleph_1$. We eventually intend for δ to become the continuum, but this is not relevant for the time being.

The first part of the preparation is the forcing $\mathbb{A} = \mathrm{Add}(\aleph_1, \delta)$, adding δ subsets of $\aleph_1$. Later on we will use these to produce almost disjoint subsets of ω_1 that will allow us to properly separate different clique-forcing posets, but this too is not relevant for the time being.

We view conditions in $\mathbb{A}$ as countable partial functions from $\omega_1 \times \delta$ into 2. The *support* of a condition a, denoted $\mathrm{Sp}(a)$, is the smallest (countable) set S so that $\mathrm{dom}(a) \subseteq \omega_1 \times S$. Given a generic A for $\mathbb{A}$, we write A_ξ, $\xi < \delta$, for the ξth subset of ω_1 added by A. We use $\dot{A}_\xi$ for the canonical name for the ξth set added by A.

By $\mathbb{A}\restriction X$, for $X \subseteq \delta$, we mean the restriction of $\mathbb{A}$ to conditions a with $\mathrm{dom}(a) \subseteq \omega_1 \times X$. Given a generic A, we write $A\restriction X$ for $A \cap \mathbb{A}\restriction X$. For a condition a we write $a\restriction X$ for the restriction of the function a to $\omega_1 \times X$. Note that $A\restriction X$ is equal to $\{a\restriction X \mid a \in A\}$, since A is a filter.

CLAIM 3.1. *Any extension of $a\restriction X$ in $\mathbb{A}\restriction X$ is compatible with a in $\mathbb{A}$. Consequently $\mathbb{A}\restriction X$ is a complete subposet of $\mathbb{A}$, and if A is generic for $\mathbb{A}$ over V, then $A\restriction X$ is generic for $\mathbb{A}\restriction X$.*

PROOF. Clear. ⊣

The forcing $\mathbb{A}$ is homogeneous under permutations of δ. We will preserve this homogeneity for the future steps of the preparation. Before we continue with the definitions of these steps, let us fix some notation, and establish some claims, related to this homogeneity.

CLAIM 3.2. *Let σ be a permutation of δ, meaning a bijection of δ with itself. Then σ extends to an automorphism of $\mathbb{A}$ by setting $\sigma(a)(\alpha, \sigma(\xi)) = a(\alpha, \xi)$. This in turn allows extending σ to act on $\mathbb{A}$ generics and names in the obvious way, with the properties that $\sigma(a) \in \sigma(A) \iff a \in A$ and $\sigma(\dot{x}) = \{\langle \sigma(\dot{y}), \sigma(b)\rangle \mid \langle \dot{y}, b\rangle \in \dot{x}\}$, so that $\sigma(\dot{x})[\sigma(A)] = \dot{x}[A]$ and $\dot{A}_{\sigma(\xi)}[\sigma(A)] = \dot{A}_\xi[A]$.*

PROOF. Clear. ⊣

CLAIM 3.3. *Let $a \in \mathbb{A}$. If σ is a permutation of δ so that σ is the identity on $\mathrm{Sp}(a) \cap \sigma'' \mathrm{Sp}(a)$, then a and $\sigma(a)$ are compatible, and $a \cup \sigma(a)$ is their largest lower bound.*

PROOF. Using the fact that σ is the identity on $\mathrm{Sp}(a) \cap \sigma'' \mathrm{Sp}(a)$, check that a and $\sigma(a)$ agree on the common part of their domains. ⊣

CLAIM 3.4. *Let σ be a permutation of δ, and extend σ in the manner of Definition 3.2. Let $\dot{\mathbb{P}}$ be a $\mathbb{A}$-name for a poset. Then σ induces an isomorphism of $\mathbb{A} * \dot{\mathbb{P}}$ into $\mathbb{A} * \sigma(\dot{\mathbb{P}})$ given by $\langle a, \dot{p}\rangle \mapsto \langle \sigma(a), \sigma(\dot{p})\rangle$. We refer to the map as σ. Note that this map fixes the interpretation of the second coordinate, meaning that if $A * P$ is generic for $\mathbb{A} * \dot{\mathbb{P}}$, then $\sigma''(A * P) = \sigma(A) * P$. As in Claim 3.2, σ then extend further to act on $\mathbb{A} * \dot{\mathbb{P}}$ names and generics.*

PROOF. Clear. ⊣

DEFINITION 3.5.
1. In the context of Claim 3.2, a name $\dot{X}$ is *invariant* if $\sigma(\dot{X}) = \dot{X}$ for all σ. The name $\dot{X}$ is invariant *modulo* $e \subseteq \delta$ if $\sigma(\dot{X}) = \dot{X}$ for all σ which are the identity on e.
2. We make the same definition for $\mathbb{A} * \dot{\mathbb{P}}$-names in the context of Claim 3.4. This makes sense when $\dot{\mathbb{P}}$ is invariant, so that the extended σ maps $\mathbb{A} * \dot{\mathbb{P}}$-names to $\mathbb{A} * \dot{\mathbb{P}}$-names.

If $\dot{X}$ names a structure, for example a poset or a poset with additional relations, $\dot{X}$ is invariant (and similarly modulo e) if the invariance condition holds for the underlying set and all implicit relations, such as the poset order.

Define the $\mathbb{A}$-*support* of an $\mathbb{A}$-name σ to be the smallest $S \subseteq \delta$ so that σ is an $\mathbb{A}{\restriction}S$-name, or, equivalently, $\mathrm{Sp}(\sigma) = \bigcup_{\langle \tau, a\rangle \in \sigma} \mathrm{Sp}(\tau) \cup \mathrm{Sp}(a)$.

Similarly, working with a composition $\mathbb{A} * \dot{\mathbb{P}}$, define the $\mathbb{A}$-support of an $\mathbb{A} * \dot{\mathbb{P}}$-name σ, denoted $\mathrm{Sp}(\sigma)$, to be $\bigcup_{\langle \tau, \langle a, \dot{p}\rangle\rangle \in \sigma} \mathrm{Sp}(\tau) \cup \mathrm{Sp}(a) \cup \mathrm{Sp}(\dot{p})$.

We refer to the $\mathbb{A}$-support simply as *support*, when the intention is clear from the context.

We say that two conditions p, u in a poset $\mathbb{P}$ are *equivalent* if $p \leq u \wedge u \leq p$. We write $p \approx u$ in this case. The conditions do not have to be equal, and in forcing compositions and iterations there are typically equivalent conditions which are not equal to each other.

DEFINITION 3.6. Let $\dot{\mathbb{P}}$ be an $\mathbb{A}$-name for a poset. Let $\kappa < \delta$. Then $\dot{\mathbb{P}}$ is *κ-appropriate* if:

1. (Invariance) $\dot{\mathbb{P}}$ is invariant.
2. (Support size) For every $\langle \dot{u}, a\rangle \in \dot{\mathbb{P}}$, there is an efficient name $\dot{p}$ for an element of $\dot{\mathbb{P}}$ so that $a \Vdash \dot{p} \approx \dot{u}$ and $|\operatorname{Sp}(\dot{p})| \leq \kappa$. (Note that $\langle a, \dot{p}\rangle$ is then an element of $\mathbb{A}*\dot{\mathbb{P}}$.) Moreover such $\dot{p}$ can be found with $\operatorname{Sp}(\dot{p}) \subseteq \operatorname{Sp}(a) \cup \operatorname{Sp}(\dot{u})$.
3. (Shift compatibility) Say that $\langle a, \dot{p}\rangle \in \mathbb{A} * \dot{\mathbb{P}}$ *shifts compatibly* if $\langle a, \dot{p}\rangle$ and $\langle \sigma(a), \sigma(\dot{p})\rangle$ are compatible for every permutation σ which is the identity on $(\operatorname{Sp}(a) \cup \operatorname{Sp}(\dot{p})) \bigcap \sigma''(\operatorname{Sp}(a) \cup \operatorname{Sp}(\dot{p}))$. Then there is $D \subseteq \mathbb{A} * \dot{\mathbb{P}}$ so that:
   (a) D is dense in $\mathbb{A} * \dot{\mathbb{P}}$.
   (b) If $\langle a, \dot{p}\rangle \in D$ and ρ is a permutation of δ then $\langle \rho(a), \rho(\dot{p})\rangle \in D$.
   (c) If $\langle a, \dot{p}\rangle \in D$ and $a \Vdash \dot{p}' \approx \dot{p}$ then $\langle a, \dot{p}'\rangle \in D$.
   (d) Every $\langle a, \dot{p}\rangle \in D$ shifts compatibly.

$\dot{\mathbb{A}}$ is κ-appropriate *modulo* $e \subseteq \delta$ if it satisfies the above conditions with invariance replaced by invariance modulo e, and condition (3) revised to permutations ρ which are the identity on e, and to conditions $\langle a, \dot{p}\rangle$ which *shift compatibly modulo* e, meaning that $\langle a, \dot{p}\rangle$ is compatible with $\langle \sigma(a), \sigma(\dot{p})\rangle$ for σ which are the identity on e and on $(\operatorname{Sp}(a) \cup \operatorname{Sp}(\dot{p})) \bigcap \sigma''(\operatorname{Sp}(a) \cup \operatorname{Sp}(\dot{p}))$.

LEMMA 3.7. *The trivial poset (having $\emptyset$ as its only element) is κ-appropriate.*

PROOF. Clear. We only note that all conditions in $\mathbb{A} * \dot{\mathbb{P}}$ shift compatibly by Claim 3.3. $\dashv$

CLAIM 3.8. *If $\dot{\mathbb{P}}$ is κ-appropriate and $\kappa' \in [\kappa, \delta)$, then $\dot{\mathbb{P}}$ is κ'-appropriate.*

PROOF. Clear. $\dashv$

CLAIM 3.9. *Let $\dot{\mathbb{P}}$ be κ-appropriate. If $\dot{x}$ is an $\mathbb{A}*\dot{\mathbb{P}}$-name of forcing-hereditary size $\leq \kappa$, then there is $\dot{x}^*$, forced to equal $\dot{x}$ and still of forcing-hereditary size at most κ, with $|\operatorname{Sp}(\dot{x}^*)| \leq \kappa$. Moreover if $\dot{x}$ is efficient (respectively efficient below $\langle a, \dot{p}\rangle$), then $\dot{x}^*$ can be picked to also be efficient (respectively efficient below $\langle a, \dot{p}\rangle$).*

PROOF. Let $\langle \dot{y}, u\rangle \in \dot{x}$. By induction there is $\dot{z}_{\dot{y},u}$, of forcing hereditary size at most κ and with support of size $\leq \kappa$, which is forced to equal $\dot{y}$. If $\dot{y}$ is efficient below u we can pick $\dot{z}_{\dot{y},u}$ to also be efficient below u. By the support size condition (2) of Definition 3.6, we can find $w_u \in \mathbb{A} * \dot{\mathbb{P}}$ which is equivalent to u, and has support of size $\leq \kappa$.

Now set $\dot{x}^* = \{\langle \dot{z}_{\dot{y},u}, w_u\rangle \mid \langle \dot{y}, u\rangle \in \dot{x}\}$. Then $\dot{x}^*$ inherits any efficiency enjoyed by $\dot{x}$, and using the fact that $|\dot{x}| \leq \kappa$ it is clear that $|\operatorname{Sp}(\dot{x}^*)| \leq \kappa$. $\dashv$

Let A be generic for $\mathbb{A}$, and let $J \subseteq \delta$. Note that $A{\restriction}J$ is generic for $\mathbb{A}{\restriction}J$ over V. Some $\mathbb{A}$-names σ have the property that $\sigma[A]$ belongs to $V[A{\restriction}J]$, and is independent of $A{\restriction}\delta - J$. We want in this case to define an $\mathbb{A}{\restriction}J$-name $(\sigma{\restriction}J)$ with the property that $(\sigma{\restriction}J)[A{\restriction}J] = \sigma[A]$.

Say that *a fixes σ at J* if there is an $\mathbb{A}{\restriction}J$-name $\bar{\sigma}$ so that $a \Vdash \sigma = \bar{\sigma}$. Recall that $\operatorname{Sp}(\sigma)$ is minimal so that σ is an $\mathbb{A}{\restriction}\operatorname{Sp}(\sigma)$-name. Then, for a generic A with $a \in A$, we have $\sigma[A] = \sigma[A{\restriction}\operatorname{Sp}(\sigma)] = \bar{\sigma}[A{\restriction}J]$, and it follows from this that $\sigma[A]$

belongs to $V[A{\restriction}(\mathrm{Sp}(\sigma)\cap J)]$. So we may assume that $\bar\sigma$ is an $\mathbb{A}{\restriction}(\mathrm{Sp}(\sigma)\cap J)$-name. We then have that already $a{\restriction}\mathrm{Sp}(\sigma)$ forces $\sigma = \bar\sigma$.

If $\langle \tau, b\rangle \in \sigma$, then any condition $c \le a{\restriction}\mathrm{Sp}(\sigma), b$ forces that $\tau \in \bar\sigma$, and therefore has an extension c' which fixes τ at J. There is therefore an antichain $X^{\sigma,a}(\tau, b)$, contained in $\mathbb{A}{\restriction}\mathrm{Sp}(\sigma)$, maximal among extensions of $a{\restriction}\mathrm{Sp}(\sigma), b$, so that all $c \in X^{\sigma,a}(\tau, b)$ fix τ at J.

Definition 3.10. (Using the notation above.) When a fixes σ at J, define $\sigma|_a J = \{\langle \tau|_c J, c{\restriction}J\rangle \mid \langle\tau, b\rangle \in \sigma \wedge c \in X^{\sigma,a}(\tau, b)\}$. (The definition is by recursion on von Neumann rank.) If the empty condition fixes σ at J, then define $\sigma|J$ to be $\sigma|_\emptyset J$.

Claim 3.11. *Suppose that a fixes σ at J. Then $\mathrm{Sp}(\sigma|_a J) \subseteq J \cap \mathrm{Sp}(\sigma)$, and $a \Vdash \sigma|_a J = \sigma$.*

Proof. Clear from the definitions by induction on rank. We only note that if c fixes σ and τ at J, and $\bar\sigma$ and $\bar\tau$ are $\mathbb{A}{\restriction}J$-names witnessing this in the sense that $c \Vdash \sigma = \bar\sigma \wedge \tau = \bar\tau$, and if $c \Vdash \tau \in \sigma$, then $c{\restriction}J \Vdash \bar\tau \in \bar\sigma$. This fact comes up in the inductive proof. ⊣

Lemma 3.12. *Let A be generic for $\mathbb{A}$. Suppose $\dot{\mathbb{P}}$ is κ-appropriate. Let $J \subseteq \delta$, and suppose that $\delta - J$ is uncountable.*

1. *$\dot{\mathbb{P}}[A] \cap V[A{\restriction}J]$ (meaning the poset, with its order) belongs to $V[A{\restriction}J]$ and is independent of $A{\restriction}(\delta - J)$.*

We write $\mathbb{P}{\restriction}J$ for $\dot{\mathbb{P}}[A] \cap V[A{\restriction}J]$. Given a generic P we write $P{\restriction}J$ for $P \cap \mathbb{P}{\restriction}J$. We write $\dot{\mathbb{P}}_J$ for the natural $\mathbb{A}$-name for $\mathbb{P}{\restriction}J$, namely $\{\langle \dot p, b\rangle \mid (\exists a)\langle \dot p, a\rangle \in \dot{\mathbb{P}}, b \le a, b \Vdash \dot p \in V[\dot A{\restriction}J]\}$. Condition (1) shows that the empty condition in $\mathbb{A}$ fixes $\dot{\mathbb{P}}_J$ at J. We set $\dot{\mathbb{P}}{\restriction}J = \dot{\mathbb{P}}_J|J$. By Claim 3.11, this is an $\mathbb{A}{\restriction}J$-name for $\mathbb{P}{\restriction}J$.

2. *Suppose that $|J| \ge \kappa^+$. Then $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}{\restriction}J$ is a complete subposet of $\mathbb{A} * \dot{\mathbb{P}}$.*

We denote $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}{\restriction}J$ as $(\mathbb{A} * \dot{\mathbb{P}}){\restriction}J$.

Proof. To prove condition (1), we show that if $a_1, a_2 \in \mathbb{A}$, $a_1{\restriction}J = a_2{\restriction}J$, and $\dot x$ is a $\mathbb{A}{\restriction}J$-name, then a_1 and a_2 cannot force incompatible information about the membership status of $\dot x$ in $\dot{\mathbb{P}}$. A similar argument applies to membership in the poset order of $\dot{\mathbb{P}}$.

Suppose otherwise, and fix witnesses $a_1, a_2, \dot x$. Since $\delta - J$ is uncountable, and since $\mathrm{Sp}(a_1)$ and $\mathrm{Sp}(a_2)$ are both countable, we can fix a permutation σ of δ, which is the identity on J, and such that $\mathrm{Sp}(a_1) - J$ and $\mathrm{Sp}(\sigma(a_2)) - J$ are disjoint. Since $a_1{\restriction}J = a_2{\restriction}J$, this implies that a_1 and $\sigma(a_2)$ are compatible, and hence so are $\sigma^{-1}(a_1)$ and a_2.

Work with a generic A for $\mathbb{A}$ which contains $\sigma^{-1}(a_1)$ and a_2. Note that since σ is the identity on J, $\dot x[A{\restriction}J] = \dot x[\sigma(A){\restriction}J]$. Call this element x. Then, by choice of a_1, a_2, and since $a_2 \in A$ while $a_1 \in \sigma(A)$, we have $x \in \dot{\mathbb{P}}[A]$ iff $x \notin \dot{\mathbb{P}}[\sigma(A)]$. In particular $\dot{\mathbb{P}}[A] \ne \dot{\mathbb{P}}[\sigma(A)]$. Since $\dot{\mathbb{P}}[A] = \sigma(\dot{\mathbb{P}})[\sigma(A)]$, this contradicts the invariance of $\dot{\mathbb{P}}$.

Next we prove condition (2) of the claim. We have to show that $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}{\restriction}J$ and $\mathbb{A} * \dot{\mathbb{P}}$ agree on compatibility, and that conditions in $\mathbb{A} * \dot{\mathbb{P}}$ have residues in $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}{\restriction}J$.

To handle compatibility, fix $\langle a_1, \dot{p}_1\rangle, \langle a_2, \dot{p}_2\rangle \in \mathbb{A}\restriction J * \dot{\mathbb{P}}\restriction J$ and suppose that $\langle a^*, \dot{p}^*\rangle$ is a lower bound for these conditions in $\mathbb{A} * \dot{\mathbb{P}}$. Using the support size condition (2) of Definition 3.6, we may replace each $\langle a_i, \dot{p}_i\rangle$ with an equivalent condition that has support of size $\leq \kappa$, still contained in J. We may similarly assume that $\langle a^*, \dot{p}^*\rangle$ has support of size $\leq \kappa$. Since $|J| \geq \kappa^+$, we can find a permutation σ of δ which is the identity on $\mathrm{Sp}(a_1) \cup \mathrm{Sp}(a_2) \cup \mathrm{Sp}(\dot{p}_1) \cup \mathrm{Sp}(\dot{p}_2)$, and maps the support of $\langle a^*, \dot{p}^*\rangle$ into J. Then $\langle \sigma(a^*), \sigma(\dot{p}^*)\rangle$ is a condition in $\mathbb{A}\restriction J * \dot{\mathbb{P}}\restriction J$. By the invariance of the poset ordering, it is a lower bound for the conditions $\langle \sigma(a_i), \sigma(\dot{p}_i)\rangle = \langle a_i, \dot{p}_i\rangle$, with the equality holding because σ fixes the support of the conditions.

Finally, we show that every condition $\langle a, \dot{p}\rangle$ in $\mathbb{A}*\dot{\mathbb{P}}$ has a residue in $\mathbb{A}\restriction J*\dot{\mathbb{P}}\restriction J$, meaning $\langle \bar{a}, \dot{\bar{p}}\rangle \in \mathbb{A}\restriction J * \dot{\mathbb{P}}\restriction J$ so that every extension of $\langle \bar{p}, \dot{\bar{a}}\rangle$ in $\mathbb{A}\restriction J * \dot{\mathbb{P}}\restriction J$ is compatible with $\langle a, \dot{p}\rangle$ in $\mathbb{A} * \dot{\mathbb{P}}$.

Fix $\langle a, \dot{p}\rangle$. By the support size condition (2) of Definition 3.6, we may assume there is $U \subseteq \delta$ of size κ so that $\mathrm{Sp}(a), \mathrm{Sp}(\dot{p}) \subseteq U$. We need to find a residue for $\langle a, \dot{p}\rangle$, and we will do this by taking the image of the condition under a permutation that shifts $U - J$ into J.

Since $|J| \geq \kappa^+$ and $|U| = \kappa$, we have $|J - U| \geq \kappa$. We can therefore find $Y \subseteq J - U$ of size $|U - J| \leq \kappa$. Let σ be a permutation of δ which sends $U - J$ to Y, sends Y to $U - J$, and is the identity otherwise. Let $\tau = \sigma^{-1}$, and note that $J \cap \tau''J = J - Y$, and hence τ is the identity on $J \cap \tau''J$. We also have that $\sigma''U \subseteq J$.

Let $\bar{a} = \sigma(a)$ and let $\dot{\bar{p}} = \sigma(\dot{p})$. Then $\mathrm{Sp}(\bar{a}), \mathrm{Sp}(\dot{\bar{p}}) \subseteq \sigma''U \subseteq J$. So $\langle \bar{a}, \dot{\bar{p}}\rangle \in \mathbb{A}\restriction J * \dot{\mathbb{P}}\restriction J$. We will prove that any extension of $\langle \bar{a}, \dot{\bar{p}}\rangle$ in $\mathbb{A}\restriction J*\dot{\mathbb{P}}\restriction J$ is compatible with $\langle a, \dot{p}\rangle$.

Fix $\langle b, \dot{q}\rangle \in \mathbb{A}\restriction J * \dot{\mathbb{P}}\restriction J$ extending $\langle \bar{a}, \dot{\bar{p}}\rangle$.

Let D witness the shift compatibility condition (3) of Definition 3.6. Note first that $\langle b, \dot{q}\rangle$ has an extension $\langle b', \dot{q}'\rangle$ which still has support contained in J, and belongs to D. This is a consequence of the density and invariance properties of D: By the support size condition, modifying $\dot{q}$ but keeping its support contained in J, we may assume that $|\mathrm{Sp}(\dot{q})| \leq \kappa$. Let $\langle b', \dot{q}'\rangle$ be any extension of $\langle b, \dot{q}\rangle$ in D. Again using the support size condition, we can find $\dot{q}''$, forced by b' to be equivalent to $\dot{q}'$, with $|\mathrm{Sp}(\dot{q}'')| \leq \kappa$. Then $\langle b', \dot{q}''\rangle \in D$, and $|\mathrm{Sp}(b') \cup \mathrm{Sp}(\dot{q}'')| \leq \kappa$. Since $|J| \geq \kappa^+$ we can find a permutation ρ of δ that sends $\mathrm{Sp}(b') \cup \mathrm{Sp}(\dot{q}'')$ into J, without moving any of the ordinals in $\mathrm{Sp}(b) \cup \mathrm{Sp}(\dot{q})$. Then $\langle \rho(b'), \rho(\dot{q}'')\rangle \leq \langle \rho(b), \rho(\dot{q})\rangle = \langle b, \dot{q}\rangle$, $\mathrm{Sp}(\rho(b')) \cup \mathrm{Sp}(\rho(\dot{q}''')) \subseteq J$, and, by the permutation invariance of D, $\langle \rho(b'), \rho(\dot{q}'')\rangle \in D$.

We can therefore assume that $\langle b, \dot{q}\rangle$ itself belongs to D, still preserving the fact that $\mathrm{Sp}(b) \cup \mathrm{Sp}(\dot{q}) \subseteq J$. By the shift compatibility condition, and since τ is the identity on $J \cap \tau''J$, it follows that $\langle \tau(b), \tau(\dot{q})\rangle$ and $\langle b, \dot{q}\rangle$ are compatible. Since $\langle \tau(b), \tau(\dot{q})\rangle$ extends $\langle \tau(\bar{a}), \tau(\dot{\bar{p}})\rangle = \langle a, \dot{p}\rangle$, this implies that $\langle a, \dot{p}\rangle$ and $\langle b, \dot{q}\rangle$ are compatible. ⊣

It is worth noting that in the situation of condition (1) of Lemma 3.12, $P\restriction J$ need not be generic for $\mathbb{P}\restriction J$ over $V[A\restriction J]$.

