Hod Pair Capturing

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Problem: Analyze HOD in models of determinacy.

Conjecture 1. Assume $\mathsf{AD}^+ + V = L(P(\mathbb{R}))$; then $\mathrm{HOD} \models \mathsf{GCH}$.

Conjecture 2. There is $M \models \mathsf{AD}^+ + V = L(P(\mathbb{R}))$ such that $\mathrm{HOD}^M \models$ "there is a huge cardinal".

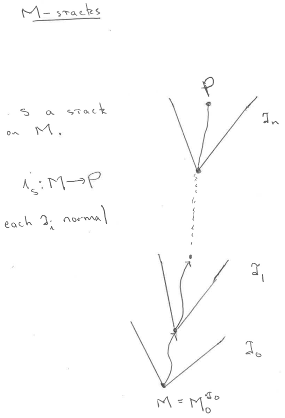
Some terminology

(a) An *extender E over M* is a system of measures on M coding an elementary $i_E \colon M \to \mathrm{Ult}(M, E)$. E is *short* iff all its component measures concentrate on $\mathrm{crit}(i_E)$.

Short extenders can capture subcompactness, but not supercompactness.

(b) A *normal iteration tree on M* is an iteration tree $\mathcal{T}$ on M in which the extenders used have increasing strengths, and are applied to the longest possible initial segment of the earliest possible model.

(c) An *M-stack* is a sequence $s = \langle \mathcal{T}_0, ..., \mathcal{T}_n \rangle$ of normal trees such that $\mathcal{T}_0$ is on M, and $\mathcal{T}_{i+1}$ is on the last model of $\mathcal{T}_i$.



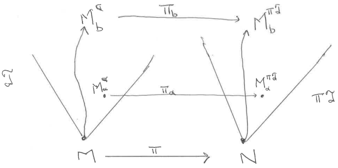
(d) An *iteration strategy* Σ for M is a function that is defined on M-stacks s that are by Σ whose last tree has limit length, and picks a cofinal wellfounded branch of that tree.

(e) If s is an M-stack by Σ with a last model P, then Σ_s is the *tail strategy* for P given by $\Sigma_s(t) = \Sigma(s^\frown t)$.

(f) It $\pi\colon M \to N$ is elementary, and Σ is an iteration strategy for N, then Σ^π is the *pullback strategy* given by: $\Sigma^\pi(s) = \Sigma(\pi s)$.

Pullback strategies

Given Σ for N, and $\pi : M \to N$



if $\quad b = \Sigma(\pi\vec{a})$

then $\quad \Sigma^{\pi}(\vec{a}) = b$

Definition

"No long extenders" (NLE) is the assertion: there is no countable, iterable pure extender mouse with a long extender on its sequence.

Theorem

(S. 2015-2021) Assume AD^+*, and suppose there is a countable, iterable pure extender mouse with a long extender on its sequence; then for any pointclass* Γ *such that* $L(\Gamma, \mathbb{R}) \models \mathsf{NLE}$*,* $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models \mathsf{GCH}$*.*

Theorem

(S. 2015-2021) Suppose that V *is uniquely iterable and that* κ *is 1-extendible; then there is a pointclass* Γ *such that* $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}}$ *and* $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models$ *"there is a subcompact cardinal".*

The proofs go by isolating the notion of *mouse pair*, and proving a general comparison theorem for them. *Modulo the existence of iteration strategies*, mouse pairs can be used to analyze HOD.

Mouse pairs

Definition

(a) A *pure extender premouse* is a structure $\mathcal{M}$ constructed from a coherent sequence $\dot{E}^{\mathcal{M}}$ of extenders.

(b) A *least branch premouse* (lpm) is a structure $\mathcal{M}$ constructed from a coherent sequence $\dot{E}^{\mathcal{M}}$ of extenders, and a predicate $\dot{\Sigma}^{\mathcal{M}}$ for an iteration strategy for $\mathcal{M}$.

Remarks

(a) $\mathcal{M}$ has a hierarchy, and a fine structure. (The *projectum-free spaces* fine structure.)

