Measures in Simple Structures

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Banach-Tarski Paradox

- **Banach-Tarski Paradox**: A ball in $\mathbb{R}^3$ can be partitioned into finitely many pieces in such a way that, after moving these pieces by translations and rotations, they may be reassembled to form two balls of equal volume as the first.

Figure: From Wikipedia

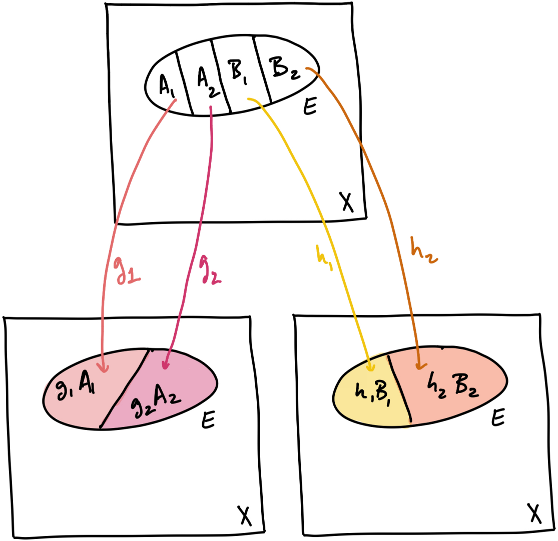
The Question

- In the Banach-Tarski Paradox, how complicated must the pieces be?

Move 1: From the geometric object to its group of symmetries

- Let G be a group of symmetries of a set X (i.e. a group acting on X) and suppose $E \subseteq X$. E is said to be *G-paradoxical* if for some m, n there exist $g_1, \ldots, g_m$ and $h_1, \ldots, h_n \in G$ and pairwise disjoint $A_1, \ldots, A_m$ and $B_1, \ldots, B_n \subseteq E$ such that $E = \bigcup g_i A_i = \bigcup h_j B_j$.

G-paradoxical



Move 1: From the geometric object to its group of symmetries

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- **Banach-Tarski Paradox**: The unit ball in $\mathbb{R}^3$ is $\mathrm{SO}(3)$-paradoxical.
- **Theorem**: A group G is G-paradoxical if and only if there is a free G-paradoxical action on some set X.

Move 2: From decompositions to measures

- A *(finitely additive probability) measure* on the group G is a function $\mu : \mathcal{P}(G) \to [0,1]$ such that
  1. $\mu(G) = 1$.
  2. If $X, Y \subseteq G$ are disjoint, then $\mu(X \cup Y) = \mu(X) + \mu(Y)$.
- A measure on G is called *G-invariant* if, for all $X \subseteq G$ and $g \in G$,

$$\mu(gX) = \mu(X).$$

- **Tarski's Theorem**: A group G is *not* G-paradoxical if and only if there is some G-invariant measure on G. Groups with such a measure are called *amenable* so a group G is not G-paradoxical if and only if it is amenable.

The Question v2

- Suppose a group G is not amenable. Must the group be very complicated?

Dividing lines

- From specific to more general: e.g. from algebraically closed fields to differentially closed fields, difference fields,...
- From general to more specific: e.g., from graphs to real-algebraic or p-adic graphs,...
- Goldilocks: Much of model theory is organized around the search for *dividing lines*: combinatorial properties of theories that divide theories into tame and wild.
- Simplicity: the key dividing line for us, which is characterized by there being a notion of independence (generalizing linear independence in vector spaces, algebraic independence in algebraically closed fields) that is symmetric and transitive, among other useful properties.

The tree property

Definition

The formula $\varphi(x; y)$ has the *tree property* if there is a tree of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ and a number $k < \omega$ so that

1. Paths are consistent – for all $\eta \in \omega^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent
2. Children of a common node are k-inconsistent – for any $\eta \in \omega^{<\omega}$, $\{\varphi(x; a_{\eta \frown \langle \alpha \rangle}) : \alpha < \omega\}$ is k-inconsistent.

The theory T has the tree property if some formula $\varphi(x; y)$ does modulo T.