Remark 3.13. Work under the assumptions of condition (2) of Lemma 3.12. Fix $\langle a, \dot{p}\rangle \in \mathbb{A} * \dot{\mathbb{P}}$ with $\mathrm{Sp}(a), \mathrm{Sp}(\dot{p}) \subseteq U$ and $|U| \leq \kappa$. Fix any permutation σ of

δ so that $\sigma''(U-J) \subseteq J-U$, and so that σ is the identity on $J-\sigma''(U-J)$. The proof of Lemma 3.12 then shows that $\langle \sigma(a), \sigma(\dot{p})\rangle$ is a residue of $\langle a, \dot{p}\rangle$ in the poset $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}{\restriction}J$, in the sense that any condition in $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}{\restriction}J$ which is compatible with $\langle \sigma(a), \sigma(\dot{p})\rangle$ in $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}{\restriction}J$, is compatible with $\langle a, p\rangle$ in $\mathbb{A} * \dot{\mathbb{P}}$.

Claim 3.14. *Let $\dot{\mathbb{P}}$ be κ-appropriate. Let $A * P$ be generic for $\mathbb{A} * \dot{\mathbb{P}}$. Let $J_1 \subseteq J_2 \subseteq \delta$, with $|J_2 - J_1| \geq \kappa^+$. Let $\langle a, \dot{p}\rangle \in A * P$, with $\mathrm{Sp}(a), \mathrm{Sp}(\dot{p}) \subseteq U$ and $|U| \leq \kappa$. Then there is a permutation σ of δ, which maps $U - J_2$ into $J_2 - J_1$, maps $\sigma''(U-J_2)$ onto $U - J_2$, is the identity otherwise, and so that $\langle \sigma(a), \sigma(\dot{p})\rangle$ belongs to $A * P$.*

Proof. Suppose not, and let $\langle b, \dot{q}\rangle \in A * P$ force this. Without loss of generality we may assume that $\langle b, \dot{q}\rangle \leq \langle a, \dot{p}\rangle$. By the support size condition (2) of Definition 3.6, we may assume that $\mathrm{Sp}(b), \mathrm{Sp}(\dot{q})$ are contained in a set U^* of size κ, and we may assume $U^* \supseteq U$. Now take any permutation σ which maps $U^* - J_2$ into $J_2 - (J_1 \cup U^*)$, maps $\sigma''(U^* - J_2)$ back onto $U^* - J_2$, and is the identity otherwise. Such a permutation exists since $|J_2 - J_1| \geq \kappa^+$. Then $\langle \sigma(b), \sigma(\dot{q})\rangle$ is compatible with $\langle b, \dot{q}\rangle$ by Remark 3.13. But any condition witnessing this is an extension of $\langle b, \dot{q}\rangle$ which forces $\langle \sigma(a), \sigma(\dot{p})\rangle$ into the generic object, a contradiction. ⊣

Claim 3.15. *Let $\dot{\mathbb{P}}$ be κ-appropriate. Let $I \subseteq \delta$ with $|I| \geq \kappa^+$ and $|\delta - I| \geq \kappa^+$. Let $A{\restriction}I * P{\restriction}I$ be generic for $\mathbb{A}{\restriction}I * \dot{\mathbb{P}}{\restriction}I$. Let $\langle a_i, \dot{p}_i\rangle \in \mathbb{A} * \dot{\mathbb{P}}$, for $i \in \{1,2\}$. Let $U_i \supseteq \mathrm{Sp}(a_i) \cup \mathrm{Sp}(\dot{p}_i)$ and suppose that $|U_i| \leq \kappa$. Suppose there are generics $A_i * P_i$ for $\mathbb{A} * \dot{\mathbb{P}}$, containing $A{\restriction}I * P{\restriction}I$, with $\langle a_i, \dot{p}_i\rangle \in A_i * P_i$. Suppose finally that $U_1 - I$ and $U_2 - I$ are disjoint. Then $\langle a_1, \dot{p}_1\rangle$ and $\langle a_2, \dot{p}_2\rangle$ are compatible.*

Proof. Let $J = I \cup U_2$. Let $\bar{A}_2 * \bar{P}_2 = A_2{\restriction}J * P_2{\restriction}J$, a generic for $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}{\restriction}J$ containing $\langle a_2, \dot{p}_2\rangle$ and $A{\restriction}I * P{\restriction}I$.

By Claim 3.14, with $J_1 = U_1$ and $J_2 = I$, we can find a permutation σ of δ, which maps $U_1 - I$ into $I - U_1$, maps $\sigma''(U_1 - I)$ back to $U_1 - I$, is the identity otherwise, and so that $\langle \sigma(a_1), \sigma(\dot{p}_1)\rangle$ belongs to $A_1 * P_1$, hence to $A{\restriction}I * P{\restriction}I$, and hence to $\bar{A}_2 * \bar{P}_2$. In particular $\langle \sigma(a_1), \sigma(\dot{p}_1)\rangle$ is compatible with $\langle a_2, \dot{p}_2\rangle$, and this can be witnessed by $\langle b, \dot{q}\rangle \in \bar{A}_2 * \bar{P}_2$. By Remark 3.13, $\langle \sigma(a_1), \sigma(\dot{p}_1)\rangle$ is a residue of $\langle a_1, \dot{p}_1\rangle$ to $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}{\restriction}J$. It follows that $\langle a_1, \dot{p}_1\rangle$ is compatible with $\langle b, \dot{q}\rangle$, and hence with $\langle a_2, \dot{p}_2\rangle$. ⊣

Definition 3.16. Let A be generic for $\mathbb{A}$, and let $\dot{\mathbb{P}}$ be an $\mathbb{A}$-name for a poset. Define $\mathbb{P}^{(\leq\kappa)}$ to be the restriction of $\dot{\mathbb{P}}[A]$ to condition $\dot{p}[A]$ with $|\mathrm{Sp}(\dot{p})| \leq \kappa$. Let $\dot{\mathbb{P}}^{(\leq\kappa)}$ name this poset.

Lemma 3.17. *Let A be generic for $\mathbb{A}$. Suppose $\dot{\mathbb{P}}$ is κ-appropriate. Let $J \subseteq \delta$. Then:*

1. *$\dot{\mathbb{P}}^{(\leq\kappa)}[A] \cap V[A{\restriction}J]$ is dense in $\dot{\mathbb{P}}[A] \cap V[A{\restriction}J]$. In particular the two posets are forcing isomorphic.*
2. *$\dot{\mathbb{P}}^{(\leq\kappa)}[A] \cap V[A{\restriction}J]$ belongs to $V[A{\restriction}J]$ and is independent of $A{\restriction}(\delta - J)$.*
3. *Suppose that $|J| \geq \kappa^+$. Then $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}^{(\leq\kappa)}{\restriction}J$ is a complete subposet of $\mathbb{A} * \dot{\mathbb{P}}$.*

Proof. Condition (1) is immediate using the support size condition (2) of Definition 3.6. Indeed every condition in $\dot{\mathbb{P}}[A] \cap V[A{\restriction}J]$ has an equivalent condition in $\dot{\mathbb{P}}^{(\leq\kappa)}[A] \cap V[A{\restriction}J]$. Conditions (2) and (3) follow from Lemma 3.12 and its proof if $\delta - J$ is uncountable. So we may assume otherwise. In particular we can write J as a strictly increasing union $\bigcup_{\xi<\kappa^+} J_\xi$.

Note that if $p = \dot{p}[A] \in \dot{\mathbb{P}}^{(\leq\kappa)}[A] \cap V[A \restriction J]$ then p has a name $\dot{p}'$ with $\mathrm{Sp}(\dot{p}') \subseteq J$ and $|\mathrm{Sp}(\dot{p}')| \leq \kappa$. This is immediate from the fact that, for any $\dot{p}$, $A \restriction \mathrm{Sp}(\dot{p})$ and $A \restriction J$ are mutually generic extensions of $A \restriction (\mathrm{Sp}(\dot{p}) \cap J)$.

It follows that $\dot{\mathbb{P}}^{(\leq\kappa)}[A] \cap V[A \restriction J]$ is equal to $\bigcup_{\xi<\kappa^+} \dot{\mathbb{P}}^{(\leq\kappa)}[A] \cap V[A \restriction J_\xi]$. Since $\delta - J_\xi$ is uncountable, each $\dot{\mathbb{P}}^{(\leq\kappa)}[A] \cap V[A \restriction J_\xi]$ belongs to $V[A \restriction J_\xi]$ and is independent of $A \restriction (\delta - J_\xi)$. Condition (2) of the current lemma follows from this.

For condition (3), fix $\langle a, \dot{p} \rangle \in \dot{\mathbb{P}}^{(\leq\kappa)}[A]$, let $\xi < \kappa^+$ be large enough that $\mathrm{Sp}(a) \cap J, \mathrm{Sp}(\dot{p}) \cap J \subseteq J_\xi$, and let $\langle \bar{a}, \dot{\bar{p}} \rangle$ be a residue of $\langle a, \dot{p} \rangle$ in the poset $\mathbb{A} \restriction J_{\xi+1} * \dot{\mathbb{P}}^{(\leq\kappa)} \restriction J_{\xi+1}$, obtained through a permutation σ as in Remark 3.13 for $J_{\xi+1}$. Then by the remark, the same condition $\langle \bar{a}, \dot{\bar{p}} \rangle$ is in fact a residue of $\langle a, \dot{p} \rangle$ in $\mathbb{A} \restriction J_\eta * \dot{\mathbb{P}}^{(\leq\kappa)} \restriction J_\eta$ for any $\eta \geq \xi + 1$. From this it follows that $\langle \bar{a}, \dot{\bar{p}} \rangle$ is a residue of $\langle a, \dot{p} \rangle$ in $\bigcup_{\eta<\kappa^+} \mathbb{A} \restriction J_\eta * \dot{\mathbb{P}}^{(\leq\kappa)} \restriction J_\eta = \mathbb{A} \restriction J * \dot{\mathbb{P}}^{(\leq\kappa)} \restriction J$. ⊣

Definition 3.18. A poset $\mathbb{Q}$ is $(\kappa, \aleph_1)$-*stemmed* if it is equipped with a well-founded pre-order s so that:

1. $\mathbb{Q}/s$ has size κ. By $\mathbb{Q}/s$ we mean the collection of equivalence classes of s, that is equivalence classes of the relation $p \; s \; q \wedge q \; s \; p$.
2. Each s-equivalence class is countably directed, meaning that every countable subset of the equivalence class has a lower bound in the class.
3. The rank of s, meaning the least ordinal into which s embeds, is ω_1.
4. For every countable α, the set of p of rank α in s is predense in $\mathbb{Q}$.
5. If q_i, $i < \omega$, are compatible in $\mathbb{Q}$, then $\bigcup_{i<\omega} q_i$ is a condition in $\mathbb{Q}$ and a largest lower bound for q_i.
6. If q is a largest lower bound for q_i, $i < \omega$, and similarly with q' and q'_i, and if for each i, q_i and q'_i are in the same s-equivalence class, then q and q' are in the same s-equivalence class.

A typical example of a $(\kappa, \aleph_1)$-stemmed poset is any "stems and commitments" poset where the stems come from a set of size κ, any countably many commitments can be joined, and stems are sequences of countable length. The relation s witnessing the conditions of Definition 3.18 is stem extension. For condition (5), take a condition with stem s and commitment H to formally be $(\{0\} \times s) \cup (\{1\} \times H)$.

Definition 3.19. Suppose $\dot{\mathbb{P}}$ is κ-appropriate. Let $\dot{\mathbb{Q}}$ be an $\mathbb{A} * \dot{\mathbb{P}}$-name for a poset. Then $\dot{\mathbb{Q}}$ is *invariantly* $(\kappa, \aleph_1)$-*stemmed* modulo e if:

1. $\dot{\mathbb{Q}}$ is forced to be $(\kappa, \aleph_1)$-stemmed, with witnessing preorder $\dot{s}$ say.
2. $\dot{\mathbb{Q}}$, together with the poset order, the preorder $\dot{s}$, and a name $\dot{f}$ forced to give an injection of $\dot{\mathbb{Q}}/\dot{s}$ into κ (viewed as an $\dot{s}$-invariant function on $\dot{\mathbb{Q}}$), is invariant modulo e.
3. For each $\dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}})$, there is $\dot{q}'$ of forcing-hereditary size $\leq \kappa$, so that $\Vdash \dot{q}' = \dot{q}$.

If $e = \emptyset$ then we say that $\dot{\mathbb{Q}}$ is *invariantly stemmed*.

Definition 3.20. Let $\dot{\mathbb{Q}}$ be invariantly stemmed modulo e, and let $\dot{s}$ and $\dot{f}$ witness this. Let $\langle a, \dot{p}, \dot{q} \rangle \in \mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{Q}}$. We say that $\langle a, \dot{p} \rangle$ *fixes* $\dot{f}$ *around* $\dot{q}$ if there are $\dot{q}_i$, $i < \omega$, so that $\langle a, \dot{p}\rangle$ forces $\dot{q}$ to be a largest lower bound for the $\dot{q}_i$s, and forces a value for $\dot{f}(\dot{q}_i)$ for each i.

Claim 3.21. *Let $\dot{\mathbb{P}}$ be κ-appropriate, and let $\dot{\mathbb{Q}}$ be invariantly $(\kappa, \aleph_1)$-stemmed modulo e, with witnesses $\dot{s}$ and $\dot{f}$ say. Then for any $\langle a, \dot{p}, \dot{q}\rangle \in \mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{Q}}$, there is $\langle a^*, \dot{p}^*\rangle \leq \langle a, \dot{p}\rangle$ which fixes $\dot{f}$ around $\dot{q}$.*

Proof. Clear, as we can simply extend $\langle a, \dot{p}\rangle$ to fix a value for $\dot{f}(\dot{q})$, and take all $\dot{q}_i$ to equal $\dot{q}$. ⊣

Claim 3.22. *Let $\dot{\mathbb{P}}$ be κ-appropriate, and let $\dot{\mathbb{Q}}$ be invariantly $(\kappa, \aleph_1)$-stemmed modulo e, with witnesses $\dot{s}$ and $\dot{f}$ say. Suppose that $\langle a, \dot{p}\rangle$ fixes $\dot{f}$ around each $\dot{q}_n$, and forces $\dot{q}$ to be a largest lower bound for the $\dot{q}_n$s. Then $\langle a, \dot{p}\rangle$ fixes $\dot{f}$ around $\dot{q}$.*

Proof. For each n we have $\dot{q}_{n,i}$, $i < \omega$, so that $\langle a, \dot{p}\rangle$ forces that $\dot{q}_n$ is the largest lower bound for conditions $\dot{q}_{n,i}$, and $\langle a, \dot{p}\rangle$ forces a value for $\dot{f}(\dot{q}_{n,i})$. Since $\langle a, \dot{q}\rangle$ forces $\dot{q}$ to be the largest lower bound for the conditions $\dot{q}_{n,i}$, $i, n < \omega$, these conditions witness that $\langle a, \dot{p}\rangle$ fixes $\dot{f}$ around $\dot{q}$. ⊣

Claim 3.23. *Let $\dot{\mathbb{Q}}$ be invariantly stemmed modulo e, and let $\dot{s}$ and $\dot{f}$ witness this. Suppose that $\langle a, \dot{p}\rangle$ fixes $\dot{f}$ around $\dot{q}$. Let σ be a permutation of δ with $\sigma\restriction e = id$. If $\langle a, \dot{p}\rangle$ and $\langle\sigma(a), \sigma(\dot{p})\rangle$ are compatible, then every lower bound for these two conditions forces $\dot{q}$ and $\sigma(\dot{q})$ to be compatible.*

Proof. Let $\langle a^*, \dot{p}^*\rangle$ be a lower bound for $\langle a, \dot{p}\rangle$ and $\langle\sigma(a), \sigma(\dot{p})\rangle$. Fix $\dot{q}_i$, $i < \omega$, so that $\langle a, \dot{p}\rangle$ forces $\dot{q}$ to be a largest lower bound for the $\dot{q}_i$s, and forces a value for $\dot{f}(\dot{q}_i)$ for each i. By the invariance of $\dot{f}$, and since $\dot{f}$ maps into V, $\langle\sigma(a), \sigma(\dot{p})\rangle$ forces the same value for $\dot{f}(\sigma(\dot{q}_i))$ that $\langle a, \dot{p}\rangle$ forces for $\dot{q}(\dot{q}_i)$. Since $\langle a^*, \dot{p}^*\rangle$ extends both $\langle a, \dot{p}\rangle$ and $\langle\sigma(a), \sigma(\dot{p})\rangle$, it forces the same values, and in particular forces $\dot{q}_i$ and $\sigma(\dot{q}_i)$ into the same $\dot{s}$-equivalence class. It follows by condition (6) of Definition 3.18 that $\langle a^*, \dot{p}^*\rangle$ forces $\dot{q}$ and $\sigma(\dot{q})$ into the same $\dot{s}$-equivalence class, and hence forces them to be compatible. ⊣

An $\mathbb{A} * \dot{\mathbb{P}}$-name $\dot{\mathbb{Q}}$ for a poset can also be viewed as a $\mathbb{A}$-name for a $\dot{\mathbb{P}}$-name for a poset, and as such it can be composed with $\dot{\mathbb{P}}$. Our next lemma is that, under some assumptions, mainly that $\mathbb{P}$ is appropriate and $\dot{\mathbb{Q}}$ is invariantly stemmed, this composition retains the property of being appropriate.

To maintain the support size condition (2) of Definition 3.6, we need the following claims.

Claim 3.24. *Suppose that $\dot{\mathbb{P}}$ is κ-appropriate, and suppose that $\mathbb{A}*\dot{\mathbb{P}}$ is κ^+-c.c. Let $\dot{\mathbb{Q}}$ be invariantly $(\kappa, \aleph_1)$-stemmed modulo e. Let $\dot{q}$ be an $\mathbb{A} * \dot{\mathbb{P}}$-name for an element of $\dot{\mathbb{Q}}$. Then there is an efficient name $\dot{q}^*$ forced in $\mathbb{A} * \dot{\mathbb{P}}$ to equal $\dot{q}$, and so that $|\operatorname{Sp}(\dot{q}^*)| \leq \kappa$.*

Proof. Let Z be a maximal antichain of conditions $\langle a, \dot{p}\rangle \in \mathbb{A} * \dot{\mathbb{P}}$ for which there is $\dot{u} \in \operatorname{dom}(\dot{\mathbb{Q}})$ so that $\langle a, \dot{p}\rangle \Vdash \dot{q} = \dot{u}$. For each $\langle a, \dot{p}\rangle \in Z$ let $\dot{u}_{a,p} \in \operatorname{dom}(\dot{\mathbb{Q}})$ be such that $\langle a, \dot{p}\rangle \Vdash \dot{q} = \dot{u}_{a,\dot{p}}$. By condition (3) of Definition 3.19, there is $\dot{u}'_{a,\dot{p}}$ of forcing-hereditary size $\leq \kappa$, forced to equal $\dot{u}_{a,\dot{p}}$. Now using the chain condition and a standard merging of names argument, we can produce $\dot{q}'$, of forcing-hereditary size $\leq \kappa$, so that each $\langle a, \dot{p}\rangle \in Z$ forces $\dot{q}' = \dot{u}'_{a,\dot{p}}$. In particular it is forced that $\dot{q}' = \dot{q}$.

By Claim 2.3, there is an efficient $\dot{q}''$, forced to equal $\dot{q}'$, and still of forcing-hereditary size $\leq \kappa$. By Claim 3.9, there is $\dot{q}^*$, forced to equal $\dot{q}''$, which is still efficient, and with $|\operatorname{Sp}(\dot{q}^*)| \leq \kappa$. ⊣

Claim 3.25. *Suppose that $\dot{\mathbb{P}}$ is κ-appropriate, and suppose that $\mathbb{A}*\dot{\mathbb{P}}$ is κ^+-c.c. Let $\langle a, \dot{p}\rangle \in \mathbb{A} * \dot{\mathbb{P}}$, let $\dot{x}, \dot{x}^*$ be $\mathbb{A} * \dot{\mathbb{P}}$-names, and let $J \subseteq \delta$ with $|J| > \kappa$. Suppose that $\operatorname{Sp}(\dot{x}) \subseteq J$, $|\operatorname{Sp}(\dot{x}^*)| \leq \kappa$, $\dot{x}^*$ is efficient, and $\langle a, \dot{p}\rangle \Vdash \dot{x}^* = \dot{x}$. Then $\langle a, \dot{p}\rangle$ forces that there is an efficient $\dot{y}$ with the same interpretation as $\dot{x}$, and with $|\operatorname{Sp}(\dot{y})| \leq \kappa$ and $\operatorname{Sp}(\dot{y}) \subseteq J$.*

Proof. Say that an $\mathbb{A} * \dot{\mathbb{P}}$-name $\dot{u}$ is *frugal* if for all $\langle \dot{w}, \langle b, \dot{r}\rangle\rangle \in \dot{u}$ we have that $|\operatorname{Sp}(b)|, |\operatorname{Sp}(\dot{r})| \leq \kappa$, and $\dot{w}$ is frugal. By the support size condition (2) of Definition 3.6, we may assume that $\dot{x}$ is frugal, while maintaining that $\operatorname{Sp}(\dot{x}) \subseteq J$.

We prove the claim by induction on the von Neumann rank of $\dot{x}^*$. In fact we prove the stronger claim, that for any $I \subseteq J$ with $|I| = \kappa^+$ and $I \supseteq J \cap \operatorname{Sp}(\dot{x}^*)$, we can obtain the conclusion of the claim with $\operatorname{Sp}(\dot{y}) \subseteq I$.

Fix I as above. We may assume, using the support size condition (2) of Definition 3.6, that $|\operatorname{Sp}(a)|, |\operatorname{Sp}(\dot{p})| \leq \kappa$. It is enough to handle a more specific case, where we add the assumption that $\operatorname{Sp}(a) \cap J, \operatorname{Sp}(\dot{p}) \cap J \subseteq I$. To see that the specific case implies the general case, consider a general $\langle a, \dot{p}\rangle$ and suppose for contradiction that $\langle a, \dot{p}\rangle$ does not force the existence of an efficient $\dot{y}$ with the same interpretation as $\dot{x}^*$ and so that $\operatorname{Sp}(\dot{y}) \subseteq I$ and $|\operatorname{Sp}(\dot{y})| \leq \kappa$. Extending $\langle a, \dot{p}\rangle$ we may assume it forces that such $\dot{y}$ does not exist. Fix any permutation σ which maps $(\operatorname{Sp}(a) \cup \operatorname{Sp}(\dot{p})) \cap (J - I)$ into $I - (\operatorname{Sp}(\dot{x}^*) \cup \operatorname{Sp}(a) \cup \operatorname{Sp}(\dot{p}))$, maps $\sigma''((\operatorname{Sp}(a) \cup \operatorname{Sp}(\dot{p})) \cap (J - I))$ back onto $(\operatorname{Sp}(a) \cup \operatorname{Sp}(\dot{p})) \cap (J - I)$, and is the identity otherwise. Note that the claim assumptions hold for $\sigma(\dot{x})$, $\sigma(\dot{x}^*) = \sigma(\dot{x})$, $\sigma'' J = J$, $\sigma(a)$, and $\sigma(\dot{p})$. Moreover $\operatorname{Sp}(\sigma(a)) \cap J, \operatorname{Sp}(\sigma(\dot{p})) \cap J \subseteq I$, and hence using only the specific case mentioned above, we have that $\langle \sigma(a), \sigma(\dot{p})\rangle$ forces the existence of an efficient $\dot{y}$ with the same interpretation as of $\dot{x}^*$, and so that $\operatorname{Sp}(\dot{y}) \subseteq I$ and $|\operatorname{Sp}(\dot{y})| \leq \kappa$. Hence $\langle a, \dot{p}\rangle$ and $\langle \sigma(a), \sigma(\dot{p})\rangle$ are incompatible. Let $\hat{J} = I \cup (\operatorname{Sp}(a) \cup \operatorname{Sp}(\dot{p}) - J)$. Without loss of generality, shrinking I if needed, we may assume that $\delta - \hat{J}$ is uncountable. Then by Remark 3.13, $\langle \sigma(a), \sigma(\dot{p})\rangle$ is a residue of $\langle a, \dot{p}\rangle$ to $\mathbb{A}{\restriction}\hat{J} * \dot{\mathbb{P}}{\restriction}\hat{J}$. In particular $\langle a, \dot{p}\rangle$ and $\langle \sigma(a), \sigma(\dot{p})\rangle$ are compatible, a contradiction.

