FORCING (LECTURE NOTES FOR MATH 223S, SPRING 2011)

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1. General theory of forcing extensions

Hilbert's 1st problem: Is there a cardinal strictly between $\aleph_0$ and $2^{\aleph_0}$? Equivalently, is there $A \subseteq \mathbb{R}$ so that $\mathbb{R}$ does not inject into A, and A does not inject into $\mathbb{N}$?

The continuum hypothesis (CH) asserts that no such A exists.

Gödel: CH is consistent with ZFC. In other words, ZFC does not prove the existence of A as above.

Cohen: negation of CH also consistent with ZFC. In other words, ZFC does not prove the *in*-existence of A as above.

How do we prove such consistency results? Construct a model.

Example 1.1. *Let $\mathsf{EU} - \mathsf{PL}$ consist of the axioms of Euclidean geometry except for the parallel lines axiom (PL). Then the negation of PL is consistent with $\mathsf{EU} - \mathsf{PL}$. To prove this, construct a model of $\mathsf{EU} - \mathsf{PL}$ in which PL fails.*

Essentially the same for the consistency results here. A couple of difficulties:

(1) Not going to construct, in ZFC, a model of $\mathsf{ZFC}+\neg\mathsf{CH}$, since we cannot even construct a model of ZFC. We have *relative* consistency results. Assuming $\mathrm{Con}(\mathsf{ZFC})$ prove $\mathrm{Con}(\mathsf{ZFC} + \neg\mathsf{CH})$.
(2) Assumption of $\mathrm{Con}(\mathsf{ZFC})$ gives us a model $\mathfrak{A}$ of ZFC. Need to produce from it a model of $\mathsf{ZFC} + \neg\mathsf{CH}$. For all we know, $\mathfrak{A}$ may be a model of "all sets are constructible", in which case it has no proper submodels of ZFC with the same ordinals. To produce a new model, at least if we want to keep the same ordinals, we must go *outside* rather than inside.

Forcing is a technique that allows us to extend models of set theory outwards. Let M be a transitive countable model (ctm) of ZFC. (The model $\mathfrak{A}$ of (2) need not be transitive. We will see later how to obtain transitive models from it, at the cost of passing to finite fragments of ZFC.) Let $\mathbb{P} \in M$ be a partially ordered set (poset), with a largest element denoted $\mathbb{1}$. We use $\leq_{\mathbb{P}}$, or simply $\leq$ when there is no room for confusion, to denote the ordering of $\mathbb{P}$. Let $G \subseteq \mathbb{P}$ with $\mathbb{1} \in G$.

Definition 1.2. For each $\tau \in M$ define $\tau[G] = \{\sigma[G] \mid (\exists p \in \mathbb{P})\langle\sigma, p\rangle \in \tau$ and $p \in G\}$.

The definition is by induction on the von Neumann rank of τ: to determine $\tau[G]$, must know $\sigma[G]$, but only for σ of smaller rank. The value of $\tau[G]$ depends on τ through the clause $\langle\sigma, p\rangle \in \tau$, and on G through the clause $p \in G$.

Might as well assume that all elements of τ are pairs of the form $\langle\sigma, p\rangle$ for $p \in \mathbb{P}$, as other elements of τ do not affect the value of $\tau[G]$. Might as well assume σ has the same property. This leads to the following definition (again a definition by induction on von Neumann rank):

Definition 1.3. τ is a $\mathbb{P}$-*name* if all elements of τ are pairs $\langle\sigma, p\rangle$ with $p \in \mathbb{P}$ and σ a $\mathbb{P}$-name.

Definition 1.4. $M[G] = \{\tau[G] \mid \tau \in M\} = \{\tau[G] \mid \tau \in M \text{ and } \tau \text{ is a } \mathbb{P}\text{-name}\}$.

We assumed that $\mathbb{P}$ belongs to M, but $G \subseteq \mathbb{P}$ was not assumed to belong to M. The next two claims show that if $G \notin M$, then $M[G]$ is a proper extension of M.

Claim 1.5. $M \subseteq M[G]$.

Proof. For each x define $\check{x} = \{\langle\check{y}, \mathbb{1}\rangle \mid y \in x\}$. The definition is by induction on rank. Then, again by induction on rank, $\check{x}$ is a $\mathbb{P}$-name for each x, and $\check{x}[G] = \{\check{y}[G] \mid y \in x\} = \{y \mid y \in x\} = x$.

M, being a model of ZFC, is closed under the operations that define $\check{x}$ from x. (This is a standard argument on the absoluteness of notions defined by induction from absolute operations.) So $x \in M$ implies $\check{x} \in M$ which in turn implies $x = \check{x}[G] \in M[G]$. $\square$

Claim 1.6. $G \in M[G]$.

Proof. Define $\dot{G} = \{\langle\check{p}, p\rangle \mid p \in \mathbb{P}\}$. Then $\dot{G}$ is a $\mathbb{P}$-name, and $\dot{G}$ belongs to M since $\mathbb{P}$ belongs to M and M, being a model of ZFC, is closed under the operations that define $\dot{G}$ from $\mathbb{P}$. So $G = \{p \mid p \in G\} = \{\check{p}[G] \mid p \in G\} = \dot{G}[G] \in M[G]$. $\square$

Claim 1.7. *$M[G]$ is transitive.*

Proof. Let $u \in M[G]$ and fix $\tau \in M$ so that $u = \tau[G]$. Then $u = \{\sigma[G] \mid (\exists p)\langle\sigma, p\rangle \in \tau, p \in G\} \subseteq \{\sigma[G] \mid \sigma \in M\} \subseteq M[G]$, where the first inclusion uses the transitivity of M, which implies that $\sigma \in M$ whenever $\langle\sigma, p\rangle \in \tau \in M$. $\square$

Claim 1.8.
(1) $\operatorname{rank}(\tau[G]) \leq \operatorname{rank}(\tau)$. *(Rank here is von Neumann rank.)*
(2) $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$.

Proof. The first item is immediate from the definitions by induction on rank. The inequality $M[G] \cap \mathrm{Ord} \leq M \cap \mathrm{Ord}$ follows from the first item, and the inequality $M \cap \mathrm{Ord} \leq M[G] \cap \mathrm{Ord}$ is clear since $M \subseteq M[G]$. $\square$

We have seen so far, in very general settings, how to extend a ctm M of ZFC to a ctm $M[G]$, with exactly the same ordinals. The extension is proper if $G \notin M$. Is the extension $M[G]$ a model of ZFC? In general it need not be. But often it is, and this, together with the meaning of often, was Cohen's great insight. To give a specific example, take $\mathbb{P}$ to be the complete binary tree, with $\mathbb{1}$ the root, and $p \leq q$ if p extends q. Some subsets G of $\mathbb{P}$ are branches through the tree, and these correspond naturally to elements of Cantor's space. Cohen showed that $M[G] \models \mathsf{ZFC}$ for a *comeager* set of branches G.[1] We will get to this later, but for now let us show that some axioms of ZFC hold in $M[G]$ without need for any further assumptions on G.

Claim 1.9. *$M[G]$ satisfies Set Existence, Extensionality, Foundations, and Infinity.*

[1]The definition of $M[G]$ that we use is due to Shoenfield. Cohen assumed that M is a countable level of the constructible universe L, and defined $M[G]$ by constructing from G up to the same level.

Proof. Immediate from the facts that $M[G]$, being a supermodel of M, is not empty, that $M[G]$ is transitive, and that $\omega \in M[G]$. $\square$

Claim 1.10. *$M[G]$ satisfies Pairing.*

Proof. Let $u, v \in M[G]$, say $u = \tau[G]$ and $v = \sigma[G]$, for $\tau, \sigma \in M$. Set $\rho = \{\langle \sigma, \mathbb{1} \rangle, \langle \tau, \mathbb{1} \rangle\}$. Then $\rho \in M$ since M, being a model of ZFC, is closed under the operations that define ρ from σ and τ. And $\rho[G] = \{\sigma[G], \tau[G]\} = \{u, v\}$ so $\{u, v\} \in M[G]$. $\square$

Claim 1.11. *$M[G]$ satisfies Union.*

Proof. Let $u = \sigma[G] \in M[G]$ with $\sigma \in M$. We have to find $Z \in M[G]$ so that $\bigcup u \subseteq Z$, that is $y \in x \in u \to y \in Z$.

Set $\rho = \bigcup\{\tau \mid (\exists p)\langle \tau, p \rangle \in \sigma\}$, and let $Z = \rho[G]$. ρ belongs to M since M, being a model of ZFC, is closed under the operations that define ρ from σ. So $Z \in M[G]$. Suppose $y \in x \in u = \sigma[G]$. Then there is $\langle \tau, p \rangle \in \sigma$ so that $y \in x = \tau[G]$. By definition of ρ, $\tau \subseteq \rho$. This implies $\tau[G] \subseteq \rho[G]$, and since $y \in x = \tau[G]$ it follows that $y \in \rho[G] = Z$. $\square$

We are still missing the axioms Powerset, Comprehension, Replacement, and Choice. We will handle Comprehension next, but before we can get to it we must add some assumptions on G.

Definition 1.12. $p, q \in \mathbb{P}$ are *compatible* if there is r with $r \le p \wedge r \le q$. Otherwise p and q are *incompatible*, denoted $p \perp q$. $D \subseteq \mathbb{P}$ is *dense* if for all $p \in \mathbb{P}$ there is $q \le p$ with $q \in D$. $A \subseteq \mathbb{P}$ is an *antichain* if no two elements of A are compatible.

Definition 1.13. $G \subseteq \mathbb{P}$ is a *filter* if:

(1) G is closed upward, meaning that $q < p \wedge q \in G \to p \in G$.
(2) Every two elements of G are compatible in G, meaning that $p, q \in G \to (\exists r \in G)(r \le p \wedge r \le q)$.

A filter G is *generic* for $\mathbb{P}$ over M if it meets all dense sets of $\mathbb{P}$ that belong to M. Equivalently, it meets all maximal antichains of $\mathbb{P}$ that belong to M.

The equivalence claimed in the definition is easy to prove, noting for one direction that any antichain which is maximal in a dense set D is also maximal in $\mathbb{P}$, and noting for the other direction that if $A \subseteq \mathbb{P}$ is a maximal antichain, then $D = \{q \mid (\exists p \in A) q \le p\}$ is dense, and any filter that meets D, also meets A, by upward closure.

Claim 1.14. *If M is countable then there are generic filters for $\mathbb{P}$ over M.*

Proof. Using the fact that M is countable, let D_n, $n < \omega$ enumerate all dense sets of $\mathbb{P}$ that belong to M. Using density, construct a descending sequence $p_0 \ge p_1 \ge \dots$ with $p_n \in D_n$, and set $G = \{r \mid (\exists n) r \ge p_n\}$. $\square$

Remark 1.15. There is a natural topology on the set of filters for $\mathbb{P}$, with basic open sets $N_p = \{G \mid p \in G\}$ for $p \in \mathbb{P}$. If M is countable, then the set of filters which are generic over M is more than just non-empty, it is in fact comeager.

We return now to the matter of proving Comprehension in $M[G]$. Comprehension produces, for each set a and each formula $\varphi(x)$ (possibly with additional parameters), the set of $x \in a$ for which $\varphi(x)$ is true. We intend to prove Comprehension in $M[G]$ by reducing it to Comprehension in M. To do this, we essentially have to reduce truth in $M[G]$ to truth in M. There is in general no reason for such a reduction to be possible, but, remarkably, it turns out that a reduction of this sort is possible if we restrict our attention to generic G.

Theorem 1.16 (Fundamental Theorem of Forcing)**.** *For every formula φ there is a relation $p \Vdash^M_{\mathbb{P}} \varphi(\tau_1, \dots \tau_n)$ on tuples $\langle p, \tau_1, \dots, \tau_n\rangle \in M$ such that:*

(1) *For generic G, $M[G] \models \varphi(\tau_1[G], \dots, \tau_n[G])$ iff $(\exists p \in G) p \Vdash^M_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$.*
(2) *$p \Vdash^M_{\mathbb{P}} \varphi(\tau_1, \dots \tau_n)$ (as a relation on $p, \tau_1, \dots, \tau_n$) is uniformly definable from $\mathbb{P}$ over M.*

The theorem reduces truth in $M[G]$ to a relation which is definable over M, though the reduction adds an extra clause of membership in G. The next lemma shows how to use the theorem, to reduce Comprehension in $M[G]$ to Comprehension in M.

The expression $p \Vdash^M_{\mathbb{P}} \varphi(\tau_1, \dots \tau_n)$ is read "p forces $\varphi(\tau_1, \dots, \tau_n)$ in $\mathbb{P}$ over M". The subscript $\mathbb{P}$ is often omitted when clear from the context.

Lemma 1.17. *Let G be generic for $\mathbb{P}$ over M. Then $M[G]$ satisfies Comprehension.*

Proof. Fix a formula φ, and sets $a, w_1, \dots, w_k \in M[G]$. We have to show that the set $z = \{x \in a \mid M[G] \models \varphi(x, a, w_1, \dots, w_k)\}$ belongs to $M[G]$.

Say $a = \sigma[G]$ and $w_i = \tau_i[G]$, with $\sigma, \tau_i \in M$.

Note that $x \in a$ implies $(\exists \rho \in \mathrm{dom}(\sigma)) x = \rho[G]$. ($\sigma$, being a name, is a set of pairs $\langle \rho, s\rangle$. The domain of σ is the set of first coordinates of pairs in σ.)

So $z = \{\rho[G] \mid \rho \in \mathrm{dom}(\sigma) \text{ and } M[G] \models \varphi(\rho[G], \tau_1[G], \dots, \tau_k[G])\}$. By the Fundamental Theorem of Forcing,

$$z = \{\rho[G] \mid \rho \in \mathrm{dom}(\sigma) \text{ and } (\exists p \in G) p \Vdash^M_{\mathbb{P}} \varphi(\rho, \tau_1, \dots, \tau_k)\}.$$

Let $\pi = \{\langle \rho, p\rangle \in \mathrm{dom}(\sigma) \times \mathbb{P} \mid p \Vdash^M_{\mathbb{P}} \varphi(\rho, \tau_1, \dots, \tau_n)\}$. Then $\pi \in M$ using the fact that M is a model of ZFC. The main axiom used is Comprehension in M, which allows identifying π as a subset of $\mathrm{dom}(\sigma) \times \mathbb{P}$, consisting of elements for which a certain formula, the formula given by the second part of the Fundamental Theorem of Forcing, holds in M.

Finally, $\pi[G] = \{\rho[G] \mid \rho \in \mathrm{dom}(\sigma) \text{ and } (\exists p \in G) p \Vdash^M_{\mathbb{P}} \varphi(\rho, \tau_1, \dots, \tau_n)\} = z$, so $z \in M[G]$. □

Uses of the Fundamental Theorem of Forcing also allow proving Replacement in $M[G]$, and help proving Powerset and Choice. We will get to these axioms later. For now we concentrate on proving the theorem.

Definition 1.18. $D \subseteq \mathbb{P}$ is *dense below p* if for every $q \leq p$ there exists $r \leq q$ with $r \in D$.

Claim 1.19. *Let $D \in M$ be dense below p. Let G be generic for $\mathbb{P}$ over M, with $p \in D$. Then $D \cap G \neq$.*

Proof. Set $D' = \{r \mid r \in D \vee r \perp p\}$. Then $D' \in M$, and D' is dense. By genericity, $G \cap D' \neq \emptyset$. Fix $r \in D' \cap G$. $r \perp p$ is impossible since $p \in G$ and any two conditions in G are compatible. So it must be that $r \in D$. □

Claim 1.20. *The following are equivalent:*

(1) *D is dense below p.*

(2) *For every* $r \leq p$, D *is dense below* r.
(3) *The set* $\{r \leq p \mid D$ *is dense below* $r\}$ *is dense below* p.

Proof. The implication (2)→(3) is clear, and the implications (1)→(2) and (3)→(1) are immediate from the definitions. □

To prove the Fundamental Theorem of Forcing, we define in V a relation $p \Vdash_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$ on tuples $\langle p, \tau_1, \dots, \tau_n \rangle$, and prove that for $\tau_1, \dots, \tau_n \in M$, $M[G] \models \varphi(\tau_1[G], \dots, \tau_n[G])$ iff $(\exists p \in G)(p \Vdash_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n))^M$. The theorem then follows by setting $\Vdash_{\mathbb{P}}^M$ to be the relativization of $\Vdash_{\mathbb{P}}$ to M, that is $p \Vdash_{\mathbb{P}}^M \varphi(\tau_1, \dots, \tau_n) \iff (p \Vdash_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n))^M$. (The formula that defines $\Vdash_{\mathbb{P}}$ in V witnesses part (2) of the theorem.)

We define $\Vdash_{\mathbb{P}}$ so as to have some additional properties, given by the next lemma.

Lemma 1.21. *The following are equivalent:*

(1) $p \Vdash_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$.
(2) *For every* $r \leq p$, $r \Vdash_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$.
(3) *The set* $\{r \leq p \mid r \Vdash_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)\}$ *is dense below* p.

In light of the lemma, particularly the implication (1)→(2), we say that r is *stronger* than p when $r \leq p$. Every statement forced by p is then forced by all stronger conditions.

We define $\Vdash_{\mathbb{P}}$, prove the Fundamental Theorem, and prove Lemma 1.21, all together, working by induction on the complexity of φ.

Case 1: φ *is atomic of the form* $v_1 = v_2$. For names τ, τ', and a condition p, define $p \Vdash_{\mathbb{P}} \tau \subseteq \tau'$ to hold iff for every $\langle \pi, s \rangle \in \tau$, the set

$$D_{\pi,s} = \{q \leq p \mid q \leq s \to (\exists \langle \pi', s' \rangle \in \tau')(q \leq s' \wedge q \Vdash_{\mathbb{P}} \pi = \pi')\}$$

is dense below p. Define $p \Vdash_{\mathbb{P}} \tau_1 = \tau_2$ iff $p \Vdash_{\mathbb{P}} \tau_1 \subseteq \tau_2$ and $p \Vdash_{\mathbb{P}} \tau_2 \subseteq \tau_1$.

