MAD families under AD

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Recall the classical definition of an almost disjoint family on ω:

Definition

$\mathcal{A} \subseteq \mathcal{P}(\omega)$ is almost disjoint if each $|A| = \omega$ for every $A \in \mathcal{A}$ and for $A \neq B \in \mathcal{A}$, $|A \cap B| < \omega$. We say $\mathcal{A}$ is a maximal almost disjoint family if it is maximal subject to being an almost disjoint family.

With AC, mad families exist in all contexts. We consider the AD context.

Theorem (Neeman, Norwood)

Assuming AD^+*, there are no mad families on* ω*.*

Answering a question of Mathias, Schrittesser and Törnquist showed the following.

Theorem (Schrittesser, Törnquist)

There are no mad families on ω assuming $\mathrm{DC}_{\mathbb{R}}$, all sets Ramsey, and Ramsey almost everywhere uniformization.

We consider mad families on $\kappa > \omega$. There are two natural definitions of an almost disjoint family.

- Let $\mathcal{B}(\kappa)$ be the ideal of bounded subsets of κ.
- Let $\mathcal{P}_\kappa(\kappa)$ be the ideal of subsets of κ of size $< \kappa$.

These ideals coincide if κ is regular.

S. Müller asked whether there are mad families on ω_1 under AD.

More generally we ask:

Question

- For which κ does AD (or AD^+ or $\mathrm{AD}_{\mathbb{R}}$) imply there are no mad families with respect to $\mathcal{B}(\kappa)$?
- For which κ does AD (or AD^+ or $\mathrm{AD}_{\mathbb{R}}$) imply there are no mad families with respect to $\mathcal{P}_\kappa(\kappa)$?
- What about other ideals on κ?

We first note the existence of the trivial mad families.

Fact

If $\lambda < cof(\kappa)$, then there are mad families of size λ for both ideals $\mathcal{B}(\kappa)$, $\mathcal{P}_\kappa(\kappa)$.

Proof.

Split κ into λ many pairwise disjoint sets. Each of these sets must have size κ. □

Remark

It is not immediately clear if the elements on a mad family must be wellorderable. We discuss this further below.

Main Results

Theorem

Assume AD^+*. If* $\kappa < \Theta$ *then there are no mad families on* κ *for* $\mathcal{B}(\kappa)$*.*

Theorem

Assume AD^+*. If* $\kappa < \Theta$ *and cof*$(\kappa) > \omega$*, then there are no mad families on* κ *for* $\mathcal{P}_\kappa(\kappa)$*.*

Theorem

Assume AD*. If* $\kappa < \Theta$ *and cof*$(\kappa) = \omega$*, then there are no wellorderable mad families on* κ*.*

Corollary (AD^+)

If $\kappa < \Theta$ is regular, then there are no mad families for either $\mathcal{B}(\kappa)$ or $\mathcal{P}_\kappa(\kappa)$.

A simple observation.

Fact (ZF)

For any κ, there are no (wellorderable) mad families of size cof(κ) for either $\mathcal{B}(\kappa)$ of $\mathcal{P}_\kappa(\kappa)$.

Proof.

Consider the case $\mathcal{P}_\kappa(\kappa)$ (other case similar). Suppose $\{A_\alpha\}_{\alpha<\mathrm{cof}(\kappa)}$ is $\mathcal{P}_\kappa(\kappa)$ almost disjoint.

Let $\rho\colon \mathrm{cof}(\kappa) \to \kappa$ be cofinal.

For $\beta < \mathrm{cof}(\kappa)$, let E_β least $\rho(\beta)$ many ordinals in $A_\beta \setminus \bigcup_{\alpha<\beta} A_\alpha$ (which has size κ).

Let $A = \bigcup_{\beta<\mathrm{cof}(\kappa)} E_\beta$.

We first show that there are no (non-trivial) wellorderable mad families.

We say the boldface GCH holds at a cardinal δ if every wellordered sequence of subsets of δ has size $< \delta^+$.

Theorem (Steel)

Assusme $\mathsf{AD} + V = L(\mathbb{R})$. *Then the boldface* GCH *holds below* Θ.

Theorem (Woodin)

Assume AD^+. *Then the boldface* GCH *holds below* Θ.