Suppose then that $\operatorname{Sp}(a) \cap J, \operatorname{Sp}(\dot{p}) \cap J \subseteq I$. Fix $J_1 \subseteq J_2 \subseteq I$ with the same properties as I (meaning that $|J_i| = \kappa$ and $J_i \supseteq J \cap (\operatorname{Sp}(\dot{x}^*) \cup \operatorname{Sp}(a) \cup \operatorname{Sp}(\dot{p})))$ and so that $|I - J_2| = |J_2 - J_1| = \kappa^+$.

Fix a generic $A * P$ containing $\langle a, p\rangle$. To prove the claim, we will show that for such generics, $\dot{x}^*[A * P]$ depends only on $A{\restriction}J_2 * P{\restriction}J_2$. We can then take $\dot{y} = \tau(\dot{x}^*)$ for a permutation τ which is the identity on J_2, maps $\operatorname{Sp}(\dot{x}^*)$ into I, and maps $\langle a, \dot{p}\rangle$ to an element of $A * P$. Such τ can be found using Claim 3.14.

By Claim 3.12, $\mathbb{A}{\restriction}J_1 * \dot{\mathbb{P}}{\restriction}J_1$ is a complete subposet of $\mathbb{A} * \dot{\mathbb{P}}$. By induction, all elements of $\dot{x}^*[A * P]$ belong to $V[A{\restriction}J_1 * P{\restriction}J_1]$.

Suppose now that there is another generic $A' * P'$ containing $\langle a, p\rangle$, with $A'{\restriction}J_2 = A{\restriction}J_2$, $P'{\restriction}J_2 = P{\restriction}J_2$, yet $\dot{x}^*[A' * P'] \neq \dot{x}^*[A * P]$. Then there is some $e = \dot{e}[A{\restriction}J_1 * P{\restriction}J_1]$ on which $\dot{x}^*[A * P]$ and $\dot{x}^*[A' * P']$ disagree. For definitiveness suppose that $e \in \dot{x}^*[A * P]$ and $e \notin \dot{x}^*[A' * P']$. This implies in particular that $e \in \dot{x}[A * P]$.

We may assume that $\dot{e}$ is frugal. Then both $\dot{e}$ and $\dot{x}$ are $\mathbb{A} * \dot{\mathbb{P}}^{(\leq\kappa)}$-names, with support contained in J. By Lemma 3.17, the fact that $\dot{e} \in \dot{x}$ must be forced by some condition $\langle b_1, \dot{q}_1\rangle \in A{\restriction}J * P^{(\leq\kappa)}{\restriction}J$.

Let $U = \mathrm{Sp}(a) \cup \mathrm{Sp}(\dot{p}) \cup \mathrm{Sp}(\dot{x}^*)$. Let $E = J_1 \cup U$. By Lemma 3.12, $\mathbb{A}{\restriction}E * \dot{\mathbb{P}}{\restriction}E$ is a complete subposet of $\mathbb{A} * \dot{\mathbb{P}}$. Since $\dot{e}[A' * P'] \notin \dot{x}^*[A' * P']$, and since $\mathrm{Sp}(\dot{x}^*), \mathrm{Sp}(\dot{e}) \subseteq E$, there must be a condition $\langle b_2, \dot{q}_2\rangle \in A'{\restriction}E * P'{\restriction}E$ forcing that $\dot{e} \notin \dot{x}^*$. Extending $\langle b_2, \dot{q}_2\rangle$, and since $\langle a, \dot{p}\rangle \in A'{\restriction}E * P'{\restriction}E$, we may assume that $\langle b_2, \dot{q}_2\rangle \leq \langle a, \dot{p}\rangle$.

By Claim 3.14 we can find a permutation ρ of δ, which maps $U - J_2$ into $J_2 - J_1$, maps $\rho''(U - J_2)$ back onto $U - J_2$, and is the identity otherwise, with $\langle \rho(b_2), \rho(\dot{q}_2)\rangle \in A' * P'$.

Since $A'{\restriction}J_2 = A{\restriction}J_2$, and $P'{\restriction}J_2 = P{\restriction}J_2$, we have $\langle \rho(b_2), \rho(\dot{q}_2)\rangle \in A * P$. In particular $\langle \rho(b_2), \rho(\dot{q}_2)\rangle$ and $\langle b_1, \dot{q}_1\rangle$ are compatible, and indeed have a lower bound in $A{\restriction}J * P^{(\leq\kappa)}{\restriction}J$. Note $U \cap J \subseteq J_1 \subseteq J_2$, and hence $U - J_2 = U - J$. So ρ maps $U - J$ into $J - (U \cap J)$, maps $\rho''(U - J)$ back onto $U - J$, and is the identity otherwise. By Remark 3.13 it follows that $\langle \rho(b_2), \rho(\dot{q}_2)\rangle$ is a residue of $\langle b_2, \dot{q}_2\rangle$ in $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}^{(\leq\kappa)}{\restriction}J$. Since $\langle \rho(b_2), \rho(\dot{q}_2)\rangle$ and $\langle b_1, \dot{q}_1\rangle$ have a lower bound in $\mathbb{A}{\restriction}J * \dot{\mathbb{P}}^{(\leq\kappa)}{\restriction}J$, this implies that $\langle b_1, \dot{q}_1\rangle$ is compatible with $\langle b_2, \dot{q}_2\rangle$. But this is impossible since the former forces $\dot{e} \in \dot{x}$ and the latter extends $\langle a, \dot{p}\rangle$ while forcing $\dot{e} \notin \dot{x}^*$. ⊣

Lemma 3.26. *Let $\dot{\mathbb{P}}$ be κ-appropriate, and let $\dot{\mathbb{Q}}$ be invariantly $(\kappa, \aleph_1)$-stemmed modulo e. Suppose $\mathbb{A} * \dot{\mathbb{P}}$ is countably closed and κ^+-c.c. Then $\dot{\mathbb{P}} * \dot{\mathbb{Q}}$ is κ-appropriate modulo e.*

Proof. The invariance condition (1) of Definition 3.6 for $\dot{\mathbb{P}} * \dot{\mathbb{Q}}$ is immediate using the invariance of $\dot{\mathbb{P}}$ and the invariance of $\dot{\mathbb{Q}}$ modulo e.

For the support size condition (2), fix a condition $\langle a, \dot{p}, \dot{q}\rangle$. Let $U = \mathrm{Sp}(a) \cup \mathrm{Sp}(\dot{p}) \cup \mathrm{Sp}(\dot{q})$. Using the support size condition for $\mathbb{A} * \dot{\mathbb{P}}$, there is an efficient name $\dot{p}'$ for an element of $\dot{\mathbb{P}}$, forced by a to be equivalent to $\dot{p}$, with support of size $\leq \kappa$ contained in U. If $|\mathrm{Sp}(\dot{q})| = \kappa$, then there is nothing further to do. Suppose $|\mathrm{Sp}(\dot{q})| > \kappa$. In particular $|U| > \kappa$. By Claim 3.24 there is an efficient $\dot{q}'$ with support of size $\leq \kappa$, which is outright forced to equal $\dot{q}$. By Claim 3.25, there is an efficient $\dot{q}''$, outright forced to equal $\dot{q}'$, with support of size $\leq \kappa$ and contained in U. Then a forces that $\langle \dot{p}, \dot{q}\rangle$ and $\langle \dot{p}', \dot{q}''\rangle$ are equivalent, and $\langle \dot{p}', \dot{q}''\rangle$ has the required support.

It remains to prove the shift compatibility condition (3) of Definition 3.6 for $\dot{\mathbb{P}} * \dot{\mathbb{Q}}$. Let D witness this condition for $\dot{\mathbb{P}}$. Let D^* consists of $\langle a, \dot{p}, \dot{q}\rangle$ so that $\langle a, \dot{p}\rangle \in D$ and $\langle a, \dot{p}\rangle$ fixes $\dot{f}$ around $\dot{q}$. Conditions (3a)–(3c) for D^* are easy to check from the same conditions for D, the invariance of $\dot{f}$ and $\dot{\mathbb{Q}}$, and Claim 3.21 in the case of density.

To prove condition (3d), fix $\langle a, \dot{p}, \dot{q}\rangle \in D^*$, and fix a permutation σ which fixes e and is the identity on $J^* \bigcap \sigma'' J^*$, where $J^* = e \cup \mathrm{Sp}(a) \cup \mathrm{Sp}(\dot{p}) \cup \mathrm{Sp}(\dot{q})$. In particular σ is the identity on $J \bigcap \sigma'' J$ where $J = e \cup \mathrm{Sp}(a) \cup \mathrm{Sp}(\dot{p})$. Since $\langle a, \dot{p}\rangle \in D$ it follows that $\langle a, \dot{p}\rangle$ and $\langle \sigma(a), \sigma(\dot{p})\rangle$ are compatible. Let $\langle a^*, \dot{p}^*\rangle$

witness this. By Claim 3.23, $\langle a^*, \dot{p}^*\rangle$ forces $\dot{q}$ and $\dot{q}^*$ to be compatible. So $\langle a, \dot{p}, \dot{q}\rangle$ and $\langle \sigma(a), \sigma(\dot{p}), \sigma(\dot{q})\rangle$ are compatible. $\dashv$

For the next lemma we will deal with products of images of $\dot{\mathbb{Q}}$ under permutations. To make the products invariant, we index them not using ordinals, but using the sequences of subsets of ω_1 added by $\mathbb{A}$. Here is the precise definition:

Recall that the generic A for $\mathbb{A}$ adds δ subsets of ω_1, and that $A_\xi = \dot{A}_\xi[A]$ is the ξth set. Given $h\colon \kappa \to \delta$, let $c(h)$ be the map $\xi \mapsto A_{h(\xi)}$. Let $\dot{c}(h)$ name this map. Note that, for a permutation σ of δ, $\dot{c}(\sigma \circ h) = \sigma(\dot{c}(h))$.

Definition 3.27. Let $\dot{\mathbb{Q}}$ be an $\mathbb{A} * \dot{\mathbb{P}}$-name for a poset, and let $h\colon \kappa \to \delta$. Suppose that $\dot{\mathbb{Q}}$ is invariant modulo range(h). For expository purposes fix a generic $A * P$ for $\mathbb{A} * \dot{\mathbb{P}}$. By $\prod_\tau^{(h)} \tau(\dot{\mathbb{Q}})[A * P]$ with countable (respectively finite) support we mean the countable (respectively finite) support product of the posets $\tau(\dot{\mathbb{Q}})[A * P]$, taken over all possible values of $\tau \circ h$ for permutations τ of δ in V, with $\tau(\dot{\mathbb{Q}})[A * P]$ indexed by $c(\tau \circ h)$. Conditions in $\prod_\tau^{(h)} \tau(\dot{\mathbb{Q}})[A * P]$ are functions r with countable (respectively finite) domain, each $k \in \mathrm{dom}(r)$ is equal to $c(\tau \circ h)$ for a permutation $\tau \in V$ of δ, and $r(k)$ is an element of $\tau(\dot{\mathbb{Q}})[A*P]$. The ordering is defined coordinate-wise in the natural way. $\prod_\tau^{(h)} \tau(\dot{\mathbb{Q}})$ itself names the poset $\prod_\tau^{(h)} \tau(\dot{\mathbb{Q}})[A * P]$, in such a way that for each $\dot{r} \in \mathrm{dom}(\prod_\tau^{(h)} \tau(\dot{\mathbb{Q}}))$, there is a countable (respectively finite) $K(\dot{r}) \in V$ so that it is outright forced that $\mathrm{dom}(\dot{r}) = \{c(\tau \circ h) \mid \tau \circ h \in K(\dot{r})\}$. We refer to $K(\dot{r})$ as the *underlying support* of $\dot{r}$. The actual support is $\{c(\tau \circ h) \mid \tau \circ h \in K(\dot{r})\}$, and does not belong to V.

When we use this notation without explicitly stating the support size, we mean the countable support product. We refer to h as the *base* of the product, and to κ as the *base height*. When we write $\prod_{\tau \in Z}^{(h)} \tau(\dot{\mathbb{Q}})$ we mean the restriction of the above product to conditions with underlying support contained in $\{\tau \circ h \mid \tau \in Z\}$.

The invariance of $\dot{\mathbb{Q}}$ modulo range(h) is used to make sense of Definition 3.27: while there may be many different permutations τ yielding the same $\tau \circ h$, they all give rise to the same name $\tau(\dot{\mathbb{Q}})$.

The forcing $\prod_\tau^{(h)} \tau(\dot{\mathbb{Q}})[A * P]$ is isomorphic to the standard product of the indexed posets. But the product as we defined it is highly symmetric, in the sense of the next claim, while the standard product is not.

Claim 3.28. *Work under the assumptions of Definition 3.27. Let σ be a permutation of δ. Then $\sigma(\prod_\tau^{(h)} \tau(\dot{\mathbb{Q}})) = \prod_\tau^{(h)} \tau(\dot{\mathbb{Q}})$, in both the countable and finite support cases.*

Proof. Clear, using the fact that $\sigma(\dot{c}(\tau \circ h)) = \dot{c}(\sigma \circ \tau \circ h)$. For each τ, in both $\sigma(\prod_\tau^{(h)} \tau(\dot{\mathbb{Q}}))$ and $\prod_\tau^{(h)} \tau(\dot{\mathbb{Q}})$, the poset indexed by $\sigma(\dot{c}(\tau \circ h)) = \dot{c}(\sigma \circ \tau \circ h)$ is $\sigma(\tau(\dot{\mathbb{Q}}))$. $\dashv$

In light of the last claim, we refer to $\prod_\tau^{(h)} \tau(\dot{\mathbb{Q}})$ as an *invariant product.* Let $\dot{\mathbb{B}}$ denote this product. In light of the invariance, we can view permutations σ of δ as acting on $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{B}}$-names. In particular, they can act on the names $\dot{B}_\tau$ for the components of the $\dot{\mathbb{B}}$ generic, with $\dot{B}_\tau$ generic for $\tau(\dot{\mathbb{Q}})$.

Claim 3.29. *(Under the assumptions of Definition 3.27 and with the notation above.)* $\sigma(\dot{B}_\tau) = \dot{B}_{\sigma\circ\tau}$.

Proof. Again clear from the fact that $\sigma(\dot{c}(\tau\circ h)) = \dot{c}(\sigma\circ\tau\circ h)$. ⊣

Remark 3.30. Note that $c(\tau\circ h)$ has von Neumann rank $\kappa+1$. If the conditions in $\tau(\dot{\mathbb{Q}})[A*P]$ have rank at most μ, then it follows that the conditions in $\prod^{(h)}_\tau \tau(\dot{\mathbb{Q}})[A*P]$ each have von Neumann rank below $\max\{\kappa,\mu\}+\omega$. This would not be true for the standard product of δ^κ factors; with a standard product indexed by ordinals, the ranks would be at least unbounded in δ.

Remark 3.31. Let $J\subseteq\delta$, and suppose that $b\in\prod^{(h)}_\tau \tau(\dot{\mathbb{Q}})[A*P]$ belongs to $V[A{\restriction}J*P{\restriction}J]$. Then b belongs to the restricted product $\prod^{(h)}_{\tau\in Z(J)}(\dot{\mathbb{Q}})[A*P]$, where $Z(J)$ consists of the permutations τ so that $(\tau\circ h)''\kappa\subseteq J$. This is clear from the definition, since for any τ so that $c(\tau\circ h)$ belongs to the domain of b, and every $\xi\in(\tau\circ h)''\kappa$, A_ξ is coded into $c(\tau\circ h)$ and hence coded into b. This remark, like Remark 3.30, would fail for the standard product.

Remark 3.32. As with all products (independently of our specific indexing), the restricted product $\prod^{(h)}_{\tau\in Z}(\dot{\mathbb{Q}})[A*P]$ is a complete subposet of the full product. The residue of a condition b in the full product is $b{\restriction}\{c(\tau\circ h)\mid\tau\in Z\}$. We refer to this residue as $b|Z$. We have that $b\le b|Z$, and as usual, any extension of $b|Z$ in the restricted product is compatible with b.

Claim 3.33. *Let $\dot{\mathbb{P}}$ be κ-appropriate. Let $\dot{\mathbb{Q}}$ be an $\mathbb{A}*\dot{\mathbb{P}}$-name which is invariantly $(\kappa,\aleph_1)$-stemmed modulo e, with witnesses $\dot{s}$ and $\dot{f}$ say. Then for every permutation τ of δ, $\tau(\dot{\mathbb{Q}})$ is invariantly $(\kappa.\aleph_1)$-stemmed modulo $\tau''e$, with witnesses $\tau(\dot{s})$ and $\tau(\dot{f})$.*

Proof. Clear from the definitions, as they are all preserved under permutations. ⊣

Lemma 3.34. *Let $\dot{\mathbb{P}}$ be κ-appropriate. Suppose $\mathbb{A}*\dot{\mathbb{P}}$ is κ^+-c.c. and countably closed. Let $\dot{\mathbb{Q}}$ be invariantly $(\kappa,\aleph_1)$-stemmed modulo e. Let $h\colon\kappa\to e$ be surjective. Let $\mathbb{B}=\prod^{(h)}_\tau \tau(\dot{\mathbb{Q}})$. Then $\dot{\mathbb{P}}*\dot{\mathbb{B}}$ is κ-appropriate.*

Proof. Fix $\dot{s}$ and $\dot{f}$ witnessing that $\dot{\mathbb{Q}}$ is invariantly $(\kappa,\aleph_1)$-stemmed modulo e.

For each permutation τ of δ, we may use Lemma 3.26 and its proof for $\tau(\dot{\mathbb{Q}})$. The assumptions of the lemma on being invariantly stemmed hold using Claim 3.33. It follows that $\dot{\mathbb{P}}*\tau(\dot{\mathbb{Q}})$ is κ-appropriate modulo $\tau''e$, for all τ. This is used implicitly below.

Let us now check that the conditions of Definition 3.6 hold for $\dot{\mathbb{P}}*\dot{\mathbb{B}}$.

The invariance condition (1) of Definition 3.6 is clear, using the invariance of $\dot{\mathbb{P}}$ and Claim 3.28.

The support size condition (2) is immediate from the same condition for $\dot{\mathbb{P}}*\tau(\dot{\mathbb{Q}})$ for each τ, using: the restriction to countable underlying support in V for the product $\prod^{(h)}_\tau \tau(\dot{\mathbb{Q}})$, the κ^+-chain condition for $\mathbb{A}*\dot{\mathbb{P}}$, and a merging of names argument in the style of Claim 3.24.

It remains to prove the shift compatibility condition (3) of Definition 3.6. Let D witness this condition for $\dot{\mathbb{P}}$ and let D^* consist of $\langle a, \dot{p}, \dot{b}\rangle$ so that $\langle a, \dot{p}\rangle \in D$, $\langle a, \dot{p}\rangle$ forces a value K for the underlying support of $\dot{b}$, and for each $\rho = \tau \circ h \in K$, $\langle a, \dot{p}\rangle$ fixes $\tau(\dot{f})$ around $\dot{b}(\dot{c}(\rho))$. (By $\dot{b}(\dot{c}(\rho))$ we mean the name for the value of the function named by $\dot{b}$, at $\dot{c}(\rho)$.)

It is clear that D^* is dense, since for any $\langle a, \dot{p}, \dot{b}\rangle$, one can strengthen $\langle a, \dot{p}\rangle$ to force a value for the underlying domain of $\dot{b}$, then strengthen in countably many steps to fix $\tau(\dot{f})$ around $\dot{b}(\dot{c}(\rho))$ for each $\rho = \tau \circ h$ in the underlying domain, and finally strengthen to get into D. It is also clear that D^* is invariant under permutations σ, and under replacing $\langle a, \dot{p}, \dot{b}\rangle$ with $\langle a, \dot{p}', \dot{b}'\rangle$ when $a \Vdash \dot{p} = \dot{p}'$ and $\langle a, \dot{p}\rangle \Vdash \dot{b} = \dot{b}'$.

Suppose now that $\langle a, \dot{p}, \dot{b}\rangle \in D^*$, and σ is a permutation which is the identity on $J \cap \sigma'' J$ where $J = \mathrm{Sp}(a) \cup \mathrm{Sp}(\dot{p}) \cup \mathrm{Sp}(\dot{b})$. Since $\langle a, \dot{p}\rangle \in D$ we have that $\langle a, \dot{p}\rangle$ and $\langle \sigma(a), \sigma(\dot{p})\rangle$ are compatible. Let $\langle a^*, \dot{p}^*\rangle$ be any common extension of these conditions.

Let K be the value that $\langle a, \dot{p}\rangle$ forces for the underlying support of $\dot{b}$. Let $\tau_i \circ h$, $i \in \omega$, enumerate K without repetitions. Let $\dot{b}_i$ be the coordinate of $\dot{b}$ that corresponds to the factor supported by $\dot{c}(\tau_i \circ h)$. Note that $\tau_i'' e \cup \mathrm{Sp}(\dot{b}_i) \subseteq J$, since $\dot{b}$ has support contained in J, and since $\tau_i'' e$ is needed to support $\dot{c}(\tau_i \circ h)$, which in turn is incorporated into $\dot{b}$. If it happens that σ fixes $\tau_i'' e$, then the assumptions on σ in the definition of shift compatibility modulo $\tau_i'' e$ all hold, and hence by Lemma 3.26 applied to the factor $\tau_i(\dot{\mathbb{Q}})$ of $\dot{\mathbb{B}}$ we have that $\langle a^*, \dot{p}^*\rangle$ forces $\dot{b}_i$ and $\sigma(\dot{b}_i)$ to be compatible.

We will therefore be done if we can show that all common coordinates of the domains of $\dot{b}$ and $\sigma(\dot{b})$ fall under this case. In other words we need to show that whenever a member $\dot{c}(\tau_i \circ h)$ of the domain of $\dot{b}$ is equal to a member $\sigma(\dot{c}(\tau_j \circ h))$ of the domain of $\sigma(\dot{b})$, we have that $i = j$ and σ fixes $\mathrm{range}(\tau_i \circ h)$.

Suppose that $\dot{c}(\tau_i \circ h) = \sigma(\dot{c}(\tau_j \circ h))$. Then $\tau_i \circ h = \sigma \circ \tau_j \circ h$. Since $\mathrm{Sp}(\dot{b}) \subseteq J$, it must be that $\mathrm{range}(\tau_i \circ h) \subseteq J$. Similarly $\mathrm{range}(\sigma \circ \tau_j \circ h) \subseteq \sigma'' J$. Using the fact that σ is the identity on $J \cap \sigma'' J$ it follows immediately that σ is the identity on $\mathrm{range}(\tau_i \circ h)$. This implies that $\sigma \circ \tau_i \circ h = \tau_i \circ h = \sigma \circ \tau_j \circ h$, hence $\tau_i \circ h = \tau_j \circ h$, and therefore $i = j$. ⊣

DEFINITION 3.35. Define $\mathbb{E}_{\text{fast}}$ to be the following version of the forcing to add a fast club in ω_1. Conditions are sets of the form $u(e, H) = (\{0\} \times e) \cup (\{1\} \times H)$ where e is a bounded closed subset of ω_1, and H is a countable collection of club subsets of ω_1. The ordering is given by $u(e^*, H^*) \leq u(e, H)$ iff $e^* \supseteq e$, $H^* \supseteq H$, and $e^* - e \subseteq \bigcap H$.

