A homogeneous model of MA which is accessible to reals, and other exotic models

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Homogeneous models accessible to reals

Definition

1. V is **accessible to reals** if there exists $x \in \mathbb{R}$ such that $\omega_1 = \omega_1^{L[x]}$.
2. V is **homogeneous** if for all $A \subset \mathbb{R}$, if A has an $^\infty$-Borel code which is OD, then A is determined.
   - $A = \{x \in \mathbb{R} \mid \mathrm{HOD}[x] = \varphi[x, a]\}$, for some φ and $a \in \mathrm{HOD}$.
3. V is **strongly homogeneous** if for all $A \subset \mathbb{R}$, if A is OD, then A is determined.

Assume MA_{ω_1} holds and V is accessible to reals. Then

$$L(\mathbb{R}) \models \mathrm{ZF} + \omega_1\text{-}\mathrm{DC} + \mathrm{MA}_{\omega_1}.$$

A natural question.

Question

Assume MA holds.

- Can V be both homogeneous and accessible to reals?
  - What if $V = L(\mathbb{R})$?

Generic MA models

Definition

Suppose $V = L[x]$ for some $x \in \mathbb{R}$ and $\mathbb{P}_{\omega_2}$ is given by a V-generic iteration of length ω_2, with finite support, of ccc partial orders of size ω_1. Suppose that

$$\left(L(\mathbb{R})\right)^{V[\mathbb{P}_{\omega_2}]^{\mathbb{P}_{\omega_2}}} \models \mathrm{AC}.$$

Then

$$\left(L(\mathbb{R})\right)^{V[\mathbb{P}_{\omega_2}]^{\mathbb{P}_{\omega_2}}}$$

is a **generic** MA **model** over $L[x]$.

- Necessarily, $V[\mathbb{P}_{\omega_2}]^{\mathbb{P}_{\omega_2}} \models \mathrm{MA} + c = \omega_2$, and so
  - $\left(L(\mathbb{R})\right)^{V[\mathbb{P}_{\omega_2}]^{\mathbb{P}_{\omega_2}}} \models \omega_1\text{-}\mathrm{DC}.$
- But can or must $\left(L(\mathbb{R})\right)^{V[\mathbb{P}_{\omega_2}]^{\mathbb{P}_{\omega_2}}} \models \mathrm{AC}$?

Generic MA models from the existence of Suslin trees

Theorem

Suppose that:

- *For all* ccc *partial orders* $\mathbb{P}$ *of size* ω_1*, there exists a Suslin tree* T *such that* $\mathbb{P} \times T$ *is* ccc*.*

Suppose $V = L[x]$ *for some* $x \in \mathbb{R}$ *and* $\mathbb{P}_{\omega_2}$ *is given by a* V*-generic iteration of length* ω_2*, with finite support, of* ccc *partial orders of size* ω_1*.*

- *Suppose* $G \subset \mathbb{P}_{\omega_2}$ *is* $V[\mathbb{P}_{\omega_2}]$*-generic. Then the following hold.*
  1. $V[G] \models \mathrm{MA} + \text{“}c = \omega_2\text{”}$.
  2. $V[G] \models \text{“There is a wellordering of } \mathbb{R} \text{ in } L(\mathbb{R})\text{”}$.
     - *The wellordering is definable from* x *in* $L(\mathbb{R})$*.*

- But is this strong existence of Suslin trees consistent if $V = L[x]$ for some $x \in \mathbb{R}$?
  - Does it hold in L?
    - If it does then the generic MA model over L satisfies $V = \mathrm{HOD}$.

Forcing the strong existence of Suslin trees

Lemma

Suppose that $G \subset \mathrm{Add}(\omega_1, \omega_2)$ is V-generic. Then the following holds in $V[G]$.

- *For all ccc partial orders $\mathbb{P}$ of size ω_1, there exists a Suslin tree T such that $\mathbb{P} \times T$ is ccc.*

- But $V[G]$ is far from satisfying $V = L[x]$ for some $x \in \mathbb{R}$.

Theorem

Suppose that $V = L[x]$ for some $x \in \mathbb{R}$. Then there exists a generic extension $V[G]$ such that the following hold in $V[G]$.