(b) We use Jensen indexing for the extenders in $\dot{E}^{\mathcal{M}}$.

(c) At strategy-active stages in an lpm, we tell $\mathcal{M}$ the value of $\dot{\Sigma}^{\mathcal{M}}(\mathcal{T})$, where $\mathcal{T}$ is the $\mathcal{M}$-least tree such that $\dot{\Sigma}^{\mathcal{M}}(\mathcal{T})$ is currently undefined. (Woodin, Schlutzenberg-Trang.)

Definition

A *mouse pair* is a pair (P, Σ) such that

(1) P is a countable premouse (pure extender or least branch),

(2) Σ is an iteration strategy defined on all countable stacks on P,

(3) Σ quasi-normalizes well, has strong hull condensation, and is internally lift consistent,

(4) if P is an lpm, then Σ is pushforward consistent; i.e. whenever Q is a Σ-iterate of P via s, then $\dot{\Sigma}^{\mathcal{Q}} \subseteq \Sigma_s$.

Elementary properties of mouse pairs

Definition

$\pi\colon (P,\Sigma) \to (Q,\Psi)$ is *elementary* iff $\pi\colon P \to Q$ is Σ_k elementary, where $k = k(P)$, and $\Sigma = \Psi^\pi$.

Lemma

An elementary submodel of a mouse pair is a mouse pair.

Definition

(Q,Ψ) is an *iterate of* (P,Σ) iff there is a stack s by Σ with last model Q, and $\Psi = \Sigma_s$.

Lemma

(Iteration maps are elementary) Let (P, Σ) *be a mouse pair, and let s be a stack by* Σ *giving rise to the iteration map* $\pi\colon P \to Q$*; then* $(\Sigma_s)^\pi = \Sigma$*.*

This property of Σ is called *pullback consistency*.

Lemma

(Dodd-Jensen) The Σ*-iteration map from* (P, Σ) *to* (Q, Ψ) *is the pointwise minimal elementary embedding of* (P, Σ) *into* (Q, Ψ)*.*

Remark. The concept of mouse pair lets us state the Dodd-Jensen in its proper generality.

Comparison

Theorem (Comparison)

Assume AD^+*, and let* (P, Σ) *and* (Q, Ψ) *be mouse pairs of the same type such that* P *and* Q *are countable; then they have a common iterate* (R, Φ) *such that* R *is countable and at least one of* P*-to-*R *and* Q*-to-*R *does not drop.*

Definition

(Mouse order) $(P, \Sigma) \leq^* (Q, \Psi)$ iff (P, Σ) embeds elementarily into some iterate of (Q, Ψ).

Corollary

Assume AD^+*; then the mouse order* $\leq^*$ *on mouse pairs of a fixed type is a prewellorder.*

Remark. In general, there is no mouse order on mice. How P and Q compare depends on which iteration strategies are used to compare them.

Phalanx comparisons work too. From this we get

Theorem

Assume AD^+*, and let* (P, Σ) *be a mouse pair; then the standard parameter of* P *is solid and universal, and hence* (P, Σ) *has a core.*

Theorem

Assume AD^+*, and let* N *be a countable, iterable, coarse* Γ*-Woodin model; then the hod pair construction of* N *does not break down.*

Theorem

Suppose that V *is uniquely iterable, and there are arbitrarliy large Woodin cardinals; then the hod pair construction of* V *does not break down.*

Phalanx comparisons show that the lpm component of an lbr hod pair has Condensation, Dodd solidity, and other fine structural properties of pure extender mice. (S., Trang.)

Concerning the strategy component of mouse pairs, comparison yields

Theorem

Assume AD^+*, and let* (P, Σ) *be a mouse pair; then*

(1) Σ *is positional,*

(2) Σ *has very strong hull condensation, and*

(3) Σ *fully normalizes well.*

Hod pair capturing

Least branch hod pairs can be used to compute HOD, provided that there are enough of them.

