ON THE EQUIVALENCE RELATIONS OF HJORTH-KECHRIS-LOUVEAU

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Abstract. Hjorth-Kechris-Louveau [1], as part of their investigations of equivalence relations induced by actions of S_∞, defined equivalence relations $\cong^*_{\gamma,\beta}$ and $\cong^*_{\lambda+1,<\lambda}$ for successor ordinals γ and β with $\beta+2 \leq \gamma$, and limit ordinals λ. They conjectured that for each fixed first coordinate, these equivalence relations are strictly increasing in Borel reducibility with respect to the second coordinate. (The non-strict part of the conjecture is clear.) They proved the case of $\cong^*_{\alpha+3,\beta} <_B \cong^*_{\alpha+3,\alpha+1}$ for $\beta \leq \alpha$, using a forcing and absoluteness argument. The remaining cases (and some additional cases) were later proved by Shani [4] using methods which involve extensions of some very elaborate symmetric choiceless models of Monro [3]. We provide a new proof of this result, building on and expanding the original approach of forcing and absoluteness used by Hjorth-Kechris-Louveau, together with some key insights from Shani's work. The new proof avoids the need for the models of Monro, and instead only uses very basic symmetric choiceless models.

1. Introduction

This paper concerns a sequence of equivalence relations introduced by Hjorth-Kechris-Louveau [1]. We begin this section by defining the relations. We will then state the main theorem and some of its history. The main theorem has been known before: one case through the results of [1], and the rest through the work of Shani [4]. Our main contribution is a new proof of the full theorem which builds on the structure used in [1] for the one case they proved. Our proof incorporates some key insights from [4], but avoids the elaborate symmetric models needed in [4], which trace back to work of Monro [3], using only very basic symmetric choiceless models instead. Along the way we identify the basic missing ingredient that prevented Hjorth-Kechris-Louveau from using their methods on the remaining cases.

Let $\mathcal{P}$ denote the powerset operation. For a cardinal τ, recall that $\mathcal{P}_\tau(X) = \{U \subseteq X \mid \mathrm{card}(U) = \tau\}$. Following set theoretic conventions, let $\mathcal{P}^\alpha(X)$, the αth iterated power of X, be defined by transfinite recursion on α as follows:

- $\mathcal{P}^0(X) = X$.
- $\mathcal{P}^{\alpha+1}(X) = \mathcal{P}(\mathcal{P}^\alpha(X))$.
- $\mathcal{P}^\lambda(X) = \bigcup_{\alpha<\lambda} \mathcal{P}^\alpha(X)$ for limit ordinals λ.

We refer to the third condition as *continuity at limits*. Define $\mathcal{P}^\alpha_\tau(X)$ similarly, using $\mathcal{P}_\tau$ instead of $\mathcal{P}$.

We will work below with $\mathcal{P}^\alpha(\omega)$ and $\mathcal{P}^\alpha_{\aleph_0}(\omega)$. Notice that these are transitive sets, and moreover each element of $\mathcal{P}^\alpha_{\aleph_0}(\omega)$ is hereditarily countable. This is easily proved

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by induction on α. It follows by induction on γ that $\beta \leq \gamma \to \mathcal{P}^\beta(\omega) \subseteq \mathcal{P}^\gamma(\omega)$ and similarly with $\mathcal{P}_{\aleph_0}$.

Remark 1.1. It is easy to see (in ZF) that for every α there is an injection of $\mathcal{P}^\alpha(\omega) \times \mathcal{P}^\alpha(\omega)$ into $\mathcal{P}^\alpha(\omega)$, definable from α by a formula which is absolute between transitive models with the same $\mathcal{P}^\alpha(\omega)$.

We use $\mathrm{tc}(A)$ to denote the transitive closure of A.

Definition 1.2. Define the set $\mathrm{Inv}(\gamma, \beta)$ to consists of pairs $\langle A, R\rangle$ where $A \in \mathcal{P}^\gamma(\omega)$ and R is a ternary relation on $A \times A \times (\mathrm{tc}(A) \cap \mathcal{P}^\beta(\omega))$ with the property that for each $a \in A$, the relation $R_a^{-1} = \{\langle u, b\rangle \mid \langle a, b, u\rangle \in R\}$ is the graph of a surjection of a subset of $\mathrm{tc}(A) \cap \mathcal{P}^\beta(\omega)$ onto A.

Define $\mathrm{Inv}_{\aleph_0}(\gamma, \beta)$ similarly, using $\mathcal{P}_{\aleph_0}$ instead of $\mathcal{P}$.

If γ is a successor ordinal, one can think of $\langle A, R\rangle \in \mathrm{Inv}(\gamma, \beta)$, and similarly with $\mathrm{Inv}_{\aleph_0}(\gamma, \beta)$, as consisting of a subset A of $\mathcal{P}^{\gamma-1}(\omega)$ of size at most $\mathcal{P}^\beta(\omega)$, together with an A-indexed family of surjections witnessing this size restriction, namely the surjections R_a^{-1} for $a \in A$. The size restriction is vacuous if $\beta \geq \gamma - 1$, so we will concentrate on cases where $\beta + 2 \leq \gamma$.

Notice that the elements of $\mathrm{Inv}_{\aleph_0}(\gamma, \beta)$ are hereditarily countable, and can therefore be coded by reals.

Definition 1.3. For $\beta, \gamma < \omega_1$, let $C_{\gamma,\beta}$ be the set of reals coding elements of $\mathrm{Inv}_{\aleph_0}(\gamma, \beta)$, and define an equivalence relation $\cong^{**}_{\gamma,\beta}$ on $C_{\gamma,\beta}$ setting $x \cong^{**}_{\gamma,\beta} y$ iff x and y code the same element of $\mathrm{Inv}_{\aleph_0}(\gamma, \beta)$.

The sets $\mathrm{Inv}_{\aleph_0}(\gamma, \beta)$ provide invariants for the equivalence relations $\cong^{**}_{\gamma,\beta}$, and this is the reason for the choice of the term Inv to denote these sets.

We neglected to specify the precise coding of elements of $\mathrm{Inv}_{\aleph_0}(\gamma, \beta)$ for the above definition. Hjorth-Kechris-Louveau [1] view $\langle A, R\rangle$ as a countable model in a certain language, coded by reals using enumerations of its universe. For example one can take the model to have universe $\mathrm{tc}(A)$, with relations specifying A, R, and $\in$. A real code for this model is a real which codes an isomorphic model with universe ω. The equivalence relation $\cong^{**}_{\gamma,\beta}$ is then induced by an action of the group S_∞, permuting the universe ω of the coding model. The relation is Borel because an isomorphism of a model $(\omega; U)$ with $(\mathrm{tc}(A); \in)$ is uniquely determined from U. In fact the complexity level of the relation in the Borel hierarchy is closely related to γ, see [1, Corollary 6.4].

For the purpose of the reducibility results in this paper, the specific coding is not so important. Any reasonable coding would do, provided it is uniformly definable from codes for γ and β in a sufficiently absolute way, and results in a Borel equivalence relation.

Remark 1.4. The definitions above are taken from Hjorth-Kechris-Louveau [1] with a couple of changes related to the iterated powerset.

The first change involves the successor case in the definition of the iterated powerset, but it makes no difference to the outcome. [1] defines $\mathcal{P}^{\alpha+1}(\mathbb{N})$ to be $\mathcal{P}(\mathcal{P}^\alpha(\mathbb{N}) \cup \mathbb{N})$ rather than $\mathcal{P}(\mathcal{P}^\alpha(\mathbb{N}))$. Following the set theoretic convention that $\mathbb{N}$ is equal to ω we automatically get that $\mathcal{P}^\alpha(\mathbb{N}) \supseteq \mathbb{N}$, so that $\mathcal{P}(\mathcal{P}^\alpha(\mathbb{N}) \cup \mathbb{N})$ and $\mathcal{P}(\mathcal{P}^\alpha(\mathbb{N}))$ are the same.

The more effectful change involves the definition at limit. [1] defines the powerset operation in a discontinuous manner at limit ordinals λ, setting $\mathcal{P}^\lambda_{\aleph_0}(\mathbb{N}) =$

$\mathcal{P}_{\aleph_0}(\bigcup_{\alpha<\lambda} \mathcal{P}^{\alpha}_{\aleph_0}(\mathbb{N}))$. This definition provides a good indexing match for the results in [1] associating the equivalence relations to Borel complexities of actions of S_∞, because the Borel hierarchy is discontinuous at limits. In contrast, our definitions above follow set theoretic conventions and take the powerset operation to be continuous at limits. This provides a good indexing match with cardinals in the irreducibility proofs, and allows for uniformity in the statement of the main theorem, 1.6 below.

Because of the second change mentioned in Remark 1.4, the connection between the equivalence relations we defined above and the ones of Hjorth-Kechris-Louveau [1] is the following:

Definition 1.5.
- For successor ordinals γ and β, with $\beta+2 \leq \gamma$, the equivalence relation $\cong^{*}_{\gamma,\beta}$ of Hjorth-Kechris-Louveau [1] is equal to the equivalence relation $\cong^{**}_{\gamma',\beta'}$ defined above where $\gamma' = \gamma$ if $\gamma < \omega$ and $\gamma' = \gamma - 1$ if $\gamma > \omega$, and similarly with β'. This can be written uniformly as $\gamma' = 1 + \gamma - 1$, and similarly with β'.
- For limit ordinals λ, the equivalence relation $\cong^{*}_{\lambda+1,<\lambda}$ of Hjorth-Kechris-Louveau [1] is equal to the equivalence relation $\cong^{**}_{\lambda+2,\lambda}$ defined above.

It is clear that if $\bar{\beta} \leq \beta$ then $\cong^{**}_{\gamma,\bar{\beta}} \leq_B \cong^{**}_{\gamma,\beta}$, meaning that $\cong^{**}_{\gamma,\bar{\beta}}$ is Borel reducible to $\cong^{**}_{\gamma,\beta}$. In the next two sections we will prove that the resulting hierarchy of reducibility is strict for countable double successor ordinals γ. In other words we will prove:

Theorem 1.6. *Let γ be a countable double successor ordinal, and let $\bar{\beta} < \beta$ with $\beta \leq \gamma - 2$. Then $\cong^{**}_{\gamma,\beta}$ is not Borel reducible to $\cong^{**}_{\gamma,\bar{\beta}}$.*

Notice that the restriction on γ to be a double successor is necessary. It is easy to see that $\cong^{**}_{\gamma,\beta} \leq_B \cong^{**}_{\gamma,0}$ when γ is a countable limit ordinal or the successor of a countable limit ordinal. We leave the details of this to the reader, and only note that, for a countable limit $\gamma = \sup_{n<\omega} \delta_n$, any subset A of $\mathcal{P}^{\gamma}_{\aleph_0}(\omega)$, of any size, is coded by $\{\{A \cap \mathcal{P}^{\delta_n}_{\aleph_0}(\omega)\} \mid n < \omega\}$, which is a countable subset of $\mathcal{P}^{\gamma}_{\aleph_0}(\omega)$.

Remark 1.7. If $\bar{\beta} \leq \bar{\beta}' < \beta' \leq \beta$, then the theorem for $\bar{\beta}$ and β is a consequence of the theorem for $\bar{\beta}'$ and β'. It follows in particular that the restriction of the theorem to successor β implies the full theore: the statement of the theorem at a limit ordinal β follows from the theorem used with any successor ordinal $\beta' < \beta$ which is still above $\bar{\beta}$.

The case of Theorem 1.6 where $\bar{\beta} = 0$, $\beta = 1$, and $\gamma = 3$ was proved in [1, Theorem 6.6]. The proof trivially generalizes to the case where (in our indexing) $\beta = \bar{\beta} + 1$ and $\gamma = \beta + 2 = \bar{\beta} + 3$. ([1] noted this fact for the equivalence relations occurring in their indexing, namely the ones with $\bar{\beta}$ not a limit.) We will refer to this case as the Hjorth-Kechris-Louveau case. It trivially implies all cases where $\gamma = \beta + 2$ and β is a successor ordinal.

At the end of Section 6 of [1], Hjorth-Kechris-Louveau raise the conjecture that all cases of Theorem 1.6 that occur in their indexing are true. In our indexing this translates to saying that all cases of Theorem 1.6 where $\bar{\beta}$ and β are not limit ordinals are true. By Remark 1.7 this implies all cases of Theorem 1.6 except the ones where $\bar{\beta}$ is a limit and $\beta = \bar{\beta} + 1$.