Lemma

Suppose the boldface GCH *holds at cof*(κ)*. Then there are no wellorderable mad families at* κ *for either* $\mathcal{B}(\kappa)$ *or* $\mathcal{P}_\kappa(\kappa)$*.*

Corollary

Assume AD^+. Then for any $\kappa < \Theta$, there are no wellorderable mad families at κ for either $\mathcal{B}(\kappa)$ or $\mathcal{P}_\kappa(\kappa)$.

Proof.

We consider the case $\mathcal{P}_\kappa(\kappa)$, the case $\mathcal{B}(\kappa)$ being similar.

Let $\{A_\alpha\}_{\alpha<\lambda}$ be an almost disjoint family of size λ, where we may assume λ is a cardinal with $\lambda > \mathrm{cof}(\kappa)$. By boldface GCH we may assume $\lambda \leq \kappa$.

Let $\rho\colon \mathrm{cof}(\kappa) \to \kappa$ be cofinal.

We consider $\{A_\alpha\}_{\alpha<\mathrm{cof}(\kappa)}$ and $\{B_\beta\}_{\beta<\lambda}$ where $B_\beta = A_{\mathrm{cof}(\kappa)+\beta}$.

For each $\beta < \lambda$, let $f_\beta\colon \mathrm{cof}(\kappa) \to \mathrm{cof}(\kappa)$ be defined by:

$$f_\beta(\alpha) = \text{ least } \gamma < \mathrm{cof}(\kappa)\ [\text{o.t.}(A_\alpha \cap B_\beta) < \rho(\gamma)]$$

By the boldface GCH at $\mathrm{cof}(\kappa)$ we may enumerate the $\{f_\beta\}_{\beta<\lambda}$ as $\{g_\alpha\}_{\alpha<\mathrm{cof}(\kappa)}$.

We define a set $C \subseteq \kappa$ which is almost disjoint from all the A_α and B_β, a contradiction.

Consider first the case $\lambda = \kappa$.

Let $g(\alpha) = \sup_{\gamma<\alpha} g_\gamma(\alpha)$, for $\alpha < \mathrm{cof}(\kappa)$.

For $\alpha < \mathrm{cof}(\kappa)$, let

$$E_\alpha = \cup\{A_\alpha \cap B_\beta : \beta < \rho(\alpha) \wedge \text{o.t.}(A_\alpha \cap B_\beta) < \rho(g(\alpha))\} \\ \cup \ \cup\{A_\alpha \cap A_\gamma : \gamma < \alpha\}$$

Then $|E_\alpha| < \kappa$.

Let F_α be the least $\rho(\alpha)$ many elements of $A_\alpha \setminus E_\alpha$.

Let $C = \bigcup_{\alpha < \mathrm{cof}(\kappa)} F_\alpha$. Then $C \subseteq \kappa$ with $|C| = \kappa$.

- For $\alpha < \mathrm{cof}(\kappa)$, $|C \cap A_\alpha| < \kappa$ from the second line in the definition of E_α.
- For $\beta < \lambda = \kappa$, $|C \cap B_\beta| < \kappa$ from the first line in the definition of E_α.

The case $\lambda < \kappa$ is similar.

Now we consider general (not wellorderable) mad families.

Lemma

Assume AD + $\mathrm{DC}_{\mathbb{R}}$. *If* $\kappa < \Theta$ *and* $\mathrm{cof}(\kappa) > \omega$, *then every* $\mathcal{B}(\kappa)$ *or* $\mathcal{P}_\kappa(\kappa)$ *almost disjoint family is wellorderable.*

Since $\mathrm{cof}(\kappa) > \omega$, the filter $\mathcal{F}$ of $A \subseteq \kappa$ with $|\kappa \setminus A| < \kappa$ is countably complete.

Since $\kappa < \Theta$, by AD there is, by an argument of Kunen, an ultrafilter (measure) μ on κ which extends $\mathcal{F}$.

Using the coding lemma, $DC_{\mathbb{R}}$ is enough to show that ${}^{\kappa}\kappa/\mu$ is wellordered.

Let $\mathcal{A}$ be an almost disjoint family (for either of the two ideals). Consider first the case $\mathcal{P}_{\kappa}(\kappa)$.

For each $A \in \mathcal{A}$, let $f_A \colon \kappa \to A$ be the increasing enumeration of A.