1. *$V[G] = L[y]$ for some $y \in \mathbb{R}^{V[G]}$.*
2. *For all ccc partial orders $\mathbb{P}$ of size ω_1, there exists a Suslin tree T such that $\mathbb{P} \times T$ is ccc.*

Proof sketch

Let

$$\langle \sigma_\alpha : \alpha < \omega_2 \rangle$$

be a sequence of uncountable subsets of ω_1 such that for all $\alpha < \beta < \omega_2$, $\sigma_\alpha \cap \sigma_\beta$ is countable. Similarly, let

$$\langle \tau_\alpha : \alpha < \omega_1 \rangle$$

be a sequence of infinite subsets of ω such that for all $\alpha < \beta < \omega_1$, $\sigma_\alpha \cap \sigma_\beta$ is finite.

- Let $G_0 \subset \mathrm{Add}(\omega_1, \omega_2)$ be V-generic.
- Let A be a $V[G_0]$-generic subset of ω_2.

Let $\sigma \subset \omega_1$ be $V[G_0][A]$-generic for almost disjoint coding A relative to the sequence $\langle \sigma_\alpha : \alpha < \omega_2 \rangle$ such that

$$A = \{\beta < \omega_2 \mid \sigma_\beta \cap \sigma \text{ is countable}\}.$$

Thus $V[G_0][A][\sigma] = L[x][\sigma]$.

- Let $\tau \subset \omega$ be $V[G_0][A][\sigma]$-generic for almost disjoint coding σ relative to $\langle \tau_\alpha : \alpha < \omega_1 \rangle$ such that

  $$\sigma = \{\beta < \omega_1 \mid \tau_\beta \cap \tau \text{ is finite}\}.$$

  - Thus $V[G_0][A][\sigma][\tau] = L[x][\tau]$.
    - $L[x][\tau]$ witnesses the conclusion of the theorem.

Truth on a Turing cone: the power of the Martin measure

Theorem (Martin)

Assume PD. *Then* $\mathrm{Th}(L[x])$ *is constant on a Turing cone.*

Theorem (Kechris, Woodin)

Assume Σ^1_2*-determinacy and that* $x^\#$ *exists for all* $x \in \mathbb{R}$. *Then* $\mathrm{Th}(L[x])$ *is constant on a Turing cone.*

Theorem

Assume Σ^1_2*-determinacy and that* $x^\#$ *exists for all* $x \in \mathbb{R}$. *Then for a Turing cone of* $x \in \mathbb{R}$,

- *if* $\mathbb{P} \in L[x]$ *and* $L[x] \models$ " $\mathbb{P}$ is ccc and $|\mathbb{P}| = \omega_1$"

then

$L[x] \models$ "There exists a Suslin tree T such that $\mathbb{P} \times T$ is ccc".

The main theorem: version 1

Theorem

Assume Σ^1_2-determinacy and that $x^\#$ exists for all $x \in \mathbb{R}$. Then for a Turing cone of $x \in \mathbb{R}$, if N is a generic MA-model over $L[x]$ then the following hold.

1. *N is homogeneous but N is not strongly homogeneous.*
2. *$\mathrm{Th}(N)$ does not depend on the choice of N.*

- Let D be the set of all $t \in \mathbb{R}^N$ such that in N, every Suslin tree of $L[t]$ which is definable from parameters in the structure $L_{\omega_1}[t]$, is specialized.
  - D is Turing invariant.
  - D and $\mathbb{R}^N \setminus D$ are both cofinal in the Turing degrees of N.
    - D is OD^N and so OD-Determinacy fails in N.
- By (2), the theory of N is constant on a cone.

A question of Enayat and Kanovei

Question

Suppose every finite nonempty OD set contains only OD members. Must $V = \mathrm{HOD}$?

Theorem (Solovay)

Suppose x is Sacks generic over L. Then in $L[x]$ there is an OD set with exactly 2 members, and neither is OD.

- What if x is Cohen (or random) over L?
- What if one assumes MA or even MM^{++}?

The main theorem: version 2

Theorem

Assume Σ^1_2-determinacy and that $x^{\#}$ exists for all $x \in \mathbb{R}$. Then for a Turing cone of $x \in \mathbb{R}$, if N is a generic MA-model over $L[x]$ then the following hold.

1. *N is homogeneous but N is not strongly homogeneous.*
2. *$\mathrm{HOD}^N \subset \mathrm{HOD}^{L[x]}$ and does not depend on the choice of N.*
3. *Suppose $Z \in N$, $Z \in \mathrm{OD}^N$, and $|Z|^N \leq \omega_1^N$. Then $Z \subset \mathrm{OD}^N$.*

- In the context of $\mathrm{ZFC} +$ "$V = L(\mathbb{R})$", and with $N = V$:
  - (3) implies that either $V = \mathrm{HOD}$ or that $\neg\mathrm{CH}$ holds.
- By (2), HOD^N should be a "canonical" model.