This conjecture, and indeed the full Theorem 1.6, was proved by Shani [4]. Subsection 6.3 of [4] handles finite γ, Subsection 7.1 handles the case where $\gamma = \omega+2$, $\beta = \omega$, and $\bar{\beta} < \omega$, Subsection 7.2 handles the case (which is part of Theorem 1.6 but not part of the conjecture of [1]) where $\gamma = \omega+3$, $\beta = \omega+1$, and $\bar{\beta} = \omega$, and a proof of the general case is outlined in Section 8. The proofs use the symmetric models of Monro [3], and higher level generalizations of these models introduced in [4]. The models and proofs become increasingly more involved as γ increases.

Here we give a proof of Theorem 1.6 which builds on the methods of [1], rather than the methods of [4]. In particular we avoid the need for the complicated models of [3], and instead use only basic symmetric choiceless models.

Our proof identifies the key "missing ingredient" that prevented Hjorth-Kechris-Louveau from using their methods when $\beta < \gamma - 2$. This missing ingredient is the fact that forcing a symmetric choiceless model can satisfy greater distributivity, and be much more limited in adding subsets of V, than the same forcing would without retreating to the symmetric part. The particular instance of this fact that we will use is presented in Lemma 3.3.

The insight that the greater distributivity obtained through uses of symmetric choiceless models can be useful in resolving the conjecture of [1] is due to Shani, and is ultimately one of the key driving forces behind his proof in [4], though he uses it in much more complicated settings than ours.

We will use one more key insight from Shani [4]: that one can generate elements of $\mathrm{Inv}(\gamma, \beta)$ from a function $f\colon \mathcal{P}^{\gamma-2}(\omega) \to \mathcal{P}^{\beta}(\omega)$ and a small set P of permutations of $\mathcal{P}^{\beta}(\omega)$ by taking (up to appropriate coding) $A = \{\sigma \circ f \mid \sigma \in P\}$ and $R = \{\langle g, \rho \circ g, \rho\rangle \mid g \in A, \rho \in P\}$, and moreover that invariants of this kind are enough to prove Theorem 1.6.

In the next section we present what is essentially the structure of the proof that Hjorth-Kechris-Louveau used for their case of Theorem 1.6, though we streamline and generalize it in ways that will help applying its methods to the remaining cases. We end the section with a proof of the Hjorth-Kechris-Louveau case. It is essentially the same as the proof of Hjorth-Kechris-Louveau [1], but we use the kind of invariants mentioned in the previous paragraph, instead of the slightly more complicated invariants used in [1].

Then in Section 3 we formulate what is missing in order to apply the methods from Section 2 to additional cases, present some basic facts about symmetric choiceless models, including Lemma 3.3 which will a key ingredient in all the remaining cases of Theorem 1.6, and proceed to prove these cases.

2. The structure of the argument

Fix throughout countable ordinals $\bar{\beta} < \beta$ and γ so that $\beta + 2 \leq \gamma$. Fix also a universal (lightface) Σ^1_1 set $A \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, meaning that for every $\mathbf{\Sigma}^1_1$ sets $Q \subseteq \mathbb{R} \times \mathbb{R}$, there is a real e so that $Q = A_e = \{\langle x, y\rangle \mid \langle e, x, y\rangle \in A\}$. In particular, if there is a Borel reduction of $\cong^{**}_{\gamma,\beta}$ to $\cong^{**}_{\gamma,\bar{\beta}}$, then there is a real e so that A_e is such a reduction. In such a situation, for a real x, we will use $A_e(x)$ to denote the unique y so that $\langle x, y\rangle \in A_e$.

The following claim will be very useful to us. Its proof is well-known, but we include it, so as to be clear that it does not require the axiom of choice.

Claim 2.1. *Let M be transitive and satisfy* ZF*, let $\mathbb{P} \in M$ be a forcing notion, and let $G_L \times G_R$ be generic over M for $\mathbb{P} \times \mathbb{P}$. Suppose $X \in M[G_L] \cap M[G_R]$. Then $X \in M$.*

Proof. Work over M throughout. Suppose the claim fails, and fix a counterexample X of minimal von-Neumann rank. This implies that $X \subseteq M$.

Fix names $\dot{X}_L, \dot{X}_R \in M$ so that $X = \dot{X}_L[G_L] = \dot{X}_R[G_R]$. Fix a condition $\langle p_L, p_R\rangle \in G_L \times G_R$ forcing that $\dot{X}_L[\dot{G}_L] = \dot{X}_R[\dot{G}_R]$. Note that this implies that for any $x \in M$, $\langle p_L, p_R\rangle \Vdash \check{x} \in \dot{X}_L[\dot{G}_L]$ iff $\langle p_L, p_R\rangle \Vdash \check{x} \in \dot{X}_R[\dot{G}_R]$, and hence $p_L \Vdash \check{x} \in \dot{X}_L$ iff $p_R \Vdash \check{x} \in \dot{X}_R$. The same is true for any extension of $\langle p_L, p_R\rangle$.

Now for any extension q of p_L and any $x \in M$, using the above with the conditions $\langle q, p_R\rangle$ and $\langle p_L, p_R\rangle$ we have that $q \Vdash \check{x} \in \dot{X}_L$ iff $p_R \Vdash \check{x} \in \dot{X}_R$ iff $p_L \Vdash \check{x} \in \dot{X}_L$. Using the fact that $X \subseteq M$ this implies that $X = \{x \in M \mid p_L \Vdash \check{x} \in \dot{X}_L\}$, and hence $X \in M$. □

Lemma 2.2. *Suppose that $e \in \mathbb{R}$ is such that A_e is a Borel reduction of $\cong^{**}_{\gamma,\beta}$ to $\cong^{**}_{\gamma,\bar{\beta}}$. Let c be a real coding a surjection of ω onto γ. Then in every transitive $M \supseteq \{e, c\}$ which satisfies* ZF *(existing in V or in a forcing extension of V), there is an injection i_M of $\mathrm{Inv}(\gamma, \beta)$ into $\mathrm{Inv}(\gamma, \bar{\beta})$.*

Moreover i_M is definable over M from the parameters e, c in a manner which is: (i) uniform, meaning that the same formula θ defines i_M over all models M; and (ii) absolute, meaning that if transitive $N \supseteq M \supseteq \{e, c\}$ satisfy ZF *then $i_M = i_N \restriction M$.*

Proof. Fix M. Fix $\langle A, R\rangle \in \mathrm{Inv}(\gamma, \beta)^M$. We describe $i_M(A, R)$.

Let G be generic over M for $\mathrm{Col}(\omega, \mathrm{tc}(A))$. Let $A^{M[G]}$ be the interpretation of the lightface Σ^1_1 definition of A in $M[G]$. The statement that A_e is a Borel reduction of $\cong^{**}_{\gamma,\beta}$ to $\cong^{**}_{\gamma,\bar{\beta}}$ is Π^1_2. By Shoenfield absoluteness this statement holds in all forcing extensions of V. Since Σ^1_2 statements reflect up between transitive models, this statement also holds in M. By Shoenfield absoluteness over M, the same statement holds of $A_e^{M[G]}$ in $M[G]$, and indeed in any forcing extension of M.

Notice that in $M[G]$, $\langle A, R\rangle$ belongs to $\mathrm{Inv}_{\aleph_0}(\gamma, \beta)$, and let x_G be the real coding $\langle A, R\rangle$ determined from the enumeration of $\mathrm{tc}(A)$ given by G. Let $y_G = A_e^G(x_G)$. Then y_G codes an element of $\mathrm{Inv}_{\aleph_0}(\gamma, \bar{\beta})$ in $M[G]$. Let $i^G_M(A, R)$ be this element of $\mathrm{Inv}_{\aleph_0}^{M[G]}(\gamma, \bar{\beta})$.

We first claim that $i^G_M(A, R)$ belongs to M and is independent of the choice of G. Indeed, fix another generic H, and let G' be generic for $\mathrm{Col}(\omega, \mathrm{tc}(A))$ over both $M[G]$ and $M[H]$. Follow the above definitions with G', leading to reals $x_{G'}$ and $y_{G'}$, and to the invariant $i^{G'}_M(A, R)$. By Shoenfield absoluteness used over the model M, the interpretations $A_e^{M[G]}$, $A_e^{M[G']}$, and $A_e^{M[G \times G']}$ all agree on any common parts of their domains. Then $x_G \cong^{**}_{\gamma,\beta} x_{G'}$ in $M[G \times G']$ since the two reals code the same invariant $\langle A, R\rangle$. This implies that $y_G \cong^{**}_{\gamma,\bar{\beta}} y_{G'}$ since $A_e^{M[G \times G']}$ is a reduction of $\cong^{**}_{\gamma,\beta}$ to $\cong^{**}_{\gamma,\bar{\beta}}$. This in turn implies that $i^G_M(A, R) = i^{G'}_M(A, R)$, and hence using Claim 2.1, that $i^G_M(A, R) \in M$. Similarly $i^H_M(A, R) = i^{G'}_M(A, R)$, and hence $i^G_M(A, R) = i^H_M(A, R)$.

In light of the independence proved in the previous paragraph we can define i_M by setting $i_M(A, R) = i^G_M(A, R)$ for some, or equivalently any, generic G. It is clear that the resulting map is definable over M, and uniformly so. Using the fact that membership in the Σ^1_1 set A reflects up between transitive models it is also clear that the definition of i_M is absolute.

It remains to check that i_M is an injection. Suppose $(A', R') \neq (A, R)$ are both in $\mathrm{Inv}(\gamma, \beta)$. Let G, x_G, y_G and $G', x'_{G'}, y'_{G'}$ be defined as above, for (A, R) and (A', R') respectively, with G and G' mutually generic. Then, in $M[G \times G']$, $x_G \not\cong^{**}_{\gamma,\beta} x'_{G'}$. Since $A_e^{M[G\times G']}$ is a reduction of $\cong^{**}_{\gamma,\beta}$ to $\cong^{**}_{\gamma,\bar{\beta}}$, this implies that $y_G \not\cong^{**}_{\gamma,\bar{\beta}} y'_{G'}$. So the invariants $i_M(A, R) = i^G_M(A, R)$ and $i_M(A', R') = i^{G'}_M(A', R')$ are distinct. $\square$

Lemma 2.3. *Suppose the conclusion of Lemma 2.2 holds. Let $N \supseteq \{e, c\}$ be transitive and satisfy* ZF*. Let $\mathbb{Q} \in N$ be a forcing notion and let H be generic for $\mathbb{Q}$ over N. Then $i_{N[H]}$ maps elements of $N[H] - N$ to elements of $N[H] - N$.*

Proof. Suppose not, and fix $(A, R) \in N[H] - N$ so that $i_{N[H]}(A, R) \in N$. Fix $\dot{A}, \dot{R} \in N$ naming A, R, let $u = i_{N[H]}(A, R) \in N$, and fix $p \in H$ forcing that $i_{N[\dot{H}]}(\dot{A}, \dot{R}) = \check{u}$. This implicitly uses the formula which defines the map $i_{N[H]}$ over $N[H]$. Let H' be generic for $\mathbb{Q}$ over $N[H]$, with $p \in H'$. Let $A' = \dot{A}[H']$ and $R' = \dot{R}[H']$. Then $i_{N[H']}(A', R') = u$. This implicitly uses the uniformity given by the conclusion of Lemma 2.2: the formula defining $i_{N[H']}$ over $N[H']$ is the same as the formula defining $i_{N[H]}$ over $N[H]$.

Using the absoluteness given by the conclusion of Lemma 2.2, the maps $i_{N[H]}$, $i_{N[H']}$, and $i_{N[H\times H']}$ agree on the common parts of their domains. It follows that $i_{N[H\times H']}(A, R) = i_{N[H]}(A, R) = u = i_{N[H']}(A', R') = i_{N[H\times H']}(A', R')$.

Since $i_{N[H\times H']}$ is an injection, this implies that $(A, R) = (A', R')$. By Claim 2.1, this in turn implies that $(A, R) \in N$, a contradiction. $\square$

Lemma 2.4. *Suppose that γ is a double successor ordinal. Let c be a real coding a surjection of ω onto γ. Let $N \supseteq \{c\}$ be transitive and satisfy* ZF*. Let $\mathbb{Q} \in N$ be a poset and let H be generic for $\mathbb{Q}$ over N. Let $\dot{y} \in N$ be a $\mathbb{Q}$-name for an element of $\mathrm{Inv}(\gamma, \bar{\beta})$, and let $y = \dot{y}[H]$. Let $\mathcal{F}$ be a set of automorphisms of $\mathbb{Q}$ in N. Suppose that:*

1. *$\mathcal{P}^{\bar{\beta}}(\omega)$ can be wellordered in N. Let $\tau = (\mathrm{card}(\mathcal{P}^{\bar{\beta}}(\omega)))^N$.*
2. *N satisfies τ-*DC *(allowing for a sequence of choices of length τ).*
3. *(Closure.) $\mathbb{Q}$ is τ-closed in N, meaning that descending sequences of length τ have lower bounds.*
4. *(Control of $\mathcal{P}^{\gamma-2}(\omega)$.) $\mathrm{tc}(y) \cap \mathcal{P}^{\gamma-2}(\omega) \subseteq N$.*
5. *(Automorphisms.) For any $p, q \in \mathbb{Q}$, there is $\pi \in \mathcal{F}$ so that $\pi(p)$ and q are compatible.*
6. *(Symmetry) For every $\pi \in \mathcal{F}$, $\dot{y}[\pi[H]] = \dot{y}[H]$.*

Then y belongs to N.