The definition of $p \Vdash_{\mathbb{P}} \tau_1 = \tau_2$ is by induction on $\min(\text{rank}(\tau_1), \text{rank}(\tau_2))$. To determine whether $p \Vdash_{\mathbb{P}} \tau_1 \subseteq \tau_2$ we have to look at $D_{\pi,s}$ for $\langle \pi, s \rangle \in \tau_1$, and this involves determining whether $q \Vdash_{\mathbb{P}} \pi = \pi'$ for $\langle \pi', s' \rangle \in \tau_2$. Since $\langle \pi, s \rangle \in \tau_1$ and $\langle \pi', s' \rangle \in \tau_2$, $\text{rank}(\pi) < \text{rank}(\tau_1)$ and $\text{rank}(\pi') < \text{rank}(\tau_2)$. Hence $\min(\text{rank}(\pi), \text{rank}(\pi')) < \min(\text{rank}(\tau_1), \text{rank}(\tau_2))$ and the truth value of $q \Vdash_{\mathbb{P}} \pi = \pi'$ is known by induction. Similarly determining whether $p \Vdash_{\mathbb{P}} \tau_2 \subseteq \tau_1$ is possible by induction.

Since the only dependence of the definition of $p \Vdash_{\mathbb{P}} \tau_1 = \tau_2$ on p is through density below p of various sets $D_{\pi,s}$, Lemma 1.21 for the formula $\tau_1 = \tau_2$ is immediate using Claim 1.19.

It remains to prove that $\tau_1[G] = \tau_2[G]$ iff $(\exists p \in G)(p \Vdash_{\mathbb{P}} \tau_1 = \tau_2)^M$, whenever $\tau_1, \tau_2 \in M$ and G is generic for $\mathbb{P}$ over M.

Consider first the right-to-left direction. Suppose $p \in G$ and $(p \Vdash_{\mathbb{P}} \tau_1 \subseteq \tau_2)^M$. We prove that $\tau_1[G] \subseteq \tau_2[G]$. Fix $x \in \tau_1[G]$. Then by definition of $\tau_1[G]$ there is $\langle \pi, s \rangle \in \tau_1$, with $s \in G$, so that $x = \pi[G]$. Since p and s both belong to G, and G is a filter, there is $r \in G$ with $r \leq p$, $r \leq s$. By assumption, $(p \Vdash_{\mathbb{P}} \tau_1 \subseteq \tau_2)^M$, and it follows from this that the set $D_{\pi,s}^M = \{q \leq p \mid q \leq s \to (\exists \langle \pi', s' \rangle \in \tau_2)(q \leq s' \wedge (q \Vdash_{\mathbb{P}} \pi = \pi')^M)\}$ is dense below p. The set is then also dense below r. By genericity of G and since $r \in G$ it follows that there is $q \leq r$ which belongs to both $D_{\pi,s}^M$ and G. Now $q \leq r \leq s$, and since $q \in D_{\pi,s}^M$ it follows that there exists $\langle \pi', s' \rangle \in \tau_2$ so that $q \leq s'$ and $(q \Vdash_{\mathbb{P}} \pi = \pi')^M$. Since G is closed upward and $q \in G$, it must be that

$s' \in G$, and since $\langle \pi', s' \rangle \in \tau_2$ this in turn implies that $\pi'[G] \in \tau_2[G]$. By induction on rank the fact that $q \in G$ and $(q \Vdash_{\mathbb{P}} \pi = \pi')^M$ implies that $\pi[G] = \pi'[G]$. So $x = \pi[G] = \pi'[G] \in \tau_2[G]$, as required.

Consider next the left-to-right direction. Suppose $\tau_1[G] = \tau_2[G]$. We prove that there is $p \in G$ which forces $\tau_1 = \tau_2$ over M. To do this we define a set $D = \{p \in \mathbb{P} \mid (p \Vdash_{\mathbb{P}} \tau_1 = \tau_2)^M\} \cup K_1 \cup K_2$ in M, show that D is dense, whence it has non-empty intersection with G, and argue that G does not meet K_1 and K_2, so that it must be that G meets D on the part $\{p \in \mathbb{P} \mid (p \Vdash_{\mathbb{P}} \tau_1 = \tau_2)^M\}$.

Let K_1 be the set of conditions r for which $(\exists \langle \pi, s \rangle \in \tau_1)$ so that $r \leq s$ and $(\forall \langle \pi', s' \rangle \in \tau_2)(\forall q \in \mathbb{P})$, if $q \leq s'$ and $(q \Vdash_{\mathbb{P}} \pi = \pi')^M$ then $q \perp r$.

Claim 1.22. $K_1 \cap G = \emptyset$.

Proof. Suppose $r \in G \cap K_1$. Fix $\langle \pi, s \rangle \in \tau_1$ witnessing that $r \in K_1$. Then $r \leq s$, and by upward closure of G, it must be that $s \in G$. Hence $\pi[G] \in \tau_1[G]$.

Since $\tau_1[G] = \tau_2[G]$, there must be some $\langle \pi', s' \rangle \in \tau_2$, with $s' \in G$, so that $\pi[G] = \pi'[G]$. By induction from the fact that $\pi[G] = \pi'[G]$, there must be some $p^* \in G$ so that $(p^* \Vdash_{\mathbb{P}} \pi = \pi')^M$.

We have $s', p^* \in G$. Since G is a filter there is $q \in G$ with $q \leq s', q \leq p^*$. By Lemma 1.21, $(q \Vdash_{\mathbb{P}} \pi = \pi')^M$. From this and the fact that $q \leq s'$ it follows by the definition of K_1 that $q \perp r$. But this is impossible since both q and r belong to G. □

Define K_2 by taking the definition of K_1 and switching τ_1, τ_2. Then K_2 also has empty intersection with G. Let $D = \{p \in \mathbb{P} \mid (p \Vdash_{\mathbb{P}} \tau_1 = \tau_2)^M\} \cup K_1 \cup K_2$.

Claim 1.23. *D is dense.*

Proof. Fix $p \in \mathbb{P}$. If $(p \Vdash_{\mathbb{P}} \tau_1 = \tau_2)^M$ then $p \in D$ and there is nothing to prove. So suppose $(p \nVdash_{\mathbb{P}} \tau_1 = \tau_2)^M$. Suppose for definitiveness that $(p \nVdash_{\mathbb{P}} \tau_1 \subseteq \tau_2)^M$. Then by definition there is $\langle \pi, s \rangle \in \tau_1$ so that the set $D_{\pi,s}$ is not dense below p. Fix then some $r \leq p$ so that $D_{\pi,s}$ has no elements $\leq r$. We claim that $r \in K_1$ (so $r \in D$ and we are done).

Indeed, we claim that $\langle \pi, s \rangle$ witnesses that $r \in K_1$. That $r \leq s$ is clear since otherwise trivially r belongs to $D_{\pi,s}$, but $D_{\pi,s}$ has no elements $\leq r$. Fix then $\langle \pi', s' \rangle \in \tau_2$ and $q \in \mathbb{P}$, and suppose that $q \leq s'$ and $(q \Vdash_{\mathbb{P}} \pi = \pi')^M$. We prove $q \perp r$. If not, then there is q^* such that $q^* \leq q$, $q^* \leq r$. From $q^* \leq q$ and the properties of q above we get that $q^* \in D_{\pi,s}$. But $D_{\pi,s}$ has no elements $\leq r$. □

Claim 1.24. *D belongs to M.*

Proof. All components of D are defined using formulas that are either absolute or relativized to M. So D is definable over M, and by Comprehension in M it is an element of M. □

Using the last two claims and genericity of G, there must be $p \in G$ which belong to G. p cannot belong to K_1 or K_2 since both have empty intersection with G. So it must be that $(p \Vdash_{\mathbb{P}} \tau_1 = \tau_2)^M$. This completes the proof of Theorem 1.16 for the case of atomic formulas $v_1 = v_2$. □

Case 2: φ is atomic of the form $v_1 \in v_2$. For names τ_1, τ_2, and a condition p, define $p \Vdash_{\mathbb{P}} \tau_1 \in \tau_2$ to hold iff the set

$$E = \{q \mid (\exists \langle \pi, s \rangle \in \tau_2) q \leq s \text{ and } q \Vdash_{\mathbb{P}} \tau_1 = \pi\}$$

is dense below p. (The definition refers to the forcing relation, but only on formulas $v_1 = v_2$ for which the meaning of forcing is given by Case 1.)

Lemma 1.21 is then immediate from Claim 1.19 since the only dependence of the definition on p comes through the requirement that a certain set is dense below p.

Suppose G is generic for $\mathbb{P}$ over M, $p \in G$, and $(p \Vdash_{\mathbb{P}} \tau_1 \in \tau_2)^M$. We prove that $\tau_1[G] \in \tau_2[G]$. The set $E^M = \{q \mid (\exists \langle \pi, s\rangle \in \tau_2) q \leq s \text{ and } (q \Vdash_{\mathbb{P}} \tau_1 = \pi)^M\}$ belongs to M, and by assumption it is dense below p. By genericity G meets this set. Fix then some $q \in G$ which belongs to E^M, and let $\langle \pi, s\rangle \in \tau_2$ witness this. Since $q \leq s$, it must be that $s \in G$, and therefore $\pi[G] \in \tau_2[G]$. Since $q \in G$ and $(q \Vdash_{\mathbb{P}} \tau_1 = \pi)^M$, by Case 1, $\tau_1[G] = \pi[G]$. So $\tau_1[G] = \pi[G] \in \tau_2[G]$.

Suppose for the converse that $\tau_1[G] \in \tau_2[G]$. We prove that there is $p \in G$ so that $(p \Vdash_{\mathbb{P}} \tau_1 \in \tau_2)^M$. By assumption and definition of $\tau_2[G]$ there must be $\langle \pi, s\rangle \in \tau_2$, with $s \in G$, so that $\tau_1[G] = \pi[G]$. By Case 1, there is some $r \in G$ so that $(r \Vdash_{\mathbb{P}} \tau_1 = \pi)^M$. Since s, r both belong to G, and G is a filter, we can find $p \in G$ so that $p \leq s$ and $p \leq r$. It is now easy to check that *all* $q \leq p$ belong to E^M, so in particular the set is dense below p and $(p \Vdash \tau_1 \in \tau_2)^M$. To see this, fix $q \leq p$. Then certainly $q \leq s$, and since q is also $\leq r$, by Lemma 1.21 for formulas in Case 1, $(q \Vdash_{\mathbb{P}} \tau_1 = \pi)^M$. □

Case 3: $\varphi = \neg\psi$. Define $p \Vdash_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$ iff there is no $q \leq p$ so that $q \Vdash_{\mathbb{P}} \psi(\tau_1, \dots \tau_n)$.

Lemma 1.21 is as usual easy to check: If there are no conditions below p forcing ψ, then there are no such conditions below any $r \leq p$. Also, if there are no such conditions below densely many $r \leq p$, then by Lemma 1.21 no $q \leq p$ forces ψ.

Suppose $p \in G$ and $(p \Vdash_{\mathbb{P}} \varphi)^M$. We have to show $M[G] \models \varphi$. Suppose it does not, in other words suppose $M[G] \models \psi$. Then by induction (on complexity, as ψ is of lower complexity than φ), there is $s \in G$ so that $(s \Vdash_{\mathbb{P}} \psi)^M$. Since p, s both belong to G there is $q \in G$ with $q \leq p$, $q \leq s$. By Lemma 1.21, $(q \Vdash_{\mathbb{P}} \psi)^M$. But then by definition $(p \not\Vdash_{\mathbb{P}} \varphi)^M$.

Conversely, suppose that $M[G] \models \varphi$. Let $D = \{p \mid (p \Vdash_{\mathbb{P}} \varphi)^M\} \cup \{p \mid (p \Vdash_{\mathbb{P}} \psi)^M\}$. Then $D \in M$, and it is easy to check that D is dense: If p is such that $(p \not\Vdash_{\mathbb{P}} \varphi)^M$ then by definition there is $q \leq p$ so that $(q \Vdash_{\mathbb{P}} \psi)^M$. By genericity then there must be $p \in G$ that belongs to D. If $(p \Vdash_{\mathbb{P}} \psi)^M$ then by induction $M[G] \models \psi$, contradicting the assumption that $M[G] \models \varphi$. So it must be that $(p \Vdash_{\mathbb{P}} \varphi)^M$. □

Case 4: $\varphi = \psi_1 \wedge \psi_2$. Set $p \Vdash_{\mathbb{P}} \psi$ iff $p \Vdash_{\mathbb{P}} \psi_1$ and $p \Vdash_{\mathbb{P}} \psi_2$. Both Lemma 1.21 and Theorem 1.16 are easy to check for ψ, using the lemma and theorem for ψ_1 and ψ_2. □

Case 5: $\varphi(v_1, \dots, v_n) = (\exists x)\psi(x, v_1, \dots, v_n)$. Set $p \Vdash_{\mathbb{P}} (\exists x)\psi(x, \tau_1, \dots, \tau_n)$ iff the set

$$\{q \mid (\exists \sigma) q \Vdash_{\mathbb{P}} \psi(\sigma, \tau_1, \dots, \tau_n)\}$$

is dense below p.

As in previous cases, Lemma 1.21 is immediate from Claim 1.19, since the only dependence of the definition on p comes through the requirement that a certain set is dense below p.

The right-to-left direction of condition (1) in Theorem 1.16 for φ is clear by induction on complexity, which allows using the theorem for ψ.

For the left-to-right direction, suppose $M[G] \models (\exists x)\psi(x, \tau_1[G], \dots, \tau_n[G])$, and let $D = \{p \mid (\exists \sigma \in M)(p \Vdash_{\mathbb{P}} \psi(\sigma, \tau_1, \dots, \tau_n))^M\} \cup \{p \mid (\forall q \leq p)(\forall \sigma \in M)(q \not\Vdash_{\mathbb{P}}$

$\psi(\sigma, \tau_1, \dots, \tau_n))^M\}$. Then D belongs to M, and it is easy to check that D is dense. By genericity there is $p \in D \cap G$. Check, using Theorem 1.16 for ψ and the assumption that for some $x \in M[G]$, $M[G] \models \psi(x, \tau_1[G], \dots, \tau_n[G])$, that p cannot belong to the right-hand set in the union in the definition of D. It must then be that for some $\sigma \in M$, $(p \Vdash_{\mathbb{P}} \psi(\sigma, \tau_1, \dots, \tau_n))^M$. The same is then true (with the same σ) for all $q \leq p$, and in particular for a dense set of q below p, so $(p \Vdash_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n))^M$. □

The cases above complete the definition of the relation $p \Vdash_{\mathbb{P}} \varphi$ in V, the proof of Lemma 1.19, and the proof of the Fundamental Theorem of Forcing, Theorem 1.16 (with $p \Vdash^M_{\mathbb{P}} \varphi$ defined to hold iff $(p \Vdash_{\mathbb{P}} \varphi)^M$).

With the Fundamental Theorem at hand we can complete the proof that for generic G, $M[G]$ is a model of ZFC. We have only the axioms Replacement, Powerset and Choice left to verify.

Lemma 1.25. *Let G be generic for $\mathbb{P}$ over M. Then $M[G]$ satisfies Replacement.*

Proof. Let $z, a_1, \dots, a_n \in M[G]$, say $z = \dot{z}[G]$ and $a_i = \dot{a}_i[G]$, with $\dot{z}, \dot{a}_i$ names in M. Suppose that $M[G]$ satisfies that for all $x \in y$, there exists a unique $y = y_x$ so that $\varphi(x, y, z, a_1, \dots, a_n)$. We have to show that there is a set $u \in M[G]$ so that the range of the function $x \mapsto y_x$, $x \in z$, is contained in u. We do this by finding a set in M that contains names for all y_x. The Fundamental Theorem will allow us to reduce the existence of such a set to Replacement in M.

For $\langle \rho, s \rangle \in \dot{z}$ and $p \in \mathbb{P}$, let $f(\rho, s, p)$ be the least ordinal α so that $(\exists \tau \in V^M_\alpha) p \Vdash^M_{\mathbb{P}} \varphi(\rho, \tau, \dot{z}, \dot{a}_1, \dots, \dot{a}_n)$, if such α exists, and otherwise set $f(\rho, s, p) = 0$. Then f is definable over M, and by Replacement in M the image of $\sigma \times \mathbb{P}$ under f is a set in M. It follows that the supremum of this image is an ordinal in M, call it δ.

Let $\dot{u} = \{\langle \tau, \mathbb{1} \rangle \mid \tau \in V^M_\delta\}$. Then $\dot{u} \in M$. Set $u = \dot{u}[G] \in M[G]$. We claim that for each $x \in z$, there is $y \in u$ so that $M[G] \models \varphi(x, y, z, a_1, \dots, a_n)$. Fix $x \in z$. Since $z = \dot{z}[G]$, there is $\langle \rho, s \rangle \in \dot{z}$ so that $x = \rho[G]$. By initial the assumption in the proof, there is $y \in M[G]$ so that $M[G] \models \varphi(x, y, z, a_1, \dots, a_n)$. This means that there is $\tau \in M$ and (by the Fundamental Theorem) $p \in G$ so that $p \Vdash^M_{\mathbb{P}} \varphi(\rho, \tau, \dot{z}, \dot{a}_1, \dots, \dot{a}_n)$. By definition of f and δ, such τ can be found in V^M_δ. Then $\tau[G] \in u$, and since $p \in G$, $M[G] \models \varphi(x, \tau[G], z, a_1, \dots, a_n)$. □

Lemma 1.26. *Let G be generic for $\mathbb{P}$ over M. Then $M[G]$ satisfies Powerset.*

Proof. Let $x \in M[G]$, say $x = \dot{x}[G]$ with $\dot{x} \in M$. We have to find $u \in M[G]$ so that all subsets of x in $M[G]$ belong to u.

