Math 33A, Midterm 1 solutions

1. We first write the augmented coefficient matrix and then perform Gauss-Jordan eliminations (row operations):

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 3 \\ 3 & 6 & -1 & 0 & 5 \end{array}\right)$$

subtract 3 times row I from row II

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 3 \\ 0 & 0 & 2 & -6 & -4 \end{array}\right)$$

divide row II by 2

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 3 \\ 0 & 0 & 1 & -3 & -2 \end{array}\right)$$

add row II to row I

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & -3 & -2 \end{array}\right)$$

From the RREF we see that variables y and w are going to be arbitrary parameters, while x and z are going to be expressed in terms of these parameters. We successively write:

$$w = s, \quad z = 3s - 2, \quad y = t, \quad x = -2t + s + 1,$$

for arbitrary real parameters s and t. We can also write the solution in the form

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2t + s + 1 \\ t \\ 3s - 2 \\ s \end{pmatrix}.$$

2. The result is:

$$\begin{pmatrix} 0 & -1 & -2 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{pmatrix}$$

3. First observe that

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

although this is not crucial and we could have left A in the trigonometric form. Now we compute both products:

$$BA = \frac{1}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a + b\sqrt{3} & -a\sqrt{3} + b \\ c + d\sqrt{3} & -c\sqrt{3} + d \end{pmatrix}$$

$$AB = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a - c\sqrt{3} & b - d\sqrt{3} \\ a\sqrt{3} + c & b\sqrt{3} + d \end{pmatrix}$$

Comparing corresponding entries in the first column, we obtain $a + b\sqrt{3} = a - c\sqrt{3}$ and $c + d\sqrt{3} = a\sqrt{3} + c$, which gives $b = -c$ and $d = a$. In that case entries in the second column are automatically equal. We conclude that B has the form

$$B = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}$$

for arbitrary numbers a and c.

As in class we conclude that this matrix represents the composition of a rotation and a dilation. To see this, it is enough to take $r = \sqrt{a^2 + c^2}$, and find an angle θ so that $a = r\cos\theta$, $c = r\sin\theta$. Then we have:

$$B = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

4. We perform the algorithm given in class:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

subtract 3 times row I from row II

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

multiply row II by -1

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

subtract row II from row I

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 3 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

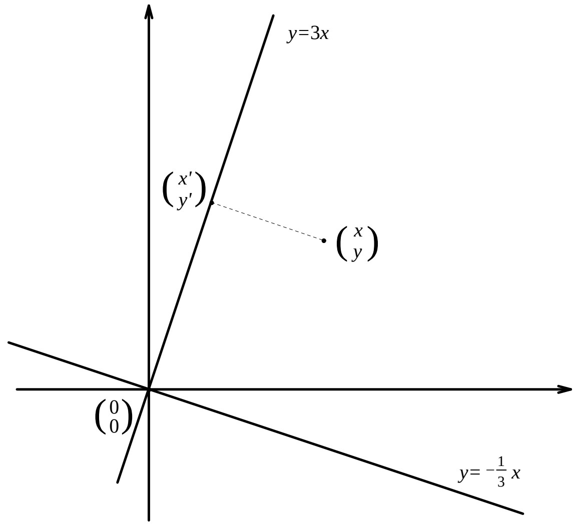
add 2 times row III to row I
subtract 3 times row III from row II

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 3 & -1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Therefore the inverse is:

$$\begin{pmatrix} -2 & 1 & 2 \\ 3 & -1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. The transformation A maps an arbitrary point $\begin{pmatrix} x \\ y \end{pmatrix}$ to a point $\begin{pmatrix} x' \\ y' \end{pmatrix}$ on the line $y = 3x$. (We do not need the actual formula for x' and y'.) Since the two lines are perpendicular, B maps every point from the line $y = 3x$ to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and in particular it maps $\begin{pmatrix} x' \\ y' \end{pmatrix}$ to the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.



In short, we can write

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{A}{\mapsto} \begin{pmatrix} x' \\ y' \end{pmatrix} \overset{B}{\mapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

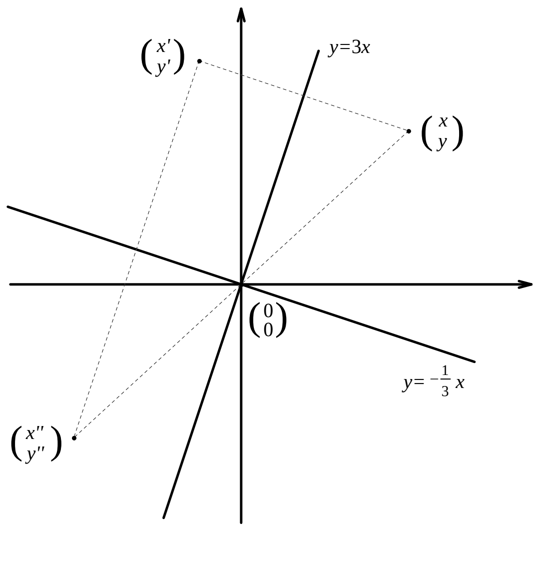
Since the matrix product BA corresponds to the composition of A followed by B, we conclude

$$BA\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and BA must be the zero-matrix $\mathbf{0}$, i.e.

$$BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

6. The transformation A maps an arbitrary point $\begin{pmatrix} x \\ y \end{pmatrix}$ to some point $\begin{pmatrix} x' \\ y' \end{pmatrix}$, and then B maps it further to some point $\begin{pmatrix} x'' \\ y'' \end{pmatrix}$.