A convenient hypothesis: $M_1^\#$ exists

Theorem

The following are equivalent.

1. *There is an iterable inner model with a Woodin cardinal, restricting to normal strongly closed iteration trees.*
2. *There is an inner model with a Woodin cardinal and $X^\#$ exists for all $X \subset \mathrm{Ord}$.*

Theorem (after Mitchell-Steel et al)

The following are equivalent.

1. *$M_1^\#$ exists.*
2. *There is an iterable inner model with a Woodin cardinal.*
3. *There is an inner model with a Woodin cardinal and $X^\#$ exists for all $X \subset \mathrm{Ord}$.*

Strategic enlargements of M_1

- δ_{M_1} is the Woodin cardinal of M_1.

Definition

Suppose that $M_1^\#$ exists. Then for each regular cardinal γ of M_1 such that $\delta_{M_1} \leq \gamma$, $M_1^+[\gamma]$ is the smallest inner model N such that

1. $M_1 \subset N$.
2. For every 0-maximal normal iteration tree on M_1 of length θ for some $\theta \leq \gamma$, if $\mathcal{T} \in M_1$ and if $b_\mathcal{T}$ is the unique cofinal wellfounded branch of $\mathcal{T}$, then $b_\mathcal{T} \in N$.

Theorem

Suppose that $M_1^\#$ exists. Then the following hold for each regular cardinal γ of M_1 such that $\delta_{M_1} \leq \gamma$.

1. $M_1^+[\gamma] \subset L[M_1^\#]$ *for all* $\gamma \in \mathrm{Ord}$.
2. *For all sufficiently large* γ, $M_1^+[\gamma] = L[M_1^\#]$.

The Schlutzenberg-Steel reduction

Theorem

Suppose that $M_1^\#$ exists. Let η_0 be the least Silver indiscernible of M_1 and suppose that γ is a regular cardinal of M_1 such that $\delta_{M_1} \leq \gamma < \eta_0$. Then the following hold where $\delta = \delta_{M_1}$.

1. $M_1^+[\gamma] \cap V_\delta = M_1 \cap V_\delta$.
2. *δ is a Woodin cardinal in $M_1^+[\gamma]$.*
3. $M_1^+[\delta] \not\subset M_1$.

Theorem (Schlutzenberg, Steel)

Assume that $M_1^\#$ exists. Let η_0 be the least Silver indiscernible of M_1 and γ is a regular cardinal of M_1 such that $\delta_{M_1} \leq \gamma < \eta_0$. Then there is a 0-maximal normal iteration tree T on M_1 such that

- $M_1^+[\gamma] = M_1[b_T]$

where b_T is the unique cofinal wellfounded branch of $\mathcal{T}$.

The main theorem: version 3

Theorem

Suppose that $M_1^\#$ *exists,* $x \in \mathbb{R}$*, and*

$$M_1^\# \in L[x].$$

Suppose N *is a generic* MA*-model over* $L[x]$*. Then the following hold.*

1. N *is homogeneous but* N *is not strongly homogeneous.*
2. $\mathrm{HOD}^N \subset \mathrm{HOD}^{L[x]}$ *and does not depend on the choice of* N*.*
3. *Suppose* $Z \in N$*,* $Z \in \mathrm{OD}^N$*, and* $|Z|^N \leq \omega_1^N$*. Then* $Z \subset \mathrm{OD}^N$*.*
4. HOD^N *is an iterate of* $M_1^+[\delta_{M_1}]$*.*

$\mathrm{HOD}^{L[x]}$ on a Turing cone

Theorem (Martin)

Assume Σ^1_2-determinacy and that $x^\#$ exists for all $x \in \mathbb{R}$. Then for a Turing cone of $x \in \mathbb{R}$,

- $\mathrm{HOD}^{L[x]} \models$ "There is a Δ^1_3-wellordering of $\mathbb{R}$".
- $\mathrm{HOD}^{L[x]} \cap \mathbb{R} = Q_3$.

Theorem

Assume Σ^1_2-determinacy and that $x^\#$ exists for all $x \in \mathbb{R}$. Then for a Turing cone of $x \in \mathbb{R}$, the following hold.

1. $\omega_2^{L[x]}$ *is a Woodin cardinal in* $\mathrm{HOD}^{L[x]}$.
2. $\omega_2^{L[x]}$ *is the only Woodin cardinal in* $\mathrm{HOD}^{L[x]}$.