Claim 1.27. *Let $y \in M[G]$ be a subset of x. Then there is a name $\tau \in M$ so that $y = \tau[G]$ and $\tau \subseteq \mathrm{dom}(\dot{x}) \times \mathbb{P}$.*

Proof. Fix a name $\dot{y} \in M$ so that $y = \dot{y}[G]$. Set $\tau = \{\langle \rho, p \rangle \mid \rho \in \mathrm{dom}(\dot{x})$ and $p \Vdash^M_{\mathbb{P}} \rho \in \dot{y}\}$. Then certainly $\tau \subseteq \mathrm{dom}(\dot{x}) \times \mathbb{P}$, and it is easy to check that $\tau[G] \subseteq \dot{y}[G]$ and $y \subseteq \tau[G]$, using the Fundamental Theorem and (for the second inclusion) the fact that $y \subseteq x$. □

Now set $\dot{u} = \{\langle \tau, \mathbb{1} \rangle \mid \tau \in M$ and $\tau \subseteq \mathrm{dom}(\dot{x}) \times \mathbb{P}\}$. By Powerset in M, the collection of $\tau \in M$ so that $\tau \subseteq \mathrm{dom}(\dot{x}) \times \mathbb{P}$ is a set in M, and it follows that $\dot{u} \in$M. Set $u = \dot{u}[G]$. Then u belongs to $M[G]$, and for each $\tau \in M$ with $\tau \subseteq \mathrm{dom}(\dot{x}) \times \mathbb{P}$, $\tau[G] \in u$. By the last claim then, all subsets of x in $M[G]$ belong to u. □

Lemma 1.28. *Let G be generic for $\mathbb{P}$ over M. Then $M[G]$ satisfies Choice.*

Proof. The Axiom of Choice is equivalent in ZF to the statement: for every x, there exists a function f so that $\mathrm{dom}(f)$ is an ordinal, and $\mathrm{range}(f) \supseteq x$. Since $M[G]$ is a model of ZF, it is enough to show that this statement holds in $M[G]$.

Fix $x \in M[G]$, say $x = \dot{x}[G]$ with $\dot{x} \in M$. Let $u = \mathrm{dom}(x)$, so that for every $y \in x$, there is a name $\rho \in u$ with $\rho[G] = y$. Using Choice in M, fix a function $h \in M$ so that $\mathrm{dom}(h)$ is an ordinal, and $\mathrm{range}(h) \supseteq u$. Since $M \subseteq M[G]$, h belongs to $M[G]$.

Let f be the function $\alpha \mapsto h(\alpha)[G]$. Then $\mathrm{range}(f) \supseteq x$. f is the composition of h with the class function $\rho \mapsto \rho[G]$. Since this class function is definable in ZF from G over $M[G]$, since $M[G] \models \mathsf{ZF}$, and since h and G belong to $M[G]$, the function f belongs to $M[G]$. $\square$

Corollary 1.29. *Let M be a ctm of* ZFC*, let $\mathbb{P}$ be a poset in M, and let G be generic for $\mathbb{P}$ over M. Then $M[G]$ is also a ctm of* ZFC*, $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$, and $M \cup \{G\} \subseteq M[G]$. $M[G]$ is the smallest model of* ZFC *which contains $M \cup \{G\}$.*

Proof. We had already proved everything claimed in the corollary, except for the minimality of $M[G]$. Suppose N is a ctm of ZFC and $M \cup \{G\} \subseteq N$. Then for every name $\tau \in M$, we have $\tau \in N$. Since $G \in N$, N is a model of ZFC, and $\tau[G]$ can be produced from τ and G in ZFC, it follow that $\tau[G] \in N$. So $M[G] = \{\tau[G] \mid \tau \in M\} \subseteq N$. $\square$

2. Initial applications

2.1. **Consistency of $V \neq L$.** As a first application of the theory of forcing we prove that, assuming ZFC is consistent, so is $\mathsf{ZFC} + V \neq L$ (that is ZFC+"*not* all sets are constructible"). We use forcing to produce a ctm of ZFC where not all sets are constructible, given a ctm of ZFC. We then convert this to a proof of the consistency of $\mathsf{ZFC} + V \neq L$ from the consistency of ZFC.

Lemma 2.1. *Suppose there is a ctm M of* ZFC*. Then there is a ctm of* ZFC*+$V \neq L$.*

Proof. We will construct a forcing extension $M[G]$ of M, which satisfies $\mathsf{ZFC} + V \neq L$. To do this we simply have to ensure that $M[G] \supsetneq M$. Then, letting $\alpha = \sup(M \cap \mathrm{Ord}) = M \cap \mathrm{Ord}$ we have by Claim 1.8 that $\sup(M[G] \cap \mathrm{Ord}) = M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord} = \alpha$, hence $L^{M[G]} = L_\alpha = L^M$. Since $M[G] \supsetneq M \supseteq L^M$ we have in particular that $M[G] \neq L^{M[G]}$, and hence $M[G] \models$"$V \neq L$".

It is easy to see that in fact any extension $M[G]$ by a generic for a sufficiently non-trivial poset $\mathbb{P}$ will properly contain M. Nonetheless we present two specific constructions.

Construction 1. Let $\mathbb{P}$ be the poset of all finite partial functions from ω into 2, ordered by reverse inclusion, meaning that $p \leq q$ iff $p \supseteq q$. Note that two conditions p and q are compatible in $\mathbb{P}$ iff $p \cup q$ is a function.

$\mathbb{P}$ belongs to M (since it is definable and the definition is absolute for any transitive model of ZFC). Let G be generic for $\mathbb{P}$ over M. Define $f_G = \bigcup\{p \mid p \in G\}$. Then f_G belongs to $M[G]$. Since every $p \in G$ is a (finite) subset of $\omega \times 2$, $f_G \subseteq \omega \times 2$. f_G is a *function*, meaning that for each n there is at most one i such that $\langle n, i\rangle \in f_G$. Otherwise there must be $n \in \omega$ and $p, q \in G$ so that $\langle n, 0\rangle \in p$ and $\langle n, 1\rangle \in q$. But then $p \cup q$ is not a function, hence p and q are not compatible in $\mathbb{P}$, contradicting the fact that G is a filter.

We claim that f_G is a total function on ω. To see this, fix $n \in \omega$, and let $D_n = \{p \in \mathbb{P} \mid n \in \mathrm{dom}(p)\}$. Then D_n belongs to M, and D_n is dense in $\mathbb{P}$ since every finite partial function can be extended to a finite partial function which has n in its domain. By genericity, $D_n \cap G$ is not empty. Let $p \in D_n \cap G$. Then $n \in \mathrm{dom}(p)$ and $p \subseteq f_G$, so $n \in \mathrm{dom}(f_G)$.

Thus f_G is an element of 2^ω. We claim further that it is distinct from each element of 2^ω that belongs to M. This will establish in particular that $M[G] \neq M$ as required. Fix then some $h \in 2^\omega$ which belongs to M. Let $D_h = \{p \in \mathbb{P} \mid (\exists n \in \mathrm{dom}(p))p(n) \neq h(n)\}$. Then (for $h \in M$) D_h belongs to M since it is defined in an absolute way from h which belongs to M. And D_h is dense, since every *finite* partial function can be extended to be different from h. By genericity, G meets D_h, and it follows that $f_G \neq h$. □

Construction 2. Let $\delta = (\omega_1)^M$. (δ is countable in V, since M is countable and $\delta \subseteq M$. But $M \models$"δ is not countable", meaning that there is no surjection of ω onto δ that belongs to M, and in fact $M \models$"δ is the first uncountable cardinal".) Let $\mathbb{P}$ be the poset of finite partial functions from ω into δ, again ordered by reverse inclusion. Then $\mathbb{P} \in M$. (From the point of view of M, $\mathbb{P}$ is the poset of all finite partial functions from ω into ω_1.)

Let G be generic for $\mathbb{P}$ over M, and let $f_G = \bigcup\{p \mid p \in G\}$. Then $f_G \in M[G]$. As in the previous construction, f_G is a total function from ω into $\delta = (\omega_1)^M$. We claim that f_G is a surjection. To see this, for each $\xi < \delta$ let $D_\xi = \{p \in \mathbb{P} \mid \xi \in \mathrm{range}(p)\}$. D_ξ belongs to M, and is dense since every finite partial function into δ can be extended to a finite partial function that has ξ in its range. By genericity, G meets D_ξ, and it follows that $\xi \in \mathrm{range}(f_G)$.

Thus, $f_G \in M[G]$ is a surjection of ω onto $\delta = (\omega_1)^M$. So δ is countable in $M[G]$. (We say that δ, a cardinal of M, has been *collapsed to* ω in the forcing extension $M[G]$.) Since δ is not countable in M it follows in particular that $M[G] \neq M$. □

The first construction above shows how a forcing extension can add a real (element of 2^ω). The second shows how a forcing extension can collapse a cardinal, adding a surjection of a smaller ordinal onto it. In both cases, since the extension adds no ordinals, the constructible sets of the extension are the same as the constructible sets of M, and so the new added set is not constructible in the extension, and in particular the extension satisfies $V \neq L$. This completes the proof of the lemma. □

In each of the constructions above, we have a proof that $M[G] \models \mathsf{ZFC} + V \neq L$, relying on the fact that $M \models \mathsf{ZFC}$ and G is generic. The proof is "local" in the sense that for each axiom of $\mathsf{ZFC} + V \neq L$ in the extension, we only needed a finite fragment of ZFC in M. Thus we have:

Lemma 2.2. *Let Φ be a finite fragment of* ZFC. *Then there is a finite fragment Ψ of* ZFC *such that (in* ZFC *it is provable that) if there is a ctm M satisfying Ψ then there is a ctm M^* satisfying $\Phi + V \neq L$.*

Recall the following facts from set theory. The first is the reflection schema. The second is proved by taking the Mostowski collapse of a countable elementary substructure of V_α.

Fact 2.3. For every formula ψ in the language of set theory, $\mathsf{ZFC} \vdash (\exists\alpha)(\psi \leftrightarrow \psi^{V_\alpha})$.

Fact 2.4. If $(\exists\alpha)\psi^{V_\alpha}$, then there is a ctm M so that $M \models \psi$.

The first fact is a schema. Formally, $(\forall\psi)\mathsf{ZFC} \vdash (\exists\alpha)(\psi \leftrightarrow \psi^{V_\alpha})$ is a theorem, but $(\forall\psi)(\exists\alpha)\psi \leftrightarrow \psi^{V_\alpha}$ is *not*, that is $\mathsf{ZFC} \not\vdash (\forall\psi)(\exists\alpha)\psi \leftrightarrow \psi^{V_\alpha}$.

Combining the two facts, using them with $\psi = \bigwedge \Psi$ for $\Psi \subseteq \mathsf{ZFC}$, and using the fact that then $\mathsf{ZFC}\vdash\psi$, we get:

Corollary 2.5. *For all finite $\Psi \subseteq \mathsf{ZFC}$, $\mathsf{ZFC}\vdash$ "there exists a ctm M which satisfies Ψ".*

Note that again, the corollary is that $(\forall\Psi \subseteq \mathsf{ZFC}$ finite$)\mathsf{ZFC}\vdash$"there exists a ctm M which satisfies Ψ", but it is *not* the case that $\mathsf{ZFC} \vdash (\forall\Psi \subseteq \mathsf{ZFC}$ finite)(there exists a ctm M which satisfies Ψ).

Theorem 2.6. *If ZFC is consistent, then so is $\mathsf{ZFC} + V \neq L$.*

Proof. Assume ZFC is consistent. Then there is a model $\mathfrak{A} = (A; \in_{\mathfrak{A}})$ of ZFC. By Corollary 2.5 and since $\mathfrak{A}$ is a model of ZFC, for all finite $\Psi \subseteq \mathsf{ZFC}$, $\mathfrak{A} \models$"there exists a ctm M which satisfies Ψ". Applying Lemma 2.2 (with the final part of the lemma applied inside $\mathfrak{A}$) we then get that for all finite $\Phi \subseteq \mathsf{ZFC}$, $\mathfrak{A} \models$"there exists a ctm M^* which satisfies $\Phi + V \neq L$". Since being a model for a formula is absolute between $\mathfrak{A}$ and V, it follows that for all finite $\Phi \subseteq \mathsf{ZFC}$, there really is a model $(M^*, \in_{\mathfrak{A}})$ which satisfies $\Phi + V \neq L$. Hence by compactness, $\mathsf{ZFC} + V \neq L$ is consistent. □

Note in the last proof that M^* is transitive in $\mathfrak{A}$, and the membership relation of M^* is $\in_{\mathfrak{A}}$. This relation may be different from $\in$, and need not even be wellfounded in V. In particular there is no need for the model we obtained at the end, $(M^*, \in_{\mathfrak{A}})$, to be wellfounded or transitive in V.

2.2. Consistency of $\neg\mathsf{CH}$.

Definition 2.7. $\mathrm{Add}(\kappa, \delta)$ denotes the poset of partial functions of size $< \delta$ from $\kappa \times \delta$ into 2, ordered by reverse inclusion.

Theorem 2.8. *Let M be a ctm of ZFC, Let $\mathbb{P} = (\mathrm{Add}(\omega_2, \omega))^M$, and let G be generic for $\mathbb{P}$ over M. Then $M[G] \models 2^\omega \geq \omega_2$.*

The following corollary, settling the consistency of $\neg\mathsf{CH}$ relative to the consistency of ZFC, is immediate from the theorem and the arguments at the end of the previous subsection.

Corollary 2.9. *If ZFC is consistent, then so is $\mathsf{ZFC}+\neg\mathsf{CH}$.*

Proof of Theorem 2.8. Fix M. Let $\delta_1 = (\omega_1)^M$ and let $\delta_2 = (\omega_2)^M$. Let $\mathbb{P} = (\mathrm{Add}(\omega_2, \omega))^M = (\mathrm{Add}(\delta_2, \omega))^V$. Let G be generic for $\mathbb{P}$ over M.

Set $f_G = \bigcup\{p \mid p \in G\}$. Then f_G belongs to $M[G]$, and (using the fact that any two conditions in G are compatible) f_G is a function from $\delta_2 \times \omega$ into 2. A genericity argument shows that f_G is total. Define x_ξ for $\xi < \delta_2 = (\omega_2)^M$ to be the real (that is element of 2^ω) given by $x_\xi(n) = f_G(\xi, n)$. Then since $f_G \in M[G]$, each real x_ξ belongs to $M[G]$, and indeed the sequence $\langle x_\xi \mid \xi < \delta_2\rangle$ belongs to $M[G]$.

Claim 2.10. *The reals x_ξ, $\xi < \delta_2$, are distinct.*

Proof. Fix $\xi \neq \zeta$ below δ_2. Let $D_{\xi,\zeta} = \{p \mid (\exists n) p(\xi, n) \neq p(\zeta, n)\}$. Then $D_{\xi,\zeta}$ belongs to M. D is dense, since any condition in $\mathbb{P}$ is finite, and therefore does not determine $p(\xi, n)$ and $p(\zeta, n)$ for all n. By genericity there is $p \in G$ which belongs to $D_{\xi,\zeta}$. Let n witness this. Then $x_\xi(n) \neq x_\zeta(n)$, hence $x_\xi \neq x_\zeta$. $\square$

The claim and the fact that $\langle x_\xi \mid \xi < \delta_2 \rangle \in M[G]$ together imply that $M[G] \models 2^\omega \geq \delta_2$. But this is not enough to ensure that $M[G] \models \neg\mathsf{CH}$. We must also show that $(\omega_2)^M[G] = \delta_2$. In other words, we must show that $\delta_1 = (\omega_1)^M$ and $\delta_2 = (\omega_2)^M$ are *not collapsed* in the extension $M[G]$, meaning that they remain cardinals in the extension.

We will in fact show that no cardinals are collapsed in the extension, in other words that all cardinals of M remain cardinals in $M[G]$.

Definition 2.11. A poset $\mathbb{P}$ has the *countable chain condition* (c.c.c.) if it has no uncountable antichains. A poset $\mathbb{P}$ has the κ *chain condition* if it has no antichains of size κ.

Remark 2.12. The least κ such that $\mathbb{P}$ has no antichains of size κ is denoted c.c.(κ). It is always a regular cardinal, see Problem 5 in Day 3 of Palumbo's lecture notes.

Lemma 2.13. *Suppose $M \models$ "$\mathbb{P}$ is κ-c.c.". Let G be generic for $\mathbb{P}$ over M. Then all cardinals $\tau \geq \kappa$ of M remain cardinals in $M[G]$. In particular, if $\mathbb{P}$ is c.c.c in M, then all cardinals of M remain cardinals in $M[G]$.*

Proof. Since c.c.$(\mathbb{P})^M$ is a regular cardinal of M, we may by reducing κ if necessary assume that it is regular in M.

Fix $\tau \geq \kappa$, a cardinal of M. Let $f \in M[G]$ be a function from $\alpha < \tau$ into τ. We prove that f is not a surjection. We do this by finding a set $X \in M$ that contains the range of f, and is small enough that it cannot contain τ. The bound on the size of X will be computed in M, using the chain condition for $\mathbb{P}$.

Let $\dot{f} \in M$ be a name so that $\dot{f}[G] = f$. Let $p_0 \in G$ force over M that "$\dot{f}$ is a function". Such p_0 exists by the Fundamental Theorem since $f = \dot{f}[G]$ is a function.

Working in M, let $U_\xi = \{p \leq p_0 \mid (\exists \delta) p \Vdash^M \dot{f}(\check{\xi}) = \check{\delta}\}$, and let $\mathcal{A}_\xi \subseteq U_\xi$ be an antichain of $\mathbb{P}$ which is maximal among antichains contained in U_ξ. Continuing to work in M, for each $p \in U_\xi$ let $\delta_{\xi,p}$ be the unique δ so that $p \Vdash^M \dot{f}(\check{\xi}) = \check{\delta}$. (Uniqueness follows from the fact that p_0, and hence p, forces $\dot{f}$ to be a function.) Finally, let $X = \{\delta_{\xi,p} \mid \xi < \alpha, p \in \mathcal{A}_\xi\}$. Note that X belongs to M, since all the definitions were made in M.

Claim 2.14. *The cardinality of X in M is smaller than τ. In particular, $X \not\supseteq \tau$.*

Proof. We work in M throughout, and all cardinalities are computed in M. By definition of X, $|X| = \Sigma_{\xi<\alpha} |\mathcal{A}_\xi|$. For each $\xi < \alpha$, $|\mathcal{A}_\xi| < \kappa$, since $\mathbb{P}$ has the κ-c.c. (in M). If $\alpha < \kappa$, then by regularity of κ it follows that $|X| < \kappa \leq \tau$. If $\alpha \geq \kappa$, then $|X| = |\alpha| < \tau$. $\square$

Claim 2.15. range$(f) \subseteq X$.