The tree property

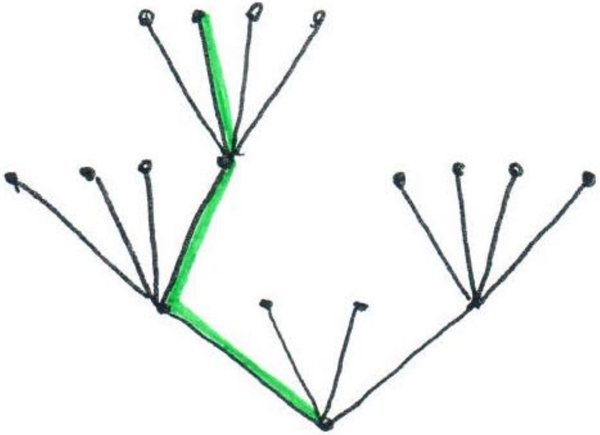

Figure: paths are consistent

The tree property

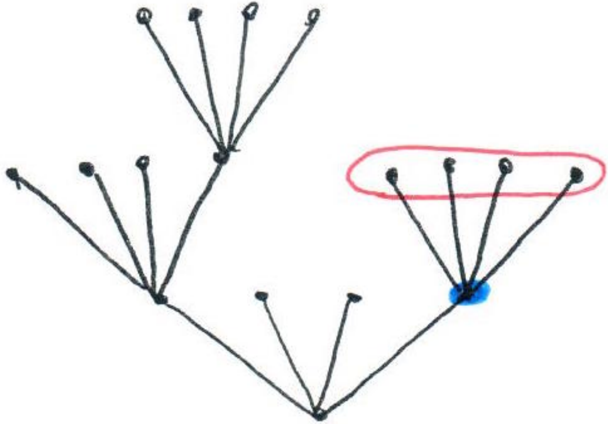

Figure: children of a common node are k-inconsistent

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1. Paths are consistent – for all $\eta \in \omega^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent
2. Children of a common node are k-inconsistent – for any $\eta \in \omega^{<\omega}$, $\{\varphi(x; a_{\eta \frown \langle \alpha \rangle}) : \alpha < \omega\}$ is k-inconsistent.

- The complete theory T is *simple* if and only if T does not have the tree property. Examples of simple theories include random graphs, difference closed fields. Non-examples include dense linear orders and triangle-free random graphs.

The map

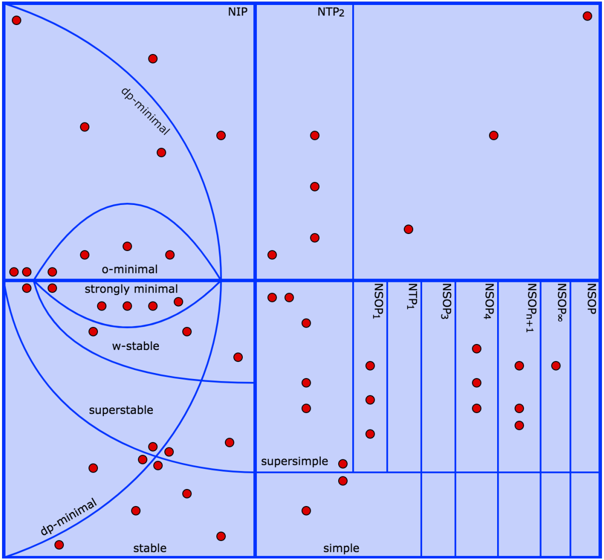


Figure: From Gabe Conant's `forkinganddividing.com`

Forking and dividing

Definition

Suppose A is a set of parameters.

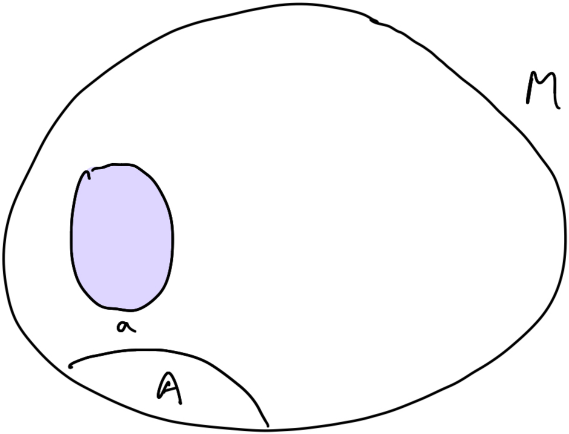
1. $\varphi(x; a)$ *divides over* A if there is an A-indiscernible sequence $\langle a_i : i < \omega \rangle$ with $a_0 = a$ such that $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.
2. $\varphi(x; a)$ *forks over* A if