Proof. Let H^* be generic for $\mathbb{Q}$ over $N[H]$. Suppose for contradiction that $\dot{y}[H] \notin N$. Then by Claim 2.1, $\dot{y}[H] \notin N[H^*]$, and this implies in particular that $\dot{y}[H] \neq \dot{y}[H^*]$. Fix a condition $\langle p, q\rangle \in H \times H^*$ which forces this over N.

We plan to extend $\langle p, q\rangle$ to force enough explicit disagreement between $\dot{y}[H]$ and $\dot{y}[H^*]$ that this disagreement would no longer be dependent on the *mutual* genericity of H and H^*. We then plan to replace H^* with a filter $\pi[H]$ and obtain a contradiction to the symmetry condition (6).

Let $\langle A, R\rangle = \dot y[H]$ and let $\dot A, \dot R \in N$ be $\mathbb{Q}$-names for A and R. We have that $\langle \dot A[H], \dot R[H]\rangle \neq \langle \dot A[H^*], \dot R[H^*]\rangle$. Suppose for simplicity that $\dot A[H] \neq \dot A[H^*]$. The proof in case $\dot R[H] \neq \dot R[H^*]$ is similar with slightly more cumbersome notation.

Suppose for definitiveness that $\dot A[H] - \dot A[H^*] \neq \emptyset$, and fix $\dot u \in N$ so that $u = \dot u[H] \in \dot A[H]$ yet $u \notin \dot A[H^*]$.

Since $A \in \mathcal{P}^{\gamma}(\omega)$, we have $u \in \mathcal{P}^{\gamma-1}(\omega)$, and hence $u \subseteq \mathcal{P}^{\gamma-2}(\omega)$. By the control of $\mathcal{P}^{\gamma-2}(\omega)$ condition (4) it follows that $u \subseteq N$. Similarly every $w \in \dot A[H^*]$ is a subset of N. Since u is distinct from every element of $\dot A[H^*]$, we have that for every $w \in \dot A[H^*]$, there exists some $z \in N$ which belongs to $u - w$ or to $w - u$.

By definition 1.2, $\mathcal{P}^{\bar\beta}(\omega)$ surjects onto $\dot A[H^*]$ in $N[H^*]$, and hence by condition (1) there is $\dot f \in N$ so that $\dot f[H^*]$ is a surjection of τ onto $\dot A[H^*]$. Extending q if needed we may assume that this is forced by q.

Set $p_0 = p$ and $q_0 = q$. By recursion, working in N using condition (2) to make the necessary choices, and using the closure of $\mathbb{Q}$ given by condition (3), define z_ξ and a descending chain of conditions p_ξ, q_ξ, for $\xi \leq \tau$, so that $\langle p_{\xi+1}, q_{\xi+1}\rangle$ forces either $\check z_\xi \in \dot u[\dot H] - \dot f[\dot H^*](\check\xi)$ or $\check z_\xi \in \dot f[\dot H^*](\check\xi) - \dot u[\dot H]$. Notice this is equivalent to the statement that either $p_{\xi+1} \Vdash \check z_\xi \in \dot u \wedge q_{\xi+1} \Vdash \check z_\xi \notin \dot f(\check\xi)$, or $p_{\xi+1} \Vdash \check z_\xi \notin \dot u \wedge q_{\xi+1} \Vdash \check z_\xi \in \dot f(\check\xi)$. Phrased this way it is clear that this statement does not involve any *mutual* genericity.

Let $p^* = p_\tau$ and $q^* = q_\tau$. Using condition (5), fix $\pi \in \mathcal{F}$ so that p^* and $\pi^{-1}(q^*)$ are compatible. Modifying H if needed we may assume without loss of generality that a lower bound for these conditions belongs to H. Let $H' = \pi[H]$. Then $q^* \in H'$. By the conclusion of the previous paragraph it follows that for every $\xi < \tau$, z_ξ belongs to $\dot u[H] - \dot f[H'](\xi)$ or to $\dot f[H'](\xi) - \dot u[H]$. This implies that $u \neq \dot f[H'](\xi)$. Since q^* forces $\dot f$ to be a surjection of τ onto $\dot A$, it follows that $u \notin \dot A[H']$, and hence $\dot y[H] = \langle \dot A[H], \dot R[H]\rangle \neq \langle \dot A[H'], \dot R[H']\rangle = \dot y[H']$. Since $H' = \pi[G]$ this contradicts the symmetry condition (6). $\square$

Let $\mathrm{Bjct}(\kappa)$ be the poset which adds a bijection of κ with itself, using conditions of size less than κ. More precisely, conditions are injective partial functions of size less than κ from κ into itself, ordered by reverse inclusion.

Let $\mathcal{B} = \mathcal{B}(\kappa)$ denote the set of bijection from κ to itself which are the identity on all but fewer than κ points. Each $\sigma \in \mathcal{B}$ induces an automorphism of $\mathrm{Bjct}(\kappa)$, sending $p \in \mathrm{Bjct}(\kappa)$ to $\sigma \circ p$. We use π_σ to denote this automorphism. It is easy to check that for any two conditions p, q, there is σ so that $\pi_\sigma(p)$ and q are compatible.

Theorem 2.5 (The Hjorth-Kechris-Louveau case)**.** *Let $\bar\beta < \beta < \gamma$ be countable ordinals with $\gamma = \beta + 2$, and β a successor ordinal. Then $\cong^{**}_{\gamma,\beta}$ is not Borel reducible to $\cong^{**}_{\gamma,\bar\beta}$.*

Proof. Let N be a forcing extension of V which satisfies the GCH at $\aleph_\alpha$ for all countable α. We work over N throughout.

Let $\kappa = \mathrm{card}(\mathcal{P}^{\beta}(\omega))$ and note that κ is regular. This implies that $\mathrm{Bjct}(\kappa)$ is $<\kappa$-closed, and (using the GCH) that $\mathrm{card}(\mathcal{B}(\kappa)) = \kappa$.

Let $\mathcal{B} = \mathcal{B}(\kappa)$, let $\mathbb{Q} = \mathrm{Bjct}(\kappa)$, and let H be generic for $\mathbb{Q}$ over N. Our plan is to show that H adds a new element x of $\mathrm{Inv}(\gamma, \beta)$ which is symmetric under the automorphisms π_σ for $\sigma \in \mathcal{B}$. We will then use Lemma 2.2 to map x to an element y of $\mathrm{Inv}(\gamma, \bar\beta)$ which is also symmetric, use Lemma 2.4 to argue that that y belongs to N, and use Lemma 2.3 to argue that y cannot belong to N.

Let h be the bijection of κ with itself added by H. Let $A = \{\sigma \circ h \mid \sigma \in \mathcal{B}\}$, and let $\dot{A} \in N$ name A. Notice that $\dot{A}$ is symmetric, by which we mean that $\dot{A}[H] = \dot{A}[\pi_\sigma[H]]$ for every $\sigma \in \mathcal{B}$. Notice further that for each $u \in A$, the map $\sigma \mapsto \sigma \circ u$ is a bijection of $\mathcal{B}$ onto A. Let R code these bijections by setting $R = \{\langle u, \sigma \circ u, \sigma\rangle \mid u \in A \wedge \sigma \in \mathcal{B}\}$, so that for each $u \in A$, R_u^{-1} is a bijection of $\mathcal{B}$ onto A. Let $\dot{R} \in N$ name R, and notice that $\dot{R}$ too is symmetric. This is immediate from the symmetry of $\dot{A}$.

Each element of A is a subset of $\kappa \times \kappa$, and each element of R consists of two subsets of $\kappa \times \kappa$ and one element of $\mathcal{B}$. Let ψ biject $\kappa \times \kappa$ with $\mathcal{P}^{\gamma-2}(\omega)$ in N, and let φ inject $\mathcal{B}$ into $\mathcal{P}^{\beta}(\omega)$ in N. Let A' and R' be the translations of A and R by these maps, namely $A' = \{\psi[u] \mid u \in A\}$ and $R' = \{\langle \psi[u], \psi[v], \varphi(\sigma)\rangle \mid \langle u, v, \sigma\rangle \in B\}$. It is easy to check that $\langle A', R'\rangle$ then belongs to $\operatorname{Inv}(\gamma, \beta)$.

Let $x = \langle A', B'\rangle$ and let $\dot{x} \in N$ name x. We saw that $x \in \operatorname{Inv}(\gamma, \beta)$, and that $\dot{x}$ is symmetric. It is clear that $x \in N[H] - N$, since h is one of the elements of A and therefore H can be recovered from x.

Suppose for contradiction that $\cong^{**}_{\gamma,\beta}$ is Borel reducible to $\cong^{**}_{\gamma,\bar{\beta}}$, and let $i_{N[H]}$ be the injection given by Lemma 2.2. Let $y = i_{N[H]}(x)$ and let $\dot{y}$ name y. We have $y \in \operatorname{Inv}(\gamma, \bar{\beta})$. The symmetry of $\dot{x}$ and the uniform definability of $i_{N[H]}$ imply that $\dot{y}$ is symmetric. Our intention now is to apply Lemma 2.4. It is easy to check that the conditions of the lemma are satisfied here. Let us only note that $N[H] \cap \mathcal{P}^{\beta}(\omega) \subseteq N$ by the closure of $\mathbb{Q}$, and this trivially implies the control of $\mathcal{P}^{\gamma-2}$ condition (4). Using the lemma, we have that $y \in N$. But by Lemma 2.3, $y \in N[H] - N$. This is a contradiction. □

3. THE REMAINING CASES

Lemma 3.1. *Let $\bar{\beta} < \beta < \gamma$ be countable ordinals, with β a successor ordinal and $\leq \gamma - 2$. Suppose $N \supseteq V$ is transitive and satisfies* ZF*. Suppose that f, φ_1, and φ_2 belong to N and that:*

1. *In N, $\kappa = \operatorname{card}(\mathcal{P}^{\beta}(\omega)) = \operatorname{card}(\mathcal{B}(\kappa))$ is regular and $\varphi_1 \colon \kappa \to \mathcal{P}^{\beta}(\omega)$, $\varphi_2 \colon \kappa \to \mathcal{B}(\kappa)$ are bijections.*
2. *$f \in N$ is a function from a (non-empty) subset of $\mathcal{P}^{\gamma-2}(\omega)$ into κ.*

Let H be generic for $\operatorname{Bjct}(\kappa)^N$ over N, and let $h \colon \kappa \to \kappa$ be the bijection added by H. Suppose there is a transitive $M \supseteq V$ satisfying ZF*, so that:*

3. *The bijections φ_1, φ_2 belong to M, $\mathcal{P}^{\beta}(\omega)^M = \mathcal{P}^{\beta}(\omega)^N$, and $\mathcal{B}(\kappa)^M = \mathcal{B}(\kappa)^N$.*
4. *$h \circ f \in M$.*
5. *$\mathcal{P}^{\gamma-2}(\omega)^M \subseteq N$.*

Then $\cong^{**}_{\gamma,\beta}$ is not Borel reducible to $\cong^{**}_{\gamma,\bar{\beta}}$.

Proof. Using Remark 1.1 and the bijection φ_1, construct an injection $\psi \colon \operatorname{dom}(f) \times \kappa \to \mathcal{P}^{\gamma-2}(\omega)$ and arrange that $\operatorname{range}(\psi) \supseteq \operatorname{range}(\varphi_1)$. The construction is done in N independently of H, and it is absolute enough that ψ belongs to M.

Our plan is to follow the proof of Theorem 2.5, that is the Hjorth-Kechris-Louveau case, but modifying the definition of A to be $\{\sigma \circ h \circ f \mid \sigma \in \mathcal{B}(\kappa)\}$, working inside the model M to obtain y from x, and using condition (5) to secure the control of $\mathcal{P}^{\gamma-2}(\omega)$ condition of Lemma 2.4.