In short, we can write

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{A}{\mapsto} \begin{pmatrix} x' \\ y' \end{pmatrix} \overset{B}{\mapsto} \begin{pmatrix} x'' \\ y'' \end{pmatrix}.$$

Since the two lines are perpendicular, we see from the picture that these 3 points are vertices of a right-angled triangle and that the origin is at the midpoint of its hypotenuse. Thus

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}.$$

Since the matrix product BA corresponds to the composition of A followed by B, we conclude

$$BA\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix},$$

so BA is the rotation by $180°$. Now we can write the matrix:

$$BA = \begin{pmatrix} \cos(180°) & -\sin(180°) \\ \sin(180°) & \cos(180°) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This can also be seen from

$$BA\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

7. (a) An example of such matrix is $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. To verify the property we first find the kernel by solving the linear system

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Its solution can be read off immediately:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so

$$\text{kernel}(A) = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

On the other hand

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1\begin{pmatrix} 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = x_2\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so also

$$\text{image}(A) = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b) Here we have to find a linear system whose solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5t \\ 2t \\ 3t \end{pmatrix}.$$

From the last row we read off $t = \frac{1}{3}x_3$ so that $x_1 = 5t = \frac{5}{3}x_3$, and $x_2 = 2t = \frac{2}{3}x_3$. This system can be written more nicely as

$$\begin{cases} 3x_1 \qquad\quad - 5x_3 = 0 \\ \qquad 3x_2 - 2x_3 = 0 \end{cases}$$

and corresponds to the matrix (i.e. linear transformation)

$$T = \begin{pmatrix} 3 & 0 & -5 \\ 0 & 3 & -2 \end{pmatrix}.$$

8. We first write the augmented coefficient matrix and then perform Gauss-Jordan eliminations (row operations):

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & k & 2 \\ 1 & 4 & k^2 & 3 \end{array}\right)$$

subtract row I from row II
subtract row I from row III

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & k-1 & 1 \\ 0 & 3 & k^2-1 & 2 \end{array}\right)$$

subtract row II from row I
subtract 3 times row II from row III

$$\left(\begin{array}{ccc|c} 1 & 0 & -k+2 & 0 \\ 0 & 1 & k-1 & 1 \\ 0 & 0 & k^2-3k+2 & -1 \end{array}\right)$$

Let us observe that $k^2 - 3k + 2 = 0$ has the solutions $k = 1$ and $k = 2$.

Case 1. $k \neq 1, 2$
In this case we can divide the third row by $k^2 - 3k + 2$, and then use the obtained 1 to annihilate all other elements in the third column. The first 3 columns of the RREF are thus the identity 3×3 matrix, and so the system has a **unique solution**.

Case 2. $k = 1$ or $k = 2$
For both of these values of k the last row of RREF reads

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & -1 \end{array}\right),$$

which shows that the system is inconsistent, i.e. has **no solutions**.

9. Let $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, $\overrightarrow{v_3}$ be columns of A, i.e. $A = [\overrightarrow{v_1}\ \overrightarrow{v_2}\ \overrightarrow{v_3}]$.

Since $A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is just $\overrightarrow{v_3}$, from the first equation we get $\overrightarrow{v_3} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

After that, since $A\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 3\overrightarrow{v_1} + \overrightarrow{v_3}$, we obtain from the second equation

$$\overrightarrow{v_1} = \frac{1}{3}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3}\overrightarrow{v_3} = \frac{1}{3}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Finally, from $A\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 2\overrightarrow{v_1} + \overrightarrow{v_2}$, and the third equation we get:

$$\overrightarrow{v_2} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - 2\overrightarrow{v_1} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - 2\begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{13}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Therefore

$$A = \begin{pmatrix} -\frac{2}{3} & \frac{13}{3} & 2 \\ -\frac{1}{3} & \frac{2}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

10. The line $y = 5x$ is spanned (determined) for instance by the vector $\overrightarrow{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. The general formula for the matrix of the orthogonal projection onto the line spanned by $\overrightarrow{w}$ is

$$\frac{1}{w_1^2 + w_2^2}\begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix},$$

so in our particular case the matrix becomes

$$\frac{1}{26}\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} = \begin{pmatrix} \frac{1}{26} & \frac{5}{26} \\ \frac{5}{26} & \frac{25}{26} \end{pmatrix}.$$

This can also be derived using the formula for the orthogonal projection:

$$\mathrm{proj}_{\overrightarrow{w}}(\overrightarrow{v}) = \frac{1}{|\overrightarrow{w}|^2}(\overrightarrow{v} \cdot \overrightarrow{w})\overrightarrow{w}.$$

V. K.