If $A \neq B \in \mathcal{A}$, then $[f_A]_\mu \neq [f_B]_\mu$, so $A \mapsto [f_A]_\mu$ is an injection of $\mathcal{A}$ into On.

Consider now $\mathcal{B}(\kappa)$.

Let $f_A(\alpha)$ be the least element of A greater than α.

Again we have $A \mapsto [f_A]_\mu$ is an injection of $\mathcal{A}$ into On.

Combining these lemmas we have shown:

Theorem

Assume AD^+. *If* $\kappa < \Theta$ *and* $\mathrm{cof}(\kappa) \neq \omega$, *then there are no mad families on* κ *for either* $\mathcal{B}(\kappa)$ *of* $\mathcal{P}_\kappa(\kappa)$.

We now turn to the case $\mathrm{cof}(\kappa) = \omega$. Here we can only get the result for $\mathcal{B}(\kappa)$.

Here we adapt the argument of Schrittesser and Törnquist.

Definition

We say Ramsey uniformization holds at κ if for all relations $R \subseteq \omega^\omega \times \mathcal{P}(\kappa)$ there is an infinite $A \subseteq \omega$ and a function $\Phi \colon [A]^\omega \cap \mathrm{dom}(R) \to \mathcal{P}(\kappa)$ so that for all $f \in [A]^\omega$, $R(f, \Phi(f))$.

Lemma

Assume AD^+*. Then for every* $\kappa < \Theta$ *we have that Ramsey uniformization holds at* κ*.*

We also need the following ordinal continuity result.

Lemma

Assume AD^+. *Let* $\kappa \in On$ *and* $\Phi \colon [\omega]^\omega \to \mathcal{P}(\kappa) \equiv {}^\kappa 2$. *There there is an infinite* $B \subseteq \omega$ *such that* $\Phi \restriction [B]^\omega$ *is continuous: for all* $f \in [B]^\omega$, *for all* $\alpha < \kappa$, *there is an* $n \in \omega$ *such that for all* $g \in [B]^\omega$ *with* $f \restriction n = g \restriction n$ *we have* $\Phi(f)(\alpha) = \Phi(g)(\alpha)$.

We assume these lemmas for now, and complete the $\mathrm{cof}(\kappa) = \omega$ case for $\mathcal{B}(\kappa)$.

The argument is similar to that of Schrittesser and Törnquist for the $\kappa = \omega$ case. We give a sketch.

Fix $\rho : \omega \to \kappa$ increasing and cofinal. Let $\mathcal{A}$ be a mad family for $\mathcal{B}(\kappa)$.

We consider $\{A_n\}_{n\in\omega}$, an ω-sequence of elements of $\mathcal{A}$.

For $i < j < \omega$, let $\eta_{i,j}$ be the least $\eta \in A_i$ with $\eta > \rho(j)$ and $\eta \notin \bigcup_{m<i} A_m$. For $f \in \omega^\omega$, let $B_f = \{\eta_{f(n),f(n+1)}\}$.

Let $R(f, A)$ iff $A \in \mathcal{A}$ and $|A \cap B_f|$ is unbounded in κ. By Ramsey uniformization, let $\Phi\colon [C_0]^\omega \to \mathcal{A}$ be such that for all for $f \in [C_0]^\omega$ we have $R(f, \Phi(f))$.

By the ordinal continuity result, let $C_1 \subseteq C_0$ be such that $\Phi \restriction [C_1]^\omega$ is continuous in the sense of the lemma.

It is not hard to check that Φ is not constant on $[C_1]^\omega$. Let $f, g \in [C_1]^\omega$ with $\Phi(f) \neq \Phi(g)$. Fix α so that $\Phi(f)(\alpha) \neq \Phi(g)(\alpha)$. Fix $s_0 \subseteq f$, $t_0 \subseteq g$ which "force" this.

Suppose s_n, t_n have been defined. Define $P \colon [C_1]^2 \to 2$ by:

- $P(i,j) = 1$ iff $\exists f \in [C_1]^\omega$ with $\min(f) > \sup(s_n)$ so that $\eta_{i,j} \in \Phi(s_n {}^\frown f)$.

Easily, P cannot be homogeneous for the 0 side, so fix $D \subseteq C_1$ homogeneous for the 1 side.

Fix $g \in [D]^\omega$ with $\min(g) > \sup(t_n)$.