Definition

(AD^+) HOD *pair capturing* (HPC) is the statement: for every Suslin, co-Suslin set of reals A, there is an lbr hod pair (P, Σ) with scope HC such that A is definable over $(\mathrm{HC}, \in, \Sigma)$.

Remark.

(a) Under AD^+, if (P, Σ) is a mouse pair, then $\mathrm{Code}(\Sigma)$ is Suslin and co-Suslin.

(b) HPC implies that every Suslin-co-Suslin set of reals A is in a derived model of some hod pair (P, Σ). So the theory of $L(A, \mathbb{R})$ is definable over P.

Theorem
Assume AD^+*, and that there is an iterable premouse with a long extender. Let* $\Gamma \subseteq P(\mathbb{R})$ *be such that* $L(\Gamma, \mathbb{R}) \models \mathsf{NLE}$ *; then* $L(\Gamma, \mathbb{R}) \models \mathsf{HPC}$*.*

In light of this theorem, the following is almost certainly true:

Conjecture. $(\mathsf{AD}^+ + \mathsf{NLE}) \Rightarrow \mathsf{HPC}$.

HPC holds in the minimal model of $\mathsf{AD}_{\mathbb{R}} + \theta$ is regular, and somewhat beyond, by Sargsyan's work.

Theorem (Sargsyan, S. 2018)
Assume $\mathsf{AD}^+ + \neg\mathsf{HPC}$*; then there is an lbr hod pair* (P, Σ) *such that* $P \models \mathsf{ZFC} +$ *"there is a strong cardinal with a Woodin cardinal above it".*

Sargsyan (WIP) has strengthened the conclusion to "$P \models$ 'there is a Woodin limit of Woodin cardinals' "

HOD pair constructions and HPC

Assume AD^+, and let

$$\Delta_{\max} = \{A \subseteq \mathbb{R} \mid A \text{ is captured by an lbr hod pair}\}.$$

Let A be Suslin-co-Suslin and $A \notin \Delta_{\max}$. Let $(N^*, \tau, \delta^*, \Sigma^*)$ coarsely capture $A^\sharp$:

(a) N is countable, $N \models \mathsf{ZFC} +$ "δ is Woodin",

(b) Σ is an iteration strategy for N defined on all $s \in \mathsf{HC}$, and $\Sigma \restriction V_\delta^N \in N$, and

(c) if $i \colon N \to M$ is an iteration map by Σ, and g is $\mathrm{Col}(\omega, i(\delta))$-generic over M, then $i(\tau)_g = A^\sharp \cap M[g]$.

Theorem (Woodin, late 1980s)

(AD^+)) For any Suslin-co-Suslin set B, there is an $(N, \tau, \delta, \Sigma)$ that coarsely captures B.

Inside N, we have the maximal hod pair construction $\langle (M_{\nu,k}, \Omega_{\nu,k}) \mid \langle \nu, k\rangle \leq_{\text{lex}} \langle \delta, 0\rangle\rangle$:

(a) each $(M_{\nu,k}, \Omega_{\nu,k})$ is an lbr hod pair,

(b) an E gets added to the sequence of $M_{\nu,0}$ whenever doing so produces a premouse, and E extends to a nice extender E^* in N,

(c) $\Omega_{\nu,k}$ is the strategy for $M_{\nu,k}$ that is induced by Σ,

(d) information about $\Omega_{\nu,k}$ is inserted at strategy-active stages, and

(e) $(M_{\nu,k+1}, \Omega_{\nu,k+1}) = \text{core}(M_{\nu,k}, \Omega_{\nu,k})$.

This construction never breaks down; all levels are lbr hod pairs whose cores exist, and the E added in (b) is unique.

We want to show that there is $\nu < \delta$ and E such that E is long and $(M_{\nu,0}, E)$ is iterable. Let

$$(H, \Omega) = (M_{\delta,0}, \Omega_{\delta,0}).$$

It is enough to find in N a club of $\eta < \delta$ on which $P(\eta) \cap H$ is uniformly definable in $L(A, \mathbb{R})$ from A and V_η^N.