$$\varphi(x; a) \vdash \bigvee_{i<k} \psi_i(x; c_i),$$

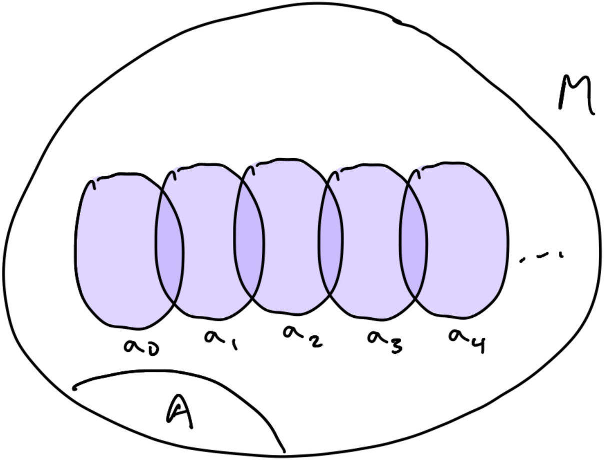
   where each $\psi_i(x; c_i)$ divides over A.

3. We say a (partial) type p forks or divides over A if it implies a formula that does.
4. We write $a \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{f}_{C} b$ to indicate that $\mathrm{tp}(a/bC)$ does not fork over C.

Forking and Dividing



Forking and Dividing



Forking and Dividing

From forking we get three things:

1. Notion of *independence*: We say a is *independent* from b over A, denoted $a \mathop{\smile\!\!\!\!\!\!\,|}_A b$ if a is contained in no Ab-definable set that forks over A.
2. Notion of *generic point*: If $A \subseteq B$ and $a \mathop{\smile\!\!\!\!\!\!\,|}_A B$, then $\mathrm{tp}(a/B)$ is a generic extension of $\mathrm{tp}(a/A)$.
3. Notion of *dimension*: Have the foundation rank on extensions that fork.

Stability

Theorem

(Harnik-Harrington) The theory T is stable if and only if there is an $Aut(\mathbb{M})$-invariant ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ on small subsets of $\mathbb{M}$ satisfying:

1. *Extension: If $a \mathop{\smile\!\!\!\!\!\mid}_C b$, then for all c, there is $a' \equiv_{Cb}$ such that $a' \mathop{\smile\!\!\!\!\!\mid}_C bc$.*
2. *Symmetry: $a \mathop{\smile\!\!\!\!\!\mid}_C b \iff b \mathop{\smile\!\!\!\!\!\mid}_C a$.*
3. *Finite character: $a \mathop{\smile\!\!\!\!\!\mid}_C b$ if and only if $a' \mathop{\smile\!\!\!\!\!\mid}_C b'$ for all finite subtuples $a' \subseteq a$, $b' \subseteq b$.*
4. *Transitivity: If $B \subseteq C \subseteq D$, $a \mathop{\smile\!\!\!\!\!\mid}_B C$, and $a \mathop{\smile\!\!\!\!\!\mid}_C D$ then $a \mathop{\smile\!\!\!\!\!\mid}_B D$.*
5. *Base monotonicity: If $B \subseteq C$ then $a \mathop{\smile\!\!\!\!\!\mid}_B Cd$ implies $a \mathop{\smile\!\!\!\!\!\mid}_C d$.*
6. *Local character: For any a and C, there is $B \subseteq C$ with $|B| \leq |T|$ such that $a \mathop{\smile\!\!\!\!\!\mid}_B C$.*
7. *Stationarity: If $C = \mathrm{acl}^{eq}(C)$, then if $a \equiv_C a'$, $a \mathop{\smile\!\!\!\!\!\mid}_C b$ and $a' \mathop{\smile\!\!\!\!\!\mid}_C b$, then $a \equiv_{Cb} a'$.*

If there is such a relation, it agrees with $\mathop{\smile\!\!\!\!\!\mid}^f$.

