Math 33A, Midterm 2 solutions

Question 1. Find a basis for the kernel of the matrix $A = \begin{pmatrix} -1 & 2 & -3 & 0 \\ 1 & 0 & 2 & -1 \end{pmatrix}$.

We have to find a basis for the space of solutions to the system $A\vec{x} = \vec{0}$. We first find the reduced row echelon form of the augmented coefficient matrix:

$$\left(\begin{array}{cccc|c} -1 & 2 & -3 & 0 & 0 \\ 1 & 0 & 2 & -1 & 0 \end{array}\right) \begin{array}{c} +R_1 \\ \\ \end{array} \longrightarrow \left(\begin{array}{cccc|c} 0 & 2 & -1 & -1 & 0 \\ 1 & 0 & 2 & -1 & 0 \end{array}\right) \textit{swap} \longrightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right) \begin{array}{c} \\ \div 2 \end{array} \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & \frac{-1}{2} & \frac{-1}{2} & 0 \end{array}\right)$$

Using the reduced row echelon form we see that variables x_3 and x_4 remain unknown, and the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2s & + & t \\ \frac{1}{2}s & + & \frac{1}{2}t \\ s & & \\ & & t \end{pmatrix} = s \begin{pmatrix} -2 \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}.$$

The vectors $\begin{pmatrix} -2 \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$ form a basis for the kernel.

Question 2. Find k so that the matrices $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & -4 \\ 2 & 0 & k \end{pmatrix}$ have the same image.

We need k so that $\mathit{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}\} = \mathit{span}\{\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ k \end{pmatrix}\}$. In particular $\begin{pmatrix} 0 \\ -4 \\ k \end{pmatrix}$ must belong to $\mathit{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}\}$, meaning that there must be c_1, c_2, c_3 so that $c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ k \end{pmatrix}$. In other words the system $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 0 & -2 & -4 \\ 1 & 1 & -1 & k \end{array}\right)$ must have a solution. We find the reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 0 & -2 & -4 \\ 1 & 1 & -1 & k \end{array}\right) \cdots\cdots\cdots \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & k+2 \end{array}\right)$$

For the system to have a solution, it must be that $k+2=0$, in other words $k=-2$.

We know so far that $k=-2$ is the only candidate that could possibly make the matrices $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & -4 \\ 2 & 0 & k \end{pmatrix}$ have the same image. It remains to check that indeed this k works, namely that with $k=-2$ the two matrices have the same image.

The vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ span the image of $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \end{pmatrix}$, because the first column is $\vec{v}_1+\vec{v}_2$, the second column is $2\vec{v}_1+\vec{v}_2$, and the third column is $-\vec{v}_2$. The same vectors also span the image of $\begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & -4 \\ 2 & 0 & -2 \end{pmatrix}$, because the first column is $3\vec{v}_1+2\vec{v}_2$, the second columns is $\vec{v}_1$, and the third columns is $-2\vec{v}_2$. So the two images are equal; both are equal to $span\{\vec{v}_1, \vec{v}_2\}$.

Question 3. Let T from $\mathbb{R}^5$ to $\mathbb{R}^5$ be orthogonal projection to $span\{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}\}$. Find the dimension of $kernel(T)$.

The vectors $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ are linearly independent because they are not scalar multiples of each other. Their span is precisely $image(T)$, so $rank(T) = dim(image(T)) = 2$. By the rank-nullity theorem we conclude

$$nullity(T) = 5 - rank(T) = 5 - 2 = 3,$$

so $dim(kernel(T)) = 3$.