We can show this if H does not have a Woodin cardinal η such that $\kappa < \eta < \delta$, where κ is the least strong cardinal of H.

The proof also shows that $P(\mathbb{R}) \cap L(\Delta_{\max}, \mathbb{R}) = \Delta_{\max}$ and $L(\Delta_{\max}, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}} +$ "θ is regular".

One can also show that $\Delta_{\max} = \Gamma \cap \check{\Gamma}$, where Γ is nonselfdual and *not* closed under real quantifiers.

HOD as a mouse limit

Definition
(AD$^+$) For (P, Σ) a mouse pair, $M_\infty(P, \Sigma)$ is the direct limit of all nondropping Σ-iterates of P, under the maps given by comparisons.

$M_\infty(P, \Sigma)$ is well-defined by the Dodd-Jensen lemma. Moreover, it is OD from the rank of (P, Σ) in the mouse order. Thus $M_\infty(P, \Sigma) \in \mathrm{HOD}$. It is an initial segment of the lpm hierarchy of HOD *if* (P, Σ) is "full".

Definition
A mouse pair (P, Σ) is full iff for all mouse pairs (Q, Ψ) such that $(P, \Sigma) \leq^* (Q, \Psi)$, we have $M_\infty(P, \Sigma) \trianglelefteq M_\infty(Q, \Psi)$.

Theorem

Assume $\mathrm{AD}_{\mathbb{R}}$ + HPC*; then* $\mathrm{HOD}\,|\theta$ *is the union of all* $M_\infty(P,\Sigma)$ *such that* (P,Σ) *is a full lbr hod pair.*

Theorem

Assume $\mathrm{AD}^+ + V = L(P(\mathbb{R})) + \mathrm{HPC}$*; then* $\mathrm{HOD}\,|\theta$ *is an lpm. Thus* $\mathrm{HOD} \models \mathrm{GCH}$*.*

The Woodins of HOD

Recall the *Solovay sequence*: θ_0 is the sup of the lengths of OD prewellorders of $\mathbb{R}$, $\theta_{\alpha+1}$ is the sup of the OD(A) prewellorders, for any and all A of Wadge rank θ_α, and $\theta_\lambda = \bigcup_{\alpha<\lambda} \theta_\alpha$ for λ a limit.

Definition

κ is a *cutpoint* of a premouse M iff there is no extender E on the M-sequence such that $\mathrm{crit}(E) < \kappa \leq \mathrm{lh}(E)$.

Theorem

Assume $\mathsf{AD}^+ + V = L(P(\mathbb{R})) + \mathsf{HPC}$*; then equivalent are:*

(a) δ *is a cutpoint Woodin cardinal of* HOD*,*

(b) $\delta = \theta_0$*, or* $\delta = \theta_{\alpha+1}$ *for some* α*.*

Thus θ_0 *is the least Woodin cardinal of* HOD*.*

Remark. Woodin showed θ_0 and the $\theta_{\alpha+1}$ are Woodin in HOD. He proved an approximation to their being cutpoints.

Theorem

Assume $\mathsf{AD}_{\mathbb{R}}$ + HPC*, and let* κ *be a successor cardinal of* HOD *such that* $\kappa < \theta$*. Let*

$$\delta = \sup(\{|S| \mid S \text{ is an OD prewellorder of } {}^{\omega}\kappa\}).$$

Then δ *is the least Woodin cardinal of* HOD *above* κ*.*

Remark. This was conjectured by Sargsyan.

The construction of Suslin representations for the iteration strategies in mouse pairs plays an important role in many of the proofs above.

Suslin representations for mouse pairs

Let (P, Σ) be a mouse pair. A tree $\mathcal{T}$ by Σ is *M_∞-relevant* iff there is a normal $\mathcal{U}$ by Σ extending $\mathcal{T}$ with last model Q such that the branch P-to-Q does not drop. Σ^{rel} is the restriction of Σ to M_∞-relevant trees.
Recall that A is *κ-Suslin* iff $A = p[T]$ for some tree T on $\omega \times \kappa$.