Proof. Fix $\xi < \alpha$ and let $\delta = f(\xi)$. Let $q \in G$ force over M that $\dot{f}(\check{\xi}) = \check{\delta}$. Extending q if needed we may assume that $q \leq p_0$. (q and p_0 both belong to G, so there is a condition $q^* \in G$ below both. We may replace q by q^*.) In particular then q belongs to U_ξ, and by maximality of $\mathcal{A}_\xi$ it follows that there is $p \in \mathcal{A}_\xi$ that is compatible with q. Let s witness the compatibility, so $s \leq p$, and $s \leq q$.

Since $p \Vdash^M \dot{f}(\check{\xi}) = \delta_{\xi,p}$, $s \leq p$ must force the same. Since $q \Vdash^M \dot{f}(\check{\xi}) = \delta$, $s \leq q$ must force the same. Since $s \leq p_0$ and p_0 forces that $\dot{f}$ is a function, it follows that $\delta = \delta_{\xi,p}$. By definition of X, $\delta_{\xi,p} \in X$, so $\delta \in X$. □

It follows from the last two claims that f is not a surjection of α onto τ. f was arbitrary in $M[G]$, so there are no surjection of $\alpha < \tau$ onto τ in $M[G]$, and hence τ is a cardinal of $M[G]$. □

Remark 2.16. The argument above shows that if $\mathbb{P}$ is c.c.c. in M, and $p \Vdash^M$ "$\dot{f}$ is a function from α into the ordinals", then there is a function $F \in M$ with domain α so that (1) $\text{card}(F(\xi)) = \omega$ for each ξ; and (2) $p \Vdash^M (\forall \xi) f(\xi) \in \check{F}(\xi)$. In other words there is a function in M that provides countable bounds for the possible values of f.

We complete the proof of Theorem 2.8 by showing that $\text{Add}(\delta_2, \omega)$ is c.c.c. in M.

Definition 2.17. A family $\mathcal{F}$ of sets is a Δ-*system* if there is r, called the *root* of the system, so that for any $a, b \in \mathcal{F}$, $a \cap b = r$.

We will show that every "large family of small sets" can be thinned to a Δ-system of the same size. The precise details of what this means are in Lemma 2.19 below. For now we just say that in particular, any family of size ω_1 of finite sets has a subfamily, still of size ω_1, which is a Δ-system. But before going into the proof of this, let us see how it implies that $\text{Add}(\delta, \omega)$ is c.c.c.

Lemma 2.18. *For any cardinal δ, the poset $\text{Add}(\delta, \omega)$ is c.c.c.*

Proof. Suppose not, and let $\mathcal{A}$ be an antichain of size ω_1. Let $\mathcal{F} = \{\text{dom}(p) \mid p \in \mathcal{A}\}$. Note that for any finite $d \subseteq \delta \times \omega$, the number of conditions with domain d is finite: it is $2^{|d|}$. Thus $|\mathcal{A}| \leq |\mathcal{F}| \times \omega$ and hence it must be that $|\mathcal{F}| = \omega_1$.

By the Δ-system Lemma, 2.19, there is $\bar{\mathcal{F}} \subseteq \mathcal{F}$, of size ω_1, which forms a Δ-system, say with root d. Let $\bar{\mathcal{A}} = \{p \in \mathcal{A} \mid \text{dom}(p) \in \bar{\mathcal{F}}\}$. Then $\bar{\mathcal{A}} = \omega_1$, but the set $\{p \restriction d \mid p \in \bar{\mathcal{A}}\}$ is finite (of size $2^{|d|}$), so there must be two conditions $p \neq q$ in $\bar{\mathcal{A}}$ so that $p \restriction d = q \restriction d$. Since $\text{dom}(p) \cap \text{dom}(q) = d$ it follows that p and q are compatible, contradicting the fact that $\mathcal{A}$ is an antichain. □

Applying Lemma 2.18 inside M to the poset $\mathbb{P} = \text{Add}(\delta_2, \omega) = \text{Add}(\omega_2, \omega)^M$ we see that $M \models \mathbb{P}$ has the c.c.c. By Lemma 2.13 it follows that all cardinals of M remain cardinals of $M[G]$, and in particular $(\omega_2)^{M[G]} = (\omega_2)^M$. By Claim 2.10 and the paragraph following its proof, $M[G] \models 2^\omega \geq \delta_2$ where $\delta_2 = (\omega_2)^M$. Putting all this together we have that $M[G] \models 2^\omega \geq \omega_2$. This completes the proof of Theorem 2.8 (modulo proof of the Δ-system lemma below). □

Lemma 2.19 (Δ-system lemma)**.** *Let κ be an infinite cardinal. Let $\theta > \kappa$ be regular. Let $\mathcal{F}$ be a family of size $\geq \theta$, with each $x \in \mathcal{F}$ of size $< \kappa$.*

Suppose that $(\forall \alpha < \theta)|\alpha^{<\kappa}| < \theta$.

Then there is a family $\bar{\mathcal{F}} \subseteq \mathcal{F}$, still of size $\geq \theta$, which forms a Δ-system.

Note that the assumptions on κ and θ are satisfied by $\kappa = \omega$ and $\theta = \omega_1$. Thus every family of ω_1 finite sets can be thinned to a Δ-system of size ω_1.

Proof of Lemma 2.19. Shrinking $\mathcal{F}$ if needed we may assume it has size θ. Then $|\bigcup \mathcal{F}| \leq \kappa \times \theta = \theta$, and by replacing the family with an isomorphic copy we may assume that $\bigcup \mathcal{F} \subseteq \theta$, meaning that each $x \in \mathcal{F}$ is a subset of θ.

For each $x \in \mathcal{F}$, the order type of x is $< \kappa$. By shrinking the family further we may assume that there is a fixed $\rho < \kappa$ so that all $x \in \mathcal{F}$ have order type ρ. Let $\langle x(\xi) \mid \xi < \rho \rangle$ enumerate the elements of x in increasing order.

Claim 2.20. *$\bigcup \mathcal{F}$ is unbounded in θ.*

Proof. Suppose not, and let $\alpha < \theta$ be a bound. Then each element of $\mathcal{F}$ is a subset of α of size $< \kappa$. So $|\mathcal{F}| \leq \alpha^{<\kappa}$. By assumption of the lemma this implies that $\mathcal{F}$ has size $< \theta$, contradiction. $\square$

$\bigcup \mathcal{F}$ is equal to $\bigcup_{\xi < \rho} \{x(\xi) \mid x \in \mathcal{F}\}$. By regularity of θ and the previous claim there must be some $\xi < \rho$ so that $\{x(\xi) \mid x \in \mathcal{F}\}$ is unbounded in θ. Let ξ_0 be the least one.

Let $\nu = \sup\{x(\xi) + 1 \mid x \in \mathcal{F}, \xi < \xi_0\}$. Then since $\{x(\xi) \mid x \in \mathcal{F}\}$ is bounded in θ for each $\xi < \xi_0$, and since $\xi_0 < \theta$, it follows from the regularity of θ that $\nu < \theta$. By assumption of the lemma, $|\nu^{<\kappa}| < \theta$. So the set $\{x'' \xi_0 \mid x \in \mathcal{F}\}$, which consists of subsets of ν of size $< \kappa$, must have size less than θ. Thus, by thinning $\mathcal{F}$ we may assume that all elements of $\mathcal{F}$ have the same ξ_0-initial segment. In other words there is r of order type ξ_0 so that $x'' \xi_0 = r$ for all $x \in \mathcal{F}$.

We now build a subfamily of $\mathcal{F}$, whose elements are the same as r below ν, and disjoint above ν. This subfamily is then a Δ-system with root r.

Let x_0 be an element of $\mathcal{F}$ with $x_0(\xi_0) > \nu$. This is possible since $\{x(\xi_0) \mid x \in \mathcal{F}\}$ is unbounded in θ. For each $\gamma < \theta$ let x_γ be an element of $\mathcal{F}$ with $x_\gamma(\xi_0) > \sup\{x_\beta(\xi) \mid \beta < \gamma, \xi < \rho\}$. This again is possible since $\{x(\xi_0) \mid x \in \mathcal{F}\}$ is unbounded in θ.

By construction, $x_\gamma \cap \nu = r$ for each γ, and $(x_\gamma - \nu) \cap (x_\beta - \nu) = \emptyset$ for $\beta \neq \gamma$. It follows that $\{x_\gamma \mid \gamma < \theta\}$ is a Δ-system with root r. $\square$

2.3. **Consistency of CH using forcing.**

Definition 2.21. $\mathrm{Col}(\kappa, X)$ denotes the poset of partial functions of size $< \kappa$ from κ into X, ordered by reverse inclusion.

Theorem 2.22. *Let M be a ctm of ZFC. Let $\mathbb{P} = (\mathrm{Col}(\omega_1, \mathbb{R}))^M$, and let G be generic for $\mathbb{P}$ over M. Then $M[G] \models 2^\omega = \omega_1$.*

As usual, this leads to the corollary that if ZFC is consistent, then so is $\mathsf{ZFC} + 2^\omega = \omega_1$, that is $\mathsf{ZFC} + \mathsf{CH}$.

It is clear using compatibility of conditions in a filter, and genericity, that if G is generic for $(\mathrm{Col}(\omega_1, \mathbb{R}))^M$ over M, then $f_G = \bigcup\{p \mid p \in G\}$ is a function, the domain of f_G is $(\omega_1)^M$, and the range of f_G is $\mathbb{R}^M$. Since f_G belongs to $M[G]$ this means that in $M[G]$, $\mathbb{R}^M$ has cardinality $(\omega_1)^M$. To prove the theorem we need only show that $(\omega_1)^M = (\omega_1)^{M[G]}$ and that $\mathbb{R}^M = \mathbb{R}^{M[G]}$. Both facts will follow from Claim 2.24 and Lemma 2.25 below.

Definition 2.23. A poset $\mathbb{P}$ is $< \kappa$-closed if every decreasing sequence of conditions $\langle p_\xi \mid \xi < \alpha \rangle$ of length $\alpha < \kappa$ has a lower bound in $\mathbb{P}$. (The sequence is decreasing if $p_\xi \leq p_\zeta$ for $\xi \geq \zeta$. A lower bound is a condition q so that $q \leq p_\xi$ for all ξ.)

Claim 2.24. *If κ is regular, then $\mathrm{Col}(\kappa, X)$ is $< \kappa$-closed.*

Proof. Let $\langle p_\xi \mid \xi < \alpha\rangle$ be a decreasing sequence of conditions. Each condition is a partial function of size $< \kappa$. Since the sequence is decreasing, $\bigcup\{p_\xi \mid \xi < \alpha\}$ is also a partial function. By regularity of κ it has size $< \kappa$, so it is a condition. It is clearly a lower bound for the sequence. $\square$

Lemma 2.25. *Suppose ($\mathbb{P}$ is $< \kappa$-closed)^M. Let G be generic for $\mathbb{P}$ over M. Let $f\colon \alpha \to \beta$ (α, β ordinals of M) be a function in $M[G]$, with $\alpha < \kappa$. Then $f \in M$.*

Proof. Let $Z = \{f \in M \mid f\colon \alpha \to \beta\}$. Note $Z \in M$. We must show that $f \in Z$. Suppose for contradiction $f \notin Z$. Let $\dot{f} \in M$ be such that $\dot{f}[G] = f$. Let $p \in G$ force over M that "$\dot{f}$ is a function from $\check{\alpha}$ into $\check{\beta}$ and $\dot{f} \notin \check{K}$". We will get a contradiction by finding a condition $p^* \leq p$ forcing that $\dot{f} \in \check{K}$.

Claim 2.26. *For every $q \leq p$, and every $\xi < \alpha$, there is $\eta < \beta$ and $r \leq q$ so that $r \Vdash^M$ "the value of $\dot{f}$ at $\check{\xi}$ is $\check{\eta}$". (r is said to* force a value for $\dot{f}$ at ξ *in M.)*

Proof. This can be done using the definition of the forcing relation, but it is easier to use the fundamental theorem of forcing.

Let H be generic over M with $q \in H$. Since $q \leq p$, we have $p \in H$ and therefore by choice of p, $\dot{f}[H]$ is a function from α into β. Let $\eta = \dot{f}[H](\xi)$. Then there is a condition $r \in H$ forcing that $\dot{f}$ takes value $\check{\eta}$ and $\check{\xi}$. Since r and q are both in H, they are compatible, and extending r if needed we may assume $r \leq q$. $\square$

Working by transfinite recursion *inside* M, define a sequence of conditions $\langle p_\xi \mid \xi \leq \alpha\rangle$ setting:

(1) $p_0 = p$.
(2) For limit $\gamma \leq \alpha$, p_γ is a lower bound for $\langle p_\xi \mid \xi < \gamma\rangle$.
(3) $p_{\xi+1} \leq p_\xi$ and p_ξ forces a value for $\dot{f}$ at ξ.

$p_{\xi+1}$ satisfying the third condition can be found using the last claim. p_γ satisfying the second condition can be found using the closure of $\mathbb{P}$ in M. (It is important for this that the sequence is constructed in M, in particular it must be constructed with no reference to the original generic G.)

Let $p^* = p_\alpha$. Then $p^* \leq p_{\xi+1}$ for each ξ, so p^* forces a value for $\dot{f}$ at ξ. The value is unique since p^* extends p which forces that $\dot{f}$ is a function. Continuing to work in M, let $h\colon \alpha \to \beta$ be the function $\xi \mapsto$unique η which is forced by p^* to be the value of $\dot{f}$ at ξ.

It is easy to check that p^* forces $\dot{f} = \check{h}$ and that (since $h \in M$ and hence $h \in Z$) p^* forces $\dot{f} \in \check{Z}$. Since $p^* \leq p_0 = p$ this is a contradiction. $\square$

Corollary 2.27. *Suppose ($\mathbb{P}$ is $< \kappa$-closed)^M. Let G be generic for $\mathbb{P}$ over M. Then every cardinal $\tau \leq \kappa$ of M is a cardinal of $M[G]$.*

Proof. Fix τ, and let $f \in M[G]$ be a function from $\alpha < \tau$ into τ. Then $\alpha < \kappa$, so by Lemma 2.25, $f \in M$. Since τ is a cardinal in M, f cannot be a surjection. As f was arbitrary in $M[G]$, this establishes τ is a cardinal in $M[G]$. $\square$

We can now complete the proof of Theorem 2.22. Let M be a ctm of ZFC. Let $\mathbb{P} = (\mathrm{Col}(\omega_1, \mathbb{R}))^M$, and let G be generic for $\mathbb{P}$ over M. Let $f_G = \bigcup p \mid p \in G$, so that $f_G \in M[G]$ is a surjection of $(\omega_1)^M$ into $\mathbb{R}^M$. By Corollary 2.27, $(\omega_1)^{M[G]} = (\omega_1)^M$. By Lemma 2.25, every function from ω into ω that belongs to $M[G]$, belongs to M. It follows that $\mathbb{R}^{M[G]} \subseteq \mathbb{R}^M$, and since the converse inclusion is clear, $\mathbb{R}^{M[G]} = \mathbb{R}^M$. Thus, in $M[G]$, f_G is a surjection of ω_1 onto the reals, and hence $2^\omega = \omega_1$.

Remark 2.28. Since $\mathrm{Col}(\omega_1, \mathbb{R})$ has size $|(\omega_1 \times \mathbb{R})^\omega| = 2^\omega$, the poset is $(2^\omega)^+$-c.c. By Lemma 2.13 it follow that M and $M[G]$ have the same cardinals above $(2^\omega)^M$. Cardinals of M in the interval $(\omega_1, 2^\omega]^M$ do *not* remain cardinals in $M[G]$. They are *collapsed* in $M[G]$ to $(\omega_1)^M$.

2.4. **Extras.** We saw in Subsection 2.2 that the extension of a ctm M by a generic for the poset $\mathrm{Add}(\omega_2, \omega)^M$ satisfies $2^\omega \geq \omega_2$. In fact, assuming GCH in M, the extension satisfies $2^\omega = \omega_2$. To see this we need to count names for subsets of ω.

Lemma 2.29. *Let M be a ctm of $\mathsf{ZFC} + \mathsf{GCH}$, and let G be generic over M for $\mathbb{P} = \mathrm{Add}(\omega_2, \omega)^M$. Then $M[G] \models 2^\omega = \omega_2$.*

Proof. We already know $M[G] \models 2^\omega \geq \omega_2$. We prove $M[G] \models 2^\omega \leq \omega_2$.

The proof of the Powerset axiom in $M[G]$, Lemma 1.26, shows that every $y \subseteq \omega$ in $M[G]$ has a name τ which is contained in $\mathrm{dom}(\check{\omega}) \times \mathbb{P}$. We improve it a little bit in the following claim.

Claim 2.30. *Let $y \subseteq \omega$ belong to $M[G]$. Then there is a name $\tau \in M$ so that $\tau[G] = y$, $\tau \subseteq \{\check{n} \mid n < \omega\} \times \mathbb{P}$, and for each $n < \omega$, $\{q \mid \langle \check{n}, q\rangle \in \tau\}$ is an antichain in $\mathbb{P}$.*

Proof. Fix a name $\dot{y} \in M$ so that $y = \dot{y}[G]$. Working in M, for each $n < \omega$ let $B_n = \{p \mid p \Vdash^M \check{n} \in \dot{y}\}$, and let A_n be a maximal antichain in B_n, that is A_n is an antichain of $\mathbb{P}$, $A_n \subseteq B_n$, and A_n is maximal among such antichains. Let $\tau = \{\langle \check{n}, p\rangle \mid p \in A_n\}$. It is clear that $\tau[G] \subseteq \dot{y}[G] = y$. For the converse, fix $n \in y$. The set D of all q which either extend some $p \in A_n$ or are incompatible with all elements of B_n is dense in $\mathbb{P}$ and belongs to M. By genericity some $q \in D$ belongs to D. Since $n \in y$ there is some $r \in G$ which forces $\check{n} \in \dot{y}$, so it is impossible for q to be incompatible with all elements of B_n. Hence q extends some $p \in A_n$, so $p \in G$ and by definition of τ, $n \in \tau[G]$. □

Remark 2.31. More generally, for each ordinal $\alpha \in M$, and every $y \subseteq \alpha$ that belongs to $M[G]$, there is a name τ so that $\tau[G] = y$, $\tau \subseteq \{\check{\xi} \mid \xi < \alpha\} \times \mathbb{P}$, and for each $\xi < \alpha$, $\{q \mid \langle \check{\xi}, q\rangle \in \tau\}$ is an antichain in $\mathbb{P}$. Such names are called *good*.