Each element of the modified A is a subset of $\operatorname{dom}(f) \times \kappa$, and each element of R (defined as in the proof of Theorem 2.5 but for the modified A) consists of two subsets of $(\mathrm{dom}(f) \times \kappa)$ and one element of $\mathcal{B}(\kappa)$. As in the proof of Theorem 2.5, A and R generate an element of $\mathrm{Inv}(\gamma, \beta)$, using the maps ψ and $\varphi_1 \circ \varphi_2^{-1}$. The fact that $\mathrm{range}(\psi) \supseteq \mathcal{P}^\beta(\omega)$ ensures that $\mathrm{tc}(A') \cap \mathcal{P}^\beta(\omega) = \mathcal{P}^\beta(\omega)$, so that R' matches its requirements in Definition 1.2.

Set $x = \langle A', R' \rangle$ and $y = i_{N[H]}(x)$. Let $\dot{y}$ name y. As in the proof of Theorem 2.5, $\dot{y}$ is symmetric, and all the conditions of Lemma 2.4, except possibly for the control of $\mathcal{P}^{\gamma-2}(\omega)$ condition, are satisfied, with the poset $\mathbb{Q} = \mathrm{Bjct}(\kappa)^N$. The symmetry relies on the fact that ψ and φ_2 are independent of H.

By condition (4), and since $\psi, \varphi_2 \in M$, we have that A' and R' belong to M. It follows that $x \in M$, and hence $i_M(x)$ is defined. By the absoluteness and uniformity given by Lemma 2.2, $i_{N[H]}(x) = i_M(x)$. Hence $y \in M$. Using condition (5) it follows that $\mathrm{tc}(y) \cap \mathcal{P}^{\gamma-2}(\omega) \subseteq N$, so that the control of $\mathcal{P}^{\gamma-2}(\omega)$ condition of Lemma 2.4 holds.

We can now finish the argument as in the proof of Theorem 2.5: Lemma 2.4 implies that $y \in N$, while Lemma 2.3 implies that $y \notin N$, a contradiction. □

In the case that $\beta = \gamma - 2$, one can obtain the conditions of Lemma 3.1 with $M = N[H]$. The key point is that condition (5) holds for the simple reason that H does not add elements of $\mathcal{P}^{\gamma-2}(\omega)$ in this case. Of course in this case the argument reduces to the proof of the Hjorth-Kechris-Louveau case.

In the case that $\beta < \gamma - 2$, forcing to add H *does* add new elements of $\mathcal{P}^{\gamma-2}(\omega)$. It would seem impossible in this case to secure condition (5) for an M which includes $h \circ f$. What makes the seemingly impossible possible in this case is *symmetry* in appropriate *choiceless models*.

Let $\mathbb{A}$ be a poset obtained as a product $\prod_{i \in I} \mathbb{P}_i$ with support some ideal $S \subseteq \mathcal{P}(I)$, meaning that the conditions in $\mathbb{A}$ are functions p with $\mathrm{dom}(p) \in S$ and $p(i) \in \mathbb{P}_i$ for each $i \in \mathrm{dom}(p)$.

If G is generic for $\mathbb{A}$, and $s \in \mathcal{S}$, then we will use $G{\restriction}s$ to mean $\{p{\restriction}s \mid p \in G\} = \{p \mid p \in G \wedge \mathrm{dom}(p) \subseteq s\}$. This is a generic for $\prod_{i \in s} \mathbb{P}_i$. Notice that this restricted product has full support, since every subset of s belongs to $\mathcal{S}$.

For $i \in I$ we will use $G(i)$ to mean $\{p(i) \mid p \in G \wedge i \in \mathrm{dom}(p)\}$. This is a generic for $\mathbb{P}_i$.

Let $\mathcal{E}$ be a collection of permutations of I with the properties that (i) for every $\nu \in \mathcal{E}$, and every $i \in I$, $\mathbb{P}_{\nu(i)} = \mathbb{P}_i$, and (ii) for every $\nu \in \mathcal{E}$ and every $s \subseteq I$, $s \in S \iff \nu[s] \in S$. Property (i) holds trivially if, for example, all the factors $\mathbb{P}_i$ are equal to the same poset $\mathbb{P}$. Property (ii) holds trivially if, for example, S consists of all subsets of I of size at most a certain δ. These and similar situations will be the ones we work with below.

Under these assumptions, every $\nu \in \mathcal{E}$ generates an automorphism π_ν of $\mathbb{A}$, defined by setting $\pi_\nu(p) = p \circ \nu$. Let $\mathcal{F} = \{\pi_\nu \mid \nu \in \mathcal{E}\}$. Each automorphism $\pi \in \mathcal{F}$ of $\mathbb{A}$ generates a map on $\mathbb{A}$-names $\dot{x}$ by setting $\pi(\dot{x}) = \{\langle \pi(\dot{y}), \pi(q) \rangle \mid \langle \dot{y}, q \rangle \in \dot{x}\}$. For every generic G we then have that $\pi(\dot{x})[\pi[G]] = \dot{x}[G]$.

Let $U \subseteq \mathcal{F}$. We say that an $\mathbb{A}$-name $\dot{x}$ is *U-symmetric* if for all $\pi \in U$, $\pi(\dot{x}) = \dot{x}$. This implies in particular that $\dot{x}[\pi[G]] = \dot{x}[G]$. For a collection $\mathcal{U}$ of subsets of $\mathcal{F}$, we say that $\dot{x}$ is *$\mathcal{U}$-symmetric* if there exists some $U \in \mathcal{U}$ so that $\dot{x}$ is U-symmetric. We say that $\dot{x}$ is *hereditarily* $\mathcal{U}$-symmetric if $\dot{x}$ itself is $\mathcal{U}$-symmetric, and for every $\langle \dot{y}, q \rangle \in \dot{x}$, $\dot{y}$ is hereditarily $\mathcal{U}$-symmetric. Given a generic G for $\mathbb{A}$, define $V_{\mathcal{U}}^{\mathrm{sym}}[G]$ to be the transitive submodel of $V[G]$ consisting only of $\dot{x}[G]$ for hereditarily $\mathcal{U}$-symmetric $\dot{x} \in V$.

The following result is a reformulations to our context of the well known fact that symmetric submodels of forcing extensions satisfy ZF:

Theorem 3.2. *Let $\mathcal{U}$ be a filter consisting of subsets of $\mathcal{F}$ and suppose that $\mathcal{U}$ is invariant under $\mathcal{F}$-conjugation, meaning that for every $\pi \in \mathcal{F}$, the filter consisting of the sets $\{\pi \circ \rho \circ \pi^{-1} \mid \rho \in U\}$ for $U \in \mathcal{U}$ is exactly equal to $\mathcal{U}$. Let G be generic for $\mathbb{A}$. Then $V^{\mathrm{sym}}_{\mathcal{U}}[G]$ satisfies ZF.*

We refer the reader to Jech [2] for the methods that go into the proof of Theorem 3.2. The key points are that $V^{\mathrm{sym}}_{\mathcal{U}}[G]$ is invariant under $\pi \in \mathcal{F}$ (meaning that $V^{\mathrm{sym}}_{\mathcal{U}}[\pi[G]] = V^{\mathrm{sym}}_{\mathcal{U}}[G]$), and that for any $x_1, \ldots, x_n \in V^{\mathrm{sym}}_{\mathcal{U}}[G]$ there are names $\dot{x}_1, \ldots, \dot{x}_n$ and $U \in \mathcal{U}$ so that $\dot{x}_1, \ldots, \dot{x}_n$ are invariant under all $\pi \in U$. This allows generating U-symmetric names for all sets obtained from $x_1, \ldots, x_n$ via pairing, union, the powerset operation relativized to $V^{\mathrm{sym}}_{\mathcal{U}}[G]$, comprehension in $V^{\mathrm{sym}}_{\mathcal{U}}[G]$, and replacement in $V^{\mathrm{sym}}_{\mathcal{U}}[G]$.

Typically the filter $\mathcal{U}$ used in applications of Theorem 3.2 is obtained by looking at permutations that fix various sets. For example, in our context, it would be typical to take $\mathcal{U}$ to be the filter generated by $U_a = \{\pi_\nu \mid \nu \in \mathcal{E} \text{ fixes } a\}$ for $a \in \mathcal{S}$. In these situations it is easy to secure the hypothesis of the next lemma. The lemma reduces distributivity for the symmetric extension to the distributivity of the restricted products $\prod_{i \in a} \mathbb{P}_i$. Since the restricted products are taken with full support, they inherit distributivity from the individual factors $\mathbb{P}_i$. This will be key to our plans to secure condition (5) of Lemma 3.1.

Lemma 3.3. *Let $\mathcal{U}$ and G be as in Theorem 3.2. Suppose that for each $U \in \mathcal{U}$, there is $a = a_U$ so that for any $s, t \in \mathcal{S}$, there is $\nu \in \mathcal{E}$ with $\nu \restriction a$ the identity, $\nu[s] \cap t \subseteq a$, and $\pi_\nu \in U$. Then any subset of V which belongs to $V^{\mathrm{sym}}_{\mathcal{U}}[G]$, belongs to $\bigcup_{a \in S} V[G\restriction a]$.*

In particular, if each of the individual posets $\mathbb{P}_i$ is $< \tau$ closed, then any subset of V which belongs to $V^{\mathrm{sym}}_{\mathcal{U}}[G]$ and has size $< \tau$ in $V^{\mathrm{sym}}_{\mathcal{U}}[G]$ belongs to V.

Proof. The second part of the lemma follows from the first using the fact that the restricted products $\prod_{i \in a} \mathbb{P}_i$ are taken with full supports. We prove the first part.

Fix $x \subseteq V$ in $V^{\mathrm{sym}}_{\mathcal{U}}[G]$, and let $\dot{x}$ be a U-symmetric name for x, for some $U \in \mathcal{U}$. Let $a = a_U$. We claim that $x = \{z \mid (\exists r \in G\restriction a) r \Vdash \check{z} \in \dot{x}\}$, and therefore $x \in V[G\restriction a]$. The right-to-left inclusion is clear. To prove the left-to-right inclusion, fix $z \in x$, and let $p \in G$ force that $\check{z} \in \dot{x}$. We will show that $p\restriction a$ forces $\check{z} \in \dot{x}$.

Suppose otherwise. Then there is $q \leq p\restriction a$ forcing that $\check{z} \notin \dot{x}$. Using the assumptions of the lemma, fix $\nu \in \mathcal{E}$, with $\pi = \pi_\nu \in U$, so that $\nu\restriction a$ is the identity, and $\nu[\mathrm{dom}(p)] \cap \mathrm{dom}(q) \subseteq a$. Then $\pi(q) = q \circ \nu$ and p are compatible, as witnessed by the condition $\pi(q) \cup p\restriction(\mathrm{dom}(p) - a)$. But this is impossible since $\pi(q) \Vdash \check{z} \notin \pi(\dot{x}) = \dot{x}$, while $p \Vdash \check{z} \in \dot{x}$. □

Theorem 3.4. *Let γ be a countable double successor ordinal, and let $\bar{\beta} < \beta$ with $\beta < \gamma - 2$ a successor ordinal. Suppose further that $\gamma - 2$ is a double successor ordinal. Then $\cong^{**}_{\gamma,\beta}$ is not Borel reducible to $\cong^{**}_{\gamma,\bar{\beta}}$.*

Proof. By passing to a forcing extension of V if necessary, we may assume that the GCH holds at $\aleph_\alpha$ for all countable α.

Let $\kappa = \mathrm{card}(\mathcal{P}^\beta(\omega))$, and let $\lambda = \mathrm{card}(\mathcal{P}^{\gamma-3}(\omega))$. Then $\kappa \leq \lambda$, and both are regular. Fix bijections $\varphi_1 \colon \kappa \to \mathcal{P}^\beta(\omega)$ and $\varphi_2 \colon \kappa \to \mathcal{B}(\kappa)$, and a further bijection $\chi \colon \lambda \to \mathcal{P}^{\gamma-3}(\omega)$, all in V.

Let $\mathbb{P}$ be the forcing to add a Cohen subset of λ. Precisely, conditions are partial functions from λ into $\{0,1\}$, with domain of size $< \lambda$, ordered by reverse inclusion. Let $\dot{C}$ name the Cohen subset of λ added by a generic for $\mathbb{P}$.

Let $I = \kappa \times \kappa$, let $S \subseteq \mathcal{P}(I)$ consist of all subsets of I of size $< \kappa$, and let $\mathbb{A} = \Pi_{i \in I} \mathbb{P}_i$, taken with supports in S, where $\mathbb{P}_i = \mathbb{P}$ for all i. Let G be generic for $\mathbb{A}$ over V, and let $N = V[G]$. The product $\mathbb{A}$ is $< \kappa$-closed, and therefore κ remains regular in N, and φ_1, φ_2 continue to be bijections into $\mathcal{P}^{\beta}(\omega)$ and $\mathcal{B}(\kappa)$ in N.