LEMMA 3.36. *Let $\dot{\mathbb{P}}$ be κ-appropriate. Suppose that $\mathbb{A} * \dot{\mathbb{P}}$ is countably closed and κ^+-c.c., and suppose the* CH *holds in V. Let $\dot{\mathbb{E}}$ name $\mathbb{E}_{\text{fast}}^{V[A*P]}$. Then:*

1. *$\dot{\mathbb{E}}$ is invariantly $(\kappa, \aleph_1)$ stemmed.*
2. *$\dot{\mathbb{P}} * \dot{\mathbb{E}}$ is κ-appropriate.*

PROOF. The second condition follows from the first using Lemma 3.26. So it is enough to prove that $\dot{\mathbb{E}}$ is invariantly $(\kappa, \aleph_1)$-stemmed.

It is clear that $\dot{\mathbb{E}}$ is forced to be $(\aleph_1, \aleph_1)$-stemmed, and hence $(\kappa, \aleph_1)$-stemmed, with $\dot{s}$ naming the order of stem extension. By the chain condition, every subset of ω_1 in the extension by $\mathbb{A} * \mathbb{P}$ has a name of forcing-hereditary size $\leq \kappa$, and so do countable sets of subsets of ω_1. So $\dot{\mathbb{E}}$ satisfies condition (3) in Definition 3.19. The invariance condition (2) is clear from the invariance of $\dot{\mathbb{P}}$ and the fact that $V[A * P] = V[\sigma(A * P)]$ for permutations σ, so long as we take $\dot{s}$ to invariantly name the order of stem extension, and take $\dot{f}$ to invariantly name the function that maps a condition $u(e, H)$ with stem e to $\tilde{f}(e)$, where $\tilde{f}$ is a bijection in V of $\omega_1^{<\omega_1}$ into ω_1. ⊣

Definition 3.37. By $\mathbb{C}(\aleph_1, \kappa)$ we mean the forcing to add a surjection of ω_1 onto κ, by countable approximations with ordinal domain.

Lemma 3.38. *Let $\dot{\mathbb{P}}$ be κ-appropriate. Suppose that $\mathbb{A} * \dot{\mathbb{P}}$ is countably closed and κ^+-c.c., and suppose the* CH *holds in V. Let $\mathbb{C} = \mathbb{C}(\aleph_1, \kappa)$. Then:*

1. $\check{\mathbb{C}}$ *is invariantly $(\kappa, \aleph_1)$-stemmed.*
2. $\dot{\mathbb{P}} * \check{\mathbb{C}}$ *is κ-appropriate.*

Proof. The first condition is obvious with s being the order of extension. The second condition follows from the first using Lemma 3.26. ⊣

Next we want to prove that countable support iterations of posets as in Lemmas 3.34, 3.36, and 3.38 preserve appropriateness, and preserve the κ^+-c.c., which we need for the successor cases of the iterations since it is assumed in these lemmas. In handling the limit stages, we will need to preserve a bit more.

We say that a map π between two posets $\mathbb{C}, \mathbb{D}$ *preserves incompatibility*, if whenever x, y are incompatible in $\mathbb{C}$, $\pi(x), \pi(y)$ are incompatible in $\mathbb{D}$. We do not require the map to preserve anything else, and in particular it need not preserve the poset ordering. The existence of an incompatibility preserving map from $\mathbb{C}$ to $\mathbb{D}$ is enough to ensure that any chain condition enjoyed by $\mathbb{D}$ is also enjoyed by $\mathbb{C}$.

For a set X, let $\mathbb{F}_{\mathrm{ctbl}}(X, \kappa)$ denote the poset of countable partial functions from X into κ, ordered by reverse inclusion. If $\kappa^\omega = \kappa$, then this poset has the κ-chain condition, by a standard Δ-system argument.

Definition 3.39. $\dot{\mathbb{P}}$ is *neatly κ-appropriate* modulo e if it is κ-appropriate modulo e, and there is a witness D for the shift compatibility condition (3) of Definition 3.6, so that:

1. Every countable descending chain in D has a largest lower bound, and this lower bound is itself in D.
2. There is an incompatibility preserving $\pi \colon D \to \mathbb{F}_{\mathrm{ctbl}}(X, \kappa)$ for some X.
3. For σ as in the shift compatibility condition (3) of Definition 3.6, and $\langle a, \dot{p} \rangle \in D$, the conditions $\pi(a, \dot{p})$ and $\pi(\sigma(a), \sigma(\dot{p}))$ are compatible in $\mathbb{F}_{\mathrm{ctbl}}(X, \kappa)$.

If $e = \emptyset$ we say that $\mathbb{P}$ is neatly κ-appropriate.

Remark 3.40. If $\kappa^\omega = \kappa$, then condition (2) of Definition 3.39 implies that $\mathbb{A} * \dot{\mathbb{P}}$ is κ^+-c.c. Condition (3) of the definition implies condition (3d) of Definition 3.6.

CLAIM 3.41. *Work under the assumptions of Lemma 3.26, and suppose that* $\dot{\mathbb{P}}$ *is neatly* κ*-appropriate modulo* e*. Let* D*,* X*, and* π *witness this. Then* $\dot{\mathbb{P}} * \dot{\mathbb{Q}}$ *is neatly* κ*-appropriate modulo* e*, and moreover there are witnesses* D^**,* X^**, and* π^* *which* neatly extend D*,* X*, and* π*, meaning that:*

1. $X^* \supseteq X$.
2. *If* $\langle a, \dot{p}, \dot{q}\rangle \in D^*$ *then* $\langle a, \dot{p}\rangle \in D$*, and* $\pi^*(a, \dot{p}, \dot{q})\restriction X = \pi(a, \dot{p})$.
3. *If* $\pi^*(a, \dot{p}, \dot{q})$ *and* $\pi^*(a', \dot{p}', \dot{q}')$ *are compatible in* $\mathbb{F}_{\mathrm{ctbl}}(X^*, \kappa)$*, then for any common extension* $\langle a^*, \dot{p}^*\rangle$ *of* $\langle a, \dot{p}\rangle$ *and* $\langle a', \dot{p}'\rangle$*, there is* $\dot{q}^*$ *so that* $\langle a^*, \dot{p}^*, \dot{q}^*\rangle$ *is a common extension of* $\langle a, \dot{p}, \dot{q}\rangle$ *and* $\langle a', \dot{p}', \dot{q}'\rangle$

Similarly for the posets $\dot{\mathbb{B}}$*,* $\dot{\mathbb{E}}$*, and* $\mathbb{C}$ *(modulo the empty set) under the assumptions of Lemmas 3.34, 3.36, and 3.38.*

PROOF. Define D^* from D exactly as in the proof of Lemma 3.26. Let X^* be the disjoint union of X and (a copy of) ω. For $\langle a, \dot{p}, \dot{q}\rangle \in D^*$, there are by the definition of D^* some $\dot{q}_i$ so that $\langle a, \dot{p}\rangle$ forces $\dot{q}$ to be a largest lower bound for the $\dot{q}_i$s, and forces a value for $\dot{f}(\dot{q}_i)$ for each i. Let $e \in \kappa^\omega$ map i to the value that $\langle a, \dot{p}\rangle$ forces for $\dot{f}(\dot{q}_i)$. Let $\pi^*(a, \dot{p}, \dot{q}) = \pi(a, \dot{p}) \cup e$. Choose $\dot{q}_i$ above so that the map associating e to $\langle a, \dot{p}, \dot{q}\rangle$ is invariant under permutations σ as in the shift compatibility condition (3) of Definition 3.6. This is possible (using a well ordering of κ^ω and the invariance of $\dot{f}$) since the proof of Lemma 3.26 shows that the map giving the set of possible $\{\dot{q}_i\}$ for each $\langle a, \dot{p}, \dot{q}\rangle \in D^*$ is invariant.

It is clear from the proof of Lemma 3.26 that D^*, X^*, and π^* satisfy the conditions of the current claim, including in particular condition (3), which gives condition (2) Definition 3.39, and also satisfy condition (3) of the Definition. It remains to check that D^* satisfies the condition on closure under largest lower bounds.

Let $\langle a_i, \dot{p}_i, \dot{q}_i\rangle$ be descending in D^*. Since D is neat, there is a largest lower bound $\langle a^*, \dot{p}^*\rangle$ for $\langle a_i, \dot{p}_i\rangle$ in D. Let $\dot{q}^*$ name the union of $\dot{q}_i$. By condition (5) of Definition 3.18, $\langle a^*, \dot{p}^*\rangle$ forces $\dot{q}^*$ to be a largest lower bound for $\dot{q}_i$. By Claim 3.22, $\langle a^*, \dot{p}^*\rangle$ fixes $\dot{f}$ around $\dot{q}^*$. So $\langle a^*, \dot{p}^*, \dot{q}^*\rangle$, which is a largest lower bound for $\langle a_i, \dot{p}_i, \dot{q}_i\rangle$, belongs to D^*.

This proves the current lemma in the case of a poset $\dot{\mathbb{Q}}$ as in Lemma 3.26. The situation in Lemmas 3.36 and 3.38 is a special case of this. For $\mathbb{B}$ as in Lemma 3.34 the proof is similar: Take X^* to be the disjoint union of X and $I \times \omega$, where $I = \{\tau \circ h \mid \tau \in V \text{ is a permutation of } \delta\}$. Define D^* as in the proof of Lemma 3.34. For $\langle a, \dot{p}, \dot{b}\rangle \in D^*$, and ρ in the underlying support K of $\dot{b}$, define $e_\rho \colon \omega \to \kappa$ as we defined e above, working with the factor of $\dot{\mathbb{B}}$ at coordinate $\dot{c}(\rho)$. Then define $e \colon K \times \omega \to \kappa$ by $e(\rho, i) = e_\rho(i)$, and set $\pi^*(a, \dot{p}, \dot{b}) = \pi(a, \dot{p}) \cup e$. The proof of Lemma 3.34 shows that this yields condition (3) of the current claim, and condition (3) of Definition 3.39. Closure under largest lower bounds can be proved using the argument above for each factor of the product $\dot{\mathbb{B}}$ in the domain of the conditions handled. ⊣

LEMMA 3.42. *Let* $\langle \dot{\mathbb{P}}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \le \gamma, \beta < \gamma\rangle$ *name a countable support iteration of limit length* γ *over the extension by* $\mathbb{A}$*. Suppose that for each* $\alpha < \gamma$*,* $\dot{\mathbb{P}}_\gamma$ *is countably closed and neatly* κ*-appropriate. Let* $D_\alpha, X_\alpha, \pi_\alpha$ *witness that* $\dot{\mathbb{P}}_\alpha$ *is neatly* κ*-appropriate. Suppose that for every* $\alpha < \beta < \gamma$*:*

1. *$X_\alpha \subseteq X_\beta$.*
2. *If $\langle a, \dot{p}\rangle \in D_\beta$ then a forces a value $d(\dot{p})$ for $\mathrm{dom}(\dot{p})$, and $d(\dot{p})$ is closed under ordinal successor below β.*
3. *If $\langle a, \dot{p}\rangle \in D_\beta$, and $\alpha \in d(\dot{p})$ or α is least so that $d(\dot{p}) \subseteq \alpha$, then $\langle a, \dot{p}\restriction\alpha\rangle \in D_\alpha$ and $\pi_\alpha(a, \dot{p}\restriction\alpha) = \pi_\beta(a, \dot{p})\restriction X_\alpha$.*
4. *If $\pi_\beta(a, \dot{p})$ and $\pi_\beta(a', \dot{p}')$ are compatible in $\mathbb{F}_{\mathrm{ctbl}}(X_\beta, \kappa)$, then for any common extension $\langle a^*, \dot{\bar{p}}\rangle$ of $\langle a, \dot{p}\restriction\alpha\rangle$ and $\langle a', \dot{p}'\restriction\alpha\rangle$ in $\mathbb{A}*\dot{\mathbb{P}}_\alpha$, there is a common extension $\langle a^*, \dot{p}^*\rangle$ of $\langle a, \dot{p}\rangle$ and $\langle a', \dot{p}'\rangle$ in $\mathbb{A}*\dot{\mathbb{P}}_\beta$ with $\dot{p}^*\restriction\alpha = \dot{\bar{p}}$.*

Then $\dot{\mathbb{P}}_\gamma$ is neatly κ-appropriate, with witnesses $D_\gamma, X_\gamma, \pi_\gamma$ which secure conditions (1)–(4) for the case $\beta = \gamma$.

Proof. The invariance and support conditions (1) and (2) of Definition 3.6 are clear. We work on the shift compatibility condition (3) of Definition 3.6, the neatness conditions of Definition 3.39, and the conditions of the current lemma with $\beta = \gamma$.

If γ has uncountable cofinality, then set $X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$, put $\langle a, \dot{p}\rangle$ in D_γ iff a forces a limit ordinal value μ for $\sup(\mathrm{dom}(\dot{p}))$ and $\langle a, \dot{p}\rangle \in D_\mu$, and set $\pi_\gamma(a, \dot{p}) = \pi_\mu(a, \dot{p})$. Most of the necessary conditions are easy to check in this case. We only comment on condition (4) of the current lemma. (Note that this condition implies condition (2) of Definition 3.39.) Fix $\langle a, \dot{p}\rangle, \langle a, \dot{p}'\rangle \in D_\gamma$ with $\pi_\gamma(a, \dot{p})$ and $\pi_\gamma(a', \dot{p}')$ compatible. Let μ, μ' be the values forced by a, a' for $\sup(\mathrm{dom}(\dot{p}))$ and $\sup(\mathrm{dom}(\dot{p}'))$. Then $\pi_\mu(a, \dot{p})$ and $\pi_{\mu'}(a', \dot{p}')$ are compatible. We work by induction on $\max(\mu, \mu')$ to show that this implies the conclusion of condition (4). If $\alpha \geq \max(\mu, \mu')$, then $\langle a^*, \dot{p}^*\rangle$ itself is already a common extension of $\langle a, \dot{p}\rangle, \langle a', \dot{p}'\rangle$. So suppose $\alpha < \max(\mu, \mu')$. If $\mu = \mu'$ we can use condition (4) with $\beta = \mu$. So suppose $\mu < \mu'$. If $\alpha \geq \mu$, then we can take $\dot{p}^*$ which agrees with $\dot{\bar{p}}$ up to α, and agrees with $\dot{p}'$ from α onwards, to get a common extension $\langle a^*, \dot{p}^*\rangle$ of $\langle a, \dot{p}\rangle$ and $\langle a', \dot{p}'\rangle$. So suppose $\alpha < \mu$. Let β be the least element of $d(\dot{p}') - \mu$. Let $\bar{\mu}$ be least so that $d(\dot{p}') \cap \beta \subseteq \bar{\mu}$. Then $\pi_{\mu'}(a, \dot{p}'\restriction\mu')\restriction X_{\bar{\mu}} = \pi_\beta(a, \dot{p}'\restriction\beta)\restriction X_{\bar{\mu}} = \pi_{\bar{\mu}}(a, \dot{p}'\restriction\bar{\mu})$, so $\pi_\mu(a, \dot{p})$ and $\pi_{\bar{\mu}}(a', \dot{p}'\restriction\bar{\mu})$ are compatible. By induction we can find a common extension as in condition (4) for $\langle a, \dot{p}\rangle$ and $\langle a', \dot{p}'\restriction\bar{\mu}\rangle$. Since $d(\dot{p})$ and $d(\dot{p}')$ are disjoint from $\bar{\mu}$ upwards, this can be completed to a common extension of $\langle a, \dot{p}\rangle$ and $\langle a', \dot{p}'\rangle$.

Suppose now that $\mathrm{cof}(\gamma) = \omega$. Fix α_n, $n < \omega$, increasing and cofinal in γ. Put $\langle a, \dot{p}\rangle \in D_\gamma$ iff a forces a value $d(\dot{p})$ for $\mathrm{dom}(\dot{p})$, $d(\dot{p})$ is closed under ordinal successor, and $(\forall n)\alpha_n \in d(\dot{p})) \wedge \langle a, \dot{p}\restriction\alpha_n\rangle \in D_{\alpha_n}$. It is clear that D_γ inherits conditions (3b) and (3c) of Definition 3.6 from the D_{α_n}s. Using the fact that each D_{α_n} is dense and closed under largest lower bounds, and using conditions (3) and (4) of the current lemma, it is easy to check that D_γ is dense, and closed under largest lower bounds.

Set $X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha = \bigcup_{n<\omega} X_{\alpha_n}$. For $\langle a, \dot{p}\rangle \in D_\gamma$, set $\pi_\gamma(a, \dot{p})$ equal to $\bigcup_{n<\omega} \pi_{\alpha_n}(a, \dot{p}\restriction\alpha_n)$. This makes sense, and is a condition in $\mathbb{F}_{\mathrm{ctbl}}(\bigcup_{n<\omega} X_{\alpha_n}, \kappa) = \bigcup \mathbb{F}_{\mathrm{ctbl}}(X_\gamma, \kappa)$, by the definitions and condition (3) of the current lemma. It is clear using this condition that $\pi_\gamma(a, \dot{p})\restriction\alpha = \pi_\alpha(a, \dot{p}\restriction\alpha)$ for all $\alpha \in d(\dot{p})$.

It is also clear from the definition that π_γ inherits condition (3) of Definition 3.39 from the π_{α_n}s.

It remains to check condition (4) of the current lemma for $\beta = \gamma$. (This, plus the fact that each π_α is incompatibility preserving, implies that π_γ is incompatibility preserving, so that there is no need to separately check condition (2) of Definition 3.39.) But this is clear. Fix any n large enough that $\alpha < \alpha_n$, and using condition (4) at the α_is, find $\dot{p}_i$ for $i \geq n$ so that $\langle a^*, \dot{p}_i \rangle$ is a common extension of $\langle a, \dot{p} \restriction \alpha_i \rangle$ and $\langle a', \dot{p}' \restriction \alpha_i \rangle$, with $\dot{p}_n \restriction \alpha = \dot{\bar{p}}$ and (if $i > n$) $\dot{p}_i \restriction \alpha_{i-1} = \dot{p}_{i-1}$. Then take $\dot{p}^* = \bigcup \dot{p}_i$. $\dashv$

§4. One step. We continue to work with δ, κ, and $\mathbb{A}$ as in Section 3, and we fix an $\mathbb{A}$-name $\dot{\mathbb{P}}$ which is κ-appropriate, countably closed, and κ^+-c.c.. Fix further an $\mathbb{A} * \dot{\mathbb{P}}$-name $\dot{\mathbb{S}}$ for a poset.

For expository simplicity, fix a generic $A * P$ for $\mathbb{A} * \dot{\mathbb{P}}$, and a further generic S for $\mathbb{S} = \dot{\mathbb{S}}[A * P]$.

It will be convenient sometimes to think of $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$-names in V as $\mathbb{S}$-names in $V[A * P]$. To avoid confusion, given an $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$-name $\dot{r} \in V$ we will use $\dot{r}[A * P]$ to denote the resulting $\mathbb{S}$-name. We will use similar notation with other poset compositions.

Fix $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$-names $\dot{X}$ for a set of reals and $\dot{U}$ for a graph on $\dot{X}$ which is open in the relative topology. Assume that these properties of $\dot{X}$ and $\dot{U}$ are outright forced to hold. Let $X = \dot{X}[A * P * S]$ and $U = \dot{U}[A * P * S]$.

Fix $I \subseteq \delta$ of size κ, an injection $g \colon \kappa \to I$, and an $\mathbb{A} \restriction I * \dot{\mathbb{P}} \restriction I$-name $\dot{\bar{\mathbb{S}}}$. Suppose that the following hold, and that conditions (6) and (8) are outright forced in $\mathbb{A} * \dot{\mathbb{P}}$.

1. $\dot{\mathbb{S}}$ is invariant (under permutations of δ, as a $\mathbb{A} * \dot{\mathbb{P}}$-name).
2. $\kappa^{\aleph_0} = \kappa$.
3. $\dot{\mathbb{S}}$ is forced to be c.c.c.
4. For every $\dot{s} \in \mathrm{dom}(\dot{\mathbb{S}})$, there is $\dot{s}'$ forced to be equivalent to $\dot{s}$ in $\dot{\mathbb{S}}$, and of forcing-hereditary size $\leq \kappa$.
5. $\mathbb{A} \restriction I * \dot{\mathbb{P}} \restriction I$ is a complete subposet of $\mathbb{A} * \dot{\mathbb{P}}$.
6. $\bar{\mathbb{S}} = \dot{\bar{\mathbb{S}}}[A \restriction I * P \restriction I]$ is a complete subposet of $\mathbb{S}$, of hereditary size at most κ. Moreover for every $\dot{s} \in \mathrm{dom}(\dot{\bar{\mathbb{S}}})$ there is an $\mathbb{A} \restriction I * \dot{\mathbb{P}} \restriction I$-name $\dot{s}'$ of forcing hereditary size $\leq \kappa$, forced to be equivalent to $\dot{s}$. Let $\bar{S}$ denote $S \cap \bar{\mathbb{S}}$.
7. $\dot{X}$ and $\dot{U}$ are $\mathbb{A} \restriction I * \dot{\mathbb{P}} \restriction I * \dot{\bar{\mathbb{S}}}$-names. In particular X and U belong to $V[A \restriction I * P \restriction I * \bar{S}]$.
8. If $\dot{t}_i \in V[A * P]$, for $i < \omega$, are $\mathbb{S}$-names for closed sets of reals so that $\dot{t}_i \cap \dot{X}[A * P]$ are forced in $\mathbb{S}$ to be 1-colorable in $\dot{U}[A * P]$, then there is an $\bar{\mathbb{S}}$-name $\dot{x} \in V[A \restriction I * P \restriction I]$, forced in $\mathbb{S}$ to belong to $\dot{X}[A * P] - \bigcup_{i<\omega} \dot{t}_i$.

Remark 4.1. The final assumption implies in particular that U, which belongs to $V[A \restriction I * P \restriction I * \bar{S}]$, is not countably colorable in $V[A * P * S]$. If we allowed $\dot{x}$ in the condition to be an $\mathbb{S}$-name in $V[A * P]$, the assumption would be equivalent to U not being countably colorable. But we require $\dot{x}$ to be a $\bar{\mathbb{S}}$-name, and this makes the assumption stronger.

Definition 4.2. An $\bar{\mathbb{S}}$-name $\dot{u}$ is *careful* if for every $\langle \dot{v}, s \rangle \in \dot{u}$:

1. There is an $\mathbb{A} \restriction I * \dot{\mathbb{P}} \restriction I$-name $\dot{y}$ so that $s = \dot{y}[A \restriction I * P \restriction I]$ and $\dot{y}$ is of forcing-hereditary size at most κ.

2. $\dot{v}$ is careful.

Similarly an $\mathbb{S}$-name is *careful* if it satisfies the above conditions with $\mathbb{A}{\restriction}I * \dot{\mathbb{P}}{\restriction}I$ relaxed to $\mathbb{A} * \dot{\mathbb{P}}$.

Claim 4.3. *Every $\bar{\mathbb{S}}$-name in $V[A{\restriction}I * P{\restriction}I]$ is equivalent to a careful $\bar{\mathbb{S}}$-name in $V[A{\restriction}I * P{\restriction}I]$. Moreover every $\bar{\mathbb{S}}$-name in $V[A{\restriction}I * P{\restriction}I]$ for a set of hereditary size at most κ has a careful $\bar{\mathbb{S}}$-name in $V[A{\restriction}I * P{\restriction}I]$ of forcing-hereditary size (in $\bar{\mathbb{S}}$) at most κ. Similarly with $\bar{\mathbb{S}}$ relaxed to $\mathbb{S}$ and $V[A{\restriction}I * P{\restriction}I]$ relaxed to $V[A * P]$.*

Proof. Immediate from assumptions (6) and (4), with the addition of assumption (3) for the second part of the claim. $\dashv$

Claim 4.4. *If $\dot{u} \in V[A{\restriction}I * P{\restriction}I]$ is a careful $\bar{\mathbb{S}}$-name of forcing hereditary size (in $\bar{\mathbb{S}}$) at most κ, then $\dot{u} = \dot{y}[A{\restriction}I * P{\restriction}I]$ for some $\dot{y}$ of forcing hereditary size (in $\mathbb{A}{\restriction}I * \dot{\mathbb{P}}{\restriction}I$) at most κ. Similarly with $\bar{\mathbb{S}}$ relaxed to $\mathbb{S}$, and $V[A{\restriction}I * P{\restriction}I]$ relaxed to $V[A * P]$.*

Proof. Clear from definition 4.2, and the κ^+-chain condition for $\mathbb{A}{\restriction}I * \dot{\mathbb{P}}{\restriction}I$ and $\mathbb{A}{\restriction}\dot{\mathbb{P}}$. $\dashv$

For a permutation σ of δ, let $\sigma(X) = \sigma(\dot{X})[A{*}P{*}S]$ and $\sigma(U) = \sigma(\dot{U})[A{*}P{*}S]$. This makes sense using the invariance of $\mathbb{A}$, $\dot{\mathbb{P}}$, and $\dot{\mathbb{S}}$. We have that $\sigma(X)$ is a set of reals, and $\sigma(U)$ is an open graph on this set. Similarly let $\sigma(\bar{\mathbb{S}}) = \sigma(\dot{\bar{\mathbb{S}}})[A * P]$. Then $\sigma(\bar{\mathbb{S}})$ is a complete poset of $\mathbb{S}$, and $\sigma(X), \sigma(U) \in V[A{\restriction}\sigma'' I{*}P{\restriction}\sigma'' I][S{\cap}\sigma(\bar{\mathbb{S}})]$.