Let $R \in M$ be the set of names τ as in the claim, that is names $\tau \subseteq \{\check{n} \mid n < \omega\} \times \mathbb{P}$ with $\{p \mid \langle \check{n}, p\rangle \in \tau\}$ and antichain for each n. Since $\mathbb{P}$ is c.c.c. in M, each $\tau \in R$ is countable in M. Hence $M \models |R| \leq |\omega \times \mathbb{P}|^\omega = (\omega_2)^\omega$.

We assumed that the GCH holds in M, and it follows from this and the fact that $\mathrm{cof}(\omega_2) > \omega$ that $M \models (\omega^2)^M = \omega_2$. So $M \models |R| = \omega_2$.

By the last claim, $\{y \in M[G] \mid x \subseteq \omega\} \subseteq \{\tau[G] \mid \tau \in R\}$, so there is a surjection in $M[G]$ of R onto the powerset of ω. Thus $(2^\omega)^{M[G]} \leq |R|^{M[G]} \leq |R|^M = (\omega_2)^M = (\omega_2)^{M[G]}$. □

Since the consistency of ZFC implies the consistency of $\mathsf{ZFC} + \mathsf{GCH}$, the results above lead to a proof that the consistency of ZFC implies the consistency of $\mathsf{ZFC} + 2^\omega = \omega_2$.

There is nothing specific to ω_2 in the arguments above, except that we used the fact that $\mathrm{cof}(\omega_2) > \omega$ in concluding that $(\omega_2)^\omega = \omega_2$. Similar arguments can thus be used to produce cardinal preserving generic extensions satisfying $2^\omega = \kappa$, for any cardinal κ of M so that $\mathrm{cof}(\kappa) > \omega$ in M.

The assumption $\mathrm{cof}(\kappa) > \omega$ is necessary, since by König's theorem, $\mathrm{cof}(2^\omega)$ is always greater than ω.

More generally, using the poset $\mathrm{Add}(\kappa,\delta)^M$ over a ctm M of $\mathsf{ZFC}+\mathsf{GCH}$, with δ a regular cardinal of M, and κ a cardinal of M with cofinality greater than δ in M, the arguments above adapt to show that the generic extension satisfies $2^\delta=\kappa$. We end with a claim showing that this poset preserves cardinals, followed by some remarks on preservation of cofinalities. The poset provides a very flexible way to violate the GCH, for example getting the consistency of $2^{\aleph_1 2}=\aleph_5 7$ among many other possibilities.

Claim 2.32. *Let δ be regular in M, and suppose that $\delta^{<\delta}=\delta$ in M. Then* $\mathrm{Add}(\kappa,\delta)^M$ *is $<\delta$ closed and δ^+-c.c. in M. In particular, by Lemma 2.13 and Corollary 2.27, all cardinals of M remain cardinals in a generic extension of M by this poset.*

Proof. The arguments in Subsection 2.2 adapt, from the case $\delta=\omega$ to the general case, to show that $\mathrm{Add}(\kappa,\delta)$ is δ^+-c.c. (the assumption that $\delta^{<\delta}=\delta$ is needed for the use of the Δ-system lemma). The proof that the poset is $<\delta$ closed is similar to the proof for the collapse poset in Claim 2.24 (the assumption that δ is regular is needed in that proof). □

Remark 2.33. Often we are interested not only in preservation of cardinals, but in preservation of cofinalities. That is we want to know that $\mathrm{cof}(\tau)^M=\mathrm{cof}(\tau)^{M[G]}$. Our preservation proofs easily give this. Precisely, if $\mathbb{P}$ is κ-c.c. in M, G is generic for $\mathbb{P}$ over M, and $\mathrm{cof}(\tau)^M\geq\kappa$, then $\mathrm{cof}(\tau)^{M[G]}=\mathrm{cof}(\tau)^M$. If $\mathbb{P}$ is $<\kappa$ closed in M, and $\mathrm{cof}(\tau)^{M[G]}<\kappa$, then $\mathrm{cof}(\tau)^{M[G]}=\mathrm{cof}(\tau)^M$ (this in particular means that if $\mathrm{cof}(\tau)^M\leq\kappa$, then $\mathrm{cof}(\tau)^{M[G]}=\mathrm{cof}(\tau)^M$, since otherwise $\mathrm{cof}(\tau)^{M[G]}<\mathrm{cof}(\tau)^M=\kappa$).

3. Products

We can violate the GCH at more than one cardinal by iterating forcing constructions. Starting with a ctm M, we can pass to a generic extension $M[G]$ of M by a poset $\mathbb{P}\in M$ that violates the GCH at a cardinal δ_1, then pass to a generic extension $M[G][H]$ of $M[G]$ by a poset $\mathbb{Q}\in M[G]$ that violates the GCH at a second cardinal δ_2.

If $\mathbb{Q}$ belongs not just to $M[G]$ but to M, then the iteration can be viewed as a single-step extension by the product of $\mathbb{P}$ and $\mathbb{Q}$. By looking at the product we will see that in this case the order of the iteration ($\mathbb{P}$ first followed by $\mathbb{Q}$, or the other way around) does not matter. More importantly, by looking at transfinite products, we will be able to deal with an infinite sequence of extensions. (When dealing with an infinite sequence of extensions, some way to view it as a single extension is necessary, since an arbitrary union of models of ZFC need not itself be a model of ZFC.)

Definition 3.1. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets. Define $\mathbb{P}\times\mathbb{Q}$ to consist of pairs $\langle p,q\rangle$ so that $p\in\mathbb{P}$ and $q\in\mathbb{Q}$, ordered coordinate-wise by $\langle p^*,q^*\rangle\leq\langle p,q\rangle$ iff $p^*\leq p$ and $q^*\leq q$.

For filters $G\subseteq\mathbb{P}$ and $H\subseteq\mathbb{Q}$, define $G\times H=\{\langle p,q\rangle\mid p\in G,q\in H\}$.

Remark 3.2. Every filter $K\subseteq\mathbb{P}\times\mathbb{Q}$ can be written as a product $G\times H$. To see this, set $G=\{p\mid(\exists q)\langle p,q\rangle\in K\}$ and $H=\{q\mid(\exists p)\mid\langle p,q\rangle\in K\}$. It is clear that $K\subseteq G\times H$. For the converse, suppose $p\in G$ and $q\in H$. Let q' and p' respectively witness this, meaning that $\langle p,q'\rangle\in K$ and $\langle p',q\rangle\in K$. Since K is a filter there is $\langle p'', q'' \rangle \in K$ extending both $\langle p, q' \rangle$ and $\langle p', q \rangle$. Then $p'' \leq p$ and $q'' \leq q$, so $\langle p'', q'' \rangle \leq \langle p, q \rangle$, and since K is a filter it follows that $\langle p, q \rangle \in K$.

Lemma 3.3. *Let M be a ctm of* ZFC*, and let $\mathbb{P}, \mathbb{Q} \in M$. Then the following are equivalent:*

1. *G is generic for $\mathbb{P}$ over M and H is generic for $\mathbb{Q}$ over $M[G]$.*
2. *H is generic for $\mathbb{Q}$ over M and G is generic for $\mathbb{P}$ over $M[H]$.*
3. *$G \times H$ is generic for $\mathbb{P} \times \mathbb{Q}$ over M.*

Moreover, if (1)–(3) hold, then $M[G][H] = M[H][G] = M[G \times H]$.

Proof. For the equivalence, it is enough to prove that (1) is equivalent to (3). By symmetry then also (2) is equivalent to (3).

We prove (1) implies (3). Let G be generic for $\mathbb{P}$ over M, and let H be generic for $\mathbb{Q}$ over $M[G]$. It is easy to check that $G \times H$ is a filter. We prove it is generic. Let $D \in M$ be a dense subset of $\mathbb{P} \times \mathbb{Q}$. Let $E = \{q \mid (\exists p \in G)\langle p, q \rangle \in D\}$. Then $E \in M[G]$; it is named by the set $\dot{E} = \langle \check{q}, p \rangle \mid \langle p, q \rangle \in D\}$. Moreover E is dense: Otherwise there is $p \in G$ and $q \in \mathbb{Q}$ so that p forces $\dot{E}$ to have no elements below $\check{q}$. But by density of D there is $\langle p^*, q^* \rangle \in D$ below $\langle p, q \rangle$ and then p^* forces $\check{q}^*$ to be an element of $\dot{E}$ below $\check{q}$. (The genericity of G is used in this argument through appeals to the fundamental theorem of forcing.) By genericity of H over $M[G]$, there must be $q \in H$ that belongs to E. By definition of E this means that there is $p \in G$ so that $\langle p, q \rangle \in D$. Since $p \in G$ and $q \in H$, $\langle p, q \rangle \in G \times H$.

Next we prove that (3) implies (1). Suppose $G \times H$ is generic for $\mathbb{P}$ over M. Again it is easy to see that G and H are filters. We prove they are generic.

Let $D \subseteq \mathbb{P}$ be dense and an element of M. Then $D \times \mathbb{Q}$ is dense in $\mathbb{P} \times \mathbb{Q}$ and therefore has non-empty intersection with $G \times H$. It follows that G meets D.

Let $E \subseteq \mathbb{Q}$ be dense and an element of $M[H]$. Let $\dot{E} \in M$ name E, and let $r \in G$ force that $\dot{E}$ is dense. Let $D^* = \{\langle p, q \rangle \mid p \Vdash^M \check{q} \in \dot{E}$, or p is incompatible with $r\}$. We claim that D^* is dense in $\mathbb{P} \times \mathbb{Q}$. To see this, fix $\langle p, q \rangle \in \mathbb{P} \times \mathbb{Q}$. If p is incompatible with r then $\langle p, q \rangle \in D^*$ and we are done. Otherwise, extending p if necessary, we may assume $p \leq r$. Since $p \Vdash^M$ "$\dot{E}$ is dense", there is $p^* \leq p$, and $q^* \leq q$, so that $p^* \Vdash^M \check{q}^* \in \dot{E}$. Then $\langle p^*, q^* \rangle \leq \langle p, q \rangle$ and $\langle p^*, q^* \rangle \in D^*$.

By genericity of $G \times H$, there is $\langle p, q \rangle \in D^*$ which belongs to $G \times H$. Then $p \in G$, and since $r \in G$ it is impossible that p and r are incompatible. So it must be that $p \Vdash^M \check{q} \in \dot{E}$, and hence $q = \check{q}[G] \in \dot{E}[G] = E$. So H meets E, as required. This completes the proof that (3) implies (1).

Finally, to prove that $M[G][H] = M[H][G] = M[G \times H]$, notes simply that, by Corollary 1.29 and since having G and H in a model of ZFC is equivalent to having $G \times H$ in the model, each of these models is the minimal model of ZFC containing $M \cup \{G, H\}$. In particular they are equal. □

As a simple application of Lemma 3.3, let M be a ctm of ZFC + GCH, and consider the task of forcing over M to obtain a model of $2^{\aleph_5} = \aleph_8 \wedge 2^{\aleph_{12}} = \aleph_{17}$. Let $\mathbb{P} = \mathrm{Add}(\aleph_8, \aleph_5)^M$, and let $\mathbb{Q} = \mathrm{Add}(\aleph_{17}, \aleph_{12})^M$. Let G be generic for $\mathbb{P}$ over M, and let H be generic for $\mathbb{Q}$ over $M[G]$.

It is useful in this case to view the product in reverse, that is starting from the top. By our work in the previous section, $M[H] \models 2^{\aleph_{12}} = \aleph_{17}$, and the extension does not collapse any cardinals. Moreover, H does not add any bounded subsets to $(\aleph_{12})^M$, and hence (a) $M[H]$ satisfies the GCH below $\aleph_{12}$; and (b)

$\mathrm{Add}(\aleph_8, \aleph_5)^{M[H]} = \mathrm{Add}(\aleph_8, \aleph_5)^M = \mathbb{P}$. Thus, again by our work in the previous section, applied over $M[H]$, the further extension $M[H][G]$ does not collapse any cardinals, and $M[H][G] \models 2^{\aleph_5} = \aleph_8$.

Since the further extension by G can at most add subsets of $\aleph_{12}$, and since $M[H] \models 2^{\aleph_{12}} = \aleph_{17}$, it must be that $M[H][G] \models 2^{\aleph_{12}} \geq \aleph_{17}$. In fact with a name counting argument one can check that the further extension does not increase the size of the powerset, so $M[H][G] \models 2^{\aleph_{12}} = \aleph_{17}$.

Putting everything together, $M[G][H] = M[H][G]$ is a model of $2^{\aleph_5} = \aleph_8 \wedge 2^{\aleph_{12}} = \aleph_{17}$. It was important to force with H first, since $\mathrm{Add}(\aleph_8, \aleph_5)^{M[H]} = \mathrm{Add}(\aleph_8, \aleph_5)^M$. In contrast, $\mathrm{Add}(\aleph_{17}, \aleph_{12})^{M[G]} \neq \mathrm{Add}(\aleph_{17}, \aleph_{12})^M$.

As a second example, let M be a ctm of $\mathsf{ZFC} + \mathsf{GCH}$, and consider the task of finding a generic extension which satisfies $(\forall i < \omega) 2^{\aleph_{3i}} = \aleph_{3i+2}$. In particular in the extension GCH fails at infinitely many cardinals.

Let $\mathbb{P}_i = \mathrm{Add}(\aleph_{3i+2}, \aleph_{3i})^M$. It is natural to consider forcing with the infinite product of the posets $\mathbb{P}_i$. Notice that we have several options in taking a product. In particular we have the following two:

(1) Product with finite support. $\mathbb{P}^{\mathrm{fin}}$ consists of sequences $\langle p_i \mid i < \omega \rangle$ so that
   (a) for each i, $p_i \in \mathbb{P}_i$, and
   (b) the set $\{i \mid p_i \neq \mathbb{1}\}$ is finite.
   Conditions are ordered in the natural way, $\langle p_i^* \mid i < \omega \rangle \leq \langle p_i \mid i < \omega \rangle$ iff $(\forall i) p_i^* \leq p_i$.
(2) Product $\mathbb{P}^{\mathrm{ctbl}}$ with full (equivalently in this case countable) support. Same as (1) but dropping the requirement that $p_i = \mathbb{1}$ for all but finitely many i.

As we are working over M, the product is taken inside M, meaning that we only take sequences $\langle p_i \mid i < \omega \rangle$ that belong to M. (In the case of finite support, the product taken in M is the same as the product taken in V. But in the case of full support, the former is smaller.)

A generic K for each of these posets can be decomposed into a sequence $\langle G_i \mid i < \omega \rangle$ of generics for the posets $\mathbb{P}_i$. But this does not mean that the products act the same. They differ substantially on the limit behavior of the sequence of generics added.

To see this, recall that each G_i is a function from $\aleph_{3i+2} \times \aleph_{3i}$ into 2. Let $\dot{d}_i$ name the first value taken by this function, that is the value at $\langle 0, 0 \rangle$.

Let K^{fin} and K^{ctbl} be generic over M for $\mathbb{P}^{\mathrm{fin}}$ and $\mathbb{P}^{\mathrm{ctbl}}$ respectively. Let $d_i^{\mathrm{fin}} = \dot{d}_i[K^{\mathrm{fin}}]$ and $d_i^{\mathrm{ctbl}} = \dot{d}_i[K^{\mathrm{ctbl}}]$. Let $f^{\mathrm{fin}} = \langle d_i^{\mathrm{fin}} \mid i < \omega \rangle$ and let $f^{\mathrm{ctbl}} = \langle d_i^{\mathrm{ctbl}} \mid i < \omega \rangle$. Then f^{fin} belongs to $M[K^{\mathrm{fin}}]$ and f^{ctbl} belongs to $M[K^{\mathrm{ctbl}}]$, but:

- f^{fin} does *not* belong to M. The reason is that, because of the use of finite support, no condition in K^{fin} can force values for $\dot{d}_i$ for all i.
- f^{ctbl} belongs to M. The reason is that, because of the use of countable support, every condition in $\mathbb{P}^{\mathrm{ctbl}}$ can be extended to a condition that forces values for $\dot{d}_i$ for all i.

In fact the behavior of the two posets is drastically different. It is easy to see that forcing with $\mathbb{P}^{\mathrm{fin}}$ collapses $\aleph_\omega$. (Let $\dot{g}$ name the function that assigns for each i the first ordinal $\alpha < \aleph_{3i}$ so that $(\bigcup G_i)(\langle 0, \alpha \rangle) = 1$. By genericity and the use of finite support, $\dot{g}[K^{\mathrm{fin}}]$ is a surjection of ω onto $\aleph_\omega$.) On the other hand we prove next that $\mathbb{P}^{\mathrm{ctbl}}$ does not collapse any cardinals.

For the rest of the discussion, let $\mathbb{P}$ be the full support product of the posets $\mathbb{P}_i$, computed in M. Let K be generic for $\mathbb{P}$ over M. For each j let $G_j = \{r \in \mathbb{P}_j \mid (\exists \langle p_i \mid i < \omega\rangle \in K) p_j = r\}$.

It is easy to check that $K = (\Pi_{i<\omega} G_i) \cap M$. The inclusion $K \subseteq (\Pi_{i<\omega} G_i) \cap M$ is immediate. For the converse, fix $p = \langle p_i \mid i < \omega\rangle \in (\Pi_{i<\omega} G_i) \cap M$, and let $D \in M$ be the set of all conditions $q \in \mathbb{P}$ which are either incompatible with p, or extend p. By genericity, K meets D, say at a condition $q = \langle q_i \mid i < \omega\rangle$. Then for each i, $q_i \in G_i$, so q_i is compatible with p_i. It follows that q is compatible with p, and by the definition of D it must therefore be that q extend p, so $p \in K$.