Abusing notation slightly, we will write $G(\alpha,\xi)$ for $G(\langle \alpha,\xi \rangle)$. In N, let θ be the function on I sending $\langle \alpha,\xi \rangle$ to $\chi[\dot{C}[G(\alpha,\xi)]]$. Then θ is an injection of I into $\mathcal{P}^{\gamma-2}(\omega)$. Let $f = \mathrm{proj} \circ \theta^{-1}$, where proj is the projection map $\langle \alpha,\xi \rangle \mapsto \alpha$. This is a map from a subset of $\mathcal{P}^{\gamma-2}(\omega)$ into κ. Let $\dot{f}$ be the canonical name for f.

Our plan is to appeal to Lemma 3.1. We have already secured conditions (1) and (2) of the lemma.

Let H be generic for $\mathrm{Bjct}(\kappa)^N = \mathrm{Bjct}(\kappa)^V$ over N, and let $h\colon \kappa \to \kappa$ be the bijection added by H. Let $h^{-1} \times id$ denote the map $\langle \alpha,\xi \rangle \mapsto \langle h^{-1}(\alpha),\xi \rangle$. Let π be the map on $\mathbb{A}$ defined by $p \mapsto p \circ (h^{-1} \times id)$. Then $p \in \mathbb{A} \to \pi(p) \in \mathbb{A}$, because $\mathrm{dom}(p)$ has size $< \kappa$, and restrictions of h to sets of size $< \kappa$ belong to V. It follows that π is an automorphism of $\mathbb{A}$.

Claim 3.5. *$\pi[G]$ is generic for $\mathbb{A}$ over V.*

Proof. By mutual genericity, we have that G is generic for $\mathbb{A}$ over $V[H]$. Working in $V[H]$ and using the fact that π is an automorhpism of $\mathbb{A}$, it follows trivially that $\pi[G]$ is generic for $\mathbb{A}$ over $V[H]$, and hence in particular generic over V. $\square$

Notice that $\dot{f}[\pi[G]] = h \circ f$, and therefore $h \circ f \in V[\pi[G]]$. Our plan is to enclose $h \circ f$ in a symmetric submodel of $V[\pi[G]]$, and use Lemma 3.3 to show that this submodel has the same elements of $\mathcal{P}^{\gamma-2}(\omega)$ as N.

In V, let $\mathcal{E}$ be the collection of permutations ν of I which fix the first coordinate, meaning that $\nu[\{\alpha\} \times \kappa] = \{\alpha\} \times \kappa$, and fix all but fewer than κ elements of I. Let $\mathcal{F} = \{\pi_\nu \mid \nu \in \mathcal{E}\}$. For each $a \subseteq I$ of size $< \kappa$, let $U_a = \{\pi_\nu \mid \nu \in \mathcal{E} \wedge \nu{\restriction}a = id\}$. Let $\mathcal{U}$ be the filter generated by $\{U_a \mid a \subseteq I \wedge |a| < \kappa\}$. Note that these assignments satisfy the assumptions of Theorem 3.2 and Lemma 3.3.

Let $M = V^{\mathrm{sym}}_{\mathcal{U}}[\pi[G]]$. This is a model of ZF by Theorem 3.2. The name $\dot{f}$ is fixed by all $\pi_\nu \in \mathcal{F}$. This is clear from the definition of f and the fact that $\nu[\{\alpha\} \times \kappa] = \{\alpha\} \times \kappa$. It follows that $\dot{f}[\pi[G]]$ belongs to M. It is clear that φ_1, φ_2 belong to M, since they in fact belong to V. Since forcing with $\mathbb{A}$ does not add elements of $\mathcal{P}^{\beta}(\omega)$ and $\mathcal{B}(\kappa)$, we have that $\mathcal{P}^{\beta}(\omega)^V \subseteq \mathcal{P}^{\beta}(\omega)^M \subseteq \mathcal{P}^{\beta}(\omega)^{V[\pi[G]]} = \mathcal{P}^{\beta}(\omega)^V$. Thus $\mathcal{P}^{\beta}(\omega)^M = \mathcal{P}^{\beta}(\omega)^V = \mathcal{P}^{\beta}(\omega)^N$. Similarly with $\mathcal{B}(\kappa)$. It remains to prove condition (5) of Lemma 3.1.

Claim 3.6. *$\mathcal{P}^{\gamma-2}(\omega)^M \subseteq N$.*

Proof. By Lemma 3.3, every subset of λ in M belongs to $V[\pi[G]{\restriction}a]$ for some $a \in S$. Letting $\bar{a} = (h \times id)[a]$ we have that $\pi[G]{\restriction}a$ is equal to the image of $G{\restriction}\bar{a}$ under the map $p \mapsto p \circ ((h^{-1} \times id){\restriction}a)$. Since restrictions of h to sets of size $< \kappa$ belong to V, it follows that $V[\pi[G]{\restriction}a] = V[G{\restriction}\bar{a}] \subseteq N$. Thus all subsets of λ in M belong to N.

Recall that, in V, $\lambda = \mathrm{card}(\mathcal{P}^{\gamma-3}(\omega))$, and that $\chi\colon \lambda \to \mathcal{P}^{\gamma-3}(\omega)$ is a bijection. Since $\mathbb{A}{\restriction}a$, for $a \in S$, is a full support product of $< \lambda$-closed posets, it does not add bounded subsets of λ, and hence does not add elements of $\mathcal{P}^{\gamma-3}(\omega)$. It follows that χ remains a bijection from λ into $\mathcal{P}^{\gamma-3}(\omega)$ in M.

Combining the conclusions of the last two paragraphs, it follows that all elements of $\mathcal{P}^{\gamma-2}(\omega) = \mathcal{P}(\mathcal{P}^{\gamma-3}(\omega))$ in M belong to N. □

We have now shown that all the conditions in Lemma 3.1 are satisfied. Applying the lemma we conclude that $\cong^{**}_{\gamma,\beta}$ is not Borel reducible to $\cong^{**}_{\gamma,\bar{\beta}}$. □

Theorem 3.7. *Let γ be a countable double successor ordinal, and let $\bar{\beta} < \beta$ with $\beta < \gamma - 2$ a successor ordinal. Suppose further that $\gamma - 2$ is the successor of a limit ordinal. Then $\cong^{**}_{\gamma,\beta}$ is not Borel reducible to $\cong^{**}_{\gamma,\bar{\beta}}$.*

Proof. As before we may assume that V satisfies the GCH at $\aleph_\alpha$ for all countable α. Note that $\beta < \gamma - 3$ since $\gamma - 3$ is a limit ordinal and β is not. Let $\kappa = \mathrm{card}(\mathcal{P}^{\beta}(\omega))$ and let $\lambda = \mathrm{card}(\mathcal{P}^{\gamma-3}(\omega))$. Then κ is regular, λ is singular of cofinality ω, and $\kappa < \lambda$. Fix bijections φ_1, φ_2, and χ as in the proof of Theorem 3.4.

Let λ_n, $n < \omega$, be an increasing sequence of regular cardinals converging to λ, with $\lambda_0 > \kappa$. Let $\mathbb{P}_n$ be the forcing to add a Cohen subset of $\lambda_{n+1} - \lambda_n$, and let $\dot{C}_n$ name the Cohen set added by $\mathbb{P}_n$. Let $\mathbb{P} = \prod_{n<\omega} \mathbb{P}_n$, taken with full supports. A generic $\prod_{n<\omega} G_n$ for $\mathbb{P}$ adds the subset $\bigcup_{n<\omega} \dot{C}_n[G_n]$ of λ. We wish to follow the proof of Theorem 3.4, but using $\mathbb{P}$ as defined here to add subsets of λ. This will require some extra symmetry in the argument. To obtain this extra symmetry we will view the $\mathbb{P}_n$ as individual factors in $\mathbb{A}$.

Let $I = \kappa \times \kappa$ and let $J = \kappa \times \kappa \times \omega$. Let $S \subseteq \mathcal{P}(J)$ consist of all subsets of J of size $< \kappa$, and let $\mathbb{A} = \prod_{\langle \alpha,\xi,n\rangle \in J} \mathbb{P}_{\alpha,\xi,n}$, taken with supports in S, where $\mathbb{P}_{\alpha,\xi,n} = \mathbb{P}_n$. Let G be generic for $\mathbb{A}$ over V. For $\langle \alpha,\xi\rangle \in I$ let $C(\alpha,\xi) = \bigcup_{n<\omega} \dot{C}_n[G(\alpha,\xi,n)]$. This is a subset of λ, added by the generic $\prod_{n<\omega} G(\alpha,\xi,n)$ for $\mathbb{P}$. As in the proof of Theorem 3.4, let $\theta(\alpha,\xi) = \chi[C]$. Before proceeding, we symmetrize these objects under changes to initial segments. Precisely, let $\Gamma_n = \{\dot{C}_n[G(\alpha,\xi,n)] \mid \alpha,\xi \in \kappa\}$, let $\Gamma = \{\bigcup_{n<\omega} C_n \mid \{C_n\}_{n<\omega} \in \prod_{n<\omega} \Gamma_n\}$, and let $\Theta(\alpha,\xi)$ consist of the sets $\chi[C]$ for all $C \in \Gamma$ which agree with $C(\alpha,\xi)$ on a tail-end. Let $s(\theta^{-1})$, the *saturation* of θ^{-1} under modifications to initial segments, be the function with domain $\bigcup_{\alpha,\xi\in\kappa} \Theta(\alpha,\xi)$ which maps elements of $\Theta(\alpha,\xi)$ to the pair $\langle\alpha,\xi\rangle$. Let $f^* = \mathrm{proj} \circ s(\theta^{-1})$, and let $\dot{f}^*$ be the canonical name for f^*.

We will use f^* instead of the function f used in the proof of Theorem 3.4. Again we will use Lemma 3.1, and f^*, being a function from a subset of $\mathcal{P}^{\gamma-2}(\omega)$ into κ, has the right format.

Let H be generic for $\mathrm{Bjct}(\kappa)^N = \mathrm{Bjct}(\kappa)^V$ over N, let $h\colon \kappa \to \kappa$ be the bijection added by H, and let π be the map defined by $p \mapsto p \circ (h^{-1} \times id \times id)$. As in the proof of Theorem 3.4, π is an automorphism of $\mathbb{A}$, $\pi[G]$ is generic for $\mathbb{A}$ over V, and $\dot{f}^*[\pi[G]] = h \circ f^*$.

In V let $\mathcal{E}$ be the collection of permutations ν of J which:

1. Fix all but fewer than κ elements of J.
2. Fix the third coordinate, meaning that $\nu[\kappa \times \kappa \times \{n\}] = \kappa \times \kappa \times \{n\}$.
3. Fix the first coordinate for all but finitely many n, meaning that there exists $k < \omega$ so that $\nu[\{\alpha\} \times \kappa \times \{n\}] = \{\alpha\} \times \kappa \times \{n\}$ whenever $n \geq k$.

In contrast with the definition in the proof of Theorem 3.4, here we allow ν to change the first coordinate, but only in a limited way.

Let $\mathcal{F} = \{\pi_\nu \mid \nu \in \mathcal{E}\}$. For each $a \subseteq J$ of size $< \kappa$ let $U_a = \{\pi_\nu \mid \nu \in \mathcal{E} \wedge \nu\restriction a = id\}$. Let $\mathcal{U}$ be the filter generated by $\{U_a \mid a \subseteq J \wedge |a| < \kappa\}$. Let $M = V^{\mathrm{sym}}_{\mathcal{U}}[\pi[G]]$. These assignments are similar to the ones in the proof of Theorem 3.4, but using the current, richer collection $\mathcal{E}$. Notice that the name $\dot{f}^*$ is fixed by all $\pi_\nu \in \mathcal{F}$, even with this richer collection $\mathcal{E}$, because saturating θ^{-1} under modifications to initial segments made the resulting function invariant under permutations of any finitely many of the maps $\alpha, \xi \mapsto G_{\alpha,\xi,n}$. So $h \circ f^* = \dot{f}^*[\pi[G]] \in M$.

As in the proof of Theorem 3.4, these assignments satisfy conditions (1)–(4) of Lemma 3.1, and it now remains only to prove that $\mathcal{P}^{\gamma-2}(\omega)^M \subseteq N$.

For $k < \omega$ let $\mathbb{A}{\restriction}k$ denote $\prod_{n<k\wedge\alpha,\xi\in\kappa} \mathbb{P}_{\alpha,\xi,n}$. Let $G{\restriction}k$ denote the restriction of G to $\mathbb{A}{\restriction}k$. We say that $\dot{x}$ is a *restricted* $\mathbb{A}$-name if it is an $\mathbb{A}{\restriction}k$-name for some k. Abusing notation slightly we write $\dot{x}[G]$ for $\dot{x}[G{\restriction}k]$.