Simplicity

Definition

T is simple if $\mathop{\smile\!\!\!\!\!\mid}^f$ satisfies local character: for any a and C, there is $B \subseteq C$ with $|B| \leq |T|$ such that $a \mathop{\smile\!\!\!\!\!\mid}^f_B C$.

Theorem

(Kim-Pillay) The theory T is simple if and only if there is an $Aut(\mathbb{M})$-invariant ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ on small subsets of $\mathbb{M}$ satisfying:

1. *Extension, Symmetry, Finite character, Transitivity, Base monotonicity, Local character*
2. *The Independence Theorem: If $M \models T$, then if $a \equiv_M a'$, $a \mathop{\smile\!\!\!\!\!\mid}_M b$, $a' \mathop{\smile\!\!\!\!\!\mid}_M c$ and $b \mathop{\smile\!\!\!\!\!\mid}_M c$, then there is a_* such that $a_* \equiv_{Mb} a$, $a_* \equiv_{Mc} a'$, and $a_* \mathop{\smile\!\!\!\!\!\mid}_M bc$.*

If there is such a relation, it agrees with $\mathop{\smile\!\!\!\!\!\mid}^f$.

Keisler measures

Definition

A *Keisler measure* over A is a finitely additive probability measure on $\mathrm{Def}_x(A)$, where $\mathrm{Def}_x(A)$ denotes the Boolean algebra of definable sets in the free variables x and parameters coming from A. We will often omit the x.

Example

In $(\mathbb{Q}, <)$, for each formula defined with parameters in $\mathbb{Q}$, we can define

$$\mu(\varphi(x; a)) = \begin{cases} 1 & \text{if } \varphi(\pi; a) \text{ is satisfied in } \mathbb{R} \\ 0 & \text{otherwise.} \end{cases}$$

Examples of Keisler measures

Example

- Lebesgue measure on $[0,1]^n$: we may define a Keisler measure over $\mathbb{R}$, viewed as a field, by stipulating that for any $X \in \mathrm{Def}(\mathbb{R})$,

$$\mu(X) = \lambda(X \cap [0,1]^n).$$

- Nonstandard counting measure: If μ is a $\{0,1\}$-valued finitely additive probability measure on $\mathbb{N}$ and $(M_i)_{i\in\mathbb{N}}$ is a sequence of finite structures, we may form the *ultraproduct* $M = \prod M_i/\mu$, identifying elements of the product that disagree on a set of measure zero. We may define a Keisler measure μ_{count} by defining, for $X \in \mathrm{Def}(M)$,

$$\mu_{\mathrm{count}}(X) = \lim_{\mu} \frac{|X(M_i)|}{|M_i|}.$$

Measures and forking

Definition

Suppose μ is a global Keisler measure. We say μ is *A-invariant* if $\mu(X) = \mu(\sigma(X))$ for all definable sets X (with parameters) and $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$. Equivalently, μ is A-invariant if, given any $\varphi(x; y)$ and $b \equiv_A b'$,

$$\mu(\varphi(\mathbb{M}; b)) = \mu(\varphi(\mathbb{M}; b')).$$

Definition

We say a definable set X is *universally of measure zero* over A if $\mu(X) = 0$ for all global A-invariant measures μ. We refer to the collection of sets universally of measure zero as the *universal measure zero ideal*.

Measures and forking

Observation

A formula that forks over A defines a set that is universally of measure zero over A.

Proof.

As a finite union of sets universally of measure zero is universally of measure zero, it suffices to show that if $\varphi(x; a)$ divides over A, then $\mu(\varphi(\mathbb{M}; a)) = 0$. Let $\langle a_i : i < \omega \rangle$ be an A-indiscernible sequence such that $a_0 = a$ and $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent. If $\mu(\varphi(\mathbb{M}; a)) > 0$ for some A-invariant μ, then there is some maximal k such that $\mu\left(\bigwedge_{i<k} \varphi(x; a_i)\right) > 0$. Then for all $j < \omega$, the sets defined by $\bigwedge_{i<k} \varphi(x; a_{k \cdot j+i})$ have pairwise intersection of measure zero and (by A-indiscernibility) constant positive measure. This contradicts the fact that μ is a probability measure. □

Measures and groups

Definition

Suppose G is a definable group.