Question 4. Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$. Let $W = span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(a) Using Gram-Schmidt, find an orthonormal basis $\vec{w}_1, \vec{w}_2, \vec{w}_3$ for W.

$$\vec{w}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{\vec{v}_1}{2} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \tag{1}$$

$$\vec{w}_2 = \frac{\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1}{\|\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1\|} = \frac{\vec{v}_2 - 0\vec{w}_1}{2} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \tag{2}$$

$$\vec{w}_3 = \frac{\vec{v}_3 - (\vec{v}_3 \cdot \vec{w}_1)\vec{w}_1 - (\vec{v}_3 \cdot \vec{w}_2)\vec{w}_2}{\|\vec{v}_3 - (\vec{v}_3 \cdot \vec{w}_1)\vec{w}_1 - (\vec{v}_3 \cdot \vec{w}_2)\vec{w}_2\|} = \frac{\vec{v}_3 - 1\vec{w}_1 + 1\vec{w}_2}{2} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \tag{3}$$

(b) Write each $\vec{v}_i$ as a linear combination of $\vec{w}_1, \vec{w}_2, \vec{w}_3$, and find the QR decomposition of $\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$.

From equation (1) above we see that $\vec{v}_1 = 2\vec{w}_1$.
From equation (2) above we see that $\vec{v}_2 = 0\vec{w}_1 + 2\vec{w}_2$.
From equation (3) above we see that $\vec{v}_3 = 1\vec{w}_1 - 1\vec{w}_2 + 2\vec{w}_3$.

Collecting these equations into matrix form we get $(\vec{v}_1\vec{v}_2\vec{v}_3) = (\vec{w}_1\vec{w}_2\vec{w}_3)\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$.

In other words

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}.$$

This is the QR decomposition.

(c) Find a (non-zero) vector $\vec{x}$ in $W^{\perp}$.

We need a vector $\vec{x}$ so that $\begin{cases} \vec{w}_1 \cdot \vec{x} = 0 \\ \vec{w}_2 \cdot \vec{x} = 0 \\ \vec{w}_3 \cdot \vec{x} = 0 \end{cases}$. In matrix form this is the system $(\vec{w}_1\vec{w}_2\vec{w}_3)^T\vec{x} = \vec{0}$, namely $\begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{pmatrix}\vec{x} = \vec{0}$. The general solution for this system is $\vec{x} = s\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$. So $\vec{x} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ (or any scalar multiple of this vector) works.

Question 5. In this question, $\vec{v}_1$ and $\vec{v}_2$ are vectors in $\mathbb{R}^2$, and you are told that $\vec{v}_i \cdot \vec{v}_j$ is the entry a_{ij} of the matrix $\begin{pmatrix} 3 & 5 \\ 5 & 7 \end{pmatrix}$. L is the line spanned by $\vec{v}_1$.

(a) Find the $\{\vec{v}_1, \vec{v}_2\}$ coordinates of $proj_L(\vec{v}_2)$.

From the matrix we read off $\vec{v}_2 \cdot \vec{v}_1 = 5$, and $\|\vec{v}_1\|^2 = \vec{v}_1 \cdot \vec{v}_1 = 3$. So

$$proj_L(\vec{v}_2) = \frac{1}{\|\vec{v}_1\|^2}(\vec{v}_2 \cdot \vec{v}_1)\vec{v}_1 = \frac{5}{3}\vec{v}_1 = \frac{5}{3}\vec{v}_1 + 0\vec{v}_2,$$

and the $\{\vec{v}_1, \vec{v}_2\}$ coordinate vector of $proj_L(\vec{v}_2)$ is $\begin{pmatrix} \frac{5}{3} \\ 0 \end{pmatrix}$.

(b) Find the $\{\vec{v}_1, \vec{v}_2\}$ coordinates of the reflection of $\vec{v}_2$ about L.

We have $refl_L(\vec{v}_2) = 2proj_L(\vec{v}_2) - \vec{v}_2 = 2\frac{5}{3}\vec{v}_1 - \vec{v}_2 = \frac{10}{3}\vec{v}_1 - \vec{v}_2$. The $\{\vec{v}_1, \vec{v}_2\}$ coordinate vector of $refl_L(\vec{v}_2)$ is $\begin{pmatrix} \frac{10}{3} \\ -1 \end{pmatrix}$.

(c) Find the $\{\vec{v}_1, \vec{v}_2\}$ matrix of reflection about L.