Since $B_{t_n{}^\frown g} \cap \Phi(t_n{}^\frown g)$ is unbounded, there is an $n > \mathrm{dom}(t_n)$ so that $\eta_{g(n),g(n+1)} \in \Phi(t_n{}^\frown g)$.

By the homogeneity of D (with $i = g(n)$, $j = g(n+1)$), there is an $f \in [C_1]^\omega$ such that $\min(f) > \sup(s(n)$ and $\eta_{g(n),g(n+1)} \in \Phi(s_n{}^\frown f)$.

By the continuity property of C_1 there are s_{n+1}, t_{n+1} extending s_n, t_n which force that $\eta_{g(n),g(n+1)} \in \Phi(s_n{}^\frown f) \cap \Phi(t_n{}^\frown g)$.

If we let $f = \bigcup_n s_n$, $g = \bigcup_n t_n$, then $\Phi(f) \neq \Phi(g)$, $\Phi(f), \Phi(g) \in \mathcal{A}$, and $\Phi(f) \cap \Phi(g)$ is unbounded in κ, a contradiction.

The $\operatorname{cof}(\kappa) = \omega$ case for $\mathcal{P}_\kappa(\kappa)$ is still open.

Question

Assume AD^+. Can there exist a mad family on κ, where $\operatorname{cof}(\kappa) = \omega$, for the ideal $\mathcal{P}_\kappa(\kappa)$?

This is open even for $\kappa = \omega_\omega$.

Sketch of the lemmas

Ramsey uniformization at κ

Let $\kappa < \Theta$ and $R \subseteq [\omega]^\omega \times \mathcal{P}(\kappa)$.

By the coding lemma, let $\pi \colon \mathbb{R} \to \mathcal{P}(\kappa)$. Let $R'(f, x)$ iff $R(f, \pi(x))$.

It is enough to get $A \in [\omega]^\omega$ and $F \colon [A]^\omega \to \mathbb{R}$ uniformizing R'.

But Ramsey uniformization for $\mathbb{R}$ follows from AD^+ (Using a result of Woodin giving Σ_1-reflection to the Suslin, co-Suslin sets).

Ramsey continuity at κ

Let $\kappa < \Theta$.

We use the facts from AD^+ that $\omega \to (\omega)^2$ and the Ramsey null ideal is fully additive.

First show continuity for $\Psi \colon \omega^\omega \to \kappa$.

- Partition $\mathcal{P}$: set $P(f) = 0$ iff $\exists n$ so that $\forall g \in [\mathrm{ran}(f) \setminus \sup(f \restriction n)]^\omega$ we have $\Phi(f) = \Phi(f \restriction n ^\frown g)$.

On the homogeneous side this must hold (use fact that a wellordered union of Ramsey null sets is Ramsey null).

By a standard ω sequence construction guessing initial sequences we construct a set $H \in [\omega]^\omega$ witnessing continuity.

If continuity fails, then for all $h \in [\omega]^\omega$ there is an $\alpha < \kappa$ and an $f \in [h[\omega]]^\omega$ such that for all n there is a $g \in [h[\omega]^\omega]$ with $g \restriction n = f \restriction n$ and $\Phi(f)(\alpha) \neq \Phi(g)(\alpha)$.

Let $\Psi \colon [\omega]^\omega \to \kappa$ be such that $\Psi(h)$ is the least such α for h (so there is some such $f \in [h[\omega]^\omega)$.

Let $C \in [\omega]^\omega$ such that $\Phi \restriction [C]^\omega$ is continuous.

Fix $h^* \in [C]^\omega$, let m be given by continuity of Ψ. Let $\alpha^* = \Psi(h^*)$. Let $F = \{h^*(0), \dots, h^*(m-1)\}$. For each $t \in F^{<\omega}$ successively thin out $C \setminus F$ to be homogeneous for $\Phi(t^\frown g)(\alpha^*)$.

This produces $D \subseteq C \setminus F$. Let $h'[\omega] = F \cup D$.

Then $\Psi(h') = \Psi(h^*) = \alpha^*$ But for any $f \in [h'[\omega]]^\omega$, $f = t^\frown p$ and if we let $n = |t|$, then for any $g \in [h'[\omega]]^\omega$ with $g \restriction n = f \restriction n$ we have $\Phi(f)(\alpha^*) = \Phi(g)(\alpha^*)$, a contradiction.