We work under the assumptions above to force to add uncountable cliques simultaneously in *all* the graphs $\sigma(U)$, with σ ranging over permutations of δ.

We will use a variation of Todorcevic's [11] clique forcing poset on each graph $\sigma(U)$. One part of the variation involves working with $\bar{\mathbb{S}}$-names for reals, and using the extra strength referenced in Remark 4.1. Producing this extra strength will eventually use homogenizing ideas that trace back to work of Farah [5] and internalizing ideas that are new to this work. Another part of the variation uses almost disjoint subsets of ω_1 to handle all graphs $\sigma(U)$ uniformly over σ. This will be important in maintaining the invariance of the overall forcing, and the appropriateness of the preparation.

The first step is to collapse κ to $\aleph_1$. Let $\mathbb{C} = \mathbb{C}(\aleph_1, \kappa)$. Let C be generic for $\mathbb{C}$ over $V[A * P * S]$. Note that C collapses the set of canonical $\bar{\mathbb{S}}$-names for elements of X to have size $\aleph_1$, since $\bar{\mathbb{S}}$ is c.c.c. of size κ, and since $\kappa^{\aleph_0} = \kappa$. Fix $h \in V[A{\restriction}I * P{\restriction}I * C]$ which enumerates this set in ordertype ω_1. Let $\dot{h} \in V$ name h, and suppose that the above property of h is outright forced to hold. For a permutation σ of δ, let $\sigma(h) = \sigma(\dot{h})[A * P * C]$. Then $\sigma(h)$ is an enumeration of the canonical $\sigma(\bar{\mathbb{S}})$-names for elements of $\sigma(X)$. $\sigma(h)$ depends only on $\sigma{\restriction}I$, and we will use the notation below for any injection $\sigma\colon I \to \delta$.

Work in $V[A * P * C]$. By a *Farah-Todorcevic commitment* (*commitment* for short) we mean an $\mathbb{S}$-name $\dot{t}$ for a closed set of reals so that $\dot{t} \cap \dot{X}$ is forced in $\mathbb{S}$ to be 1-colorable in $\dot{U}[A * P]$. Since $\mathbb{C}$ is countably closed, and $\mathbb{S}$ is c.c.c., the former does not add canonical $\mathbb{S}$-names for closed sets of reals. So we may always assume $\dot{t}$ belongs to $V[A * P]$.

Let $\dot{F} \in V[A * P]$ be an $\mathbb{S}$-name for a closed subset of $\mathbb{R}^l$, for some $l \geq 1$. Let $\sigma_1, \ldots, \sigma_k$ be injections of I into δ in V. Let $\alpha < \omega_1$. We say that a commitment

$\dot{t}$ is *generated* by $\dot{F}$ and $\sigma_1, \dots, \sigma_k$ below α if there are $\dot{x}_1, \dots, \dot{x}_{l-1} \in \{\sigma_i(h)(\xi) \mid i \leq k, \xi < \alpha\}$ so that $\dot{t}$ is the canonical $\mathbb{S}$-name for $\{x \mid \langle x, \dot{x}_1[S], \dots, \dot{x}_{l-1}[S]\rangle \in \dot{F}[S]\}$. Note that only countably many commitments can be generated by F and $\sigma_1, \dots, \sigma_k$ below each α.

Recall from the definition of symmetric products in Section 3 that, for an injection $\sigma\colon I \to \delta$, we use $c(\sigma \circ g)$ to denote the map $\xi \mapsto A_{\sigma(g(\xi))}$, from κ into $\mathcal{P}(\omega_1)$. A_ι, $\iota < \delta$, are the subsets of ω_1 added by the generic A. $c(\sigma \circ g)$ codes σ in the sense that the map $\sigma \mapsto c(\sigma \circ g)$ is one-to-one, since the sets A_ι are pairwise distinct. Below when we say *code* for an injection $\sigma\colon I \to \delta$ we mean $c(\sigma \circ g)$.

By a *code* for a closed set we mean a real coding the closed set, in some fixed standard manner.

DEFINITION 4.5. Let $\mathbb{T}_I$ be the following poset in $V[A * P * C]$. Conditions are pairs $\langle u, \mathcal{F}\rangle$, viewed formally as $(\{0\} \times u) \cup (\{1\} \times \mathcal{F})$, where:

- u is a function on a countable ordinal.
- For each $\alpha \in \mathrm{dom}(u)$, $u(\alpha)$ is a careful canonical $\bar{\mathbb{S}}$-name in $V[A{\restriction}I * P{\restriction}I]$ for a real which belongs to $\dot{X}$. (Note the restriction to $\bar{\mathbb{S}}$.)
- $\mathcal{F}$ is a countable collection of codes for injections of I into δ in V, and careful canonical $\mathbb{S}$-names in $V[A * P]$ for codes for closed subsets of finite powers of $\mathbb{R}$.

The ordering is given by $\langle u^*, \mathcal{F}^*\rangle \leq \langle u, \mathcal{F}\rangle$ iff:

- $u^* {\restriction} \mathrm{dom}(u) = u$
- $\mathcal{F}^* \supseteq \mathcal{F}$.
- For every $\alpha \in \mathrm{dom}(u^*) - \mathrm{dom}(u)$, every $\dot{F} \in \mathcal{F}$, every list of injections $\sigma_1, \dots \sigma_k$ with $c(\sigma_1 \circ g), \dots, c(\sigma_k \circ g) \in \mathcal{F}$, and every commitment $\dot{t}$ generated by $\dot{F}$ and $\sigma_1, \dots, \sigma_k$ below α, it is forced in $\mathbb{S}$ that $\dot{u}^*(\alpha) \notin \dot{t}$.

Let $\dot{\mathbb{T}}_I$ name $\mathbb{T}_I$ in $\mathbb{A} * \dot{\mathbb{P}} * \mathbb{C}$.

REMARK 4.6. For the purpose of analyzing the poset $\mathbb{T}_I$, we could have used the actual injections as commitments rather than their codes, and similarly with the closed sets. The use of codes reduces the von Neumann rank of the objects used in conditions, and hence the rank of the conditions themselves, from roughly δ to roughly κ. This will allow us to bound the von Neumann rank of names for conditions, in Claim 4.19. The bounds are needed later on, in Section 5.

CLAIM 4.7. *$\dot{\mathbb{T}}_I$ is invariantly $(\kappa, \aleph_1)$-stemmed modulo I.*

PROOF. This is immediate from the definitions. The pre-order s witnessing definition 3.19 is the pre-order of stem extension. There are at most κ stems u by assumptions (2), (3), and (6), and the restriction to canonical names in Definition 4.5. Since the stems all exist in $V[A{\restriction}I * P{\restriction}I]$, we can find a bijection of κ with the set of stems which belongs to $V[A{\restriction}I * P{\restriction}I]$, and hence is in particular invariant modulo I. The fact that every condition can be extended to have stem of arbitrary countable length is proved using assumption (8), and using Claim 4.3 to convert the arbitrary name given by assumption (8) to a canonical careful name. The size requirement in condition (3) of Definition 3.19 can be proved using the restriction to careful canonical names in Definition 4.5, the fact that the objects comprising conditions in $\mathbb{T}_I$ are small enough (canonical names for reals, and functions from κ into $\mathcal{P}(\omega_1)$), Claim 4.4, and the κ^+-chain condition for $\mathbb{A} * \dot{\mathbb{P}}$. ⊣

Definition 4.8. Let $\dot{\mathbb{T}} = \prod_\tau^{(g)} \tau(\dot{\mathbb{T}}_I)$. This refers to Definition 3.27 over the extension by $\mathbb{A} * \dot{\mathbb{P}} * \check{\mathbb{C}}$.

Let $\dot{\mathbb{E}}$ name the poset $\mathbb{E}_{\text{fast}}$ of Definition 3.35 as computed in the extension by $\mathbb{A} * \dot{\mathbb{P}} * \check{\mathbb{C}} * \dot{\mathbb{T}}$. Our full forcing for this one step of the preparation is $\check{\mathbb{C}} * \dot{\mathbb{T}} * \dot{\mathbb{E}}$.

Lemma 4.9. *$\dot{\mathbb{P}} * \check{\mathbb{C}} * \dot{\mathbb{T}} * \dot{\mathbb{E}}$ is κ-appropriate. $\mathbb{A} * \dot{\mathbb{P}} * \check{\mathbb{C}} * \dot{\mathbb{T}} * \dot{\mathbb{E}}$ is κ^+-c.c. and countably closed. If $\dot{\mathbb{P}}$ is neatly κ-appropriate, then so is $\dot{\mathbb{P}}*\check{\mathbb{C}}*\dot{\mathbb{T}}*\dot{\mathbb{E}}$, by a witness that neatly extends the witness for $\dot{\mathbb{P}}$.*

Proof. Immediate from the definitions, Claim 4.7, Lemmas 3.38, 3.34 and 3.36, and from Claim 3.41. ⊣

Let T be generic for $\mathbb{T} = \dot{\mathbb{T}}[A * P * C]$, and let E be generic for $\mathbb{E} = \dot{\mathbb{E}}(A * P * C * T)$. Write T as a product $\prod_\tau^{(g)}(T^\tau)$, where τ ranges over permutations of δ, and T^τ is generic for $\tau(\dot{\mathbb{T}}_I)$. Note that T^τ depends only on $\tau\restriction I$; if $\tau\restriction I = \tau'\restriction I$ then T^τ and $T^{\tau'}$ are the same factor of T. Let $T_I = T^{id}$, which is generic for $\mathbb{T}_I$.

For each τ, let $\vec{u}^\tau = \langle \dot{u}^\tau_\alpha \mid \alpha < \omega_1\rangle$ be the union of the stems occurring in T^τ, namely $\bigcup\{u \mid (\exists\mathcal{F})\langle u, \mathcal{F}\rangle \in T^\tau\}$. Let $\dot{u}_\alpha = \dot{u}^{id}_\alpha$, so that $\langle \dot{u}_\alpha \mid \alpha < \omega_1\rangle$ is the union of the stems of T_I. Note that $\tau(\dot{u}_\alpha) = \dot{u}^\tau_\alpha$, by Claim 3.29.

Claim 4.10. *For every τ, every $\dot{F}$, and every $\sigma_1, \dots, \sigma_k$, for all large enough α, $\dot{u}^\tau_\alpha$ is forced in $\mathbb{S}$ to be outside any commitment generated by F and $\sigma_1, \dots, \sigma_k$ below α.*

Proof. Clear from the genericity of T^τ for $\tau(\mathbb{T}_I)$. There is some condition $\langle u, \mathcal{F}\rangle \in T^\tau$ so that $\mathcal{F}$ includes codes for $\dot{F}, \sigma_1, \dots, \sigma_k$. (This uses Claim 4.3, to find careful names for these codes.) The claim then holds for all $\alpha \geq \text{dom}(u)$. ⊣

We will use the points $\dot{u}^\tau_\alpha[S]$ to construct the clique adding posets. But we will use only the points indexed by elements of the fast club E.

Abusing notation, we view the fast club E both as a set and as the function on ω_1 which enumerates the elements of the club in increasing order. Let $\mathbb{K}$ be the poset of finite cliques in the restriction of U to $\{\dot{u}_\alpha[\bar{S}] \mid \alpha \in E\} = \{\dot{u}_{E(\mu)}[\bar{S}] \mid \mu < \omega_1\}$, ordered by reverse inclusion. For any $d \subseteq \omega_1$, let $\mathbb{K}_d$ be the restriction of $\mathbb{K}$ to tuples contained in $\{\dot{u}_{E(\mu)}[\bar{S}] \mid \mu \in d\}$. Define $\mathbb{K}^\tau$ and $\mathbb{K}^\tau_d$ similarly using $\dot{u}^\tau_\alpha$. Let $\dot{\mathbb{K}}$, $\dot{\mathbb{K}}_d$, $\dot{\mathbb{K}}^\tau$, and $\dot{\mathbb{K}}^\tau_d$ name these posets in $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{C}} * \dot{\mathbb{T}} * \dot{\mathbb{E}} * \dot{\mathbb{S}}$. Note that $\tau(\dot{\mathbb{K}}) = \dot{\mathbb{K}}^\tau$, and $\tau(\dot{\mathbb{K}}_d) = \dot{\mathbb{K}}^\tau_d$.

Let $\psi\colon \mathcal{P}_{<\omega_1}(\omega_1 \times \omega_1) \to \omega_1$ be a bijection in V. We will use ψ to create unbounded almost disjoint subsets of ω_1. Note to this end that if $A, B \subseteq \omega_1 \times \omega_1$ are distinct and uncountable, then $\psi''\{A \cap (\gamma\times\gamma) \mid \gamma < \omega_1\}$ and $\psi''\{B \cap (\gamma\times\gamma) \mid \gamma < \omega_1\}$ are almost disjoint, and unbounded in ω_1.

Recall that A_ξ, for $\xi < \delta$, are the subsets of ω_1 added by the generic A. Recall that C collapses κ to ω_1. Abusing notation we view C as a surjection from ω_1 onto κ. For $\tau\colon I \to \delta$, let A_τ consist of the sets $A_{\tau(g(\xi))}$, $\xi < \kappa$, arranged as a single subset of $\omega_1 \times \omega_1$ using C, meaning that $\langle \nu, \eta\rangle \in A_\tau \leftrightarrow \eta \in A_{\tau(g((C(\nu)))}$. Let $\dot{A}_\tau$ name A_τ, and note that $\dot{A}_\tau = \tau(\dot{A}_{id})$. (This uses the invariance of $\dot{\mathbb{C}}$.)

Let $d_\tau = \psi''\{A_\tau \cap (\gamma \times \gamma) \mid \gamma < \omega_1\}$, and let $\dot{d}_\tau$ name d_τ. Let $\dot{d} = \dot{d}_{id}$ and note that $\tau(\dot{d}) = \dot{d}_\tau$. Note also that the sets d_τ are almost disjoint unbounded subsets of ω_1 as $\tau \restriction I$ varies, meaning that if $\tau \restriction I \neq \tau' \restriction I$ then d_τ and $d_{\tau'}$ are almost disjoint.

LEMMA 4.11. *Let $\tau_1, \ldots, \tau_n$ be permutations of δ so that $\tau_1 \restriction I, \ldots, \tau_n \restriction I$ are distinct. Then $\mathbb{K}^{\tau_1}_{d_{\tau_1}} \times \cdots \times \mathbb{K}^{\tau_n}_{d_{\tau_n}}$ is c.c.c.*

PROOF. Suppose not, and let $\langle b^1(\xi), \ldots, b^n(\xi)\rangle$, $\xi < \omega_1$, be an antichain. Let $l^i(\xi)$ be the size of $b^i(\xi)$. By thinning the antichain we may assume that these are independent of ξ. Denote the sizes by $l^1, \ldots, l^n$. Without loss of generality we may assume that $l^1 + \cdots + l^n$ is minimal.

Write $b^i(\xi)$ as a sequence of points $x^i_1(\xi), \ldots, x^i_{l^i}(\xi)$. These points form a clique in $\tau_i(U)$. Since $\tau_i(U)$ is an open graph, there are basic open neighborhoods $N^i_j(\xi)$, $j = 1, \ldots, l^i$, of $x^i_j(\xi)$ so that any two points of $\tau_i(X)$ in any two of these neighborhoods are connected in $\tau_i(U)$. By thinning the antichain we may assume that these neighborhoods are independent of ξ, equal to $N^i_1, \ldots, N^i_{l^i}$ say. This implies that an incompatibility between $b^i(\xi)$ and $b^i(\zeta)$ must be due to points $x^i_j(\xi)$ and $x^i_k(\zeta)$ with $j = k$.

By a Δ-system argument, and noting that an incompatibility between two conditions $b^i(\xi), b^i(\zeta)$ cannot be due to any points in $b^i(\xi) \cap b^i(\zeta)$, we may assume that for each i, the conditions $b^i(\xi)$, $\xi < \omega_1$, are pairwise disjoint.

By dropping an initial segment of the antichain we may now assume that $b^i(\xi) \subseteq \mathbb{K}^{\tau_i}_{d_{\tau_i} - \eta}$, where $\eta < \omega_1$ is large enough that $d_{\tau_1} - \eta, \ldots, d_{\tau_n} - \eta$ are pairwise disjoint. This point, which uses the fact that $d_{\tau_1}, \ldots, d_{\tau_n}$ are almost disjoint, is key to the rest of the argument. It will allow us to adapt Todorcevic's proof of the countable chain condition for his clique forcing poset in [11].

Recall that E is a fast club, relative to all clubs in $V[A * P * C * T]$. Since the surjections $\tau_i(h)$ and the sequences $\langle \dot{u}^{\tau_i}_\alpha \mid \alpha < \omega_1\rangle$ belong to $V[A * P * C * T]$, this implies that for all large enough $\gamma \in E$, and for all i, $\{\dot{u}^{\tau_i}_\alpha \mid \alpha < \gamma\} \subseteq \tau_i(h)''\gamma$. By increasing η, we may assume this holds for $\gamma = E(\xi)$ for all $\xi \geq \eta$.

Each point $x^i_j(\xi) \in b^i(\xi)$ is of the form $u^{\tau_i}_{E(\mu)}$ for some $\mu \in d_{\tau^i} - \eta$. Let $R^i(\xi)$ be the set of these μ, so that $b^i(\xi) = \{u^{\tau_i}_{E(\mu)} \mid \mu \in R^i(\xi)\}$. Let $R(\xi) = \bigcup_{i=1}^n R^i(\xi)$.

Since we have made sure that $d_{\tau_1} - \eta, \ldots, d_{\tau_n} - \eta$ are disjoint, each $\mu \in R(\xi)$ belongs to exactly one of $R^1(\xi), \ldots, R^n(\xi)$. In particular for each ξ, there is exactly one i so that $\max(R(\xi)) \in R^i(\xi)$. By thinning the antichain we may assume that this i is fixed independently of ξ. By re-ordering $\tau_1, \ldots, \tau_n$ we may assume this i is equal to 1. By re-ordering the enumeration $x^1_1(\xi), \ldots, x^1_{l^1}(\xi)$ of $b^1(\xi)$ we may assume that for $i = j = 1$, $x^i_j(\xi) = u^{\tau_1}_{E(\max(R(\xi)))}$, and that for all other i, j, $x^i_j(\xi) = u^{\tau_i}_{E(\mu)}$ for some $\mu < \max(R(\xi))$. Let $\dot{u}^i_j(\xi) = \dot{u}^{\tau_i}_{E(\mu)}$ for this μ. Then $\dot{u}^i_j(\xi)$ names $x^i_j(\xi)$, and, through our use above of the fact that E is fast, for all i, j other than $i = j = 1$, $\dot{u}^i_j(\xi) \in \tau^i(h)''E(\max(R(\xi)))$.

Let $b(\xi) = b^1(\xi)^\frown \cdots {}^\frown b^n(\xi)$. Let $l = l^1 + \cdots + l^n$, so that $b(\xi) \in \mathbb{R}^l$. Let $F \subseteq \mathbb{R}^l$ be the closure of the set $\{b(\xi) \mid \xi < \omega_1\}$. Let $b_{\text{tail}}(\xi)$ be obtained from $b(\xi)$ by removing its first element $x^1_1(\xi)$. For each ξ, let $t(\xi) = \{x \mid \langle x\rangle^\frown b_{\text{tail}}(\xi) \in F\}$. Notice that $x^1_1(\xi) \in t(\xi)$.

CLAIM 4.12. *For arbitrarily large $\xi < \omega_1$, $t(\xi)$ is* not *1-colorable in $\tau_1(U)$.*

PROOF. By definition F belongs to $V[A * P * C * T * E][S]$. But since F is coded by a real, $\mathbb{S}$ is c.c.c., and $\dot{\mathbb{C}} * \dot{\mathbb{T}} * \dot{\mathbb{E}}$ is forced to be countably closed, F belongs to $V[A*P][S]$. Let $\dot{F} \in V[A*P]$ name F in $\mathbb{S}$. Let $\dot{t}(\xi)$ be the canonical name for $t(\xi)$ obtained from $\dot{F}$ and the names $\dot{u}^i_j(\xi)$ for the elements of $b_{\mathrm{tail}}(\xi)$. Suppose that the claim fails, and fix a condition $s \in \mathbb{S}$ forcing this. By extending s, we may assume there is a specific ρ so that s forces $\dot{t}(\xi)$ to be 1-colorable for all $\xi > \rho$. By modifying $\dot{F}$ (and carrying these modification to $\dot{t}(\xi)$) we may further assume that any condition incompatible with s forces $\dot{F}$ to be empty.

Then for all $\xi > \rho$, $\dot{t}(\xi)$ is outright forced in $\mathbb{S}$ to be 1-colorable. The condition s forces this fact by the way it was chosen, and any condition incompatible with s forces that $\dot{t}(\xi)$ is empty, hence in particular 1-colorable.

It follows that $\dot{t}(\xi)$ is a commitment. Moreover it is generated from $\dot{F}$ and $\tau^1{\restriction}I, \dots, \tau^n{\restriction}I$ below $E(\max(R(\xi)))$, because the names $\dot{u}^i_j$ for elements of $b_{\mathrm{tail}}(\xi)$ belong to $\bigcup_{i \le n} \tau^i(h)''E(\max(R(\xi)))$. So by Claim 4.10, for all large enough ξ, $\dot{u}^{\tau_1}_{E(\max(R(\xi)))}$ is forced in $\mathbb{S}$ to *not* be an element of $\dot{t}(\xi)$. This is a contradiction since $\dot{u}^{\tau_1}_{E(\max(R(\xi)))}[S] = x^1_1(\xi) \in t(\xi)$. ⊣

For each ξ so that $t(\xi)$ is not 1-colorable, fix two points $y_1(\xi), y_2(\xi) \in t(\xi)$ which are connected in $\tau_1(U)$. Fix basic open neighborhoods $O_1(\xi), O_2(\xi)$ of $y_1(\xi), y_2(\xi)$ so that every point in $O_1(\xi) \cap \tau_1(X)$ is connected to every point in $O_2(\xi) \cap \tau_1(X)$ in $\tau_1(U)$. Fix specific O_1, O_2, and an unbounded $Z \subseteq \omega_1$, so that for every $\xi \in Z$, $O_1(\xi) = O_1$ and $O_2(\xi) = O_2$.

Let $b^1_{\mathrm{tail}}(\xi)$ be obtained from $b^1(\xi)$ by removing its first element $x^1_1(\xi)$. By the minimality of l, the conditions $\langle b^1_{\mathrm{tail}}(\xi), b^2(\xi), \dots, b^n(\xi)\rangle$, for $\xi \in Z$, do not form an uncountable antichain in $\mathbb{K}^{\tau_1}_{d_{\tau_1}} \times \cdots \times \mathbb{K}^{\tau_n}_{d_{\tau_n}}$. So there must be $\xi \ne \zeta$ in Z so that $\langle b^1_{\mathrm{tail}}(\xi), b^2(\xi), \dots, b^n(\xi)\rangle$ and $\langle b^1_{\mathrm{tail}}(\zeta), b^2(\zeta), \dots, b^n(\zeta)\rangle$ are compatible. Since $x^i_j(\xi) \ne x^i_j(\zeta)$ it follows in particular that $x^i_j(\xi)$ and $x^i_j(\zeta)$ are connected in $\tau_i(U)$, for all i, j except possibly $i = j = 1$.