The first column of the matrix is the $\{\vec{v}_1, \vec{v}_2\}$ coordinate vector of $refl_L(\vec{v}_1)$, which is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, since the reflection of $\vec{v}_1$ is $\vec{v}_1$ itself. The second column of the matrix is the $\{\vec{v}_1, \vec{v}_2\}$ coordinate vector of $refl_L(\vec{v}_2)$, which by (b) is $\begin{pmatrix} \frac{10}{3} \\ -1 \end{pmatrix}$.

So the matrix is $\begin{pmatrix} 1 & \frac{10}{3} \\ 0 & -1 \end{pmatrix}$.

Question 6. Find the orthogonal projection of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ to the plane in $\mathbb{R}^3$ spanned by $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$.

Let us first "orthonormalize" the vectors $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$, that is, find an orthonormal basis of the plane. The Gram-Schmidt process gives $\vec{w}_1 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$, $\vec{w}_2 = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$.

We now compute $proj_{span(\vec{v}_1,\vec{v}_2)}(\begin{pmatrix}1\\2\\3\end{pmatrix}) = proj_{span(\vec{w}_1,\vec{w}_2)}(\begin{pmatrix}1\\2\\3\end{pmatrix}) =$
$(\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \vec{w}_1)\vec{w}_1 + (\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \vec{w}_2)\vec{w}_2 = \frac{10}{3}\begin{pmatrix}2/3\\1/3\\2/3\end{pmatrix} + \frac{5}{3}\begin{pmatrix}-2/3\\2/3\\1/3\end{pmatrix} = \begin{pmatrix}10/9\\20/9\\25/9\end{pmatrix}.$

Question 7. Find the least square solution $\vec{x}^*$ to the system $\begin{pmatrix}1&2\\2&4\\1&4\\2&3\end{pmatrix}\vec{x} = \begin{pmatrix}0\\0\\3\\1\end{pmatrix}.$

The least squares solution to $A\vec{x} = \vec{b}$ is the vector $\vec{x}^*$ so that $\vec{b} - A\vec{x}^*$ is perpendicular to $image(A)$. This means $A^T(\vec{b} - A\vec{x}^*) = \vec{0}$. So $A^T\vec{b} - A^TA\vec{x}^* = \vec{0}$. This leads to the equations $(A^TA)\vec{x}^* = A^T\vec{b}$, and $\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$. (If you remember the final equation, you can just use it directly.)

In our case $\vec{b} = \begin{pmatrix}0\\0\\3\\1\end{pmatrix}$, $A = \begin{pmatrix}1&2\\2&4\\1&4\\2&3\end{pmatrix}$, $A^TA = \begin{pmatrix}1&2&1&2\\2&4&4&3\end{pmatrix}\begin{pmatrix}1&2\\2&4\\1&4\\2&3\end{pmatrix} = \begin{pmatrix}10&20\\20&45\end{pmatrix}$,
and $(A^TA)^{-1} = \begin{pmatrix}10&20\\20&45\end{pmatrix}^{-1} = \frac{1}{10}\begin{pmatrix}9&-4\\-4&2\end{pmatrix}$.
So:

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b} = \frac{1}{10}\begin{pmatrix}9&-4\\-4&2\end{pmatrix}\begin{pmatrix}1&2&1&2\\2&4&4&3\end{pmatrix}\begin{pmatrix}0\\0\\3\\1\end{pmatrix} =$$
$$\frac{1}{10}\begin{pmatrix}9&-4\\-4&2\end{pmatrix}\begin{pmatrix}5\\15\end{pmatrix} = \frac{1}{10}\begin{pmatrix}-15\\10\end{pmatrix} = \begin{pmatrix}-1.5\\1\end{pmatrix}.$$

Question 8. Find a point on the plane $x_1 + x_2 + x_3 = 5$ which is closest possible to the origin.

The line through the origin perpendicular to the plane is $span(\begin{pmatrix}1\\1\\1\end{pmatrix})$. We need a point $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ which lies both on this line and on the plane. In other words we need $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = c\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $x_1 + x_2 + x_3 = 5$. Solving for c we get $c = \frac{5}{3}$, and $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \frac{5}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$.