1. We say a measure μ on $\mathrm{Def}(G)$ is *G-invariant* if $\mu(X) = \mu(g \cdot X)$ for all definable subsets $X \subseteq G$.
2. We say G is *definably amenable* if there is an invariant Keisler measure on definable subsets of G.

Measures and groups

Example

1. Amenable groups are definably amenable—this includes all solvable groups.
2. All stable groups: $\mathrm{SL}_2(\mathbb{C})$, non-abelian free groups (!).
3. Pseudo-finite groups: If $(G_i)_{i\in\mathbb{N}}$ is a sequence of finite groups, μ is a $\{0,1\}$-valued finitely additive probability measure on $\mathbb{N}$ and $\tilde{G} = \prod_{i\in\mathbb{N}} G_i/\mu$, then for any definable subset $X \subseteq G(F)$, and $g = (g_i)/\mu \in \tilde{G}$, we have:

$$\mu_{\mathrm{count}}(X) = \lim_{\mu} \frac{|X(K_i)|}{|G(K_i)|} = \lim_{\mu} \frac{|g_i X(K_i)|}{|G(K_i)|} = \mu_{\mathrm{count}}(gX).$$

The Question v3

- The simple theories include the stable theories (algebraically and separably closed fields, differentially closed fields, free groups), and many of the most intensively studied examples are pseudo-finite (hence definably amenable).
- This led to the following question: is *every* group definable in a simple theory definably amenable?
- Related question: Do the universal measure zero ideal and forking ideal always agree?

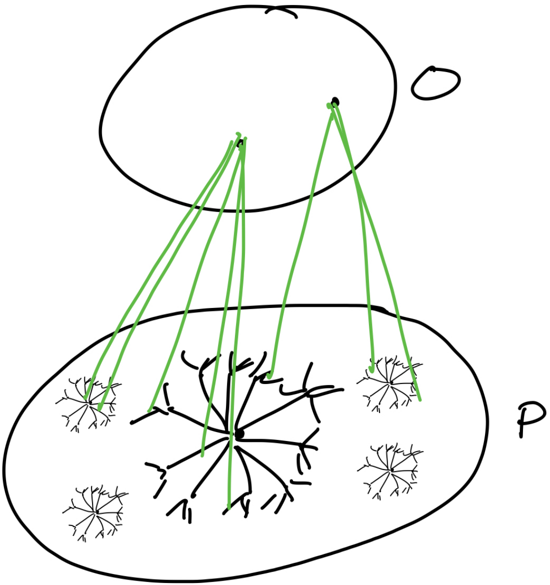
First construction

1. The language L: two sorts O and P, a binary relation $R \subseteq O \times P$, and 10 unary functions from P to P, $f_1^{\pm}$, $f_2^{\pm}$, $f_3^{\pm}$, $g_1^{\pm}$, and $g_2^{\pm}$.
2. For all h in the free group on the 5 generators $\{f_1, f_2, f_3, g_1, g_2\}$ determines a term $t_h(x)$ that defines a function from $P \to P$ by composing the functions in the obvious way.
3. The L-theory T will consist of the following axioms:
   - 3.1 We have an axiom asserting that, for each i, f_i and f_i^{-1} are inverses of each other and similarly for g_i and g_i^{-1}.
   - 3.2 We have an axiom schema asserting that the action of F_5 is free. More precisely, for each non-identity element h in the free group on 5 generators, we have

$$(\forall x \in P)[t_h(x) \neq x].$$

   - 3.3 We finally have an axiom asserting that for all $a \in P$, the sets $R(f_1(a))$, $R(f_2(a))$, and $R(f_3(a))$ are pairwise disjoint and contained in $R(g_1(a)) \cup R(g_2(a))$.

Models of T

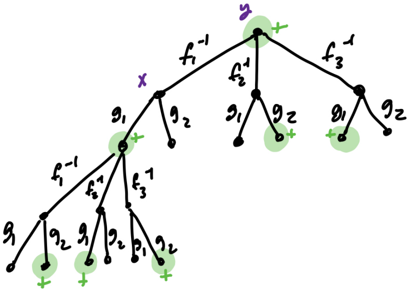


The goal

We want to do the following:

1. Show that the universal theory T has a model companion T^*.
2. Show T^* is simple with trivial forking.
3. Show the formula $R(x; a)$ for any $a \in P$ is universally of measure zero but does not fork.

