Closed groups generated by generic measure preserving transformations

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Supported by NSF grant DMS-1954069

February 2023

Polish groups

Polish group = a topological group whose group topology is Polish

There is **no** Haar measure unless the group is locally compact.

But **meagerness** and **Baire property** are translation invariant notions of smallness and regularity.

A convention:

a **generic** $g \in G$ **has property** P if
$\{g \in G \mid g \text{ has } P\}$ is **comeager**

The group of measure preserving transformations

Aut $=$ the Polish group of all Lebesgue measure λ preserving transformations of $[0,1]$

Measure preserving transformations are identified if they coincide on a set of full measure.

Aut is taken with composition and the weak topology, that is,

$$T_n \to T \text{ iff } \lambda\big(T_n(A) \triangle T(A)\big) \to 0, \text{ for each Borel } A \subseteq [0,1].$$

Some important results in Ergodic Theory can be viewed as results about the following **equivalence relation** on Aut:

$T_1 \sim T_2$ if and only if $T_1 = ST_2S^{-1}$, for some $S \in \mathrm{Aut}$.

The study of **generic** elements of Aut has a long history:

Halmos: *In general a measure preserving transformation is mixing*, Ann. of Math. 1944

Rokhlin: *A 'general' measure-preserving transformation is not mixing*, Doklady Akad. Nauk SSSR 1948

The group of measurable functions

$\mathbf{L^0}$ = the Polish group of all Lebesgue measurable functions from $[0,1]$ to the unit circle

L^0 is taken with pointwise multiplication and the topology of convergence in measure.

The unitary group

$\boldsymbol{\mathcal{U}}$ = the Polish group of unitary transformations of the separable, infinite dimensional, complex Hilbert space

$\mathcal{U}$ is taken with composition and the strong operator topology.

The question

The subject matter of the talk

For a Polish group G and $g \in G$, let

$$\langle g \rangle_c = \mathrm{closure}\big(\{g^n \mid n \in \mathbb{Z}\}\big).$$

We study **closed subgroups of** Aut **generated by generic elements of** Aut, that is, groups of the form

$$\langle T \rangle_c,$$

for a generic measure preserving transformation T.

$T_1, T_2 \in \mathrm{Aut}$

$T_1 \sim T_2$ if and only if $T_1 = ST_2S^{-1}$, for some $S \in \mathrm{Aut}$.

$\sim$ is an equivalence relation on Aut with a **complicated behavior** even for generic elements of Aut—Rokhlin, Foreman–Weiss.

$T_1 \equiv T_2$ if and only if $\langle T_1 \rangle_c$ and $\langle T_2 \rangle_c$ are isomorphic as topological groups

$\equiv$ is an equivalence relation on Aut that is more generous than $\sim$.

Hope: $\equiv$ has a much more uniform behavior than $\sim$;
maybe even there is a generic $\equiv$-class, that is, a $\equiv$-class that is comeager.

The question

Glasner–Weiss: Is it the case that for a generic $T \in \mathrm{Aut}$, $\langle T \rangle_c$ is isomorphic to L^0?

Motivation for the question

Qualifications

Glasner–Weiss: $\langle T \rangle_c$ is isomorphic to L^0 for some $T \in \mathrm{Aut}$.

Analogy

Melleray–Tsankov: $\langle U \rangle_c$ is isomorphic to L^0 for a generic $U \in \mathcal{U}$.

Structure

Ageev: For a generic $T \in \mathrm{Aut}$, each finite abelian group embeds into $\langle T \rangle_c$.

S.: For a generic $T \in \mathrm{Aut}$, there is a Polish linear space L_T and a continuous surjective homomorphism $L_T \to \langle T \rangle_c$.

Dynamics

Glasner–Weiss: For a generic $T \in \mathrm{Aut}$, the natural boolean action of $\langle T \rangle_c$ is whirly.

The theorem

Theorem (S.)

For a generic transformation $T \in \mathrm{Aut}$,
the group $\langle T \rangle_c$ *is* **not** *isomorphic to* L^0.

A rough outline of the proof

Prove the following two points.

1. If $L^0 \cong \langle T \rangle_c < \mathrm{Aut}$, for a generic $T \in \mathrm{Aut}$, then **some** ergodic boolean action of L^0 has **spectral properties** similar to spectral properties of a generic $T \in \mathrm{Aut}$.

2. No ergodic boolean actions of L^0 has **spectral properties** similar to spectral properties of a generic $T \in \mathrm{Aut}$.

Spectral behavior

Spectral behavior of a generic $T \in \mathrm{Aut}$

Building on earlier work of Choksi–Nadkarni, Katok, and Stepin, del Junco–Lemańczyk proved:

Theorem (del Junco–Lemańczyk, 1992)

For a generic $T \in \mathrm{Aut}$, a strong **orthogonality condition** *holds for convolutions of maximal spectral types $\nu(T^\ell)$ of powers of T.*

Call the condition the **del Junco–Lemańczyk orthogonality condition**.