Theorem (Kechris, Solovay)

Suppose $V = L[x]$, $x \in \mathbb{R}$, and that Σ^1_2-Determinacy holds.

- *Then* OD*-Determinacy holds.*

The $\mathrm{HOD}^{L[x]}$ problem

- Appealing to the Schlutzenberg-Steel reduction, one can reduce the $\mathrm{HOD}^{L[x]}$ problem to a specific conjecture.

Conjecture

Suppose that $M_1^\#$ exists, $x \in \mathbb{R}$, and

$$M_1^\# \in L[x].$$

Then the following hold.

1. $\mathrm{HOD}^{L[x]}|\omega_2^{L[x]}$ *is an iterate of* $M_1|\delta_{M_1}$.
2. *There exists a 0-maximal normal iteration tree $T \in M_1$ such that $\mathrm{HOD}^{L[x]}$ is an iterate of $M_1[b_T]$, where b_T is the cofinal wellfounded branch of T.*

- The conjecture is true with $L[x]$ replaced by $L[x][G]$ for a wide class of extensions, including the generic MA models.
  - There is a key empirical **monotonicity pattern**:
    - The closer $L[x][G]$ is to $L[y]$ for some $y \in \mathbb{R}^{L[x][G]}$, the closer $M_1[b_T]$ is to M_1.

Corollary of the main theorem on generic MA models

Theorem

Suppose that $M_1^{\#}$ exists, $x \in \mathbb{R}$, and $M_1^{\#} \in L[x]$. Then there exists $E \in L[x]$ such that the following hold.

1. $E \subset \omega_2^{L[x]}$.
2. $\left(\mathrm{HOD}_E\right)^{L[x]}$ *is an iterate of* $M_1^+[\delta_{M_1}]$.
3. $\omega_2^{L[x]}$ *is the Woodin cardinal of* $\left(\mathrm{HOD}_E\right)^{L[x]}$.

Definition

Suppose that $M_1^{\#}$ exists. Then

$$M_1^+\left[\underset{\sim}{\Delta}_2(M_1|\delta_{M_1})\right]$$

is the smallest inner model N such that $M_1 \subset N$ and such that for every 0-maximal normal iteration tree $\mathcal{T}$ on M_1, if

- $\mathcal{T}$ has length δ_{M_1}
- $\mathcal{T}$ is Δ_2-definable from parameters in $M_1 \cap V_{\delta_{M_1}}$;

then the cofinal wellfounded branch of $\mathcal{T}$ is in N.

More evidence about $\mathrm{HOD}^{L[x]}$

Theorem

Suppose that $M_1^\#$ exists, $x \in \mathbb{R}$, and

$$M_1^\# \in L[x].$$

Suppose G is an $L[x]$-generic subset of $\omega_1^{L[x]}$. Then there is a partial order $\mathbb{P} \in L[x][G]$ such that if $H \subset \mathbb{P}$ is $L[x][G]$-generic and if

$$\mathbb{R}_H = \mathbb{R}^{L[x][G][H]},$$

then the following hold.

1. $\mathbb{P}$ *has cardinality* ω_1 *in* $L[x][G]$.
2. $\mathbb{P}$ *is* ccc *in* $L[x][G]$.
3. $\mathrm{HOD}^{L(\mathbb{R}_H)}$ *is an iterate of* $M_1^+\left[\underset{\sim}{\Delta}_2(M_1|\delta_{M_1})\right]$.

- $\mathbb{P}$ is really an $L[x]$-generic iteration of length ω_1
  - but of "simple" ccc partial orders of cardinality ω_1.

A conjecture for $\mathrm{HOD}^{L[x]}$

- By the monotonicity pattern:

Conjecture

Suppose that $M_1^\#$ exists, $x \in \mathbb{R}$, and

$$M_1^\# \in L[x].$$

Then there exists a 0-maximal normal iteration tree $T \in M_1$ such that

- $\mathrm{HOD}^{L[x]}$ *is an iterate of $M_1[b_T]$,*
- $M_1 \subseteq M_1[b_T] \subseteq M_1^+\left[\underset{\sim}{\Delta}_2(M_1|\delta_{M_1})\right]$.

Conjecture

Suppose that $M_1^\#$ exists, $x \in \mathbb{R}$, and

$$M_1^\# \in L[x].$$

Then $\mathrm{HOD}^{L[x]}$ is an iterate of $M_1^+\left[\underset{\sim}{\Delta}_2(M_1|\delta_{M_1})\right]$.