For each $j < \omega$, set $K_{<j} = \Pi_{i<j} G_i$, and $K_{>j} = (\Pi_{j>k} G_i) \cap M$. Define $\mathbb{P}_{<j}$ and $\mathbb{P}_{>j}$ similarly. The work above on finite products shows that $M[K] = M[K_{<j}][G_j][K_{>j}] = M[K_{>j}[G_j][K_{<j}]$, where, in the order of the right-most extension, $K_{>j}$ is generic for $\mathbb{P}_{>j}$ over M, G_j is generic for $\mathbb{P}_j$ over $M[K_{>j}]$, and $K_{<j}$ is generic for $\mathbb{P}_{<j}$ over $M[K_{>j}][G_j]$. (All this could be done also in the case of a finite support product.)

Because of the use of full support for the product, and the closure of each individual poset $\mathbb{P}_i$ for $i > j$, the product $\mathbb{P}_{>j}$ is $< \aleph_{3j+3}$ closed. (Note that this would not be true with finite support. The finite support product is not even countably closed.)

It follows that M and $M[K_{>j}$ have the same cardinals up to $\aleph_{3j+3}$, and the same bounded subsets of $\aleph_{3j+3}$. In particular, $M[K_{>j}]$ satisfies the GCH below $\aleph_{3j+3}$, and $\mathrm{Add}(\aleph_{3j+2}, \aleph_{3j})^{M[K_{>j}]} = \mathrm{Add}(\aleph_{3j+2}, \aleph_{3j})^M = \mathbb{P}_j$. It follows by the work in Section 2 that $M[K_{>j}][G_j]$ satisfies $2^{\aleph_{3j}} = \aleph_{3j+2}$. As in the case of a product of two posets above, the addition of the generic $K_{<j}$ does not affect this, so $M[K] = M[K_{>j}][G_j][K_{<j}]$ satisfies $2^{\aleph_{3j}} = \aleph_{3j+2}$. This is true for each j.

We have produced an extension $M[K]$ satisfying $(\forall i < \omega) 2^{\aleph_{3i}} = \aleph_{3i+2}$. We did this by forcing with the full support product in M of posets $\mathrm{Add}(\aleph_{3i+2}, \aleph_{3i})^M$. As a final comment on this extension, we note that it has exactly the same cardinals as M. The poset has size $\aleph_{\omega+1}$ in M (using the GCH in M), so it is $\aleph_{\omega+2}$-c.c. in M and therefore preserves all cardinals $\kappa \geq \aleph_{\omega+2}$ of M. By closure, the extension by $K_{>j}$ preserves cardinals up to $\aleph_{3j+3}$ of M, and the further extensions by $G_j, G_{j-1}, \ldots, G_0$ preserve all cardinals. This implies that all cardinals of M below $\aleph_\omega$ are preserved (and hence so it $\aleph_\omega$ itself). It remains to consider $\aleph_{\omega+1}$ of M. If this cardinal is collapsed, then its cofinality in the extension must be $\aleph_n$ for some $n < \omega$. Let j be large enough that $n < 3j + 3$. By closure of $\mathbb{P}_{>j}$, $K_{>j}$ does not add functions from $\aleph_n^M$ into ordinals, and hence in $M[K_{>j}]$, the cofinality of $\aleph_{\omega+1}^M$ is greater than $\aleph_n^M$. By Remark 2.33, the individual posets $\mathbb{P}_i$ do not change cofinalities, so it follows that the cofinality of $\aleph_{\omega+1}^M$ is greater than $\aleph_n^M$ also in $M[K_{>j}][G_j] \ldots [G_0] = M[K]$.

The methods above can be used to change the powerset of regular cardinals in very flexible ways. The most general result, which we do not prove here, is the following:

Theorem 3.4. *Let M be a ctm of $\mathsf{ZFC} + \mathsf{GCH}$. Let $E \in M$ be a function that satisfies the following requirements in M:*

(1) $\mathrm{dom}(E)$ *is a set of regular cardinals, and the values taken by E are cardinals.*
(2) *If $\kappa < \kappa'$ both belong to* $\mathrm{dom}(E)$*, then $E(\kappa) \leq E(\kappa')$.*
(3) $\mathrm{cof}(E(\kappa)) > \kappa$.

Then there is a cardinal preserving generic extension $M[G]$ of M which satisfies $(\forall\kappa \in \mathrm{dom}(E))2^\kappa = E(\kappa)$.

The theorem states that the behavior of the exponential function at regular cardinals can be changed in arbitrary ways subject only to two restrictions: monotonicity, and König's theorem that $\mathrm{cof}(2^\kappa) > \kappa$.

4. Iterations

4.1. **Two step iterations.** Let M be a ctm of ZFC. We work to present a two-step forcing iteration, leading to extension $M[G][H]$, as a single step forcing extension. (If the poset for the second step belongs to M, this can be done using products. But we do not assume here that the second step poset is in M; it need only belong to $M[H]$.) There is not much gained from presenting the iteration as a single step extension in the case of a finite iteration, but later on we will build on the work here to handle an infinite number of steps. With infinitely many steps, viewing the extensions individually, for example as extensions $M[G_0]$, $M[G_0][G_1]$, $M[G_0][G_1][G_2]$, $\cdots$ does not by itself suffice for handling the limit stages, since the increasing union of models of ZFC need not be a model of ZFC. It will be important there to be able to have a single poset that adds the entire sequence of generics.

Let $\mathbb{P}$ be a poset, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a poset. Suppose that "$\dot{\mathbb{Q}}$ is a poset" is forced by $\mathbb{1}_\mathbb{P}$ in $\mathbb{P}$.

Remark 4.1. A poset $\mathbb{Q}$ formally consists of three parts, the underlying set, the poset ordering $\leq_\mathbb{Q}$, and the largest element $\mathbb{1}_\mathbb{Q}$. Though we abuse notation and talk only about $\dot{\mathbb{Q}}$, there are two additional names hiding in the notation, for $\leq_\mathbb{Q}$ and for $\mathbb{1}_\mathbb{Q}$. Thus, our precise assumption is that the statement "$\dot{\leq}_\mathbb{Q}$ is forced to be a partial order on $\dot{\mathbb{Q}}$, with largest element $\dot{\mathbb{1}}_\mathbb{Q}$" is forced by $\mathbb{1}_\mathbb{P}$ in $\mathbb{P}$.

Definition 4.2. $\mathbb{P} * \dot{\mathbb{Q}}$ is the poset consisting of pairs $\langle p, \tau\rangle$ so that $p \in \mathbb{P}$, $\tau \in \mathrm{dom}(\dot{\mathbb{Q}})$, and $p \Vdash_\mathbb{P} \tau \in \dot{\mathbb{Q}}$. The poset ordering is $\langle p^*, \tau^*\rangle \leq \langle p, \tau\rangle$ iff $p^* \leq_\mathbb{P} p$ and $p^* \Vdash_\mathbb{P} \tau^* \dot{\leq}_\mathbb{Q} \tau$.

The restriction $\tau \in \mathrm{dom}(\dot{\mathbb{Q}})$ in the definition is rather strict. When working with generics, every $x \in \dot{Q}[G]$ has a name $\tau \in \mathrm{dom}\,\mathbb{Q}$. Put another way, for every σ that is forced by some condition in a G to belong to $\dot{\mathbb{Q}}$, there is $\tau \in \mathrm{dom}(\dot{\mathbb{Q}})$ that is forced by a condition in G to be equal to σ. In that sense the restriction in the definition causes no loss. But if $p \Vdash \sigma \in \dot{\mathbb{Q}}$, it is *not* the case that there is $\tau \in \mathrm{dom}(\dot{\mathbb{Q}})$ so that $p \Vdash \sigma = \tau$. Rather it is only the case that there is $q \leq p$ and $\tau \in \mathrm{dom}(\dot{\mathbb{Q}})$ so that $q \Vdash \sigma = \tau$. In that sense the restriction in the definition does causes some loss. In some situations later on we will have to avoid this loss by ensuring that $\mathrm{dom}(\dot{\mathbb{Q}})$ is sufficiently rich, and when that situation comes, we will in fact revise the definition. (The revised definition is Definition 4.28.)

We work for the rest of the section over a ctm M of ZFC. We assume $\mathbb{P}, \dot{\mathbb{Q}} \in M$, $\mathbb{P}$ is a poset, and $\dot{\mathbb{Q}}$ is forced by $\mathbb{1}_\mathbb{P}$ in $\mathbb{P}$ over M to name a poset. For notational simplicity we for the most part drop the superscript M below, but the relevant notions (such as $\mathbb{P} * \dot{\mathbb{Q}}$) are still relativized to M, even when this is not indicated explicitly. We also drop the poset indicating subscripts $\Vdash$ and of poset relations $\leq$. The poset used is understood from the context.

Definition 4.3. Let G be generic for $\mathbb{P}$ over M, and let H be generic for $\dot{\mathbb{Q}}[G]$ over $M[G]$. Define $G * H$ to be $\{\langle p, \tau\rangle \in (\mathbb{P} * \dot{\mathbb{Q}})^M \mid p \in G \text{ and } \tau[G] \in H\}$.

Lemmas 4.4, 4.6, and 4.7 below show that a two-step extension $M[G][H]$ by posets $\mathbb{P}$ and $\dot{\mathbb{Q}}[G]$ is exactly the same as a one-step extension $M[K = G * H]$ by the poset $\mathbb{P} * \dot{\mathbb{Q}}$.

Lemma 4.4. *Let G be generic for $\mathbb{P}$ over M, and let H be generic for $\dot{\mathbb{Q}}[G]$ over $M[G]$. Then $G * H$ is generic for $\mathbb{P} * \dot{\mathbb{Q}}$ over M.*

Proof. It is easy to check that $G * H$ is a filter. We prove it meets every dense set in M.

Let $D \in M$ be dense in $\mathbb{P} * \dot{\mathbb{Q}}$. D is a set of pairs of the form $\langle p, \tau \rangle$, where $p \in \mathbb{P}$ and τ is a $\mathbb{P}$-name. So $D^{-1} = \{\langle \tau, p \rangle \mid \langle p, \tau \rangle \in D\}$ is a $\mathbb{P}$-name. Set $E = D^{-1}[G]$. Then $E \in M[G]$.

Claim 4.5. *E is dense in $\dot{\mathbb{Q}}[G]$.*

Proof. Let $a \in \dot{\mathbb{Q}}[G]$, say $a = \sigma[G]$ where $\sigma \in \operatorname{dom}(\dot{\mathbb{Q}})$. Fix $q \in G$ so that $q \Vdash \sigma \in \dot{\mathbb{Q}}$. Let $A \in M$ be the set $\{p \leq q \mid (\exists \tau) \langle p, \tau \rangle \in D \text{ and } \langle p, \tau \rangle \leq \langle q, \sigma \rangle\}$.

We prove to begin with that A is dense in $\mathbb{P}$ below q. We will then use the fact that G must meet A to complete the proof of the claim.

Fix $r \leq q$. Note that $\langle r, \sigma \rangle$ is a condition in $\mathbb{P} * \dot{\mathbb{Q}}$. Since D is dense, there must be some $\langle p, \tau \rangle \in D$ below $\langle r, \sigma \rangle$. Then $p \leq r$ and (since $r \leq q$) $\langle p, \tau \rangle$ is also below $\langle q, \sigma \rangle$, so $p \in A$. This shows that A is dense below q.

By genericity of G, there must be $p \in G$ that belongs to A. Let τ witness this. Then $\langle p, \tau \rangle \in D$, so $\langle \tau, p \rangle \in D^{-1}$ and since $p \in G$ it follows that $\tau[G] \in E$. Also by definition of A, $\langle p, \tau \rangle \leq \langle q, \sigma \rangle$ so $p \Vdash \tau \leq \sigma$ and since $p \in G$ it follows that $\tau[G] \leq \sigma[G]$. This completes the proof that E is dense. □

Now since H is generic for $\dot{\mathbb{Q}}[G]$ over $M[G]$, it must meet E. Fix then some $a \in H \cap E$. By definition of E and since $a \in E$, there is some $\langle \tau, p \rangle \in D^{-1}$, with $p \in G$ and $a = \tau[G]$. We have then $\langle p, \tau \rangle \in D$, and since $p \in G$ and $\tau[G] \in H$, $\langle p, \tau \rangle \in G * H$. So $G * H$ meets D, as required. □

Lemma 4.6. *(G, H as above.) $M[G][H]$ is precisely equal to $M[G * H]$.*

Proof. $M[G][H]$ is the minimal model of ZFC containing $M \cup \{G, H\}$, and $M[G*H]$ is the minimal model of ZFC containing $M \cup \{G * H\}$. The lemma is immediate from this using the fact that any model of ZFC can compute $G * H$ from G and H, and vice-versa. □

Lemma 4.7. *Let K be generic for $\mathbb{P} * \dot{\mathbb{Q}}$ over M. Set $G = \{p \mid (\exists \tau) \langle p, \tau \rangle \in K\}$ and $H = \{\tau[G] \mid (\exists p) \langle p, \tau \rangle \in K\}$. Then:*
1. *G is generic for $\mathbb{P}$ over M.*
2. *H is generic for $\dot{\mathbb{Q}}[G]$ over $M[G]$.*
3. *$G * H = K$.*

Proof. Part (1) is easy. The inclusion $K \subseteq G * H$ in part (3) is immediate. For the reverse inclusion, fix $\langle p, \tau \rangle \in G * H$. Then $p \in G$ and $\tau[G] \in H$. The latter implies that there are τ' and q so that $\langle q, \tau' \rangle \in K$ and $\tau[G] = \tau'[G]$. Let $p' \in G$ be stronger than p and strong enough to force that $\tau = \tau'$. Since $p' \in G$, there is σ so that $\langle p', \sigma \rangle \in K$.

Since K is a filter, there is $\langle r, \pi \rangle$ in K below both $\langle q, \tau' \rangle$ and $\langle p', \sigma \rangle$. Then $r \leq p' \leq p$, and $r \Vdash \pi \leq \tau' = \tau$, so $\langle r, \pi \rangle \leq \langle p, \tau \rangle$, and since K is a filter it follows that $\langle p, \tau \rangle \in K$.

Finally we prove (2). Let $E \in M[G]$ be dense in $\dot{\mathbb{Q}}[G]$. Let $\dot{E} \in M$ be such that $\dot{E}[G] = E$. Let $r \in G$ be strong enough to force "$\dot{E}$ is dense in $\dot{\mathbb{Q}}$". (In finding r we are using the genericity of G, given by part (1).) Working in M let $D = \{\langle p, \tau\rangle \in \mathbb{P} * \dot{\mathbb{Q}} \mid p \text{ is incompatible with } r \text{ or } p \Vdash \tau \in \dot{E}\}$.

Then D is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. (The proof of this uses the fact that $r \Vdash \dot{E}$ is dense in $\dot{\mathbb{Q}}$.) By genericity it follows that K meets D. Fix $\langle p, \tau\rangle \in K$ that belongs to D. Then $p \in G$, and since $r \in G$ and G is a filter by part (1), p and r are compatible. By definition of D it follows that p must force $\tau \in \dot{E}$. So $\tau[G] \in \dot{E}[G] = E$. Since $\langle p, \tau\rangle \in K$, $\tau[G]$ belongs also to H. This establishes the genericity of H over $M[G]$. □

We end this subsection with a proof that the composition of c.c.c. posets is c.c.c.

Claim 4.8. *Let $\mathbb{P}$ be c.c.c. and let σ be a $\mathbb{P}$-name. Suppose $\Vdash \sigma \subseteq \check{\omega}_1 \wedge \mathrm{card}(\sigma) < \check{\omega}_1$. Then there is $\beta < \omega_1$ so that $\Vdash \sigma \subseteq \check{\beta}$.*

Proof. This is a consequence of Remark 2.16 (with a function f that has domain $\{0\}$ and takes value the supremum of σ), and can also be proved directly using antichains. We take the latter approach. Let A be a maximal antichain of conditions p which force a value for $\sup(\sigma)$. For each $p \in A$ let $\delta_p < \omega_1$ be the value that p forces for the supremum, that is the unique δ so that $p \Vdash \sup(\sigma) = \check{\delta}$. Let $\beta = \sup\{\delta_p \mid p \in A\}$. Then σ is forced to be contained in β, and since $\mathbb{P}$ is c.c.c., A is countable and so $\beta < \omega_1$. □

Lemma 4.9. *Suppose that $\mathbb{P}$ is c.c.c. and that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is c.c.c.". Then $\mathbb{P} * \dot{\mathbb{Q}}$ is c.c.c.*

Proof. Suppose not, and let $A = \{\langle p_\xi, \tau_\xi\rangle \mid \xi < \omega_1\}$ be an antichain of $\mathbb{P} * \dot{\mathbb{Q}}$. Let σ be the $\mathbb{P}$-name $\{\langle \check{\xi}, p_\xi\rangle \mid \xi < \omega_1\}$. In other words σ names the set of ξ so that p_ξ belongs to the generic object.

Let G be generic for $\mathbb{P}$ over V. (We are abusing notation here. Strictly speaking one should pass to a countable submodel of V, and work over that model instead of over V. But it is convenient to pretend instead that there are generic objects external to V.)

Claim 4.10. *$\sigma[G]$ is countable.*

Proof. It is enough to show that if $\xi \neq \eta$ both belong to $\sigma[G]$, then $\tau_\xi[G]$ and $\tau_\eta[G]$ are incompatible in $\dot{\mathbb{Q}}[G]$. From the fact that $\dot{\mathbb{Q}}[G]$ is c.c.c. in $V[G]$ it then follows that $\sigma[G]$ must be countable.