Claim 3.8. *For every $\delta < \gamma - 3$, there is a hereditarily $\mathcal{U}$-symmetric, restricted $\mathbb{A}$-name $\dot{X}_\delta$, of size* $\operatorname{card}(\mathcal{P}^\delta(\omega)) = \aleph_\delta$ *in V, which is forced in $\mathbb{A}$ to name $\mathcal{P}^\delta(\omega)^{V^{\mathrm{sym}}_{\mathcal{U}}[\dot{G}]}$.*

Proof. This is an adaptation to our context of the standard "reverse analysis" of products $\prod_{n<\omega} \mathbb{P}_n$, using closure of the tail-end $\prod_{n\geq k} \mathbb{P}_n$ to show that small enough sets in the extension by the full product belong to the extension by the initial segment $\prod_{n<k} \mathbb{P}_n$.

Let δ_k be such that $\lambda_k = \aleph_{\delta_k}$. The sequence is increasing, each δ_k is a successor ordinal, $\operatorname{card}(\mathcal{P}^{\delta_k}(\omega)) = \lambda_k$ by the GCH, and $\sup(\delta_k) = \gamma - 3$. We will prove the claim by induction on δ, and maintain inductively that if $\delta \leq \delta_{k+1}$ then $\dot{X}_\delta$ is an $\mathbb{A}{\restriction}k$-name. This makes the limit case trivial, taking $\dot{X}_\delta$ to name the union of $\dot{X}_{\bar\delta}$ for $\bar\delta < \delta$.

By Lemma 3.3 and since the posets $\mathbb{A}{\restriction}a$ for a of size $< \kappa$ are full support products of $< \lambda_1$-closed factors, all bounded subsets of λ_1 in $V^{\mathrm{sym}}_{\mathcal{U}}[G]$ belong to V. It follows from this that, for $\delta \leq \delta_1$, $\mathcal{P}^\delta(\omega)$ is the same in V and in $V^{\mathrm{sym}}_{\mathcal{U}}[G]$. The claim therefore holds trivially up to δ_1.

Now suppose $\delta_k \leq \delta < \delta_{k+1}$ for $k \geq 1$, and suppose inductively that the claim holds at δ and that $\dot{X}_\delta$ is an $\mathbb{A}{\restriction}k$-name. We prove the claim at $\delta + 1$, continuing to use an $\mathbb{A}{\restriction}k$-name.

We can assume without loss of generality that $\dot{X}_\delta$ has the form $\{\langle \dot{x}, \emptyset\rangle \mid \dot{x} \in X_\delta\}$ for some set X_δ. The size condition on $\dot{X}_\delta$ allows us to take X_δ of size $\aleph_\delta$. Let E be the set of $a \subseteq I \times k$ of size $< \kappa$. For $a \in E$, let Z_a be the set of $\mathcal{U}$-symmetric $\mathbb{A}$-names which are contained in $X_\delta \times \mathbb{A}{\restriction}a$. Notice that these are all $\mathbb{A}{\restriction}k$-names. Every $\dot{y} \in Z_a$ names a subset of $\mathcal{P}^\delta(\omega)$. Let $X_{\delta+1} = \bigcup_{a\in E} Z_a$, and let $\dot{X}_{\delta+1} = \{\langle \dot{r}, \emptyset\rangle \mid \dot{r} \in X_{\delta+1}\}$. It is clear that $\dot{X}_{\delta+1}$ is an $\mathbb{A}{\restriction}k$-name, and is hereditarily $\mathcal{U}$-symmetric. It names a collection of subsets of $\mathcal{P}^\delta(\omega)$, and since it is hereditarily $\mathcal{U}$-symmetric it follows that it names a subset of $\mathcal{P}^{\delta+1}(\omega)^{V^{\mathrm{sym}}_{\mathcal{U}}[G]}$. Each Z_a has size $\mathcal{P}(X_\delta \times \mathbb{A}{\restriction}a) \leq 2^{\aleph_\delta \times \lambda_k} = 2^{\aleph_\delta} = \aleph_{\delta+1}$. So $\dot{X}_{\delta+1}$ has size $\kappa \times \aleph_{\delta+1} = \aleph_{\delta+1}$.

It remains to prove that $\dot{X}_{\delta+1}$ names the full $\mathcal{P}^{\delta+1}(\omega)^{V^{\mathrm{sym}}_{\mathcal{U}}[G]}$. Since G is an arbitrary generic, it suffices to show that every $U \subseteq \mathcal{P}^\delta(\omega)$ in $V^{\mathrm{sym}}_{\mathcal{U}}[G]$ belongs to $\dot{X}_{\delta+1}[G]$. Fix U. We will find $\dot{r} \in X_{\delta+1}$ so that $\dot{r}[G] = U$. Here at last we will use the "reverse analysis" mentioned at the start.

Let $\bar{U} = \{\dot{x} \in X_\delta \mid \dot{x}[G] \in U\}$. Since $\dot{X}_\delta$ names the full $\mathcal{P}^\delta(\omega)^{V^{\mathrm{sym}}_{\mathcal{U}}[G]}$, we have that $U = \{\dot{x}[G] \mid \dot{x} \in \bar{U}\}$. By Lemma 3.3, and since X_δ belongs to V, there is $c \subseteq J$ of size $< \kappa$ so that $\bar{U} \in V[G{\restriction}c]$. Let $c_{<k} = c \cap I \times k$ and let $c_{\geq k} = c \cap I \times (\omega - k)$. We can write $G{\restriction}c$ as $G{\restriction}c_{<k} \times G{\restriction}c_{\geq k}$. The poset $\mathbb{A}{\restriction}c_{<k}$ is of size at most λ_k, while the poset $\mathbb{A}{\restriction}c_{\geq k}$, being a full support product of $< \lambda_{k+1}$-closed posets, is itself $< \lambda_{k+1}$-closed. It follows that the latter poset does not add bounded subsets of λ_{k+1} over the extension by the former poset. Since X_δ has size $\aleph_\delta < \lambda_{k+1}$, this implies that $\bar{U} \in V[G \upharpoonright c_{<k}]$. Let $a = c_{<k}$ and let $\dot{\bar{U}}$ be an $\mathbb{A}\upharpoonright a$-name for $\bar{U}$. Using the fact that $U \in V_{\mathcal{U}}^{\mathrm{sym}}[G]$ we can pick $\dot{\bar{U}}$ which is $\mathcal{U}$-symmetric. Let $\dot{r} = \{\langle \dot{x}, q\rangle \in X_\delta \times \mathbb{A}\upharpoonright a \mid a \Vdash_{\mathbb{A}\upharpoonright a} \dot{x} \in \dot{\bar{U}}\}$. Then $\dot{r} \in Z_a \subseteq X_{\delta+1}$ and $\dot{r}[G] = \{\dot{x}[G] \mid \dot{x} \in \bar{U}\} = U$. □

Say that $\nu \in \mathcal{E}$ is the *a-fragment* of $h^{-1} \times id \times id$ if $a \subseteq I \times k$ for some $k < \omega$, a has size $< \kappa$, $\nu \upharpoonright a = (h^{-1} \times id \times id)\upharpoonright a$, and ν is the identity outside a. In that case we also say that π_ν is the *a-fragment* of π, and we have that π_ν and π agree on conditions with domain contained in $\nu[a]$. We say that a statement holds for *unboundedly many fragments of* π if for every $l < \omega$ and every $c \subseteq I \times l$ of size $< \kappa$ there exists $a \supseteq c$ so that the statement holds for the a-fragment of π.

Claim 3.9. *Let $\dot{x}$ be a restricted $\mathbb{A}$-name which is hereditarily $\mathcal{U}$-symmetric. Then for unboundedly many fragments $\tau \in \mathcal{F}$ of π, $\dot{x}[\pi[G]] = \dot{x}[\tau[G]]$. In particular, since $\tau[G] \in V[G]$, $\dot{x}[\pi[G]] \in V[G] = N$.*

Proof. The proof of this claim will make use of the fact that condition (3) in the definition of $\mathcal{E}$ above allows for changes of the first coordinate on $\kappa \times \kappa \times \{n\}$ for finitely many n. Indeed, both the permutations ν and μ below, being fragments of $h^{-1} \times id \times id$, will change the first coordinate on the support of the fragment based on the action of h^{-1}.

We prove the claim by induction on the von Neumann rank of $\dot{x}$, simultaneously for all choices of G and H.

Let k be such that $\dot{x}$ is an $\mathbb{A}\upharpoonright k$-name. Fix $a \subseteq \kappa \times \kappa \times k$ of size $< \kappa$ so that $\dot{x}$ is U_a-symmetric. Expanding a if needed, assume that it has the form $E \times E \times k$ for $E \subseteq \kappa$ of size $< \kappa$, and that $h[E] = E$. Let ν be the a-fragment of $h^{-1} \times id \times id$, so that π_ν is the a-fragment of π. Since we could have increased k, and picked a to contain any $c \subseteq I \times k$ of size $< \kappa$, it is enough to prove that $\dot{x}[\pi[G]] = \dot{x}[\pi_\nu[G]]$.

We will first show that $\dot{x}[\pi[G]] \subseteq \dot{x}[\pi_\nu[G]]$. Suppose $\langle \dot{y}, q\rangle \in \dot{x}$ and $q \in \pi[G]$, so that $\dot{y}[\pi[G]] \in \dot{x}[\pi[G]]$. We prove that $\dot{y}[\pi[G]] \in \dot{x}[\pi_\nu[G]]$. By the inductive hypothesis there is $b \subseteq J$ so that $\dot{y}[\pi[G]] = \dot{y}[\tau[G]]$ where τ is the b-fragment of π, and moreover we can find b large enough that $a \subseteq b$ and $\mathrm{dom}(q) \subseteq b$. The latter implies that $q \in \tau[G]$ and therefore $\dot{y}[\tau[G]] \in \dot{x}[\tau[G]]$. Let μ be the b-fragment of $h^{-1} \times id \times id$, so that $\tau = \pi_\mu$. Note that $\nu^{-1} \circ \mu \in \mathcal{E}$ is the identity on a, and therefore by symmetry and the choice of a, $(\tau \circ \pi_{\nu^{-1}})(\dot{x}) = \dot{x}$. This implies that $\dot{x}[\tau[G]] = \pi_{\nu^{-1}}(\dot{x})[G] = \dot{x}[\pi_\nu[G]]$. Putting all this together we get that $\dot{y}[\pi[G]] = \dot{y}[\tau[G]] \in \dot{x}[\tau[G]] = \dot{x}[\pi_\nu[G]]$. This completes the proof that $\dot{x}[\pi[G]] \subseteq \dot{x}[\pi_\nu[G]]$.

Next we handle the other direction. Since our induction is done simultaneously for all choices of G and H, we can carry out the above argument replacing H with $\tilde{H}$ which adds the bijection $(h^{-1}\upharpoonright\kappa - E) \cup (id\upharpoonright E)$, and replacing G with $\tilde{G} = \pi[G]$. (Notice that $\tilde{H}$ is generic for $\mathrm{Bjct}(\kappa)$ over V, that $V[\tilde{H}] = V[H]$, and that $\tilde{G}$, being the image of G under an automorphism of $\mathbb{A}$ in $V[H]$, is generic for $\mathbb{A}$ over $V[H]$.) We then get the exact same conclusion, but replacing G with $\tilde{G}$, replacing π with the automorphism $\tilde{\pi}$ generated using $\tilde{h}$, which is exactly $\pi_\nu \circ \pi^{-1}$, and replacing π_ν with the a-fragment of $\tilde{\pi}$, which is the identity. We get that $\dot{x}[\tilde{\pi}[\tilde{G}]] \subseteq \dot{x}[\tilde{G}]$, or equivalently, substituting the values listed above, that $\dot{x}[\pi_\nu[G]] \subseteq \dot{x}[\pi[G]]$. □

Recall that to complete the proof of Theorem 3.7, we need only show that $\mathcal{P}^{\gamma-2}(\omega)^M \subseteq N$. Fix to this end $U \in \mathcal{P}^{\gamma-2}(\omega)$, equivalently $U \subseteq \mathcal{P}^{\gamma-3}(\omega)$, which belongs to $M = V_{\mathcal{U}}^{\mathrm{sym}}[\pi[G]]$. Let δ_n, $n < \omega$, be cofinal in $\gamma - 3$, and let