Implications



Note $y = f_1(x)$ so if

$a \in R(y) = R(f_1(x))$,

then

$a \in R(g_1(x)) \cup R(g_2(x))$,

i.e.

$a \in R(g_1 f_1^{-1}(x))$ or

$a \in R(g_2 f_1^{-1}(x))$.

Axiomatizing T^*

1. Let $G = F_5 = \langle f_1, f_2, f_3, g_1, g_2 \rangle$. Suppose $G \curvearrowright X$ is a free action. We may regard X as a disjoint union of Cayley graphs of G.

2. For $u, v \in X$, we write $d(u, v)$ for the graph distance from u to v and $B_n(v)$ for the ball of radius n centered at v:

$$B_n(v) = \{u \in X \mid d(v, u) \leq n\}.$$

   Given $V \subseteq X$, we also define

$$B_n(V) = \bigcup_{v \in V} B_n(v).$$

Good colorings

Recall we have $G \curvearrowright X$ freely.

Definition

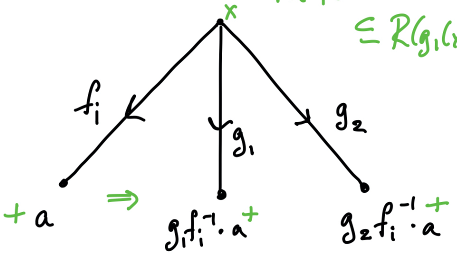
Given $D \subseteq X$, a *good coloring* of D is a function $c \colon D \to \{+, -\}$ such that for all $v \in D$:

1. If $c(v) = +$, then for all $i \in [3]$ there exists $j \in [2]$ such that if $g_j f_i^{-1} v \in D$, then $c(g_j f_i^{-1} v) = +$. (Containments)
2. If $c(v) = +$, then for all $i \neq j \in [3]$, if $f_j f_i^{-1} v \in D$, then $f_j f_i^{-1} v = -$. (Disjointness)

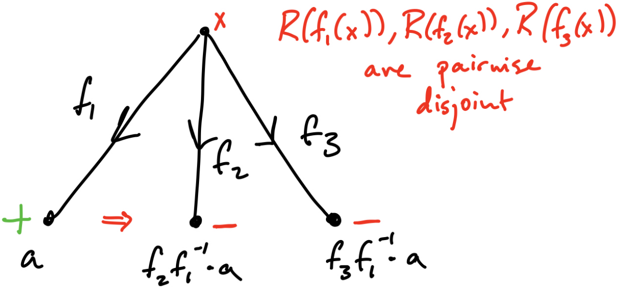
If $D = X$, then we say c is *total*.

Containments

$$R(f_1(x)) \cup R(f_2(x)) \cup R(f_3(x)) \subseteq R(g_1(x)) \cup R(g_2(x))$$



Disjointness



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If $D = X$, then we say c is *total*.

A combinatorial lemma

Recall we have a free action $G \curvearrowright X$.

Lemma

Let V and W be disjoint subsets of X with $|V| = |W| = n$, and let $c\colon V \cup W \to \{+, -\}$ be the function sending each element of V to $+$ and each element of W to $-$. Then there is a good coloring of X extending c if and only if there is a good coloring of $B_N(V)$ extending the restriction of c to $B_N(V)$, where $N = n(n+1) - 2$.

Properties of T^*

1. This bounding lets us axiomatize a model companion T^* —T^* eliminates quantifiers.
2. There are only 1-types over $\emptyset$, which are axiomatized by $x \in O$ and $x \in P$, respectively.
3. Definable closure in T^* is just closure under the action. Hence if A is a set, we have

$$\mathrm{dcl}(A) = O(A) \cup G \cdot P(A).$$

Forking in T^*

Proposition

We have $a \not\mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}_A b$ *if and only if* $a \cap (\mathrm{dcl}(Ab) \setminus \mathrm{dcl}(A)) \neq \emptyset$.

Corollary

T^* *is supersimple of SU-rank* 1.