Fix $\xi \neq \eta$ both in $\sigma[G]$. Note that then p_ξ and p_η both belong to G. Suppose for contradiction that $\tau_\xi[G]$ and $\tau_\eta[G]$ are compatible. Then there is $\pi \in \mathrm{dom}(\dot{\mathbb{Q}})$ so that $\pi[G]$ extends both $\tau_\xi[G]$ and $\tau_\eta[G]$. Using the fundamental theorem of forcing, there is a condition $r \in G$ forcing this, namely $r \Vdash_{\mathbb{P}} \pi \leq \tau_\xi \wedge \pi \leq \tau_\eta$. Extending r if necessary, and using the fact that all conditions in G are compatible in G, we may assume that $r \leq p_\xi$ and $r \leq p_\eta$. Then by definition of $\mathbb{P} * \dot{\mathbb{Q}}$, $\langle r, \pi\rangle \leq \langle p_\xi, \tau_\xi\rangle$ and $\langle r, \pi\rangle \leq \langle p_\eta, \tau_\eta\rangle$. But this contradicts the fact that A is an antichain. □

Now by Claim 4.8 and since $\mathbb{P}$ is c.c.c., there is $\beta < \omega_1$ so that $\Vdash_{\mathbb{P}} \sigma \subseteq \check{\beta}$. But this is impossible, since $p_\beta \Vdash_{\mathbb{P}} \check{\beta} \in \sigma$. This contradiction completes the proof that $\mathbb{P} * \dot{\mathbb{Q}}$ is c.c.c. □

We remark that it was important for the last lemma that $\mathbb{Q} = \dot{\mathbb{Q}}[G]$ is c.c.c. in the extension $V[G]$, rather than the ground model (in cases where $\mathbb{Q}$ belongs to the ground model). There are situation where a poset $\mathbb{Q} \in V$ that is c.c.c. in V, fails to be c.c.c. in the extension, and in that case so does the composition. We give an example below. (Note that for $\mathbb{Q} \in V$, the composition is essentially a product, so for the example below we work with products.)

Suppose $V = L$, so that there is a Suslin tree T. Let $\mathbb{P} = \mathbb{Q}$ be the poset consisting of nodes of T, ordered by extension, that is $p \leq q$ if p extends q. Since Suslin trees are c.c.c., the forcing is c.c.c. Let G be generic for $\mathbb{P}$ over V. Then G is a cofinal branch through T. But then the set of nodes $\{q \mid q$ is a successor of some $p \in G$ and $q \notin G\}$, that is the set of nodes that veer off from the branch G, is an antichain of T, of size ω_1, that belongs to $V[G]$. It follows that $\mathbb{Q} = \mathbb{P}$ is *not* c.c.c. in the extension $V[G]$.

This construction of an antichain of T in the extension can easily be rephrased to show that $\mathbb{P} \times \mathbb{P}$ is not c.c.c., even though $\mathbb{P}$ itself is. (The set $\{\langle p, q\rangle \mid p$ and q are both successors of the same node, and $q \neq p\}$ is an antichain of size ω_1 in the product.) The reason this does not contradict the lemma, is that the lemma assumes $\mathbb{Q}$ is c.c.c. in the extension, not just in the ground model.

4.2. **Transfinite iterations.**

Definition 4.11. Let α be an ordinal. An *α-stage finite support iteration* is a pair of sequences $\langle \mathbb{P}_\xi \mid \xi \leq \alpha\rangle$ and $\langle \dot{\mathbb{Q}}_\xi \mid \xi < \alpha\rangle$ so that:

(1) Each $\mathbb{P}_\xi$ is a poset.
(2) All conditions in $\mathbb{P}_\xi$ are sequences of length ξ.
(3) Each $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_\xi$-name that is forced to be a poset. (Precisely, there are three names, $\dot{\mathbb{Q}}_\xi$, $\dot{\mathbb{1}}_\xi$, and $\dot{\leq}_\xi$, so that $\dot{\mathbb{1}}_\xi \in \operatorname{dom}(\dot{\mathbb{Q}}_\xi)$ and it is forced by all conditions in $\mathbb{P}_\xi$ that "$\dot{\leq}_\xi$ is a partial order on $\dot{\mathbb{Q}}_\xi$ with largest element $\dot{\mathbb{1}}_\xi$".)
(4) (Basis) $\mathbb{P}_0$ is the trivial poset $\{0\}$.
(5) (Successor) Conditions in $\mathbb{P}_{\xi+1}$ are sequences $p = \langle p(\mu) \mid \mu < \xi + 1\rangle$ so that
    (a) $p{\restriction}\xi \in \mathbb{P}_\xi$,
    (b) $p(\xi) \in \operatorname{dom}(\dot{\mathbb{Q}}_\xi)$ and $p{\restriction}\xi \Vdash_{\mathbb{P}_\xi} p(\xi) \in \dot{\mathbb{Q}}_\xi$.
    The ordering on conditions is $p^* \leq p$ iff $p^*{\restriction}\xi \leq p{\restriction}\xi$ and $p^*{\restriction}\xi \Vdash p^*(\xi) \leq p(\xi)$.
(6) (Limit) For limit η, $\mathbb{P}_\eta$ consists of sequences $p = \langle p(\mu) \mid \mu < \eta\rangle$ so that
    (a) $(\forall \xi < \eta) p{\restriction}\xi \in \mathbb{P}_\xi$, and
    (b) For all but finitely many $\mu < \eta$, $p(\mu) = \dot{\mathbb{1}}_\mu$.
    The ordering is the natural one, $p^* \leq p$ iff $(\forall \xi < \eta) p^*{\restriction}\xi \leq p{\restriction}\xi$.

Note that in the successor case, the poset $\mathbb{P}_{\xi+1}$ is isomorphic to the composition $\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$. Indeed, the only difference is that in the composition we take pairs $\langle p, \tau\rangle$ and here we take instead an extended sequence $p^\frown\langle\tau\rangle$. The clause $\tau \in \dot{\mathbb{Q}}_\xi$ in the successor case will prove too restrictive later on, and we will revise it in Definition 4.28.

Note also that the iteration is completely determined by the sequence $\langle \dot{\mathbb{Q}}_\xi \mid \xi < \alpha\rangle$. The definition of an α-stage iteration produces the posets $\mathbb{P}_\xi$, $\xi \leq \alpha$, from the sequence $\langle \dot{\mathbb{Q}}_\xi \mid \xi < \alpha\rangle$.

Countable support iterations are defined similarly, replacing "all but finitely" with "all but countably" in condition (6b). *Full support* iterations are defined by dropping condition (6b) altogether. There are many other kinds of supports, some standard, and some tailored to specific applications.

Lemma 4.12. *Let $\langle \mathbb{P}_\xi \mid \xi \leq \alpha \rangle$ and $\langle \dot{\mathbb{Q}}_\xi \mid \xi < \alpha \rangle$ be an α length iteration (of any kind of support) in a ctm M. Let G be generic for $\mathbb{P}_\alpha$ over M.*

1. *For each $\xi < \alpha$ let $G_\xi = \{p \restriction \xi \mid p \in G\}$. Then G_ξ is generic for $\mathbb{P}_\xi$ over M.*
2. *For each $\xi < \alpha$ let $H_\xi = \{p(\xi)[G_\xi] \mid p \in G\}$. Then H_ξ is generic for $\dot{Q}_\xi[G_\xi]$ over $M[G_\xi]$, and $M[G_\xi][H_\xi] = M[G_{\xi+1}]$.*

The lemma shows that the iteration does what it is supposed to do, that is add generics for each of the posets named by the $\dot{Q}_\xi$s. In a case of a finite length iteration, it shows that the end extension is exactly $M[H_0] \dots [H_{\alpha-1}]$ (but this multi-step extension does not make sense for transfinite length iterations, since an increasing union of models of ZFC need not be a mode of ZFC).

Proof of Lemma 4.12. It is easy to check that G_ξ is a filter. We prove genericity. Let D be dense in $\mathbb{P}_\xi$. Let $D' = \{p \in \mathbb{P}_\alpha \mid p \restriction \xi \in D\}$. Then D' is dense in $\mathbb{P}_\alpha$, and the fact that G meets D' implies that G_ξ meets D. This establishes the genericity of G_ξ.

For part (2), note that $\mathbb{P}_{\xi+1}$ is by definition isomorphic to $\mathbb{P}_\xi * \dot{Q}_\xi$. (The isomorphism sends $p \in \mathbb{P}_{\xi+1}$ to $\langle p \restriction \xi, p(\xi) \rangle$ in $\mathbb{P}_\xi * \dot{Q}_\xi$.) Now apply Lemma 4.7 with K equal to (the image under the isomorphism of) $G_{\xi+1}$. □

Lemma 4.13. *Let $\langle \mathbb{P}_\xi \mid \xi \leq \alpha \rangle$ and $\langle \dot{\mathbb{Q}}_\xi \mid \xi < \alpha \rangle$ be an α length* finite support *iteration in M. Suppose that for each $\xi < \alpha$, $\Vdash_{\mathbb{P}_\xi}$ "$\dot{Q}_\xi$ is c.c.c.". Then $\mathbb{P}_\alpha$ is c.c.c.*

Proof. We prove for each $\xi \leq \alpha$ that $\mathbb{P}_\xi$ is c.c.c. The proof is by induction on ξ. The case of $\xi = 0$ is trivial, as $\mathbb{P}_0 = \{0\}$. The successor case is given by Lemma 4.9 as $\mathbb{P}_{\xi+1}$ is isomorphic to $\mathbb{P}_\xi * \dot{Q}_\xi$. Suppose then that $\gamma \leq \alpha$ is a limit ordinal, and that for all $\xi < \gamma$, $\mathbb{P}_\xi$ is c.c.c. We prove that $\mathbb{P}_\gamma$ is c.c.c.

Suppose for contradiction that $\{p^\nu \mid \nu < \omega_1\}$ is an antichain in $\mathbb{P}_\gamma$. For each p, let $\mathrm{suprt}(p)$ denote the set of ξ so that $p(\xi) \neq \mathbb{1}_\xi$. Since we are using finite support iteration, $\mathrm{suprt}(p)$ is finite for every condition p. Thus the family $\{\mathrm{suprt}(p^\nu) \mid \nu < \omega_1\}$ is an ω_1 size family of finite sets. By the Δ-system lemma (Lemma 2.19), it has a subfamily of size ω_1 which forms a Δ-system, with root r say. Shrinking the initial antichain if necessary, we may assume without loss of generality that the entire family $\{\mathrm{suprt}(p^\nu) \mid \nu < \omega_1\}$ is a Δ-system with root r.

Since r is finite, and γ is a limit ordinal, there is $\delta < \gamma$ so that $r \subseteq \delta$.

Claim 4.14. *Let $\nu, \mu < \omega_1$ be distinct. Then $p^\nu \restriction \delta$ and $p^\mu \restriction \delta$ are incompatible in $\mathbb{P}_\delta$.*

Proof. Suppose for contradiction that $q \in \mathbb{P}_\delta$ extends both $p^\nu \restriction \delta$ and $p^\mu \restriction \delta$. Define $q^* \in \mathbb{P}_\gamma$ by:

$$q^*(\xi) = \begin{cases} q(\xi) & \text{if } \xi < \delta, \\ p^\nu(\xi) & \text{if } \xi \geq \delta \text{ and } \xi \in \mathrm{suprt}(p^\nu), \\ p^\mu(\xi) & \text{if } \xi \geq \delta \text{ and } \xi \in \mathrm{suprt}(p^\mu), \\ \mathbb{1}_\xi & \text{otherwise.} \end{cases}$$

Note that there is no conflict between the clauses of the definition, since $\mathrm{suprt}(p^\mu) \cap \mathrm{suprt}(p^\nu) = r \subseteq \delta$.

Using the fact that q extends both $p^\mu \restriction \delta$ and p^ν restricted to δ it is easy to check from the definition of q^* that it extends both p^μ and p^ν. (Prove inductively for $\zeta \in [\delta, \gamma]$ that $q^* \restriction \zeta$ extends both $p^\nu \restriction \zeta$ and $p^\mu \restriction \zeta$.) But this is a contradiction, since p^μ and p^ν belong to an antichain in $\mathbb{P}_\gamma$, and are therefore incompatible. □

It follows from the last claim that $\{p^\nu \restriction \delta \mid \nu < \omega_1\}$ is an antichain of size ω_1 in $\mathbb{P}_\delta$. But by induction, as $\delta < \gamma$, $\mathbb{P}_\delta$ has the c.c.c. This contradiction completes the proof that $\mathbb{P}_\gamma$ has the c.c.c. □

4.3. **Consistency of $\mathsf{MA}(\omega_1)$.** Recall that Martin's Axiom for κ ($\mathsf{MA}(\kappa)$) states that for every c.c.c. poset $\mathbb{B}$, and every κ size family $\mathcal{F}$ of dense sets of $\mathbb{B}$, there is a filter $G \subseteq \mathbb{B}$ which meets every dense set in $\mathcal{F}$.

Let $\mathsf{bMA}(\kappa)$ be the same statement, but restricted to posets of size $\leq \kappa$. We show that the two are equivalent. This will later allow us, in the forcing construction for the consistency of $\mathsf{MA}(\omega_1)$, to restrict our attention to handling posets of size $\leq \omega_1$ (and in fact, just posets contained in ω_1, since every poset of size $\leq \omega_1$ is isomorphic to a poset contained in ω_1).

Lemma 4.15. $\mathsf{MA}(\kappa)$ *is equivalent to* $\mathsf{bMA}(\kappa)$.

Proof. The left-to-right implication is clear. In the other direction, let $\mathbb{B}$ be a c.c.c. poset (of any size), and let $\mathcal{F} = \{D_\xi \mid \xi < \kappa\}$ be a family of dense sets. We find a filter G on $\mathbb{B}$ which meets each of the sets D_ξ.

Let θ be a regular cardinal large enough that $\{\mathbb{B}\} \cup \mathcal{F} \subseteq V_\theta$. Fix an elementary substructure $H \prec V_\theta$, of size κ, with $\{\mathbb{B}\} \cup \mathcal{F} \subseteq H$. We intend to apply $\mathsf{bMA}(\kappa)$ to the poset $\mathbb{B} \cap H$.

First, note the following simple observation:

- $\mathbb{B} \cap H$ is a poset. This is immediate using the elementarity of H.
- For each ξ, $D_\xi \cap H$ is dense in $\mathbb{B} \cap H$. This is also immediate by elementarity: let $p \in \mathbb{B} \cap H$. Then $V_\theta \models (\exists q)(q \leq p \wedge q \in D_\xi)$. Since p, $\leq$, and D_ξ belong to H, by elementarity $H \models (\exists q)(q \leq p \wedge q \in D_\xi)$. So there is $q \in H$ so that $q \leq p$ and $q \in D_\xi$.
- $\mathbb{B} \cap H$ is c.c.c. To prove this, first note that if $p, q \in \mathbb{B} \cap H$ are incompatible in $\mathbb{B} \cap H$, then they are also incompatible in $\mathbb{B}$. (By elementarity of H, if there is r that is below both p and q, then there is such r in H.) It follows that every antichain of $\mathbb{B} \cap H$ is also an antichain of $\mathbb{B}$. Since $\mathbb{B}$ does not have antichains of size ω_1, neither does $\mathbb{B} \cap H$.

Since H has size κ, $\mathbb{B} \cap H$ has size $\leq \kappa$. By the observations above we may apply $\mathsf{bMA}(\kappa)$ to $\mathbb{B} \cap H$ and the family $\{D_\xi \cap H \mid \xi < \kappa\}$. We get a filter $\bar{G}$ on $\mathbb{B} \cap H$, which meets each of the sets $D_\xi \cap H$. Let G be the upward closure of $\bar{G}$ in $\mathbb{B}$. Then G is a filter, and since it contains $\bar{G}$, it meets each of the sets D_ξ. □

We begin now to work on a forcing extension that satisfies $\mathsf{bMA}(\omega_1)$. Fix a cardinal δ with $\operatorname{cof}(\delta) \geq \omega_2$. Let $f \colon \delta \to V$ be a function. Let $\langle \mathbb{P}_\xi \mid \xi \leq \delta \rangle$, $\langle \dot{\mathbb{Q}}_\xi \mid \xi < \delta \rangle$ be the finite support iteration determined by:

$$\dot{\mathbb{Q}}_\xi = \begin{cases} f(\xi) & \text{if } \Vdash_{\mathbb{P}_\xi} \text{“}f(\xi) \text{ is a c.c.c. poset contained in } \check{\omega}_1\text{”}, \\ \mathbb{P}_\xi\text{-name for the poset } \{0\} & \text{otherwise.} \end{cases}$$

We will add more assumptions on δ, and define f more precisely, later on.

Let $\mathbb{P} = \mathbb{P}_\delta$. Then $\mathbb{P}$ is c.c.c. Let G be generic for $\mathbb{P}$ over V. Set $G_\xi = G\restriction\xi = \{p\restriction\xi \mid p \in G\}$ and $H_\xi = \{p(\xi)[G_\xi] \mid p \in G\}$. Then G_ξ is generic for $\mathbb{P}_\xi$ over V, and H_ξ is generic for $\dot{\mathbb{Q}}_\xi[G_\xi]$ over $V[G_\xi]$.

Since $\mathbb{P}$ is c.c.c., V and $V[G]$ have the same cardinals. In particular $(\omega_1)^{V[G]} = (\omega_1)^V$. In the arguments below, we simply write ω_1 for $(\omega_1)^{V[G]} = (\omega_1)^V$. The fact that the two are equal is used throughout the proof, but since our notation takes their equality for granted, the uses are sometimes masked.

Claim 4.16. *Let $A \subseteq \omega_1$ belong to $V[G]$. Then there is $\nu < \delta$ so that $A \in V[G_\nu]$. Similarly for $A \subseteq \omega_1 \times \omega_1$.*

Proof. It is enough to prove the claim for $A \subseteq \omega_1$. Fix A. By Remark 2.31, there is a name $\dot{A}$ for A so that:

(1) $\dot{A} \subseteq \{\check{\alpha} \mid \alpha < \omega_1\} \times \mathbb{P}$.
(2) For all $\beta < \omega_1$, $\{q \mid \langle\check{\alpha}, q\rangle \in \dot{A}\}$ is an antichain in $\mathbb{P}$.

Since $\mathbb{P}$ is c.c.c., it follows that $|\dot{A}| \leq \omega_1 \cdot \omega = \omega_1$. Since $\text{suprt}(p)$ is *finite* for each $p \in \mathbb{P}$ this in turn implies that $Z = \bigcup\{\text{suprt}(p) \mid (\exists\alpha)\langle\check{\alpha}, p\rangle \in \dot{A}\}$ has size $\leq \omega_1$. By assumption $\text{cof}(\delta) \geq \omega_2$, so Z is bounded in δ. Let $\nu < \delta$ be a bound.

Let $\dot{A}\restriction\nu = \{\langle\check{\alpha}, p\restriction\nu\rangle \mid \langle\check{\alpha}, p\rangle \in \dot{A}\}$. (The first "check" is in the post $\mathbb{P}_n u$, the second is in the poset $\mathbb{P}$.) We claim that $(\dot{A}\restriction\nu)[G_\nu] = A$. This implies $A \in V[G_\nu]$.