What about strongly homogeneous models?

- The generic MA models over $L[x]$ are never strongly homogeneous
  - The models can never satisfy OD-Determinacy.

Question

Suppose that

- $L(\mathbb{R}) \models \mathrm{ZFC} + \text{``OD-Determinacy''}.$

Suppose that $L(\mathbb{R})$ is accessible to reals.

- *Must $L(\mathbb{R}) = L[x]$ for some $x \in \mathbb{R}$?*

More generally:

Question

Suppose that

- $L(\mathbb{R}) \models \mathrm{ZFC} + \text{``OD-Determinacy''}.$

Must CH hold?

MM^{++} versus supercompact

A widely believed conjecture:

Conjecture

The following are equiconsistent.

1. $\mathrm{ZFC} + \mathrm{MM}^{++}$.
2. $\mathrm{ZFC} + \mathrm{SC}$.
   - i. e. $\mathrm{ZFC} +$ "There is a supercompact cardinal".

- But this conjecture is just one of many. For example:

Conjecture

The following are equiconsistent

1. $\mathrm{ZFC} + \mathrm{MM}^{++} +$ "There is a proper class of huge cardinals".
2. $\mathrm{ZFC} + \mathrm{SC} +$ "There is a proper class of huge cardinals".

The δ-cover and δ-approximation properties

Definition (Hamkins)

Suppose N is an inner model of ZFC and that δ is an uncountable (regular) cardinal of V.

1. N has the δ-**cover property** if for all $\sigma \subset N$, if $|\sigma| < \delta$ then there exists $\tau \in N$ such that:
   - $\sigma \subset \tau$,
   - $|\tau| < \delta$.
2. N has the δ-**approximation property** if for all sets $X \subset N$, the following are equivalent.
   - $X \in N$.
   - For all $\sigma \in N$ if $|\sigma| < \delta$ then $\sigma \cap X \in N$.

- If V is a (set) generic extension of an inner model N then for all sufficiently large regular cardinals δ:
  - N has the δ-approximation property.
  - N has the δ-cover property.

The Hamkins Uniqueness and Universality Theorems

Theorem (Hamkins Uniqueness Theorem)

Suppose N_0 and N_1 both have the δ-approximation property and the δ-cover property. Suppose

- $N_0 \cap H(\delta^+) = N_1 \cap H(\delta^+)$

Then $N_0 = N_1$.

Theorem (Hamkins Universality Theorem)

Suppose that N is an inner model of ZFC with the δ-cover and δ-approximation properties, $\kappa > \delta$, and that κ is a supercompact cardinal.

- *Then κ is a supercompact cardinal in N.*

- The Hamkins Universality Theorem holds for almost all large cardinal notions, except the very strongest notions.
  - The theorem fails for Axiom I_0 cardinals.
- Something seems to be missing.

The δ-genericity property and strong universality

Definition

Suppose that N is an inner model of ZFC and δ is strongly inaccessible.

- Then N has the δ-**genericity property** if for all $\sigma \subset \delta$, if $|\sigma| < \delta$ then σ is N-generic for a partial order $\mathbb{P} \in N$ such that $|\mathbb{P}|^N < \delta$.

Theorem (Strong Universality)

Suppose that:

- *N has the δ-approximation property, the δ-cover property, and the δ-genericity property.*

Suppose that there is a proper class of $\mathrm{Axiom\ I}_0$ cardinals.

- *Then in N, there is a proper class of $\mathrm{Axiom\ I}_0$ cardinals.*

Inner models by approximation and cover

Theorem (Hamkins)

Suppose δ is strongly compact and that N is an inner model with the δ-approximation property and the δ-cover property.

- *Then δ is strongly compact in N.*

Theorem (Viale, Weiss)

Assume PFA and that N is an inner model with the ω_2-approximation property and the ω_2-cover property such that ω_2 is strongly inaccessible in N.

- *Then ω_2 is strongly compact in N.*

Theorem

Suppose that N is an inner model of ZFC, ω_2 is supercompact in N, and that this is witnessed by normal fine ultrafilters which concentrate on sets which are stationary in V.

- *Then N has the ω_2-approximation property and the ω_2-cover property.*

Equiconsistency by purely combinatorial methods?

Conjecture (Viale, Weiss, Foreman)

Assume MM^{++}. Then there exists an inner model N with the ω_2-approximation property and the ω_2-cover property such that ω_2 is strongly inaccessible in N.