Using the fact that the graphs $\tau_i(U)$ are all open, we can therefore find open neighborhoods P and Q of $b_{\mathrm{tail}}(\xi)$ and $b_{\mathrm{tail}}(\zeta)$ in $\mathbb{R}^{l-1}$, so that any $a \in P$ and $b \in Q$ are connected in $\tau_1(U)^{l_1 - 1} \times \tau_2(U)^{l_2} \times \cdots \times \tau_n(U)^{l_n}$. Let $P^* = O_1 \times P$ and let $Q^* = O_2 \times Q$. Since any $y_1 \in O_1$ and $y_2 \in O_2$ are connected in $\tau_1(U)$, we have that any $a \in P^*$ and $b \in Q^*$ are connected in $\tau_1(U)^{l_1} \times \cdots \times \tau_n(U)^{l_n}$.

By choice of the open neighborhoods, we have $y_1(\xi)^\frown b_{\mathrm{tail}}(\xi) \in P^*$. Since $y_1(\xi) \in t(\xi)$, we have $y_1(\xi)^\frown b_{\mathrm{tail}}(\xi) \in F$. So P^* is an open neighborhood of a point in F. By the definition of F it follows that there is ξ^* so that $b(\xi^*) \in P^*$. Similarly, using O_2 and $y_2(\zeta)^\frown b_{\mathrm{tail}}(\zeta)$, there must be ζ^* so that $b(\zeta^*) \in Q^*$. But then for each individual pair i, j, the points $x^i_j(\xi^*)$ and $x^i_j(\zeta^*)$ are connected in $\tau_i(U)$. By the refinements made at the start of the proof of the current lemma, this implies that $\langle b^1(\zeta^*), \dots, b^n(\zeta^*)\rangle$ and $\langle b^1(\xi^*), \dots, b^n(\xi^*)\rangle$ are compatible, a contradiction. ⊣

CLAIM 4.13. *There is $\rho < \omega_1$ so that in the forcing $\mathbb{K}_{d-\rho}$, every condition has extensions using points u_α for arbitrarily large $\alpha < \omega_1$. Moreover there is $\rho < \omega_1$ for which this statement is outright forced to hold in $\mathbb{S}$.*

PROOF. The second part of the claim is immediate from the first since $\mathbb{S}$ is c.c.c.

Suppose the first part fails. Then we can construct conditions $b_\xi \in \mathbb{K}_d$, for $\xi < \omega_1$, and ordinals $\beta_\xi < \omega_1$, so that b_ξ is incompatible with any condition $\langle u_\alpha \rangle$ for $\alpha > \beta_\xi$, and so that $\min\{\alpha \mid u_\alpha \in b_\xi\}$ is larger than $\sup\{\beta_\zeta \mid \zeta < \xi\}$. But then the conditions b_ξ form an uncountable antichain in $\mathbb{K}_d$, contradicting Lemma 4.11. ⊣

Let $\dot\rho$ be an $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{C}} * \dot{\mathbb{T}} * \dot{\mathbb{E}}$-name for the least ordinal witnessing Claim 4.13. For a permutation τ of δ, let $\dot\rho_\tau = \tau(\dot\rho)$. Let $\rho_\tau = \dot\rho_\tau[A * P * C * T * E]$. Then ρ_τ is the minimal witness for Claim 4.13 for the poset $\mathbb{K}^\tau_{d_\tau}$.

DEFINITION 4.14. Let $\dot{\mathbb{D}} = \prod^{(g)}_\tau \tau(\dot{\mathbb{K}}_{\dot d - \dot\rho})$, taken with finite support.

Let $\mathbb{D} = \dot{\mathbb{D}}[A * P * C * T * E][S]$. Let D be generic for $\mathbb{D}$. Write D as $\prod^{(g)}_\tau D_\tau$. Then D_τ is generic for $\mathbb{K}^\tau_{d_\tau - \rho_\tau}$.

REMARK 4.15. Note that every factor of the symmetric product $\mathbb{D}$ is a poset of size $\aleph_1$ in $V[A * P * C * T * E][S]$, whose conditions are finite sets of certain reals.

LEMMA 4.16. $\mathbb{D}$ *is c.c.c., and hence* $\mathbb{S} * \dot{\mathbb{D}}[A * P * C * T * E]$ *is c.c.c.*

PROOF. Clear from Lemma 4.11 and the use of finite support. ⊣

LEMMA 4.17. $\dot{\mathbb{S}} * \dot{\mathbb{D}}$ *(viewed as an* $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{C}} * \dot{\mathbb{T}} * \dot{\mathbb{E}}$*-name) is invariant under permutations.*

PROOF. Clear from Claim 3.28. ⊣

LEMMA 4.18. *In* $V[A * P * C * T * E][S * D]$*, each of the graphs* $\tau(U)$ *(*τ *a permutation of* δ*) has an uncountable clique.*

PROOF. The generic D_τ is a clique in $\tau(U)$. It is uncountable by Claim 4.13 and since ρ_τ witnesses the claim for $\mathbb{K}^\tau_{d_\tau}$. ⊣

CLAIM 4.19. *Let* $\mu > \kappa$ *be an inaccessible cardinal. Suppose that* $\mathbb{S}$ *has von Neumann rank* $\eta < \mu$*. Then there is* $\eta^* < \mu$ *so that every condition in* $\mathbb{C} * \dot{\mathbb{T}} * \dot{\mathbb{E}}$ *has von Neumann rank below* η^**.*

PROOF. Let $\langle c, \dot t, \dot e\rangle \in \mathbb{C} * \dot{\mathbb{T}} * \dot{\mathbb{E}}$. c is a partial function from $\aleph_1$ into κ of size $< \kappa$, and so has von Neumann rank below κ.

Fix a generic C for $\mathbb{C}$, and consider the von Neumann of $t = \dot t[C]$. Note that canonical $\mathbb{S}$-names for reals have von Neumann rank at most $\eta + 3$. Each condition in $\mathbb{T}_I$ is generated using finitely many pairing operations from such names, from countable ordinals, and from codes for injections of I into δ. The codes have von Neumann rank at most $\kappa + 1$. So every condition in $\mathbb{T}_I$ has von Neumann rank below $\max\{\kappa, \eta, \omega_1\} + \omega$. (See Remark 4.6 in connection to this.) By Remark 3.30 this implies that t, which is a condition in a symmetric product generated from copies of $\mathbb{T}_I$, has rank below $\eta' = \max\{\kappa, \eta, \omega_1\} + \omega \cdot 2$.

Since the fact that t has von Neumann rank below η' holds regardless of the choice of the generic C, Claim 2.4 gives $\eta'' < \mu$, so that the name $\dot t$ has von Neumann rank below η''.

The claim produces η'' as a function of η', independently from $\dot{t}$. Increasing η'' to a limit ordinal if necessary, it follows that $\mathbb{C} * \dot{\mathbb{T}}$ is contained in $V_{\eta''}^{V[A*P]}$.

Through another use of Claim 2.4, and since $\dot{e}$ names a set of von Neumann rank at most $\omega_1 + \omega$, it follows that there is $\eta''' < \mu$ so that the name $\dot{e}$ has von Neumann rank below η'''.

Let $\eta^* = \max\{\eta, \eta', \eta'', \eta'''\}$, and note that we obtain η^* through computations as in Claim 2.4, independently of the choice of c, $\dot{t}$, and $\dot{e}$. ⊣

Claim 4.20. *Let $\mu > \kappa$ be an inaccessible cardinal. Suppose that $\mathbb{S}$ has von Neumann rank $\eta < \mu$. Then there is $\eta^{**} < \mu$ so that, in $V[A * P * C * T * E]$, $\mathbb{S} * \dot{\mathbb{D}}$ has von Neumann rank below η^{**}.*

Proof. Let S be generic for $\mathbb{S}$. Note that every condition $d \in \mathbb{D} = \dot{\mathbb{D}}[S]$ has von Neumann rank below $\kappa + \omega$. This follows by Remark 3.30 since conditions in the individual factors of $\mathbb{D}$ are just finite sets of reals.

Using Claim 2.4 it follows that there is $\eta' < \mu$ so that every efficient $\mathbb{S}$-name for a condition in $\dot{\mathbb{D}}$ has rank at most η'.

Then $\mathbb{S} * \dot{\mathbb{D}}$ has von Neumann rank at most $\max\{\eta, \eta'\} + 3$. ⊣

Remark 4.21. Claim 4.20 continues to hold for any poset $\dot{\mathbb{D}}$ which is a symmetric product of height κ of factors whose conditions have von Neumann rank smaller than κ. In particular it holds for $\dot{\mathbb{D}} = \prod_{\tau}^{(id\restriction\kappa)} \mathrm{Add}(\aleph_0, 1)\check{}$, the symmetric product, over $\tau \colon \kappa \to \delta$, of the Cohen poset for adding one real.

Claim 4.22. *Suppose $\langle a, \dot{p}, \dot{s}\rangle \Vdash \dot{d} \in \dot{\mathbb{D}}$. Then there is $\dot{d}'$ of forcing hereditary size and support size $\leq \kappa$, so that $\langle a, \dot{p}, \dot{s}\rangle \Vdash \dot{d}' = \dot{d}$.*

Proof. Modifying $A * P * S$ if necessary, we may assume $\langle a, \dot{p}, \dot{s}\rangle \in A * P * S$.

We saw in the proof of Claim 4.7 that stems in $\dot{\mathbb{T}}_I$ have $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{C}}$-names of forcing hereditary size $\leq \kappa$. Since conditions in the factor $\dot{\mathbb{D}}_I$ of $\dot{\mathbb{D}}$ are finite sequences of reals with names occurring on stems of T_I, it follows that, for any generic $A * P * S$, and any name $\dot{u}$ for a condition in $\dot{\mathbb{D}}_I$, there is $\dot{u}'$ of forcing hereditary size $\leq \kappa$, so that $\dot{u}'[A * P * S] = \dot{u}[A * P * S]$. The same is then true for each of the factors $\tau(\dot{\mathbb{D}}_I)$. And since the product defining $\mathbb{D}$ is taken with finite supports, and indexed by codes which themselves have names of forcing hereditary size $\leq \kappa$, the same is true for the name $\dot{d}$ for a condition in $\dot{\mathbb{D}}$.

This implies that densely many conditions in $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$ below $\langle a, \dot{p}, \dot{s}\rangle$ force the existence of $\dot{d}'$ of forcing-hereditary size $\leq \kappa$ with $\dot{d}'[A * P * S] = \dot{d}[A * P * S]$. Using a merging of names argument, and the κ^+-c.c. for $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$, one can obtain $\dot{d}'$, still of forcing-hereditary size $\leq \kappa$, which is forced by $\langle a, \dot{p}, \dot{s}\rangle$ to equal $\dot{d}$.

By the appropriateness of $\mathbb{A} * \dot{\mathbb{P}}$, using the support size condition (2) of Definition 3.6, and using assumption (4) on the forcing hereditary size of conditions in $\dot{\mathbb{S}}$, each condition of $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$ used in the name $\dot{d}'$ can be replaced by an equivalent condition with support of size $\leq \kappa$. Since $\dot{d}'$ is of forcing-hereditary size $\leq \kappa$, the name resulting from these substitutions has support of size $\leq \kappa$. ⊣

Remark 4.23. If $J \supseteq \mathrm{Sp}(\dot{d})$, $\mathbb{U}$ is the restriction of $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$ to condition with support contained in J, and $\mathbb{U}$ is a complete subposet of $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$, then one can find $\dot{d}'$ as in Claim 4.22 which is a $\mathbb{U}$-name. This is clear from the proof, noticing that all coordinates in the domain of $\dot{d}$ must involve τ which map into J, and then the names $\dot{u}'$ obtained in the proof are all $\mathbb{U}$-names, since the names occurring on stems of $\tau(\dot{T}_I)$ have supports contained in $\tau''I$. The completeness of $\mathbb{U}$ in $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$ is used to carry out the merging argument while keeping to $\mathbb{U}$-names. And the support size condition of Definition 3.6 allows keeping all supports contained in J, in the final part of the proof.

Let $W \subseteq \delta$. Let $Z = Z(W)$ be the set of injections of I into W. Recall from Definition 3.27 that $\prod^{(g)}_{\tau\in Z} \tau(\dot{\mathbb{K}}_{d-\dot{\rho}})$ is the restriction of $\mathbb{D}$ to its factors indexed by $c(\tau \circ g)$ for $\tau \in Z$. Let $\mathbb{D}|_W$ denote this product, and let $D|_W = D \cap \mathbb{D}|_W$. Then $\mathbb{D}|_W$ is clearly a complete subposet of the full product $\mathbb{D}$, and $D|_W$ is generic over $V[A*P*C*T*E][S]$. The factor poset $\mathbb{D}/D|_W$ for adding D over $V[A*P*C*T*E][S][D|_W]$ is as usual the restriction of $\mathbb{D}$ to conditions d which are compatible with all elements of $D|_W$. Letting $d|_W = d\restriction\{c(\tau \circ g) \mid \tau \in Z\}$ be the restriction of a condition $d \in \mathbb{D}$ to factors indexed by $c(\tau\circ g)$ for $\tau \in Z$, it is easy to check that $d \in \mathbb{D}/D|_W$ iff $d \in \mathbb{D}$ and $d|_W \in D|_W$.

Claim 4.24. *Suppose that $d = \dot{d}[A*P*C*T*E][S] \in \mathbb{D}$, and that $\mathrm{Sp}(\dot{d}) \subseteq W$. Then $d \in \mathbb{D}|_W$.*

Proof. Each $c(\tau \circ g) \in \mathrm{dom}(d)$ codes $A\restriction\mathrm{range}(\tau)$. If it has a name with support contained in W, then it must be that $\mathrm{range}(\tau) \subseteq W$. ⊣

Claim 4.25. *Let $W \subseteq \delta$. Let d belong to the restricted product $\mathbb{D}|_W$. Then there is an $\mathbb{A}*\dot{\mathbb{P}}*\dot{\mathbb{S}}$-name $\dot{d}$ for d, of forcing hereditary size $\leq \kappa$, and with support contained in W.*

Proof. Since $\mathbb{D}$ is taken with finite supports, it is enough to prove the claim separately for each of the non-trivial coordinates in d. These coordinates are conditions in (subposets of) $\mathbb{K}^\tau$, for permutations τ which map I into W. By symmetry, and since $\tau(\dot{\mathbb{K}}) = \dot{\mathbb{K}}^\tau$, it is therefore enough to prove that every condition in $\mathbb{K}$ has a name of forcing hereditary size $\leq \kappa$ with support contained in I. But this is clear, since any condition in K is a finite subsets $\{\dot{u}_\alpha[S] \mid \alpha < \omega_1\}$, and $\dot{u}_\alpha$, by Definition 4.5 and Claim 4.4, has an $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$-name of forcing hereditary size $\leq \kappa$ with support contained in I. ⊣

Claim 4.26. *Let $W \subseteq \delta$. Let $\langle d_n \mid n < \omega\rangle$ be a sequence of conditions in $\mathbb{D}/D|_W$, which belongs to an extension of $V[A*P*C*T*E][S][D|_W]$ by F which is generic over $V[A*P*C*T*E][S][D]$. Say $d_n = \dot{d}_n[A*P*C*T*E][S]$, and suppose that the sets $\mathrm{Sp}(\dot{d}_n) - W$ are pairwise disjoint. Then there are infinitely many n so that $d_n \in D$.*

Proof. This is a consequence of the use of finite supports in the product forming $\mathbb{D}$. Suppose the claim fail, and fix $N_0 < \omega$ and $d \in D$ forcing in $\mathbb{D}/D|_W$, over $V[A*P*C*T*E][S][D|_W][F]$, that $n > N_0 \to d_n \notin D$. (This uses the fact that F is generic over $V[A*P*C*T*E][S][D]$, and consequently D is generic for $\mathbb{D}/D|_W$ over $V[A*P*C*T*E][S][D|_W][F]$.) Let $W_n = \mathrm{Sp}(\dot{d}_n)$. Then for every $c(\tau \circ g) \in \mathrm{dom}(d_n)$ it must be that $\mathrm{range}(\tau) \subseteq W_n$. Since (for $n \neq m$) $W_n \cap W_m \subseteq W$, this implies in particular that the only coordinates $c(\tau \circ g)$ which belong to both $\mathrm{dom}(d_n)$ and $\mathrm{dom}(d_m)$ are ones where $\mathrm{range}(\tau) \subseteq W$. Let

$C = \{c(\tau \circ g) \mid \mathrm{range}(\tau) \subseteq W\}$. Then each $c(\tau \circ g) \in \mathrm{dom}(d) - C$ belongs to the domain of at most one d_n. Since $\mathrm{dom}(d)$ is finite, we can pick $n > N_0$ so that none of these coordinates belongs to $\mathrm{dom}(d_n)$. So $\mathrm{dom}(d_n) \cap \mathrm{dom}(d) \subseteq C$. Since $d_n \restriction C = d_n|_W$ and $d \restriction C = d|_W$ both belong to $D|_W$, it follows that d_n and d are compatible, and indeed there is a witness d^* for this so that $d^* \restriction C \in D|_W$. But this is a contradiction, since d^* forces (in $\mathbb{D}/D|_W$) that $d_n \in \dot{D}$, while d forces that $d_n \notin \dot{D}$. ⊣

§5. Putting everything together. We are now ready to prove Theorem 1.1. Suppose that θ is Mahlo, $\delta > \theta$, and $\delta^{<\theta} = \delta$. Without loss of generality we may assume that the GCH holds below θ. If the CH holds in the ground model, this can be arranged without collapsing $\aleph_1$. By adding a Cohen subset of θ if necessary, we may also assume that there is a $\diamond$-sequence of length θ, concentrating on the strongly inaccessible cardinals. In other words we assume that there is a sequence $\vec{d} = \langle d_\kappa \mid \kappa < \theta \rangle$ so that for every $Z \subseteq \theta$, the set of strongly inaccessible κ with $d_\kappa = Z \cap \kappa$ is stationary.

Let $\mathbb{A} = \mathrm{Add}(\aleph_1, \delta)$. Define a countable support iteration $\langle \dot{\mathbb{P}}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \theta, \beta < \theta \rangle$ over the extension by $\mathbb{A}$, and a finite support iteration $\langle \dot{\mathbb{S}}_\alpha, \dot{\mathbb{D}}_\beta \mid \alpha \leq \theta, \beta < \theta \rangle$ over the extension by $\mathbb{A} * \dot{\mathbb{P}}_\theta$, through the following conditions. The conditions specify the iterands $\dot{\mathbb{Q}}_\beta$ and $\dot{\mathbb{D}}_\beta$; the posets $\dot{\mathbb{P}}_\alpha$ and $\dot{\mathbb{S}}_\alpha$ are the iteration stages determined from these iterands. The posets $\dot{\mathbb{P}}_\alpha$ will be κ-appropriate, and the posets $\dot{\mathbb{S}}_\alpha$ will be c.c.c. Each iterand-name $\dot{\mathbb{D}}_\alpha$ will belong to the extension by $\mathbb{A} * \dot{\mathbb{P}}_{\alpha+1}$, so that $\dot{\mathbb{S}}_\alpha$ belongs to the extension by $\mathbb{A} * \dot{\mathbb{P}}_\alpha$.

1. If α is not an inaccessible cardinal, then $\dot{\mathbb{Q}}_\alpha = \mathrm{Col}(\aleph_1, \alpha)\check{}$, and $\dot{\mathbb{D}}_\alpha = \prod_\tau^{(id\restriction\alpha)} \mathrm{Add}(\aleph_0, 1)\check{}$ (the symmetric product, over $\tau\colon \alpha \to \delta$, of the Cohen poset for adding one real).
2. If κ is an inaccessible cardinal, but $d_\kappa \subseteq \kappa$ does not code some I, g $\dot{\dot{\mathbb{S}}}$, $\dot{X}$, and $\dot{U}$ which satisfy assumptions (1)–(8) of Section 4 for $\dot{\mathbb{P}}_\kappa$ and $\dot{\mathbb{S}}_\kappa$, then again $\dot{\mathbb{Q}}_\kappa = \mathrm{Col}(\aleph_1, \kappa)\check{}$, and $\dot{\mathbb{D}}_\kappa = \prod_\tau^{(id\restriction\alpha)} \mathrm{Add}(\aleph_0, 1)\check{}$.
3. If κ is an inaccessible cardinal, and $d_\kappa \subseteq \kappa$ *does* code some I, g $\dot{\dot{\mathbb{S}}}$, $\dot{X}$, and $\dot{U}$ which satisfy assumptions (1)–(8) of Section 4 for $\dot{\mathbb{P}}_\kappa$ and $\dot{\mathbb{S}}_\kappa$, then set $\dot{\mathbb{Q}}_\kappa = \dot{\mathbb{C}} * \dot{\mathbb{T}} * \dot{\mathbb{E}}$ and $\dot{\mathbb{D}}_\kappa = \dot{\mathbb{D}}$, for the posets $\dot{\mathbb{C}}$, $\dot{\mathbb{T}}$, $\dot{\mathbb{E}}$, and $\dot{\mathbb{D}}$ defined in Section 4. We say in this case that stage κ is *active* using $\dot{\dot{\mathbb{S}}}$, $\dot{X}$, and $\dot{U}$.

We refer in these conditions to a subset of κ *coding* I, g, $\dot{\dot{\mathbb{S}}}$, $\dot{X}$, and $\dot{U}$ as in Section 4. The coding converts sets of hereditary size κ to subsets of κ. The precise coding used does not matter, as long as it is continuous in the following sense: If Z is of hereditary size θ, $d \subseteq \theta$ codes Z, T is the transitive closure of Z, E_α, $\alpha < \theta$, is continuous and $\subseteq$-increasing with $|E_\alpha| < \theta$ and $\bigcup_{\alpha<\theta} E_\alpha = T$, and $\pi_\alpha\colon E_\alpha \to T_\alpha$ are the transitive collapse embeddings, then for a club of α, $d \cap \alpha$ codes $\pi_\alpha''(Z \cap E_\alpha)$. Any reasonable coding will have this continuity property.

For limit γ, we take $\dot{\mathbb{S}}_\gamma$ to literally be the union $\bigcup_{\alpha<\gamma} \dot{\mathbb{S}}_\alpha$, not just an arbitrary $\mathbb{A} * \dot{\mathbb{P}}_\gamma$-name for this union. We then have that for every $\dot{s} \in \mathrm{dom}(\dot{\mathbb{S}}_\gamma)$, there is $\alpha < \gamma$ so that $\dot{s} \in \mathrm{dom}(\dot{\mathbb{S}}_\alpha)$. We do the same with $\dot{\mathbb{P}}_\alpha$ for α of uncountable cofinality.

At all stages α, we pick the name $\dot{\mathbb{S}}_\alpha$ so that every element of $\mathrm{dom}(\dot{\mathbb{S}}_\alpha)$ is outright forced to belong to $\dot{\mathbb{S}}_\alpha$.