Corollary

If $a \in P$, *then* $R(x, a)$ *does not fork over the empty set.*

universal measure zero $\neq$ forking

1. Suppose $a \in P$. We have seen $R(x, a)$ does not fork over $\emptyset$.
2. Because there is a unique 1-type in P over $\emptyset$, we have

$$f_1(a) \equiv f_2(a) \equiv f_3(a) \equiv g_1(a) \equiv g_2(a).$$

3. Suppose μ is an invariant Keisler measure with $\mu(R(a)) = \epsilon$. Then we have, by invariance and disjointness,

$$\mu(R(f_1(a)) \cup R(f_2(a)) \cup R(f_3(a))) = 3\epsilon.$$

   By containment and invariance, we have

$$3\epsilon = \mu(R(f_1(a)) \cup R(f_2(a)) \cup R(f_3(a))) \leq \mu(R(g_1(a)) \cup R(g_2(a))) \leq 2\epsilon.$$

   Hence $\epsilon = 0$.

Back to the group question: the strategy

1. Take a definable group whose definable sets are well-understood and not complicated.
2. Enrich the definable sets by adding in new symbols to the language to identify a paradoxical decomposition.
3. Argue that the resulting structure is still not complicated.

A group example

The language L will consist of the language of rings, together with 4 quaternary relations C_1, C_2, C_3, C_4. We will write SL_2 to denote the definable group of 2×2 matrices of determinant 1. It is known that the matrices

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a free group in $\mathrm{SL}_2(\mathbb{Z})$. Hence so do the matrices

$$a^{-k}ba^{k} = \begin{pmatrix} 1-4k & -8k^2 \\ 2 & 4k+1 \end{pmatrix},$$

for $k = 0, \ldots, 11$. We renumber these 12 matrices in some way as $a(i,j)$ $i \in [4]$, $j \in [3]$. We will refer to the group generated by these matrices as G, and we will treat the $a(i,j)$ as though they were individual constants in SL_2 (note that, because they are integer matrices, their entries are already named in the language).

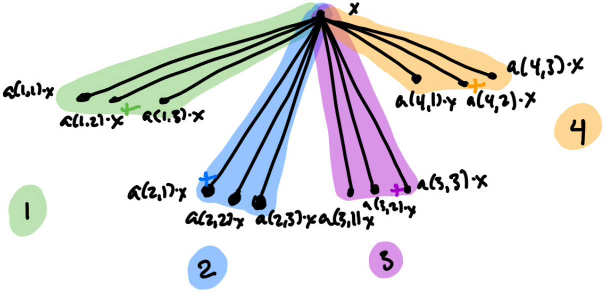
A group example

The theory T will extend the theory of $(\mathbb{C}, +, -, \times, 0, 1)$ with a sentence asserting that C_1, C_2, C_3, C_4 form a partition of SL_2, together with the following axiom:

$$(\forall x \in \mathrm{SL}_2) \left[\bigwedge_{i \in [4]} \bigvee_{j \in [3]} C_i(a(i,j) \cdot x) \right].$$

Gloss: For every group element x and for every index $i \in [4]$, there is some index $j \in [3]$ such that the translation of x by $a(i,j)$ lands inside the set C_i.

Coloring axiom



A group example

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Gloss: For every group element x and for every index $i \in [4]$, there is some index $j \in [3]$ such that the translation of x by $a(i,j)$ lands inside the set C_i.

We show that the *generic* structure satisfying these conditions is simple.

SL_2 is not definably amenable in T^*

Towards contradiction that μ is a Keisler measure on SL_2, invariant under translation. By the coloring axiom, we know that for each $i \in [4]$, we have

$$\mathrm{SL}_2 \subseteq a(i,1)^{-1}C_i \cup a(i,2)^{-1}C_i \cup a(i,3)^{-1}C_i,$$

and, hence, by translation invariance, we have

$$1 \leq 3\mu(C_i),$$

which shows $\mu(C_i) \geq \frac{1}{3}$. On the other hand, because C_1, C_2, C_3, and C_4 partition SL_2, we have

$$1 = \mu(\mathrm{SL}_2) = \sum_{i=1}^{4} \mu(C_i) \geq \frac{4}{3},$$

a contradiction.

Thanks!