- This holds in all known models of MM^{++}.
- This conjecture if true suggests that even the following conjecture might be provable by purely combinatorial methods.

Conjecture

The following are equiconsistent

1. $\mathrm{ZFC} + \mathrm{MM}^{++} +$ "There is a proper class of I_0 cardinals".
2. $\mathrm{ZFC} + \mathrm{SC} +$ "There is a proper class of I_0 cardinals".

An exotic model of MM^{++}

Theorem (ZF)

Suppose that δ is a Vopěnka cardinal and that there exists an elementary embedding

$$j : V_\delta \to V_\delta$$

such that $V_\kappa \prec V_\delta$ where $\kappa = \mathrm{CRT}(j)$. Then there is a generic extension $V[G]$ of V such that

$$V[G]_\delta \models \mathrm{ZFC}$$

and such that the following hold in $V[G]_\delta$.

1. *Vopěnka's Principle.*
2. MM^{++}.
3. *Suppose that N is an inner model with the ω_2-approximation property and the ω_2-cover property. Then $H(\omega_2) \subset N$.*

Generalizations of Axiom $(*)$

- Γ^∞ is the set of all $A \subseteq \mathbb{R}$ such that A is universally Baire.

Definition

1. Axiom $(*)^+$: For each $X \subseteq \mathbb{R}$, there exists $A \subset \mathbb{R}$ and $G \subset \mathbb{P}_{\max}$ such that
   - $L(A, \mathbb{R}) \models \mathrm{AD}^+$.
   - G is $L(A, \mathbb{R})$-generic and $X \in L(A, \mathbb{R})[G]$.
2. Axiom $(*)^{++}$: There exists $\Gamma \subset \mathcal{P}(\mathbb{R})$ and $G \subset \mathbb{P}_{\max}$ such that
   - $L(\Gamma, \mathbb{R}) \models \mathrm{AD}^+$.
   - G is $L(\Gamma, \mathbb{R})$-generic and $\mathcal{P}(\mathbb{R}) \subset L(\Gamma, \mathbb{R})[G]$.
3. Axiom $(*)^{++}_{\mathrm{UB}}$: There exists $G \subset \mathbb{P}_{\max}$ such that
   - G is $L(\Gamma^\infty, \mathbb{R})$-generic.
   - $\mathcal{P}(\mathbb{R}) \subset L(\Gamma^\infty, \mathbb{R})[G]$.
     - (And there is a proper class of Woodin cardinals)

The equivalence theorems

Theorem

The following are equivalent.

1. $\mathrm{Axiom}\ (*)^{+}$ *holds.*
2. $\mathrm{Axiom}\ (*)^{++}$ *holds.*

Theorem

Assume there is a proper class of Woodin cardinals and that MM^{++} *holds. Then following are equivalent.*

1. $\mathrm{Axiom}\ (*)^{+}$ *holds.*
2. $\mathrm{Axiom}\ (*)^{++}$ *holds.*
3. $\mathrm{Axiom}\ (*)^{++}_{\mathrm{UB}}$ *holds.*

- As a corollary, $\mathrm{Axiom}\ (*)^{+}$ fails in all the known models of MM^{++}.

Two key questions about MM^{++}

Definition (A convenient strengthening of Axiom I_0)

Axiom I_0^+ holds at λ if:

1. Axiom I_0 holds at λ.
2. (scheme) $V_\lambda \prec V$.

Question

Assume Axiom I_0^+ holds at λ and that MM^{++} holds.

- Must $\mathbb{R} \subset \mathrm{HOD}$?

Theorem (Aspéro, Schindler)

Assume MM^{++} *holds. Then* $\mathrm{Axiom}\ (*)$ *holds.*

Question

Assume Axiom I_0^+ holds at λ and that MM^{++} holds.

- Must Axiom $(*)^+$ fail?

Is there a generalization of Axiom $(*)$ to MM^{++}?

The condition:

- $\mathrm{HOD} \models$ "$V = \mathrm{Ultimate}\text{-}L$"

is another version of homogeneity for models of MA.

Question

Assume there is a proper class of Woodin cardinals and that MM^{++} holds.

- Can $\mathrm{HOD} \models$ "$V = \mathrm{Ultimate}\text{-}L$"?

Question

Assume Axiom I_0^+ holds at λ and that MM^{++} holds.

- Can $\mathrm{HOD} \models$ "$V = \mathrm{Ultimate}\text{-}L$"?