$U_n = U \cap \mathcal{P}^{\delta_n}(\omega) \in \mathcal{P}^{\delta_n+1}(\omega)^M$. By Claim 3.8, applied with the generic $\pi[G]$, there is a restricted $\mathbb{A}$-name $\dot{X}_{\delta_n+1}$ so that $\dot{X}_{\delta_n+1}[\pi[G]] = \mathcal{P}^{\delta_n+1}(\omega)^M$. By Claim 3.9 it follows that $\mathcal{P}^{\delta_n+1}(\omega)^M$ belongs to N, and in particular so does U_n. The sequence $\langle U_n \mid n < \omega\rangle$ belongs to $V[G][H]$, since both G and π belong to this model. The posets adding G and H are both countably closed, and therefore $N = V[G]$ is countably closed in $V[G][H]$. Since each U_n individually belongs to N it follows that the entire sequence $\langle U_n \mid n < \omega\rangle$ belongs to N, and therefore $U = \bigcup_{n<\omega} U_n \in N$. This completes the proof of Theorem 3.7. $\square$

Theorem 3.10. *Let γ be a countable double successor ordinal, and let $\bar{\beta} < \beta$ with $\beta < \gamma - 2$ a successor ordinal. Suppose further that $\gamma - 2$ is a limit ordinal. Then $\cong^{**}_{\gamma,\beta}$ is not Borel reducible to $\cong^{**}_{\gamma,\bar{\beta}}$.*

Proof. As before we may assume that V satisfies the GCH at $\aleph_\alpha$ for all countable α. Let $\kappa = \text{card}(\mathcal{P}^{\beta}(\omega))$ and let $\lambda = \text{card}(\mathcal{P}^{\gamma-2}(\omega))$. Then $\kappa < \lambda$, κ is regular, and λ is singular of cofinality ω. Let δ_n, $n < \omega$, be an increasing sequence of successor ordinals converging to $\gamma - 2$, with $\delta_0 > \beta$. Let $\lambda_n = \aleph_{\delta_n} = \text{card}(\mathcal{P}^{\delta_n}(\omega))$. Fix bijections $\varphi_1 \colon \kappa \to \mathcal{P}^{\beta}(\omega)$, $\varphi_2 \colon \kappa \to \mathcal{B}(\kappa)$, and $\chi_n \colon \lambda_n \to \mathcal{P}^{\delta_n}(\omega)$, all in V.

Let $I = \kappa \times \kappa$, let $J = I \times \omega$, and let $S \subseteq \mathcal{P}(J)$ consist of all subsets of J of size $< \kappa$. Let $\mathbb{P}_n$ be the forcing to add a Cohen subset of λ_n, and let $\dot{C}_n$ name this added set. Let $\mathbb{A} = \prod_{i \in I, n<\omega} \mathbb{P}_{i,n}$, taken with supports in S, where $\mathbb{P}_{i,n} = \mathbb{P}_n$. Let G be generic for $\mathbb{A}$ over V and let $N = V[G]$. We will use $\mathbb{A}{\restriction}k$ and $G{\restriction}k$ to denote the restrictions to $I \times k$.

Let θ_n be the function on I sending $\langle \alpha, \xi\rangle$ to $\chi_n[\dot{C}_n[G(\alpha,\xi,n)]$, and let $f_n = \text{proj} \circ \theta_n^{-1}$. This parallels the definitions in the proof of Theorem 3.4, and f_n is a function from a collection of subsets of $\mathcal{P}^{\delta_n}(\omega)$ into κ. Let $f^* = \bigcup_{n<\omega} f_n$. Then f^* is a function from a subset of $\bigcup_{n<\omega} \mathcal{P}^{\delta_n+1}(\omega) = \mathcal{P}^{\gamma-2}(\omega)$ into κ. Let $A = \{\vec{\sigma} \circ f^* \mid \vec{\sigma} \in \mathcal{B}^*(\kappa)\}$, where $\mathcal{B}^*(\kappa)$ consists of the sequences $\vec{\sigma} = \{\sigma_n\}_{n<\omega} \in \mathcal{B}(\kappa)^\omega$ which are eventually equal to the identity, and $\vec{\sigma} \circ f^* = \bigcup_{n<\omega} \sigma_n \circ f_n$. Then A is a κ-sized collection of subsets of $\mathcal{P}_{\gamma-2}(\omega) \times \kappa$. From each $g \in A$ we can define a bijection of $\mathcal{B}^*(\kappa)$ onto A, namely the map $\vec{\sigma} \mapsto \vec{\sigma} \circ g$. Let $R = \{\langle g, \vec{\sigma} \circ g, \vec{\sigma}\rangle \mid g \in A \wedge \vec{\sigma} \in \mathcal{B}^*(\kappa)\}$ code these bijections.

With appropriate bijections we can convert the pair $\langle A, R\rangle$ to an invariant in $\text{Inv}(\gamma, \beta)$, as we did in the proof of Lemma 3.1. Let $\psi \colon \mathcal{P}^{\gamma-2} \times \kappa \to \mathcal{P}^{\gamma-2}(\omega)$ be an injection obtained using Remark 1.1 and the bijection φ_1, arranging that $\psi[f_0] \supseteq \mathcal{P}^{\beta}(\omega)$. Let $\varphi_2^* \colon \kappa \to \mathcal{B}^*(\kappa)$ be a bijection obtained from φ_2. Define $A' = \{\psi[g] \mid g \in A\}$ and $R' = \{\langle \psi[g], \psi[\vec{\sigma} \circ g], \varphi_1 \circ (\varphi_2^*)^{-1}(\vec{\sigma})\rangle \mid \langle g, \vec{\sigma} \circ g, \vec{\sigma}\rangle \in R\}$. Let $x = \langle A', R'\rangle$. It is clear that x then belongs to $\text{Inv}(\gamma, \beta)$.

Let $\mathcal{E}$ be the collection of permutations ν of J which fix all but fewer than κ elements of J, and fix the first and third coordinates, meaning that $\nu[\{\alpha\} \times \kappa \times \{n\}] = \{\alpha\} \times \kappa \times \{n\}$. Let $\mathcal{F} = \{\pi_\nu \mid \nu \in \mathcal{E}\}$, let $U_a = \{\pi_\nu \mid \nu \in \mathcal{E} \wedge \nu{\restriction}a = id\}$, and let $\mathcal{U}$ be the filter generated by the sets U_a for $a \subseteq J$ of size $< \kappa$. Let $M = V^{\text{sym}}_{\mathcal{U}}[G]$.

It is clear from the definitions that the canonical names $\dot{f}_n$, $\dot{f}^*$, $\dot{A}$, $\dot{A}'$, $\dot{R}$, $\dot{R}'$, and $\dot{x}$ for f_n, f^*, A, A', R, R' and x are invariant under all automorphisms in $\mathcal{F}$. These objects therefore belong to M. Let i_M be the map given by Lemma 2.2, and let $y = i_M(x)$. Using the definability in Lemma 2.2 and the invariance of $\dot{x}$, we can find a name $\dot{y}$ for y which is also invariant under all automorphisms in $\mathcal{F}$.

For $\vec{\rho} \in \mathcal{B}(\kappa)^\omega$, let $\vec{\rho}(x)$ (respectively $\vec{\rho}(\dot{x})$) be defined as x (respectively $\dot{x}$) was defined above, but using the function $\vec{\rho} \circ f^*$ instead of f^*. It is clear from the symmetry inherent in the definition of A that $\vec{\rho}(x) = \vec{\sigma}(x)$ iff $\vec{\rho}$ and $\vec{\sigma}$ agree on a tail-end.

Notice that $\vec{\rho}(x) = \dot{x}[\pi_\nu[G]]$, where ν is the permutation $\langle \alpha, \xi, n\rangle \mapsto \langle \rho_n^{-1}(\alpha), \xi, n\rangle$ of J and π_ν is the automorphism $p \mapsto p \circ \nu$ of $\mathbb{A}$. To simplify notation we refer to π_ν as $\pi^{\vec{\rho}}$. For $\langle \rho_0, \dots, \rho_{k-1}\rangle \in \mathcal{B}(\kappa)^{<\omega}$ we write $\pi^{\rho_0,\dots,\rho_{k-1}}$ to denote $\pi^{\vec{\rho}}$ where $\vec{\rho}$ has ρ_i at its ith coordinate for $i < k$ and is the identity otherwise. For $\rho \in \mathcal{B}(\kappa)$, we write $\pi^{\rho,k}$ for $\pi^{\vec{\rho}}$ where $\vec{\rho}$ has ρ at its kth coordinate and is the identity otherwise. Since $\vec{\rho}(x) = x$ when $\vec{\rho}$ is the identity on a tail-end, we have that $\dot{x}[\pi^{\rho,k}[G]] = x$ and similarly with $\pi^{\rho_0,\dots,\rho_{k-1}}$.

Our plan is to produce $\vec{\rho}$ and $\vec{\sigma}$ which do *not* agree on a tail-end, while making sure that $\vec{\rho}(x)$ and $\vec{\sigma}(x)$ are mapped to the same object by the maps of Lemma 2.2. This will contradict the fact that the maps of Lemma 2.2 are injective. We will use the next two claims for this. The first claim will allow us to construct $\vec{\rho}\restriction k$ and $\vec{\sigma}\restriction k$ by recursion on k making sure that specific enumerations of $\dot{y}[\pi^{\vec{\rho}\restriction k}[G]]$ and $\dot{y}[\pi^{\vec{\sigma}\restriction k}[G]]$ are exactly equal. The second claim will provide some continuity, allowing us to connect the enumerations of $\dot{y}[\pi^{\vec{\rho}\restriction k}[G]]$ for $k < \omega$ with the enumeration of $\dot{y}[\pi^{\vec{\rho}}[G]]$.

Say $y = \langle C, T\rangle$. For simplicity suppose that $\emptyset \notin C$. If $\emptyset \in C$ the coding below needs to be adjusted to accommodate this, but no other changes are needed in the proof. Fix some $u \in C$. Recall from Definition 1.2 that for every $v \in C$, T_v^{-1} is the graph of a surjection of a subset of $\mathcal{P}^{\bar{\beta}}(\omega)$ onto C. We write $T_v^{-1}(e)$ to denote the value of this surjection at e, namely the unique d so that $\langle e, d\rangle \in T_v^{-1}$ if such a d exists. Letting $Z = \{\langle w, e\rangle \mid w \in T_u^{-1}(e)\} \cup \{\langle w.e_1, e_2\rangle \mid w \in T^{-1}_{T_u^{-1}(e_1)}(e_2)\}$ we then have that Z codes both C and T, in the sense that C is the collection of all non-empty sets of the form $d_e = \{w \mid \langle w, e\rangle \in Z\}$, and T is the collection of triples $\langle d_{e_1}, d_{e_1,e_2}, e_2\rangle$ with the first two coordinates non-empty, where $d_{e_1,e_2} = \{w \mid \langle w, e_1, e_2\rangle \in Z\}$. In contrast with the coded object y, the coding object Z is a subset of $\mathcal{P}^{\gamma-2}(\omega)$. Let $\dot{u}$ and $\dot{Z}$ be hereditarily $\mathcal{U}$-symmetric names for u and Z. We can pick $\dot{Z}$ which is outright forced to name the code of $i_M(\dot{x})$ as defined above from $\dot{u}$.

Claim 3.11. *For every $n < \omega$, and every $c \subseteq \kappa$ of size $< \kappa$, there are $\rho \neq \sigma \in \mathcal{B}(\kappa)$, both the identity on c, so that $\dot{Z}[\pi^{\rho,n}[G]] = \dot{Z}[\pi^{\sigma,n}[G]]$.*

Proof. As noted above, $\dot{x}[\pi^{\rho,n}[G]] = \dot{x}[\pi^{\sigma,n}[G]]$ for any $\rho, \sigma \in \mathcal{B}(\kappa)$. This implies that $\dot{y}[\pi^{\rho,n}[G]] = \dot{y}[\pi^{\sigma,n}[G]]$. Since Z is defined from y and u, it is enough to make sure that $\dot{u}[\pi^{\rho,n}[G]] = \dot{u}[\pi^{\sigma,n}[G]]$. But this is a simple use of a pigeonhole principle. Recall that $y = \langle C, T\rangle$ is an element of $\mathrm{Inv}(\gamma, \bar{\beta})$. This implies that C is a set of cardinality $\bar{\kappa} = \mathrm{card}(\mathcal{P}^{\bar{\beta}}(\omega))$. The poset $\mathbb{A}$ is $< \kappa$ closed, and therefore $\bar{\kappa}$ remains strictly smaller than $\kappa = \mathrm{card}(\mathcal{P}^{\beta}(\omega))$ in $V[G]$. Now the set of $\rho \in \mathcal{B}(\kappa)$ which are the identity on c has size κ. Since $\dot{u}[\pi^{\rho,n}[G]] \in \dot{C}[\pi^{\rho,n}[G]] = C$ for all ρ, it follows that there must be $\rho \neq \sigma \in \mathcal{B}(\kappa)$, both the identity on c, with $\dot{u}[\pi^{\rho,n}[G]] = \dot{u}[\pi^{\sigma,n}[G]]$. □

Claim 3.12. *For every $n < \omega$, there is a hereditarily $\mathcal{U}$-symmetric, restricted $\mathbb{A}$-name (meaning an $\mathbb{A}\restriction k$-name for some $k < \omega$) $\dot{Z}_n$ so that $\dot{Z}_n[G] = Z \cap \mathcal{P}^{\delta_n}(\omega)^M$.*

Proof. The proof of Claim 3.8 applies in the current context using the current $\mathcal{E}$; it did not rely on the added richness of the family $\mathcal{E}$ used in proving Theorem 3.7. This proof shows that there is a hereditarily $\mathcal{U}$-symmetric, restricted $\mathbb{A}$-name for

$\mathcal{P}^{\delta_n+1}(\omega)^M$. This implies that the same is true for every element of $\mathcal{P}^{\delta_n+1}(\omega)^M$, and hence in particular for $Z \cap \mathcal{P}^{\delta_n}(\omega)^M$ □

Claim 3.12 requires the use of a symmetric model $M = V^{\mathrm{sym}}_{\mathcal{U}}[G]$, and indirectly introduces a use of Lemma 3.3. This is the first place in the proof of Theorem 3.10 where symmetry is needed.