CLAIM 5.1. *For each $\alpha \leq \theta$, $\dot{\mathbb{P}}_\alpha$ is neatly $|\alpha|$-appropriate, and $\mathbb{A} * \dot{\mathbb{P}}_\alpha$ is countably closed. If $|\alpha|^\omega = |\alpha|$ then $\mathbb{A} * \dot{\mathbb{P}}$ is also $|\alpha|^+$-c.c.*

PROOF. Countable closure is clear. Appropriateness in the successor case at active stages is clear from Lemma 4.9. The successor case at non-active stages is clear from Lemma 3.38 and Claim 3.41. It is also clear from these results that any witness that $\dot{\mathbb{P}}_\alpha$ is neatly $|\alpha|$-appropriate can be neatly extended to a witness that $\dot{\mathbb{P}}_{\alpha+1}$ is neatly appropriate. This (with Claim 3.8) allows for an inductive construction of witnesses that fits with the assumptions of Lemma 3.42, showing that $\dot{\mathbb{P}}_\alpha$ is neatly $|\alpha|$-appropriate for all $\alpha \leq \delta$. The chain condition then follows by Remark 3.40. ⊣

CLAIM 5.2. *For each $\alpha \leq \theta$, $\dot{\mathbb{S}}_\alpha$ is c.c.c. and invariant under permutations of δ. The iterands $\dot{\mathbb{D}}_\beta$ are forced to be symmetric products of posets of size $\aleph_1$. If μ is inaccessible, and $A * P_\mu$ is generic for $\mathbb{A} * \dot{\mathbb{P}}_\mu$, then for every $\alpha < \mu$, $\dot{\mathbb{S}}_\alpha[A * P_\mu]$ has von Neumann rank below μ, and $\dot{\mathbb{S}}_\mu[A * P_\mu]$ has von Neumann rank at most μ.*

PROOF. The claims about $\dot{\mathbb{S}}_\alpha$ and $\dot{\mathbb{D}}_\alpha$ are clear by induction, using the definitions and results in Section 4, in particular Remark 4.15, Lemma 4.16, and Lemma 4.17 for active stages, and the obvious parallel results for non-active stages. The claim on the von Neumann rank of $\dot{\mathbb{S}}_\alpha$ follows from Claim 4.20 and Remark 4.21 by induction on α. Since $\dot{\mathbb{S}}_\mu[A * P_\mu] = \bigcup_{\alpha<\mu} \dot{\mathbb{S}}_\alpha[A * P_\mu]$, it then follows that $\dot{\mathbb{S}}_\mu[A * P_\mu]$ has von Neumann rank at most μ. ⊣

CLAIM 5.3. *Let $\mu \leq \theta$ be inaccessible. Let $\alpha < \mu$. If A is generic for $\mathbb{A}$, then the von Neumann rank of $\dot{\mathbb{P}}_\alpha[A]$ is below μ.*

PROOF. By induction on α, using Claim 4.19 (and Claim 2.4) for the successor case. To apply Claim 4.19 we need to verify that $\mathbb{S}_\alpha$ has von Neumann rank less than μ, when P_α is generic for $\dot{\mathbb{P}}_\alpha[A]$ and $\mathbb{S}_\alpha = \dot{\mathbb{S}}_\alpha[A * P_\alpha]$. This is given by Claim 5.2. ⊣

Let $\dot{\mathbb{P}} = \dot{\mathbb{P}}_\theta$ and $\dot{\mathbb{S}} = \dot{\mathbb{S}}_\theta$. Let $A * P * S$ be generic for $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$. Let $\mathbb{P} = \dot{\mathbb{P}}[A]$ and let $\mathbb{S} = \dot{\mathbb{S}}[A * P]$. Similarly let $\mathbb{P}_\alpha = \dot{\mathbb{P}}_\alpha[A]$ and let $\mathbb{S}_\alpha = \dot{\mathbb{S}}_\alpha[A * P_\alpha]$. We will show that the forcing extension by $A * P * S$ witnesses Theorem 1.1.

CLAIM 5.4. *$\aleph_1^{V[A*P*S]} = \aleph_1^V$, $\aleph_2^{V[A*P*S]} = \theta$, and $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$ is θ-c.c., so that all cardinals of V above θ are preserved in $V[A * P * S]$.*

PROOF. $\mathbb{A} * \dot{\mathbb{P}}$ is countably closed, so preserves $\aleph_1$. $\dot{\mathbb{P}}$ incorporates posets to collapse all $\kappa \in (\aleph_1, \theta)$ to $\aleph_1$. For cofinally many $\kappa < \theta$, $\mathbb{A} * \dot{\mathbb{P}}_\kappa$ is κ^+-c.c. and hence θ-c.c. Using a Δ-system argument this implies that $\mathbb{A} * \dot{\mathbb{P}}$ is θ-c.c. Since $\dot{\mathbb{S}}$ is forced to be c.c.c. this implies that $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$ is θ-c.c., and does not collapse any cardinals from θ upwards. ⊣

CLAIM 5.5. $\mathfrak{c}^{V[A*P*S]} = \delta$.

Proof. The direction $\mathfrak{c}^{V[A*P*S]} \geq \delta$ is clear from the definition in condition (1) above, since the symmetric product used there has $|\delta^\alpha| = \delta$ copies of the Cohen poset. The direction $\mathfrak{c}^{V[A*P*S]} \leq \delta$ is clear by a name counting argument, using the facts that $\mathbb{S}$ is a finite support iteration of length θ, each iterand $\dot{\mathbb{D}}_\alpha[A * P_\alpha * S_\alpha]$ is a (symmetric) finite support product of $\delta^{|\alpha|} \leq \delta^{<\theta} = \delta$ factors, each of size $\aleph_1$, and $\mathbb{S}$ is c.c.c. $\dashv$

To complete the proof of Theorem 1.1, it remains to show that $V[A * P * S]$ satisfies $\mathsf{OCA}_{\mathrm{Tod}}(< \mathfrak{c})$.

Let H_α, $\alpha < \theta$, be an increasing continuous chain of elementary substructures of some large enough initial segment of V, with $|H_\alpha| < \theta$, $H_\alpha \cap \theta$ an ordinal, H_α closed under sequences of length $< \mathrm{cof}(\alpha)$ (length $\leq \max\{\omega, \alpha\}$ if α is not a limit), and with all relevant objects belonging to H_0, including in particular $\theta, \delta, \mathbb{A}, \dot{\mathbb{P}}, \dot{\mathbb{S}}$. Let $H_\theta = \bigcup_{\alpha<\theta} H_\alpha$. Let M_α for $\alpha \leq \theta$ be the transitive collapse of H_α, and let $\pi_\alpha \colon H_\alpha \to M_\alpha$ be the collapse embedding. Let $\pi_{\alpha,\beta} \colon M_\alpha \to M_\beta$ be $\pi_\beta \circ \pi_\alpha^{-1}$.

Claim 5.6. *For any $Q \in M_\theta$, there are stationarily many $\alpha < \theta$ so that d_α codes $\pi_\alpha^{-1}(Q)$, via the coding used for the definition of $\mathbb{P}_\theta$.*

Proof. Clear from the fact that $\langle d_\alpha \mid \alpha < \theta\rangle$ is a $\diamond$ sequence, the fact that M_θ is transitive of cardinality θ, and the continuity of the coding. $\dashv$

Let $I_\tau = \pi_\tau(\delta) = \pi_\tau''(H_\tau \cap \delta)$.

Claim 5.7. *Let $\tau \leq \theta$ be inaccessible. Let $A * P_\alpha$ be generic for $\mathbb{A} * \dot{\mathbb{P}}_\alpha$. Then:*

1. *For every $\alpha \leq \tau$, $\mathbb{A}\restriction I_\tau * \dot{\mathbb{P}}_\alpha\restriction I_\tau$ is a complete subposet of $\mathbb{A} * \dot{\mathbb{P}}_\alpha$, and hence of $\mathbb{A} * \dot{\mathbb{P}}$.*
2. *For every $\alpha \leq \tau$, $\dot{\mathbb{S}}_\alpha[A * P_\alpha] \cap V[A\restriction I_\tau * P_\alpha\restriction I_\tau]$ belongs to $V[A\restriction I_\tau * P_\alpha\restriction I_\tau]$, and depends only on $A\restriction I_\tau * P_\alpha\restriction I_\tau$.*

Let $\dot{\dot{\mathbb{S}}}_{\alpha,\tau}$ be an $\mathbb{A}\restriction I_\tau * \dot{\mathbb{P}}_\alpha\restriction I_\tau$-name for $\dot{\mathbb{S}}_\alpha[A * P_\alpha] \cap V[A\restriction I_\tau * P_\alpha\restriction I_\tau]$, and let $\dot{\dot{\mathbb{S}}}_\tau = \dot{\dot{\mathbb{S}}}_{\tau,\tau}$.

3. *$\dot{\dot{\mathbb{S}}}_{\alpha+1}$ is (isomorphic to) $\dot{\dot{\mathbb{S}}}_\alpha * \dot{\mathbb{D}}_\alpha|_{I_\tau}$.*
4. *$\dot{\dot{\mathbb{S}}}_{\alpha,\tau}$ is forced to be a complete subposets of $\dot{\mathbb{S}}_\alpha$. In particular $\dot{\dot{\mathbb{S}}}_\tau$ is forced to be a complete subposets of $\dot{\mathbb{S}}_\tau$., and hence of $\dot{\mathbb{S}}$.*

Proof. For $\alpha < \tau$, the first condition follows from Claim 3.12, since $\dot{\mathbb{P}}_\alpha$ is α-appropriate and $|I_\tau| = \tau > \alpha$. The condition then holds also for $\alpha = \tau$, since τ is inaccessible and $\dot{\mathbb{P}}_\tau$ is defined with countable supports.

For the second condition, suppose there are two conditions $\langle a_1, \dot{p}_1\rangle$ and $\langle a_2, \dot{p}_2\rangle$ in $\mathbb{A} * \dot{\mathbb{P}}_\alpha / A\restriction I_\tau * P_\alpha\restriction I_\tau$ which force incompatible information about $\dot{\mathbb{S}}_\alpha[A * P_\alpha] \cap V[A\restriction I_\tau * P_\alpha\restriction I_\tau]$. Since τ is inaccessible, and using the restrictions to countable and finite supports in the iterations defining $\dot{\mathbb{P}}_\tau$ and $\dot{\mathbb{S}}_\tau$, we may assume that $\alpha < \tau$. Using the support size condition (2) of Definition 3.6 we may assume that $|\mathrm{Sp}(a_1) \cup \mathrm{Sp}(\dot{p}_1)| \leq \alpha$, and similarly for $\langle a_2, \dot{p}_2\rangle$. Using the invariance of $\dot{\mathbb{S}}_\alpha$, and a permutation that fixes all elements of I_τ while shifting $\mathrm{Sp}(a_1) \cup \mathrm{Sp}(\dot{p}_1) - I_\tau$ away from $\mathrm{Sp}(a_2) \cup \mathrm{Sp}(\dot{p}_2) - I_\tau$, we may assume that $\mathrm{Sp}(a_1) \cup \mathrm{Sp}(\dot{p}_1) - I_\tau$ and $\mathrm{Sp}(a_2) \cup \mathrm{Sp}(\dot{p}_2) - I_\tau$ are disjoint. Now by Claim 3.15, for $\kappa = \alpha$, $\dot{\mathbb{P}} = \dot{\mathbb{P}}_\alpha$ and $I = I_\tau$, it follows that $\langle a_1, \dot{p}_1\rangle$ and $\langle a_2, \dot{p}_2\rangle$ are compatible, a contradiction.

The third conditions holds by Claims 4.24 and 4.25, viewing $\dot{\mathbb{S}}_{\alpha+1}$ as $\dot{\mathbb{S}}_\alpha * \dot{\mathbb{D}}_\alpha$.

The fourth condition is proved by induction on $\alpha \leq \tau$. The limit case is trivial since $\dot{\mathbb{S}}$ is defined with finite supports. The successor case follows from the inductive hypothesis, the third condition, and the fact that $\dot{\mathbb{D}}_\alpha|_{I_\tau}$ names a complete subposet of $\dot{\mathbb{D}}_\alpha$. ⊣

Let $\bar{\mathbb{S}}_{\alpha,\tau} = \dot{\bar{\mathbb{S}}}_{\alpha,\tau}[A{\restriction}I_\tau * P_\alpha{\restriction}I_\tau]$, $\bar{S}_{\alpha,\tau} = S \cap \bar{\mathbb{S}}_{\alpha,\tau}$, $\bar{\mathbb{S}}_\tau = \bar{\mathbb{S}}_{\tau,\tau}$, and $\bar{S}_\tau = \bar{S}_{\tau,\tau}$.

Claim 5.8. *Let $s_n = \dot{s}_n[A * P_\tau]$ be conditions in $\mathbb{S}_\tau/\bar{S}_\tau$. Suppose that the sequence $\langle s_n \mid n < \omega\rangle$ belongs to $V[A * P_\tau * \bar{S}_\tau]$, and that the sets $\mathrm{Sp}(\dot{s}_n) - I_\tau$ are pairwise disjoint. Then there are infinitely many n so that $s_n \in S_\tau$.*

Proof. Check by induction on α that no condition in $\mathbb{S}_\alpha/\bar{S}_\alpha$ can force a value for a bound on the set of n so that $s_n{\restriction}\alpha \in S_\alpha$. The limit case is clear through the use of finite supports. The successor case follows from the inductive fact that there are infinitely many n so that $s_n{\restriction}\alpha \in S_\alpha$, using Claim 4.26 (with $W = I_\tau$) and condition (3) of Claim 5.7. Note in using Claim 4.26 that the model $V[A * P_\tau * \bar{S}_\tau * S_\alpha]$, to which the sequence $\langle s_n{\restriction}\alpha+1 \mid n < \omega, s_n{\restriction}\alpha \in S_\alpha\rangle$ belongs, is a generic extension of $V[A * P_{\alpha+1} * \bar{S}_{\alpha+1,\tau} * S_\alpha]$ by the tailends of P_τ and $\bar{S}_\tau$ from coordinate $\alpha+2$ onward, and that these tailends are generic over $V[A * P_{\alpha+1} * S_{\alpha+1}]$. ⊣

Claim 5.9. *Let $\tau \leq \theta$ be inaccessible. Let $\alpha \leq \tau$. Then:*

1. *For every $\dot{s} \in \mathrm{dom}(\dot{\mathbb{S}}_\alpha)$, there is $\dot{s}'$ forced to be equivalent to $\dot{s}$, of forcing hereditary and support sizes $\leq \alpha$.*
2. *If $\dot{s} \in \mathrm{dom}(\dot{\bar{\mathbb{S}}}_{\alpha,\tau})$, then $\dot{s}'$ as above can be found with support contained in I_τ, hence forced into $\dot{\bar{\mathbb{S}}}_{\alpha,\tau}$.*

Proof. Clear by induction on α, using Claim 4.22 and Remark 4.23 for the active successor case, and the obvious parallel results for the non-active stages. The remark is used (at successor stage $\alpha+1$) with $J = I_\tau$ and $\mathbb{U} = \mathbb{A}{\restriction}I_\tau * \dot{\mathbb{P}}_\alpha{\restriction}I_\tau * \dot{\bar{\mathbb{S}}}_{\alpha,\tau}$. ⊣

Let C be the club of $\tau < \theta$ so that $H_\tau \cap \theta = \tau$.

Claim 5.10. *Let $\tau \in C \cup \{\theta\}$ be inaccessible. Then:*

1. *For every $\alpha < \tau$, $\pi_\tau(\mathbb{A} * \dot{\mathbb{P}}_\alpha * \dot{\mathbb{S}}_\alpha)$ consists of the elements of $\mathbb{A}{\restriction}I_\tau * \dot{\mathbb{P}}_\alpha{\restriction}I_\tau * \dot{\bar{\mathbb{S}}}_{\alpha,\tau}$ which belong to M_τ.*
2. *$\pi_\tau(\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}})$ consists of the elements of $\mathbb{A}{\restriction}I_\tau * \dot{\mathbb{P}}_\tau{\restriction}I_\tau * \dot{\bar{\mathbb{S}}}_\tau$ which belong to M_τ.*

Proof. The first part is clear by induction from the absoluteness to transitive models of the notions involved in defining $\mathbb{A}$, $\dot{\mathbb{P}}_\alpha$, and $\dot{\mathbb{S}}_\alpha$, and the facts that $\pi_\tau(\alpha) = \alpha$ for $\alpha < \tau$, $\pi_\tau(\vec{d}) = \vec{d}{\restriction}\tau$, and $\pi_\tau(\delta) = I_\tau$. We only note that for the successor case, the product defining $\pi_\tau(\dot{\mathbb{D}}_\alpha)$ is (by absoluteness arguments) the restriction to I_τ of the product defining $\dot{\mathbb{D}}_\alpha$, and this coincides with the restriction to supports contained in I_τ, by Claims 4.24 and 4.25. A similar argument applies to $\dot{\mathbb{Q}}_\alpha$.

The second part of the claim is immediate from the first, since $\pi(\theta) = \tau$, and τ is inaccessible. ⊣

Let σ_τ be a permutation of δ with the property that $\sigma_\tau \restriction I_\tau$ is exactly the inverse of π_τ. We will use σ_τ to shift conditions and names, as in Claim 3.2.

Given any injection $f\colon Z \to \delta$, one can define a shift as in Claim 3.2, for names and conditions with support contained in Z. We will write $f(a)$ and $f(\dot{u})$ for these shifts. These definitions are absolute for transitive models of enough of ZFC, since they are done by transfinite recursion using operations given by formulas with only bounded quantifiers. For any permutation ρ extending f, and any $\dot{u}$ with support contained in Z, we have that $\rho(\dot{u}) = f(\dot{u})$.

Claim 5.11. *Let $\tau \in C \cup \{\theta\}$ be inaccessible.*

1. *Every $\langle a, \dot{p}, \dot{s}\rangle \in \mathbb{A}\restriction I_\tau * \dot{\mathbb{P}}_\tau \restriction I_\tau * \dot{\dot{\mathbb{S}}}_\tau$ is equivalent some $\langle a, \dot{p}', \dot{s}'\rangle \in \mathbb{A}\restriction I_\tau * \dot{\mathbb{P}}_\tau \restriction I_\tau * \dot{\dot{\mathbb{S}}}_\tau$ which belongs to M_τ.*
2. *The condition $\langle a, \dot{p}', \dot{s}'\rangle$ above has the additional property that $\sigma_\tau(\langle a, \dot{p}', \dot{s}'\rangle) = \pi_\tau^{-1}(\langle\langle a, \dot{p}', \dot{s}'\rangle)$.*

Proof. Since τ is inaccessible, we can find $\alpha < \tau$ so that $\langle a, \dot{p}, \dot{s}\rangle$ is in fact a condition in $\mathbb{A} * \dot{\mathbb{P}}_\alpha * \dot{\mathbb{S}}_\alpha$. Since $\dot{\mathbb{P}}_\alpha$ is $|\alpha|$-appropriate, and using the support size condition (2) of Definition 3.6, there is $J \subseteq I_\tau$ and $\dot{p}'$ so that $\langle a, \dot{p}'\rangle$ is equivalent to $\langle a, \dot{p}\rangle$, $\mathrm{Sp}(a), \mathrm{Sp}(\dot{p}') \subseteq J$, and $|J| \leq |\alpha|$. Using Claim 5.9 and the α^+-chain condition for $\mathbb{A} * \dot{\mathbb{P}}$, and increasing J if needed, we may further find $\dot{s}'$, forced to be equivalent to $\dot{s}$, with $\mathrm{Sp}(\dot{s}') \subseteq J$.

Since $|J| < \tau$, and since M_τ is $< \tau$-closed, there is a bijection $f \in M_\tau$ of J onto an ordinal $\bar{\tau} < \tau$.

Consider the shifted conditions $\bar{a} = f(a)$, $\dot{\bar{p}}' = f(\dot{p}')$ and $\dot{\bar{s}}' = f(\dot{s}')$ in $\mathbb{A}$, $\dot{\mathbb{P}}_\alpha$, and $\dot{\mathbb{S}}_\alpha$. These all have von Neumann rank below τ. This is clear for $\bar{a}$ since it belongs to $\mathrm{Add}(\aleph_1, \bar{\tau})$, holds for $\dot{\bar{p}}'$ by Claims 5.3 and 2.4 since $\dot{\bar{p}}'$ is an $\mathrm{Add}(\aleph_1, \bar{\tau})$-name and $\mathrm{Add}(\aleph_1, \bar{\tau})$ belongs to V_τ, and holds for $\dot{\bar{s}}'$ by Claims 5.2, 5.3, and 2.4, again using the fact that $\mathrm{Add}(\aleph_1, \bar{\tau}) \in V_\tau$. Since $M_\tau \supseteq V_\tau$, it follows that $\bar{a}, \dot{\bar{p}}', \dot{\bar{s}}' \in M_\tau$. The reverse shifts $a = f^{-1}(\bar{a})$, $\dot{p}' = f^{-1}(\dot{\bar{p}}')$, and $\dot{s}' = f^{-1}(\dot{\bar{s}}')$ then belong to M_τ by the absoluteness of the operation of shifting by f^{-1}. This proves the first part of the claim.

Apply the elementary map π_τ^{-1} to the statements that $a = f^{-1}(\bar{a})$, $\dot{p}' = f^{-1}(\dot{\bar{p}}')$, and $\dot{s}' = f^{-1}(\dot{\bar{s}}')$. (These statements are true in M by absoluteness.) The objects $\bar{a}$, $\dot{\bar{p}}'$, and $\dot{\bar{s}}'$ are not moved by π_τ^{-1}, since they have von Neumann rank below τ. So we get that $\pi_\tau^{-1}(a) = \pi_\tau^{-1}(f^{-1})(\bar{a})$, and similarly with $\dot{p}'$ and $\dot{s}'$. It is easy to check that $\pi_\tau^{-1}(f^{-1}) = \pi_\tau^{-1} \circ f^{-1} = \sigma_\tau \circ f^{-1}$, since $\mathrm{dom}(f^{-1}) = \bar{\tau} < \tau$ and $\mathrm{range}(f^{-1}) \subseteq I_\tau$. So $\pi_\tau^{-1}(a) = (\sigma_\tau \circ f^{-1})(\bar{a}) = \sigma_\tau(f^{-1}(\bar{a})) = \sigma_\tau(a)$ and similarly with $\dot{p}'$ and $\dot{s}'$. $\dashv$

Lemma 5.12. *$V[A * P * S]$ satisfies $\mathsf{OCA}_{\mathrm{Tod}}(< \mathfrak{c})$.*

Proof. Suppose not. Then the failure of $\mathsf{OCA}_{\mathrm{Tod}}(< \mathfrak{c})$ must be forced by the empty condition. This can be seen using the invariance of $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$ under permutations to separate the supports of conditions forcing contradictory truth values for $\mathsf{OCA}_{\mathrm{Tod}}(< \mathfrak{c})$, and using the fact that conditions with disjoint supports in invariant products must have disjoint domains and are therefore automatically compatible. By a similar argument there is a cardinal ρ which is outright forced to be the minimum size of a counterexample to $\mathsf{OCA}_{\mathrm{Tod}}(<\mathfrak{c})$. Since the continuum is forced to be δ, we have $\rho < \delta$.

Fix names $\dot{X}$ and $\dot{U}$, for a space and an open coloring, which are outright forced to provide a counterexample to $\mathsf{OCA}_{\mathrm{Tod}}(<\mathfrak{c})$, of size ρ.

Claim 5.13. *There is a name $\dot{X}'$ with support of size $<\delta$ which is forced to equal $\dot{X}$, and similarly with $\dot{U}$.*

Proof. This is clear for any name for a real, using the fact that every condition in $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$ is equivalent to a condition with support of size $<\theta$, and the chain conditions for $\mathbb{A}$, $\dot{\mathbb{P}}$, and $\dot{\mathbb{S}}$. The claim for $\dot{X}$ then holds because $\dot{X}$ is forced to be a set of reals of size $\rho < \delta$, and the claim for $\dot{U}$ holds because open sets can be coded by reals. $\dashv$

In light of Claim 5.13 we may assume that $\dot{X}$ and $\dot{U}$ themselves have supports of size $<\delta$.

By elementarity we can assume that the names $\dot{X}$ and $\dot{U}$ belong to H_0, and hence to H_α for all $\alpha < \theta$. Let $\dot{\bar{X}}_\alpha = \pi_\alpha(\dot{X})$ and $\dot{\bar{U}}_\alpha = \pi_\alpha(\dot{U})$. Let g_θ biject θ onto I_θ, and let $g_\alpha = \pi^{-1}_{\alpha,\theta} \circ (g_\theta \restriction \alpha)$, which for a club of α is a bijection of α onto I_α. Using Claim 5.6 on $Q = \langle I_\theta, g_\theta, \pi_\theta(\dot{\mathbb{S}}), \dot{\bar{X}}_\theta, \dot{\bar{U}}_\theta \rangle$, and since θ is Mahlo, there is an inaccessible $\tau < \theta$ so that g_τ bijects τ onto I_τ and $d_\tau = \langle I_\tau, g_\tau, \pi_\tau(\dot{\mathbb{S}}), \dot{\bar{X}}_\tau, \dot{\bar{U}}_\tau \rangle$.