Spectral behavior of L^0

A unitary representation of L^0 can be constructed as follows. Given $\phi \in L^0$, let

$$L^2(\lambda) \ni f \to \phi \cdot f \in L^2(\lambda).$$

This is a unitary representation in $\mathcal{U}\big(L^2(\lambda)\big)$.

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Spectral behavior of L^0

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$$L^2(\mu) \ni f \to \phi^k \cdot f \in L^2(\mu),$$

for $\mu \preceq \lambda$ and $k \in \mathbb{Z}$. This is a unitary representation in $\mathcal{U}\big(L^2(\mu)\big)$.

One can form a multidimensional version of the above unitary representation determined by

— a finite Borel measure μ on $[0,1]^n$ with marginals absolutely continuous with respect to λ and

— an assignment of powers to the coordinates of $[0,1]^n$: $i \to k_i$, for $1 \leq i \leq n$.

Let's call these **atomic representations**.

There is a semigroup

$$(A, \oplus)$$

each of whose elements **encodes** n **and** $i \to k_i$**, for** $1 \leq i \leq n$**, of an atomic representation**.

The semigroup A consists of all finite functions x such that

$$\emptyset \neq \mathrm{dom}(x) \subseteq \mathbb{Z}^{\times} \text{ and } \mathrm{rng}(x) \subseteq \mathbb{N}$$

Theorem (S., 2014)

$\xi : L^0 \to \mathcal{U}$ a unitary representation without non-zero fixed vectors

Then, *ξ is built from atomic representations determined by x and finite measures μ_x as x ranges over A.*

The sequence $(\mu_x)_{x \in A}$ is unique up to mutual absolute continuity of its entries.

The above is true modulo multiplicity of μ_x.

Geography of the proof of the main theorem

We study $\xi : L^0 \to \mathcal{U}$ a unitary representation without non-zero fixed vectors.

The main issue is a comparison, for $x, y \in A$, of

$$\mu_x \times \mu_y \text{ and } \mu_{x \oplus y} \text{ computed for } \xi.$$

Del Junco–Lemańczyk condition for L^0

Theorem (Etedadialiabadi, 2020)

$\xi\colon L^0 \to \mathcal{U}$ a unitary representation without non-zero fixed vectors

Assume: *for a generic $\phi \in L^0$, the del Junco–Lemańczyk orthogonality condition holds for maximal spectral types $\nu(\xi(\phi)^\ell)$ of powers of $\xi(\phi)$.*

Then

$$\mu_x \times \mu_y \perp \mu_{x \oplus y}, \text{ for all } x, y \in A.$$

Theorem on Koopman representations of L^0

A continuous homomorphism $\zeta\colon G \to \mathrm{Aut}$ is called a **boolean action**.

Given a boolean action $\zeta\colon G \to \mathrm{Aut}$, the **Koopman representation associated with** ζ is given by

$$G \ni g \to U_g \in \mathcal{U}\big(L^2(\lambda)\big),$$

where, for $f \in L^2(\lambda)$,

$$U_g(f) = f \circ \big(\zeta(g)\big)^{-1}.$$

Theorem (S.)

ξ = the Koopman representation associated with an ergodic boolean action of L^0

Then

$$\mu_x \times \mu_y \preceq \mu_{x \oplus y}, \text{ for all } x, y \in A.$$

Proof of the main theorem

The proposition below gives a connection with the Glasner–Weiss question.
It uses Etedadialiabadi's and del Junco–Lemańczyk's theorems.

Proposition (S.)

Assume, for a generic $T \in \mathrm{Aut}$, $\langle T \rangle_c$ is isomorphic to L^0.
There exists an ergodic boolean action of L^0, whose Koopman representation is such that

$$\mu_x \times \mu_y \perp \mu_{x \oplus y}, \text{ for all } x, y \in A.$$

SO, **for all ergodic Koopman** representations of L^0

$$\mu_x \times \mu_y \preceq \mu_{x \oplus y}, \text{ for all } x, y.$$

BUT, if $\langle T \rangle_c$ is isomorphic to L^0 for a generic $T \in \mathrm{Aut}$, then **there exists an ergodic Koopman** representation of L^0 with

$$\mu_x \times \mu_y \perp \mu_{x \oplus y}, \text{ for all } x, y.$$

CONTRADICTION

Questions

Is there a Polish group G such that $\langle T \rangle_c$ is isomorphic to G, for a generic $T \in \mathrm{Aut}$?

Glasner–Weiss: Is the group $\langle T \rangle_c$ a Lévy group for a generic $T \in \mathrm{Aut}$?