Working by recursion on $n < \omega$ we now construct $\rho_n, \sigma_n \in \mathcal{B}(\kappa)$, a descending sequence of conditions $p_n \in G$, and restricted $\mathbb{A}$-names $\dot{Z}_n$ and $\dot{Y}_n$, so that:

(1) $\pi^{\rho_0,\dots,\rho_{n-1}}(p_n) \Vdash \dot{Z} \cap \mathcal{P}^{\delta_n}(\omega) = \dot{Z}_n$, and $\pi^{\sigma_0,\dots,\sigma_{n-1}}(p_n) \Vdash \dot{Z} \cap \mathcal{P}^{\delta_n}(\omega) = \dot{Y}_n$.
(2) $\dot{Z}[\pi^{\rho_0,\dots,\rho_n}[G]] = \dot{Z}[\pi^{\sigma_0,\dots,\sigma_n}[G]]$.
(3) $\rho_n \neq \sigma_n$.

Having constructed the objects prior to n, we construct the objects for n as follows. First, let $\dot{Z}_n$ be given by Claim 3.12, applied with the generic $\pi^{\rho_0,\dots,\rho_{n-1}}[G]$ instead of G, so that $\dot{Z}[\pi^{\rho_0,\dots,\rho_{n-1}}[G]] \cap \mathcal{P}^{\delta_n}(\omega) = \dot{Z}_n[\pi^{\rho_0,\dots,\rho_{n-1}}[G]]$. Let $q_1 \in \pi^{\rho_0,\dots,\rho_{n-1}}[G]$ be a condition forcing that $\dot{Z} \cap \mathcal{P}^{\delta_n}(\omega) = \dot{Z}_n$. Working similarly with $\pi^{\sigma_0,\dots,\sigma_{n-1}}[G]$, find a restricted name $\dot{Y}_n$ and a condition $q_2 \in \pi^{\sigma_0,\dots,\sigma_{n-1}}[G]$ forcing that $\dot{Z} \cap \mathcal{P}^{\delta_n}(\omega) = \dot{Y}_n$. Let $p_n \in G$, extending p_{n-1} if $n > 0$, be strong enough that $\pi^{\rho_0,\dots,\rho_{n-1}}(p_n) \leq q_1$ and $\pi^{\sigma_0,\dots,\sigma_{n-1}}(p_n) \leq q_2$. This secures condition (1).

By induction we know that $\dot{Z}[\pi^{\rho_0,\dots,\rho_{n-1}}[G]] = \dot{Z}[\pi^{\sigma_0,\dots,\sigma_{n-1}}[G]]$. Extending $p_n \in G$ if needed, make sure it is strong enough to force this (as a statement about the generic G).

Let ρ_n and σ_n be given by Claim 3.11, used with $c = \{\alpha, \mid (\exists\xi)\langle \alpha, \xi, n\rangle \in \mathrm{dom}(p_n)\}$, and with the generic $\pi^{\rho_0,\dots,\rho_{n-1}}[G]$ instead of G. The choice of c guarantees that $\pi^{\rho_n,n}(p_n) = \pi^{\sigma_n,n}(p_n) = p_n$. In particular this implies that $p_n \in \pi^{\sigma_n,n}[G]$, and hence the equality of condition (2) for $n-1$, being forced by p_n, holds also with G replaced by $\pi^{\sigma_n,n}[G]$, i.e., $\dot{Z}[\pi^{\rho_0,\dots,\rho_{n-1}}[\pi^{\sigma_n,n}[G]]] = \dot{Z}[\pi^{\sigma_0,\dots,\sigma_{n-1}}[\pi^{\sigma_n,n}[G]]]$. The right-hand-side of this equation is equal to $\dot{Z}[\pi^{\sigma_0,\dots,\sigma_n}[G]]$. The left-hand-side, using the commutativity of the two embeddings involved, can be written as $\dot{Z}[\pi^{\sigma_n,n}[\pi^{\rho_0,\dots,\rho_{n-1}}[G]]]$, which is equal to $\dot{Z}[\pi^{\rho_n,n}[\pi^{\rho_0,\dots,\rho_{n-1}}[G]]]$ through the use of Claim 3.11. Since $\dot{Z}[\pi^{\rho_n,n}[\pi^{\rho_0,\dots,\rho_{n-1}}[G]]] = \dot{Z}[\pi^{\rho_0,\dots,\rho_n}[G]]$ it follows that $\dot{Z}[\pi^{\rho_0,\dots,\rho_n}[G]] = \dot{Z}[\pi^{\sigma_0,\dots,\sigma_n}[G]]$. This completes the construction.

For each $k < \omega$, we can write $\pi^{\vec{\rho}}$ as the composition $\pi^{\vec{\rho}\restriction[k,\omega)} \circ \pi^{\rho_0,\dots,\rho_{k-1}}$ where $\vec{\rho}\restriction[k,\omega)$ is the sequence which agrees with $\vec{\rho}$ on coordinates $n \geq k$, and is the identity at coordinates $n < k$. We can similarly write $\pi^{\vec{\sigma}} = \pi^{\vec{\sigma}\restriction[k,\omega)} \circ \pi^{\sigma_0,\dots,\sigma_{k-1}}$. Our construction is such that ρ_n and σ_n are the identity at α if $\langle \alpha, \xi, n\rangle \in \mathrm{dom}(p_n)$. It follows from this that $\pi^{\vec{\rho}\restriction[k,\omega)}(p_k) = p_k$ and similarly with $\vec{\sigma}\restriction[k,\omega)$. So $\pi^{\rho_0,\dots,\rho_{k-1}}(p_k) = \pi^{\vec{\rho}}(p_k) \in \pi^{\vec{\rho}}[G]$ and similarly $\pi^{\sigma_0,\dots,\sigma_{k-1}}(p_k) \in \pi^{\vec{\sigma}}[G]$.

At stage n of the construction we had the freedom to extend p_n arbitrarily inside G. In particular, for any dense open $D \subseteq \mathbb{A}$ in V, by extending p_n as needed, and noticing that the preimages of D under $\pi^{\rho_0,\dots,\rho_{n-1}}$ and $\pi^{\sigma_0,\dots,\sigma_{n-1}}$ are dense open subsets of $\mathbb{A}$ which belong to V, we can pick p_n to belong to both these preimages, so that both $\pi^{\rho_0,\dots,\rho_{n-1}}(p_n)$ and $\pi^{\sigma_0,\dots,\sigma_{n-1}}(p_n)$ belong to D. Using the conclusion of the previous paragraph it follows that both $\pi^{\vec{\rho}}[G]$ and $\pi^{\vec{\sigma}}[G]$ meet D.

Working in an extension $V[G][H]$ of $V[G]$ where the powerset of $\mathbb{A}$ in V is collapsed to have size $\aleph_0$, and using the appropriate bookkeeping, we can therefore make sure that $\pi^{\vec{\rho}}[G]$ and $\pi^{\vec{\sigma}}[G]$ are both generic for $\mathbb{A}$ over V. This genericity implies that $V^{\mathrm{sym}}_{\mathcal{U}}[\pi^{\vec{\rho}}[G]]$ is a transitive model of ZF, that $\dot{x}[\pi^{\vec{\rho}}[G]]$ belongs to $\mathrm{Inv}(\gamma,\beta)$, that $i_{V^{\mathrm{sym}}_{\mathcal{U}}[\pi^{\vec{\rho}}[G]]}(\dot{x}[\pi^{\vec{\rho}}[G]]) = \dot{y}[\pi^{\vec{\rho}}[G]]$, and that $\dot{y}[\pi^{\vec{\rho}}[G]]$ is coded by

$\dot{Z}[\pi^{\vec{\rho}}[G]]$. Using condition (1), and since $\pi^{\rho_0,\dots,\rho_{n-1}}(p_n) \in \pi^{\vec{\rho}}[G]$, we also get that $\dot{Z}[\pi^{\vec{\rho}}[G]] \cap \mathcal{P}^{\delta_n}(\omega) = \dot{Z}_n[\pi^{\vec{\rho}}[G]]$. The same equalities hold with ρ replaced by σ and $\dot{Z}_n$ replaced by $\dot{Y}_n$.

Using the fact that $\dot{Z}_n$ and $\dot{Y}_n$ are restricted $\mathbb{A}$-names, fixing k large enough that they are both $\mathbb{A}{\restriction}k$-names, and using the fact that $\pi^{\vec{\rho}\restriction[k,\omega)}$ does not move $\mathbb{A}{\restriction}k$, we have that $\dot{Z}_n[\pi^{\vec{\rho}}[G]] = \dot{Z}_n[\pi^{\rho_0,\dots,\rho_{k-1}}[G]]$, and similarly with σ and $\dot{Y}_n$.

Increasing k if necessary we can assume that $k \geq n$. Then $\pi^{\rho_0,\dots,\rho_{n-1}}(p_n) = \pi^{\rho_0,\dots,\rho_{k-1}}(p_n) \in \pi^{\rho_0,\dots,\rho_{k-1}}[G]$. Using condition (1) we get that $\dot{Z}_n[\pi^{\rho_0,\dots,\rho_{k-1}}[G]] = \dot{Z}[\pi^{\rho_0,\dots,\rho_{k-1}}[G]] \cap \mathcal{P}^{\delta_n}(\omega)$. Similarly $\dot{Y}_n[\pi^{\sigma_0,\dots,\sigma_{k-1}}[G]] = \dot{Z}[\pi^{\sigma_0,\dots,\sigma_{k-1}}[G]] \cap \mathcal{P}^{\delta_n}(\omega)$. Using condition (2) at $k-1$ it follows that $\dot{Z}_n[\pi^{\rho_0,\dots,\rho_{k-1}}[G]] = \dot{Y}_n[\pi^{\sigma_0,\dots,\sigma_{k-1}}[G]]$. Combining this with the conclusion of the last paragraph we get that $\dot{Z}_n[\pi^{\vec{\rho}}[G]] = \dot{Y}_n[\pi^{\vec{\sigma}}[G]]$. This is true for each n, and hence, using the conclusion of the paragraph before last, $\dot{Z}[\pi^{\vec{\rho}}[G]] = \dot{Z}[\pi^{\vec{\sigma}}[G]]$.

But now we have a contradiction: The invariants $\vec{\rho}(x) = \dot{x}[\pi^{\vec{\rho}}[G]]$ and $\vec{\sigma}(x) = \dot{x}[\pi^{\vec{\sigma}}[G]]$ are distinct by condition (3). They both belong to the model $V[G][H]$, and using the injectivity of the map $i_{V[G][H]}$ of Lemma 2.2, and the absoluteness of this map to the models $V_{\mathcal{U}}^{\mathrm{sym}}[\pi^{\vec{\rho}}[G]]$ and $V_{\mathcal{U}}^{\mathrm{sym}}[\pi^{\vec{\sigma}}[G]]$ respectively, it follows that $\dot{y}[\pi^{\vec{\rho}}[G]]$ and $\dot{y}[\pi^{\vec{\sigma}}[G]]$ are distinct. Yet each can be recovered in the same manner from its code, and the codes $\dot{Z}[\pi^{\vec{\rho}}[G]]$ and $\dot{Z}[\pi^{\vec{\sigma}}[G]]$ are equal. This contradiction completes the proof of Theorem 3.10. □

Theorems 2.5, 3.4, 3.7, and 3.10 cover all cases of Theorem 1.6 where β is a successor ordinal. By Remark 1.7 this completes the proof of Theorem 1.6.

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