Our goal is to show that stage τ of our poset construction is active, and that (with a possible revision to S) the generic for $(\dot{\mathbb{Q}}_\tau * \dot{\mathbb{D}}_\tau)[A * P_\tau * S_\tau]$ adds an uncountable clique through $\dot{U}[A * P * S]$. This will contradict the fact that $\dot{X}, \dot{U}$ is forced to provide a counterexample to $\mathsf{OCA}_{\mathrm{Tod}}(<\mathfrak{c})$.

It is easy to check that I_τ, g_τ, $\pi_\tau(\dot{\mathbb{S}})$, $\dot{\bar{X}}_\tau$, and $\dot{\bar{U}}_\tau$ satisfy assumptions (1)–(7) of Section 4 for $\dot{\mathbb{P}}_\tau$ and $\dot{\mathbb{S}}_\tau$. We just note the following: $\pi_\tau(\dot{\mathbb{S}})$ is forced to be dense in $\dot{\bar{\mathbb{S}}}_\tau$, and indeed the two posets are forced to be the same up to equivalence of conditions, by Claims 5.10 and 5.11. We may therefore work with $\dot{\bar{\mathbb{S}}}_\tau$ instead of $\pi_\tau(\dot{\mathbb{S}})$ throughout. $\mathbb{A}\restriction I_\tau * \dot{\mathbb{P}}_\tau \restriction I_\tau$ is a complete subposet of $\mathbb{A} * \dot{\mathbb{P}}_\tau$ by Claim 5.7. By the same claim, $\dot{\bar{\mathbb{S}}}_\tau[A\restriction I_\tau * P_\tau \restriction I_\tau]$ is a complete subposet of $\dot{\mathbb{S}}_\tau[A * P_\tau]$. Finally the size requirements on conditions in assumptions (4) and (6) hold by Claim 5.9.

To show that stage τ is active, it remains to prove assumption (8). To this end, fix $\mathbb{A} * \dot{\mathbb{P}}_\tau * \dot{\mathbb{S}}_\tau$-names $\dot{t}_i$, for $i < \omega$. Suppose it is forced in $\mathbb{S}_\tau$ that $\dot{t}_i[A * P_\tau]$ is a closed set of reals, and is 1-colorable in $\dot{\bar{U}}_\tau[A\restriction I_\tau * P_\tau \restriction I_\tau]$. We need to show that there is an $\bar{\mathbb{S}}_\tau$-name $\dot{x} \in V[A\restriction I_\tau * P_\tau \restriction I_\tau]$ which is forced in $\mathbb{S}_\tau$ to belong to $\dot{\bar{\mathbb{X}}}[A\restriction I_\tau * P_\tau \restriction I_\tau] - \bigcup_{i<\omega} \dot{t}_i[A * P_\tau]$.

Suppose this is not the case. We will obtain a contradiction by internalizing shifts of the names $\dot{t}_i$ to the model M_τ, and then using the fact that in M_τ, $\dot{\bar{U}}_i$ is forced to not be countably chromatic.

Claim 5.14. *Modifying the generic S if necessary, we can arrange that every real $x \in \dot{\bar{X}}[A\restriction I_\tau * P_\tau \restriction I_\tau * \bar{S}_\tau]$ can be forced by some condition in $\mathbb{S}_\tau/\bar{S}_\tau$ to belong to some $\dot{t}_i[A * P_\tau]$. (There is no harm at this stage in modifying S as needed, since we have not yet made any assumptions involving the specific generic S used.)*

PROOF. Suppose not. Then for densely many conditions s in $\bar{\mathbb{S}}_\tau$, there is an $\bar{\mathbb{S}}_\tau$-name $\dot{x}_s$ in $V[A\restriction I_\tau * P_\tau\restriction I_\tau]$ so that s forces $\dot{x}_s \in \dot{\dot{X}}_\tau[A\restriction I_\tau * P_\tau\restriction I_\tau]$, and s forces in $\bar{\mathbb{S}}_\tau$ that it is forced in $\mathbb{S}/\dot{\bar{S}}_\tau$ that $\dot{x}$ is outside all $\dot{t}_i[A\restriction I_\tau * P_\tau\restriction I_\tau]$. Merging these names for a maximal antichain of such s, we can obtain an $\bar{\mathbb{S}}_\tau$-name $\dot{x} \in V[A\restriction I_\tau * P_\tau\restriction I_\tau]$ which is outright forced in $\bar{\mathbb{S}}_\tau$ to have the above properties. This implies that $\dot{x}$ is forced in $\mathbb{S}_\tau$ to belong to $\dot{\dot{X}}_\tau[A\restriction I_\tau * P_\tau\restriction I_\tau] - \bigcup_{i<\omega}\dot{t}_i[A*P_\tau]$, contradicting our assumptions on $\dot{t}_i$. ⊣

In witnessing Claim 5.14 we are making an assumption on $\bar{S}_\tau$, and so we will not modify it further.

Abusing notation, we identify the names $\dot{t}_i$ for closed sets with names for reals coding these closed sets. We may assume without loss of generality that the names $\dot{t}_i$ (for reals) are canonical, meaning of the form $\{\langle \check{n}, r\rangle \mid r \in R_n\}$ for antichains R_n. Since τ is inaccessible, and using the chain conditions and support restrictions for the iterations forming $\dot{\mathbb{P}}_\tau$ and $\dot{\mathbb{S}}_\tau$, there is a cardinal $\alpha < \tau$ so that $\dot{t}_i$ are in fact $\mathbb{A} * \dot{\mathbb{P}}_\alpha * \dot{\mathbb{S}}_\alpha$-names. Each $\dot{t}_i$ then has size at most α, using the fact that $\dot{\mathbb{P}}_\alpha$ and $\dot{\mathbb{S}}_\alpha$ are α^+-c.c. and c.c.c. respectively. Without loss of generality, using Claim 5.9 and the fact that $\dot{\mathbb{P}}_\alpha$ is α-appropriate, we may assume that each condition appearing in $\dot{t}_i$ has support of size at most α. Then $\mathrm{Sp}(\dot{t}_i)$ itself has size at most α.

Fix $E \subseteq \delta$ of size α so that $\bigcup_{i<\omega}\mathrm{Sp}(t_i) \subseteq E$.

By Claim 5.14, for each $x \in \dot{\dot{X}}_\tau[A\restriction I_\tau * P_\tau\restriction I_\tau * \bar{S}_\tau]$, there is $i < \omega$ and a condition $s = \dot{s}[A*P_\tau] \in \mathbb{S}_\tau/\bar{S}_\tau$ forcing that $x \in \dot{t}_i[A*P_\tau]$. We may take $\dot{s}$ to have support of size $< \tau$. Since $\dot{\dot{X}}_\tau[A*P_\tau]$ itself has size at most τ (this is because $\dot{\dot{X}}_\tau \in M_\tau$), we can find $E^* \supseteq E$, of size τ, so that for each $x \in \dot{\dot{X}}_\tau[A\restriction I_\tau * P_\tau\restriction I_\tau * \bar{S}_\tau]$, $\dot{s}$ as above can be found with $\mathrm{Sp}(\dot{s}) \subseteq E^*$. Fix a condition $\langle \bar{a}, \dot{\bar{p}}, \dot{\bar{s}}\rangle \in A * P_\tau * \bar{S}_\tau$ which forces this statement. We may assume that $|\mathrm{Sp}(a) \cup \mathrm{Sp}(\dot{\bar{p}})| \leq \tau$, and we have $\mathrm{Sp}(\dot{\bar{s}}) \subseteq I_\tau$.

Fix permutations σ_n of δ, for $n < \omega$, which are the identity on I_τ, so that the sets $\sigma_n''(E^*) - I_\tau$ are pairwise disjoint, and so that the shifted conditions $\sigma_n(\langle \bar{a}, \dot{\bar{p}}\rangle)$ all belong to $A * P_\tau$. Permutations $\sigma_{n,1}$ fixing I_τ and securing the latter part, with $\sigma_{n,1}''(\mathrm{Sp}(a) \cup \mathrm{Sp}(\dot{\bar{p}}))$ pairwise disjoint, can be found using Claim 3.14, and then the former part can be secured by composing with additional permutations $\sigma_{n,2}$, using the fact that $|E^*| < \delta$.

CLAIM 5.15. *Every real in* $\dot{\dot{X}}[A\restriction I_\tau * P_\tau\restriction I_\tau * \bar{S}_\tau]$ *belongs to* $\bigcup_{i,n<\omega}\sigma_n(\dot{t}_i)[A * P_\tau * S_\tau]$.

PROOF. By Claim 5.14 and the subsequent choice of E^* and $\langle \bar{a}, \dot{\bar{p}}, \dot{\bar{s}}\rangle$, shifting via σ_n, and using the facts that $\sigma_n(\bar{a}, \dot{\bar{p}}\rangle) \in A * P_\tau$ and that $\sigma_n\restriction I_\tau$ is the identity, we get for each $n < \omega$ some i_n and a condition $s_n = \dot{s}_n[A * P_\tau] \in \mathbb{S}_\tau/\bar{S}_\tau$ forcing that $x \in \sigma_n(\dot{t}_{i_n})[A * P_\tau]$, with $\mathrm{Sp}(\dot{s}_n) \subseteq \sigma_n''E^*$. (Since this statement references $\bar{S}_\tau$, the sequence $\langle s_n \mid n < \omega\rangle$ belongs to the further extension $V[A * P_\tau * \bar{S}_\tau]$.)

Then $\mathrm{Sp}(\dot{s}_n) - I_\tau$ are pairwise disjoint, so that by Claim 5.8, infinitely many of the conditions s_n belong to S_τ. It follows in particular that $x \in \sigma_n(\dot{t}_{i_n})[A*P_\tau*S_\tau]$ for some (indeed infinitely many) n. ⊣

We have now shown that, in $V[A * P_\tau * S_\tau]$, the space $\dot{\bar{X}}[A\restriction I_\tau * P_\tau\restriction I_\tau * \bar{S}_\tau]$ is covered by countably many sets which are 1-colorable sets in $\dot{\bar{U}}[A\restriction I_\tau * P_\tau\restriction I_\tau * \bar{S}_\tau]$, namely the sets $\sigma_n(\dot{t}_i)[A * P_\tau * S_\tau]$. To do this we *homogenized* the original collection of sets $\dot{t}_i$, by passing to the larger collection $\sigma_n(\dot{t}_i)$. Something similar to this was done by Farah [5], though in a very different forcing notion.

We will finish the proof that stage τ is active by *internalizing*, that is shifting the sets $\sigma_n(\dot{t}_i)$ into (a generic extension of) M_τ, contradicting the fact that this model thinks that $\dot{\bar{X}}$ and $\dot{\bar{U}}$ name a counterexample to $\mathsf{OCA}_{\mathrm{Tod}}(< \mathfrak{c})$. This crucial step is our most essential use of the homogeneity of $\mathbb{A} * \dot{\mathbb{P}}$ under permutations of δ.

Recall that we have $E \subseteq \delta$, of size $\alpha < \tau$, so that $\mathrm{Sp}(\dot{t}_i) \subseteq E$ for all i. Let $\hat{E} = \bigcup_{n<\omega} \sigma_n'' E$, so that $\bigcup_{i,n<\omega} \mathrm{Sp}(\sigma_n(\dot{t}_i)) \subseteq \hat{E}$, and $|\hat{E}| = \alpha < \tau$.

Using Claim 5.13 we assumed that $\dot{X}$ and $\dot{U}$ have supports of size $< \delta$. Hence certainly $|\delta - (\mathrm{Sp}(\dot{X}) \cup \mathrm{Sp}(\dot{U}))| \geq \theta$ (which is all we need below). It follows by the elementarity of π_τ that $|I_\tau - (\mathrm{Sp}(\dot{\bar{X}}) \cup \mathrm{Sp}(\dot{\bar{U}}))| \geq \tau$.

We can therefore find a permutation $\hat{\sigma}$ of δ so that $\hat{\sigma}''\hat{E} \subseteq I_\tau$ and $\hat{\sigma}$ is the identity on $\mathrm{Sp}(\dot{\bar{X}}) \cup \mathrm{Sp}(\dot{\bar{U}})$. Let $A' * P'_\tau * S'_\tau = \hat{\sigma}(A * P_\tau * S_\tau)$. Shifting the conclusion of Claim 5.15 by $\hat{\sigma}$, and using the fact that $\hat{\sigma}$ is the identity on $\mathrm{Sp}(\dot{\bar{X}}) \cup \mathrm{Sp}(\dot{\bar{U}})$, we have that each of the sets $(\hat{\sigma}\circ\sigma_n)(\dot{t}_i)[A' * P'_\tau * S'_\tau]$ is 1-colorable in $\dot{\bar{U}}[A' * P'_\tau * S'_\tau]$, and $\bigcup_{i,n<\omega} (\hat{\sigma}\circ\sigma_n)(\dot{t}_i)[A' * P'_\tau * S'_\tau]$ covers $\dot{\bar{X}}[A' * P'_\tau * S'_\tau]$.

Let $\bar{A}' * \bar{P}'_\tau * \bar{S}'_\tau = (A * P'_\tau * S'_\tau) \cap M_\tau$. This makes sense using Claims 5.7, 5.9, 5.10, and 5.11, which also show that $\bar{A}' * \bar{P}'_\tau * \bar{S}'_\tau$ is generic over V for $\pi_\tau(\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}})$, and hence certainly generic over M_τ. Moreover these claims, and the fact that $\mathrm{Sp}((\hat{\sigma}\circ\sigma_n)(\dot{t}_i)) \subseteq I_\tau$, allow us to assume that $(\hat{\sigma}\circ\sigma_n)(\dot{t}_i)$ is a $\pi_\tau(\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}})$-name. Recall that we took $\dot{t}_i$ to name reals coding closed sets, rather than the closed sets themselves, and arranged for $\dot{t}_i$ to be canonical names, of size at most some $\alpha < \tau$. Since M_τ is closed under α-sequences, we can conclude that the names $(\hat{\sigma}\circ\sigma_n)(\dot{t}_i)$ belong to M_τ, and so does the sequence of these names. Certainly the names $\dot{\bar{X}}$ and $\dot{\bar{U}}$ belong to M_τ. Thus the conclusion of the previous paragraph implies that $M_\tau[\bar{A}' * \bar{P}'_\tau * \bar{S}'_\tau]$ satisfies that the sets $(\hat{\sigma}\circ\sigma_n)(\dot{t}_i)[\bar{A}' * \bar{P}'_\tau * \bar{S}'_\tau]$ partition $\dot{\bar{X}}[\bar{A}' * \bar{P}'_\tau * \bar{S}'_\tau]$ into countably many 1-colorable sets in $\dot{\bar{U}}[\bar{A}' * \bar{P}'_\tau * \bar{S}'_\tau]$.

But this is a contradiction, since $\dot{\bar{X}}$ and $\dot{\bar{U}}$ are forced in $\pi_\tau(\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}})$ over M_τ to name a counterexample to $\mathsf{OCA}_{\mathrm{Tod}}(< \mathfrak{c})$. This contradiction completes the proof that stage τ is active.

Since stage τ is active, and by Lemma 4.18, for every permutation σ of δ, the graph $\sigma(\dot{\bar{U}}_\tau)[A * P_\tau * S_\tau]$ on the space $\sigma(\dot{\bar{X}})[A * P_\tau * S_\tau]$ has an uncountable clique in $V[A * P_{\tau+1} * S_{\tau+1}]$. The following claim will therefore suffices to show that $\dot{U}[A * P * S]$ has an uncountable clique, and complete the proof of Lemma 5.12.

Claim 5.16. *There is a permutation σ of δ so that $\sigma(\dot{\bar{X}})[A * P_\tau * S_\tau] \subseteq \dot{X}[A * P * S]$ and so that $\sigma(\dot{\bar{U}}_\tau)[A * P_\tau * S_\tau]$ is the restriction of $\dot{U}[A * P * S]$ to $\sigma(\dot{\bar{X}})[A * P_\tau * S_\tau]$.*

Proof. Recall that σ_τ is a permutation of δ with the property that $\sigma_\tau \restriction I_\tau = \pi_\tau^{-1}$. Take $\sigma = \sigma_\tau$.

Since $\sigma(\dot{\bar{X}}_\tau)[A * P_\tau * S_\tau]$ and $\dot{\bar{X}}_\tau[\sigma^{-1}(A * P_\tau * S_\tau)]$ are equal, we may work with the latter instead of the former. Similarly with $\dot{\bar{U}}$.

By Claims 5.7, 5.9, 5.10, and 5.11, applied with $\sigma^{-1}(A*P_\tau*S_\tau)$ as a generic for $\mathbb{A}*\dot{\mathbb{P}}_\tau*\dot{\mathbb{S}}_\tau$, we have that $\sigma^{-1}(A*P_\tau*S_\tau)\cap\pi_\tau(\mathbb{A}*\dot{\mathbb{P}}*\dot{\mathbb{S}})$ is generic over M_τ (indeed over V) for $\pi_\tau(\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}})$. Moreover by Claim 5.11, for $\langle a, \dot{p}, \dot{s}\rangle \in H_\tau$ we have that $\pi_\tau(\langle a, \dot{p}, \dot{s}\rangle)$ is equivalent to $\sigma^{-1}(\langle a, \dot{p}, \dot{s}\rangle)$, and hence $\langle a, \dot{p}, \dot{s}\rangle \in A * P_\tau * S_\tau$ iff $\pi_\tau(\langle a, \dot{p}, \dot{s}\rangle) \in \sigma^{-1}(A * P_\tau * S_\tau)$. It follows that π_τ extends to an isomorphism π^*_τ from $H_\tau[A * P * S]$ to $M_\tau[\sigma^{-1}(A * P_\tau * S_\tau) \cap \pi_\tau(\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}})]$, by setting $\pi^*_\tau(\dot{x}[A * P * S]) = \pi_\tau(\dot{x})[\sigma^{-1}(A * P_\tau * S_\tau) \cap \pi_\tau(\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}})]$. We then have that $\pi^*_\tau(\dot{X}[A*P*S]) = \dot{\bar{X}}_\tau[\sigma^{-1}(A*P_\tau*S_\tau)\cap\pi_\tau(\mathbb{A}*\dot{\mathbb{P}}*\dot{\mathbb{S}})] = \dot{\bar{X}}_\tau[\sigma^{-1}(A*P_\tau*S_\tau)]$, and similarly $\pi^*_\tau(\dot{U}[A * P * S]) = \dot{\bar{U}}_\tau[\sigma^{-1}(A * P_\tau * S_\tau)]$.

So it remains to prove that $\pi^*_\tau(\dot{X}[A * P_\tau * S_\tau]) \subseteq \dot{X}[A * P * S]$, and that $\pi^*_\tau(\dot{U}[A * P * S])$ is the restriction of $\dot{U}[A * P * S]$ to $\pi^*_\tau(\dot{X}[A * P_\tau * S_\tau])$. But this is clear from elementarity and the fact that π^*_τ is the identify on reals in its domain. Indeed, $\pi^*_\tau(\dot{X}[A * P_\tau * S_\tau])$ is simply $\dot{X}[A * P * S] \cap H_\tau[A * P * S]$, and $\pi^*_\tau(\dot{U}[A * P * S])$ is the restriction of $\dot{U}[A * P * S]$ to this set. ⊣

In light of Claim 5.16 and Lemma 4.18, The generic D_τ adds an uncountable clique in the graph $\dot{U}[A * P * S]$ on $\dot{X}[A * P * S]$, contradicting the fact that $\dot{X}$ and $\dot{U}$ are forced in $\mathbb{A} * \dot{\mathbb{P}} * \dot{\mathbb{S}}$ to give a counterexample to $\mathsf{OCA}_{\mathrm{Tod}}(< \mathfrak{c})$. This contradiction completes the proof of Lemma 5.12, and with it the proof of Theorem 1.1. ⊣

Recall that our definition of the iterands $\dot{\mathbb{Q}}_\alpha$ and $\dot{\mathbb{D}}_\alpha$ divides into three cases: First when α is not inaccessible, second when α is inaccessible but d_α does not code an appropriate name for an open graph, and third, the active case, when we add cliques to all shifts of the coded graph. In the first two cases we took $\dot{\mathbb{D}}_\alpha$ to only add Cohen reals.

We can prove Theorem 1.2 by adding an additional case: when α is inaccessible, and d_α codes (a name for) a Knaster poset of size α. In this case we can take $\dot{\mathbb{D}}_\alpha$ to be the symmetric product of all shifts of the Knaster poset coded by d_α. Note that $\dot{\mathbb{D}}_\alpha$ is c.c.c., since the product of Knaster posets retains the countable chain condition. (This could fail if d_α were to code a poset which is only c.c.c., and it is because of this that we restrict to Knaster posets in Theorem 1.2.) It is easy to check that our analysis of the properties of $\mathbb{S}$ continues to apply with this change. And it is not too hard to check that this leads to a proof of Theorem 1.2. We leave the details to the reader.

In both the proof of Theorem 1.1 and the proof of Theorem 1.2, only the poset $\mathbb{S}$ adds new reals. This poset incorporates the addition of $\aleph_2$ Cohen reals, and it is a finite support iteration of finite support products of posets of size $\aleph_1$. This implies that any collection of $\aleph_2$ mutually generic Cohen reals added (for example) by $\dot{\mathbb{D}}_0$ remain unbounded in the full extension by $\dot{\mathbb{S}}$. Hence, in the full generic extension, $\mathfrak{b} \leq \aleph_2$. Since $\mathsf{OCA}_{\mathrm{Tod}}(\aleph_1)$ implies that $\mathfrak{b} > \aleph_1$, our end models satisfy that $\mathfrak{b} = \aleph_2$.

REFERENCES

[1] Uri Abraham, Matatyahu Rubin, and Saharon Shelah, *On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types*, ***Ann. Pure Appl. Logic***, vol. 29 (1985), no. 2, pp. 123–206.

[2] James E. Baumgartner, *All $\aleph_1$-dense sets of reals can be isomorphic*, ***Fund. Math.***, vol. 79 (1973), no. 2, pp. 101–106.

[3] ———, *Chains and antichains in $\mathcal{P}(\omega)$*, ***J. Symbolic Logic***, vol. 45 (1980), no. 1, pp. 85–92.

[4] Ben de Bondt, Ilijas Farah, and Alessandro Vignati, *Trivial automorphisms of reduced products*, To appear.

[5] Ilijas Farah, $\mathsf{OCA}_{\aleph_1} + \mathsf{MA}_{\aleph_1}$ *is consistent with continuum being large*, (1995), Unpublished notes.

[6] ———, *All automorphisms of the Calkin algebra are inner*, ***Ann. of Math. (2)***, vol. 173 (2011), no. 2, pp. 619–661.

[7] Thomas Gilton and Itay Neeman, *Abraham-Rubin-Shelah open colorings and a large continuum*, ***J. Math. Log.***, vol. 22 (2022), no. 1, pp. Paper No. 2150027, 55.

[8] Justin Tatch Moore, *Open colorings, the continuum and the second uncountable cardinal*, ***Proc. Amer. Math. Soc.***, vol. 130 (2002), no. 9, pp. 2753–2759.

[9] ———, *Weak diamond and open colorings*, ***Journal of Mathematical Logic***, vol. 03 (2003), no. 01, pp. 119–125.

[10] ———, *Some remarks on the Open Coloring Axiom*, ***Ann. Pure Appl. Logic***, vol. 172 (2021), no. 5, pp. Paper No. 102912, 6.

[11] Stevo Todorčević, ***Partition problems in topology***, Contemporary Mathematics, vol. 84, American Mathematical Society, Providence, RI, 1989.

[12] Boban Veličković, OCA *and automorphisms of $\mathcal{P}(\omega)/\mathrm{fin}$*, ***Topology Appl.***, vol. 49 (1993), no. 1, pp. 1–13.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT LOS ANGELES, LOS ANGELES, CA 90095-1555, USA

E-mail: ineeman@math.ucla.edu