Theorem

(AD$^+$) Let (P, Σ) be an lbr hod pair with scope HC; then Code(Σ^{rel}) is κ-Suslin, for $\kappa = |M_\infty(P, \Sigma)|$.

Remark. Code(Σ^{rel}) is not α-Suslin, for any $\alpha < |M_\infty(P, \Sigma)|$, by Kunen-Martin. So $|M_\infty(P, \Sigma)|$ is a Suslin cardinal.

Suslin cardinals and mouse limits

Theorem (Jackson, Sargsyan, S. 2018-2019)

Let (P, Σ) be a mouse pair, and let $\kappa < o(M_\infty(P, \Sigma))$; then equivalent are

(a) κ is a Suslin cardinal,

(b) $\kappa = |\tau|$ for some cutpoint τ of $M_\infty(P, \Sigma)$.

Corollary

Assume $\mathsf{AD}^+ + \mathsf{HPC}$*; then equivalent are*

(a) κ is a Suslin cardinal,

(b) $\kappa = |\tau|$, for some cutpoint τ of HOD.

Determinacy models from hod pairs

Woodin limits of Woodins have more strength than one might guess.

Theorem (Sargsyan,S.)

Assume AD^+*, and that there is an lbr hod pair* (P, Σ) *such that* $P \models \mathsf{ZFC}$ + *"*δ *is a Woodin limit of Woodin cardinals + "there are infinitely many Woodin cardinals above* δ*". Then there is a pointclass* Γ *such that*

(1) $L(\Gamma, \mathbb{R}) \models$ *"the largest Suslin cardinal exists, and belongs to the Solovay sequence" (*LSA*), and*

(2) $L(\Gamma, \mathbb{R}) \models$ *"if A is a set of reals that is* $OD(s)$ *for some* $s \colon \omega \to \theta$*, then A is Suslin and co-Suslin".*

Part (1) is due to Sargsyan, and requires weaker hypotheses on P. The insight that Woodin limits of Woodins are what you need for (2) is due to Sargsyan.

HOD pairs and Chang models

Relatives of the following theorems were proved earlier by Woodin.

Theorem (Gappo, Sargsyan 2022)

Suppose that there are arbitrarily large Woodin cardinals, and that there is an lbr hod pair (P, Σ) such that P is countable, Σ is coded by a uB set, and $P \models \mathrm{ZFC}+$ "there is a Woodin limit of Woodin cardinals"; then the Chang model $L({}^{\omega}OR)$ satisfies AD.

Let $F(\alpha, X)$ iff $X \subseteq P_{\omega_1}({}^{\omega}\alpha)$ and contains a club in $P_{\omega_1}({}^{\omega}\alpha)$.

Corollary (to proof)

Suppose that there are arbitrarily large Woodin cardinals, and that there is an lbr hod pair (P, Σ) such that P is countable, Σ is coded by a uB set, and $P \models \mathrm{ZFC}+$"there is a measurable Woodin cardinal". Let $F(\alpha, X)$ iff X contains a club in $P_{\omega_1}({}^{\omega}\alpha)$; then

(1) $L({}^{\omega}OR)[F] \models \mathrm{AD}_{\mathbb{R}}$, *and*

Remarks

(i) The model of the corollary satisfies $AD_{\mathbb{R}}$ plus "ω_1 is X-supercompact, for all sets X.

(ii) We don't see how to reduce the mouse-existence hypothesis in the corollary to that in the theorem. Both proofs lean heavily of the theory of hod mice, and on the proofs of approximations to HPC that we have now.

(iii) Woodin had already found a proof of the same conclusions from a proper class of Woodin limits of Woodins, using results of Neeman on iterability and long game determinacy at that level.

(iv) In the Gappo-Sargsyan proof, initial segments of the Chang model in question get realized as generalized derived models associated to iterates of (P, Σ).

(v) The proof of HPC may require a better understanding of models of $\mathsf{AD}_{\mathbb{R}} + V \neq L(P(\mathbb{R}))$.

Thank you!