Groups generated by generic measure preserving transformations

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$Aut(\mu)$

μ a Borel probability measure on a standard Borel space X

$\mathrm{Aut}(\mu)$ = all μ-preserving Borel bijections of X up to measure 0

Topology on $\mathrm{Aut}(\mu)$ = generated by functions

$$T \longrightarrow \mu(A \triangle TB)$$

$A, B \subseteq X$ Borel.

This is a Polish topology.

Property P holds for the generic $T \in Aut(\mu)$ $\equiv$ $\{T \in Aut(\mu): \text{P holds for } T\}$ is comeager in $Aut(\mu)$.

Notation: $\langle T \rangle = \overline{\{T^n : n \in \mathbb{Z}\}}$

$C(T) = \{S : ST = TS\}$

J. King: For the generic T

$$\langle T \rangle = C(T).$$

This is false for arbitrary T.

$L_0(\lambda, \mathbb{T})$ and $L_0(\lambda, \mathbb{R})$

λ a Borel probability measure

$\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$

$L_0(\lambda, \mathbb{T})$ = all λ-measurable functions to $\mathbb{T}$ with the topology of convergence in measure

$L_0(\lambda, \mathbb{R})$ = — " —

— " — to $\mathbb{R}$ — " —

— " —

$$\exp : L_0(\lambda, \mathbb{R}) \longrightarrow L_0(\lambda, \mathbb{T})$$

$$\exp(f)(x) = e^{i \cdot f(x)}$$

Proposition (5.)

Let $\varphi : \mathbb{R} \longrightarrow L_0(\lambda, \mathbb{T})$ be a 1-parameter subgroup. Then for a unique $f \in L_0(\lambda, \mathbb{R})$

$$\varphi(t) = \exp(t \cdot f).$$

$L_0(\lambda, \mathbb{T})$ a torus

$L_0(\lambda, \mathbb{R})$ the tangent space to the torus at 0

Theorem

Theorem (S.)

For the generic $T \in Aut(\mu)$ there exists a closed linear space $L_T < L_0(\lambda, \mathbb{R})$ s.t.

$$\langle T \rangle \cong \exp(L_T) < L_0(\lambda, \mathbb{T})$$

<u>Corollary</u> (Ageev, Stepin-Eremenko)

For the generic $T \in Aut(\mu)$, $\langle T \rangle$ contains $\mathbb{T}^{\mathbb{N}}$, so it contains an arbitrary compact abelian group.

In addition to the theorem and King's theorem, we need: for the generic T

$\forall\, U \ni Id$ open $\exists\, S \in U$ s.t.

$$ST = TS, \quad S^2 = Id, \quad S \neq Id.$$

About the proof of the theorem:

- (de la Rue - de Sam Lazaro) The generic T is an element of a 1-parameter subgroup.

- $A \subseteq Aut(\mu)$ Borel

  A is meager

  iff

  $\forall^* T \in Aut(\mu)$ $\langle T \rangle \cap A$ is meager in $\langle T \rangle$.

- (implicit in Herer–Christensen) the form of unitary representations of linear F-spaces (taken as groups with +): they factor through $L_0(\lambda, \mathbb{T})$ for some λ.

A question and unitary representations of $L_0(\lambda, \mathbb{T})$

Question. For $T_1, T_2 \in Aut(\mu)$ define $T_1 \sim T_2$ iff

$$\langle T_1 \rangle \cong \langle T_2 \rangle.$$

Does $\sim$ have a comeager equivalence class?

For the generic T,

$\langle T \rangle$ is a "maximal torus" in $Aut(\mu)$.

(King + the theorem)

Rosendal: Let $T_1 \sim_1 T_2$ iff $\langle T_1 \rangle \cong \langle T_2 \rangle$ via an isomorphism mapping T_1 to T_2. Then $\sim_1$ has meager equivalence classes.

What about: for the generic T,

$$\langle T \rangle \cong L_0(\lambda, \mathbb{T}) \; ?$$

Classification of unitary representations of $L_0(\lambda, \mathbb{T})$.
It uses: factorization theorems of Nikishin, Maurey, and Mezrag and Kwapień's theorem on the form of linear operators

$$L_0(\lambda, \mathbb{R}) \to L_0(\lambda, \mathbb{R}).$$

Theorem (S.)

Each unitary representation of $L_0(\lambda, \mathbb{T})$ decomposes on the orthogonal to the space of fixed vectors into representations of the form

$$\Sigma(\kappa, k_1, \dots, k_n),$$

κ a Borel probab. measure on X^n,

$k_1, \dots, k_n$ non-zero integers,

X = the underlying space of λ.

$\Sigma(\kappa, k_1, \ldots, k_n)$

- κ on X^n s.t.

  $\forall 1 \le i \le n \quad \pi_i \kappa \lessapprox \lambda$

  $\forall 1 \le i < j \le n \quad \kappa(\{x \in K^n : x_i = x_j\}) = 0$

- given $f \in L_0(\lambda, \mathbb{T})$ define U_f on $L_2(\kappa, \mathbb{C})$ by

$$U_f(h) = \Big(\prod_{i \le n} (f \circ \pi_i)^{k_i}\Big) \cdot h$$

for $h \in L_2(\kappa, \mathbb{C})$.