The direction $A \subseteq (\dot{A}\restriction\nu)[G_\nu]$ does not make any use of the specific choice of ν. If $\alpha \in A$ then there is $p \in G$ with $\langle\check{\alpha}, p\rangle \in \dot{A}$. Then $\langle\check{\alpha}, p\restriction\nu\rangle \in \dot{A}\restriction\nu$ and $p\restriction\nu \in G_\nu$ by definitions, so $\alpha \in (\dot{A}\restriction\nu)[G_\nu]$.

For the other direction, fix $\alpha \in (\dot{A}\restriction\nu)[G_\nu]$. Then there is $p \in G_\nu$ so that $\langle\check{\alpha}, p\restriction\nu\rangle \in \dot{A}\restriction\nu$. We then have by definitions $q_1 \in G$ so that $q_1\restriction\nu = p$, and q_2 so that $\langle\check{\alpha}, q_2\rangle \in \dot{A}$ and $q_2\restriction\nu = p$. By definition of ν, $\text{suprt}(q_2) \subseteq \nu$, hence $q_2(\xi) = \mathbb{1}_\xi$ for $\xi \geq \nu$. Since $q_2\restriction\nu = p = q_1\restriction\nu$ this implies that $q_1 \leq q_2$. Since $q_1 \in G$ this in turn implies that $q_2 \in G$ and hence $\alpha \in \dot{A}[G] = A$. □

Remark 4.17. If $A \subseteq \omega_1$ belongs to $V[G_\nu]$ and $\mu > \nu$, then $A \in V[G_\mu]$. To see this, let $\dot{A}$ name A in $\mathbb{P}_\mu$, with $\dot{A} \subseteq \{\check{\alpha} \mid \alpha < \omega_1\} \times \mathbb{P}_\nu$, and define $\dot{A}^{*\mu} = \{\langle\check{\alpha}, p\rangle \mid p \in \mathbb{P}_\mu$ and $\langle\check{\alpha}, p\restriction\nu\rangle \in \dot{A}\}$. It is easy to check that $\dot{A}^{*\mu}[G_\mu]$ is equal to $\dot{A}[G_\nu] = A$. We refer to $\dot{A}^{*\mu}$ as the *trivial stretch* of $\dot{A}$ from a $\mathbb{P}_\nu$-name to a $\mathbb{P}_\mu$-name.

Claim 4.18. *Let $\mathbb{B}$ be a c.c.c. poset contained in ω_1, in $V[G]$. Suppose there are arbitrarily large $\mu < \delta$ so that $\dot{\mathbb{Q}}_\mu[G_\mu] = \mathbb{B}$. Then all instances of $\mathsf{MA}(\omega_1)$ with poset $\mathbb{B}$ are true in $V[G]$.*

Proof. Let D_i, $i < \omega_1$, be a sequence of dense subsets of $\mathbb{B}$ in $V[G]$. Let $\dot{D}_i$, $i < \omega_1$ be a sequence in V of names for these sets. By Claim 4.16 there are ν_i, $i < \omega_1$ so that $D_i \in V[G\restriction\nu_i]$. The proof of the claim determines ν_i from the name $\dot{D}_i$, so the sequence $\langle\nu_i \mid i < \omega_1\rangle$ can be determined in V. Since $\text{cof}(\delta) \geq \omega_2$, it follows that $\sup\{\nu_i \mid i < \omega_1\}$ is smaller than δ.

By assumption of the claim, there is $\mu > \sup\{\nu_i \mid i < \omega_1\}$ so that $\dot{\mathbb{Q}}_\mu[G_\mu] = \mathbb{B}$. So H_μ is generic for $\mathbb{B}$ over $V[G_\mu]$. My choice of μ and Remark 4.17, the sets D_i all belong to $V[G_\mu]$. By genericity then H meets all these sets. So $H \in V[G]$ witnesses the instance of $\mathsf{MA}(\omega_1)$ corresponding to $\mathbb{B}$ and $\{D_i \mid i < \omega_1\}$. □

To complete the proof of $\mathsf{bMA}(\omega_1)$ in $V[G]$ (and with it the proof of $\mathsf{MA}(\omega_1)$ in $V[G]$) it is enough now to make sure that for every poset $\mathbb{B} \subseteq \omega_1$ that belongs to

$V[G]$ and is c.c.c. in $V[G]$, there are arbitrarily large $\mu < \delta$ so that $\dot{\mathbb{Q}}_\mu[G_\mu] = \mathbb{B}$. This task in turn can be reduced to bookkeeping, provided that the number of names $\dot{\mathbb{B}}$ that is required to cover all these posets is at most δ. Below we calculate the number of names, then describe the bookkeeping.

Claim 4.19. *Suppose that for each $\xi < \delta$, $|\dot{\mathbb{Q}}_\xi| \leq |\mathbb{P}_\xi| \cdot \omega_1$. Then for each $\xi < \delta$, $|\mathbb{P}_\xi| \leq |\xi| \cdot \omega_1$. In particular $|\mathbb{P}_\xi| < \delta$.*

Proof. By induction on ξ. The case $\xi = 0$ is clear, as $\mathbb{P}_0 = \{0\}$. For successors, $\mathbb{P}_{\xi+1} \cong \mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi \subseteq \mathbb{P}_\xi \times \mathrm{dom}(\dot{\mathbb{Q}}_\xi)$, so $|\mathbb{P}_{\xi+1}| \leq |\mathbb{P}_\xi| \cdot |\mathrm{dom}(\dot{\mathbb{Q}}_\xi)| \leq |\mathbb{P}_\xi| \cdot |\dot{\mathbb{Q}}_\xi| \leq |\mathbb{P}_\xi| \cdot |\mathbb{P}_\xi| \cdot \omega_1 \leq |\xi| \cdot \omega_1$, where the second to last inequality uses the assumption of the claim, and the last inequality uses induction.

Finally, for limit $\gamma < \delta$, $|\mathbb{P}_\gamma| \leq |\bigcup_{\xi<\gamma} \mathbb{P}_\xi| \leq \sum_{\xi<\gamma} |\xi| \cdot \omega_1 \leq |\gamma| \cdot \omega_1$, where the first inequality uses the fact that the iteration has finite support (so that every $p \in \mathbb{P}_\gamma$ is completely determined by $p{\restriction}\xi$ for some $\xi < \gamma$), and the second inequality uses induction. $\square$

Claim 4.20. *Let $\mathbb{B}$ be a c.c.c. poset in $V[G]$, contained in ω_1. Then there is a name $\dot{\mathbb{B}}$ in V so that:*

(1) $\dot{\mathbb{B}}[G] = \mathbb{B}$.
(2) $\Vdash_{\mathbb{P}}$ "*$\dot{\mathbb{B}}$ is a c.c.c. poset contained in $\check{\omega}_1$.*"
(3) $\dot{\mathbb{B}} \subseteq \{\check{\alpha} \mid \alpha < \omega_1\} \times \mathbb{P}$ *and* $|\dot{\mathbb{B}}| \leq \omega_1$.

(Formally $\mathbb{B}$ involves three objects, the domain of the poset, the poset order, and the largest element. Similarly $\dot{\mathbb{B}}$ involves three names. The final condition is stated for the name for the domain of the poset. A similar condition holds for the other names, using $\{\check{x} \mid x \in \omega_1 \times \omega_1\} \times \mathbb{P}$ in case of the poset order.)

Proof. Start with any name τ for $\mathbb{B}$, and any condition $p \in G$ forcing τ to name a c.c.c. poset contained in $\check{\omega}_1$. Redefine τ so that its parts below any condition q which is incompatible with p, name the trivial poset $\{0\}$. Then the redefined name satisfies (1) and (2). Finally, let $\dot{\mathbb{B}}$ be obtained from the revised τ using Remark 2.31. Then because $\mathbb{P}$ is c.c.c. and the domain of $\dot{\mathbb{B}}$ has size at most ω_1, $\dot{\mathbb{B}}$ has size at most ω_1. $\square$

Theorem 4.21. *Suppose that for every $\kappa < \delta$, $\kappa^{\omega_1} \leq \delta$. Then there is a function $f\colon \delta \to V$ so that $V[G] \models \mathsf{MA}(\omega_1)$ (where G is generic over V for the iteration defined above).*

Proof. Let U consist of all $\dot{A}$ so that for some $\xi < \delta$, $\dot{A}$ is a $\mathbb{P}_\xi$-name, $\dot{A} \subseteq \{\check{\alpha} \mid \alpha < \omega_1\} \times \mathbb{P}_\xi$, and $|\dot{A}| \leq \omega_1$.

We will define the iteration in such a way that $|\dot{Q}_\xi| \leq |\mathbb{P}_\xi| \cdot \omega_1$ for each ξ. With this size bound we have $|U| \leq \sum_{\xi<\delta} |(\omega_1 \times \mathbb{P}_\xi)^{\omega_1}| \leq \sum_{\xi<\delta} (|\xi| \cdot \omega_1)^{\omega_1} \leq \sum_{\xi<\delta} \delta = \delta$, where the first inequality uses the definition of U, the second inequality uses Claim 4.19, and the third uses the theorem assumption on δ.

Let $h\colon \delta \to V$ enumerate U with each $\dot{A} \in U$ repeated δ times. This is possible since $|U| \leq \delta$. Define $f\colon \delta \to V$ as follows: for each $\mu < \delta$, if there is $\xi < \mu$ so that $h(\mu)$ is a $\mathbb{P}_\xi$-name of size ω_1, let $f(\mu)$ be the trivial stretch of this name to a $\mathbb{P}_\mu$-name, in other words set $f(\mu) = h(\mu)^{*\mu}$; otherwise set $f(\mu) = \emptyset$. Note in the non-trivial case of the definition that $|f(\mu)| \leq |\mathbb{P}_\mu| \cdot \omega_1$ since $|h(\mu)| \leq \omega_1$. The use of f in defining the iteration then implies that $|\dot{\mathbb{Q}}_\mu| \leq |\mathbb{P}_\mu| \cdot \omega_1$ for all μ, as required for the computation of the size of U above.

It remains to prove that in the extension $V[G]$ by the iteration resulting from this function f, $\mathsf{bMA}(\omega_1)$ holds. By Lemma 4.15 the extension then satisfies $\mathsf{MA}(\omega_1)$.

Let $\mathbb{B}$ be a c.c.c. poset in $V[G]$, contained in ω_1. Let $\dot{\mathbb{B}}$ be the name for $\mathbb{B}$ given by Claim 4.20. Let $\nu < \delta$ be given by Claim 4.16 so that $(\dot{\mathbb{B}}\restriction\nu)[G\restriction\nu] = \dot{\mathbb{B}}[G]$. Note that $\dot{\mathbb{B}}\restriction\nu$ belongs to U. Fix then some $\mu > \nu$ so that $h(\mu) = \dot{\mathbb{B}}\restriction\nu$. Note that there are arbitrarily large such μ below δ, since every element of U is repeated δ time in the enumeration h. Note further that by definition of f, $f(\mu) = (\dot{\mathbb{B}}\restriction\nu)^{*\mu}$.

By Remark 4.17, $(\dot{\mathbb{B}}\restriction\nu)^{*\mu}[G\restriction\mu] = (\dot{\mathbb{B}}\restriction\nu)[G\restriction\nu] = \dot{\mathbb{B}}[G]$. Since $\dot{\mathbb{B}}[G]$ is c.c.c. in $V[G]$, and being c.c.c. reflects to submodels with the same ω_1, $(\dot{\mathbb{B}}\restriction\nu)^{*\mu}[G\restriction\mu]$ is c.c.c. in $V[G\restriction\mu]$. (${\omega_1}^{V[G\restriction\mu]} = {\omega_1}^{V[G]}$, as both are equal to ${\omega_1}^{V}$. As $V[G\restriction\mu] \subseteq V[G]$, the fact that $(\dot{\mathbb{B}}\restriction\nu)^{*\mu}[G\restriction\mu] = \dot{\mathbb{B}}[G]$ has no antichains of size ω_1 in $V[G]$ implies it has no antichains of this size in $V[G\restriction\mu]$.) Since $\Vdash_{\mathbb{P}}$“$\dot{\mathbb{B}}$ is c.c.c.”, this argument works with *any* generic, not just G, and it follows that $\Vdash_{\mathbb{P}_\mu}$“$(\dot{\mathbb{B}}\restriction\nu)^{*\mu}$ is a c.c.c. poset.” The poset is also forced to be contained in $\check{\omega}_1$, since it is a subset of $\{\check{\alpha} \mid \alpha < \omega_1\} \times \mathbb{P}_\mu$.

We now have $\mu > \nu$ so that $f(\mu) = (\dot{\mathbb{B}}\restriction\nu)^{*\mu}$, and so that $\Vdash_{\mathbb{P}_\mu}$“$(\dot{\mathbb{B}}\restriction\nu)^{*\mu}$ is a c.c.c. poset contained in $\check{\omega}_1$.” By definition of the iteration, $\dot{\mathbb{Q}}_\mu = (\dot{\mathbb{B}}\restriction\nu)^{*\mu}$, and hence $\dot{\mathbb{Q}}_\mu[G\restriction\mu] = (\dot{\mathbb{B}}\restriction\nu)^{*\mu}[G\restriction\mu] = \dot{\mathbb{B}}[G] = \mathbb{B}$. We in fact found arbitrarily large such μ below δ. By Claim 4.18 all instances of $\mathsf{MA}(\omega_1)$ corresponding to $\mathbb{B}$ are true in $V[G]$. As $\mathbb{B}$ was an arbitrary c.c.c. poset contained in ω_1 in $V[G]$, $V[G] \models \mathsf{bMA}(\omega_1)$. □

Remark 4.22. In the proof of Theorem 4.21, we are really defining U, h, f, and the iteration $\langle \mathbb{P}_\xi \mid \xi \leq \delta\rangle$, $\langle \dot{\mathbb{Q}}_\xi \mid \xi < \delta\rangle$ simultaneously, by induction on ξ. As we define more of the iteration, we identify more of U, define more of h, and obtain more of f.

Remark 4.23. The assumption that $(\forall\kappa < \delta)\kappa^{\omega_1} \leq \delta$ allows showing that the set of good names for reals in the extension by $\mathbb{P}$ has size δ. It follows that $V[G] \models 2^\omega \leq \delta$. It is easy to see that Cohen forcing is repeated δ times in the iteration so that in fact $V[G] \models 2^\omega = \delta$.

Corollary 4.24. *If* ZFC *is consistent, then so is* $\mathsf{ZFC}+2^\omega = \omega_2+\mathsf{MA}$.

Proof. It is enough to show that every ctm M of $\mathsf{ZFC}+\mathsf{GCH}$ has a forcing extension that satisfies $2^\omega = \omega_2$ and $\mathsf{MA}(\omega_1)$.

Fix M. Let $\delta = (\omega_2)^M$. Note that since M satisfies the GCH, $(\forall\kappa < \delta)\kappa^{\omega_1} \leq \delta$ in M. Theorem 4.21, applied in M, then produces a c.c.c. extension $M[G]$ that satisfies $\mathsf{MA}(\omega_1)$. By Remark 4.23 and since $(\omega_2)^{M[G]} = (\omega_2)^M = \delta$, the extension also satisfies $2^\omega = \omega_2$. □

Remark 4.25. It is easy to modify the argument above to obtain extensions satisfying $\mathsf{MA}(\rho)$ for arbitrary ρ. One has to require $\mathrm{cof}(\delta) > \rho$, so that Claim 4.18 adapts to handle families of ρ dense sets. One then has to change the definition of U, to enumerate names of size $\leq \rho$ for posets contained in ρ, and make the obvious adaptations in the proof of Theorem 4.21 and the claims leading to it.

4.4. **Countable closure.** One of the essential properties of c.c.c. posets that allowed us to obtain $\mathsf{MA}(\omega_1)$ in a generic extension is that the posets, and their (finite support) iterations do not collapse ω_1. (Without the preservation of ω_1, our proof in the previous subsection would only yield $\mathsf{MA}((\omega_1)^V)$ in $V[G]$, and if $(\omega_1)^V$ is collapsed, this is nothing more than $\mathsf{MA}(\omega)$ in the extension.) Are there other classes of posets with this property, that can lend to iterations?

Recall that countably closed posets do not add reals, and in particular they do not collapse ω_1. We aim in this section to show that they can be iterated in a manner that continues to preserve ω_1, and in fact maintains countable closure.

In our definition of a composition $\mathbb{P} * \dot{\mathbb{Q}}$, we demanded (among other things) that members $\langle p, \tau \rangle$ of $\mathbb{P} * \dot{\mathbb{Q}}$ must satisfy $\tau \in \mathrm{dom}(\mathbb{Q})$. We noted in a discussion following the definition that this can be too restrictive in some settings. For the settings of countable closure, we we need to allow a bit more.

Definition 4.26. A $\mathbb{P}$-name $\dot{\mathbb{Q}}$ is *full* if whenever $p \Vdash_{\mathbb{P}} \sigma \in \dot{\mathbb{Q}}$, there is $\tau \in \mathrm{dom}(\dot{\mathbb{Q}})$ so that $p \Vdash_{\mathbb{P}} \sigma = \tau$.

Claim 4.27. *Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name. Then there is a name $\dot{\mathbb{Q}}'$ so that:*

(1) $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}' = \dot{\mathbb{Q}}$.
(2) $\dot{\mathbb{Q}}'$ *is full.*

Proof. Let $U = \mathrm{dom}(\dot{\mathbb{Q}})$ and let $W = \{\rho \mid (\exists \sigma \in U)\rho \in \mathrm{dom}(\sigma)\}$. Note that if G is generic for $\mathbb{P}$ over V, we have $\dot{\mathbb{Q}}[G] \subseteq \{\sigma[G] \mid \sigma \in U\} \subseteq \mathcal{P}(\{\rho[G] \mid \rho \in W\})$.

Set $\langle \tau, p \rangle \in \dot{\mathbb{Q}}'$ iff:

(1) $\tau \subseteq W \times \mathbb{P}$.
(2) For each $\rho \in W$, $\{p \mid \langle \rho, p \rangle \in \tau\}$ is an antichain in $\mathbb{P}$.
(3) $p \Vdash_{\mathbb{P}} \tau \in \dot{\mathbb{Q}}$.

Previous argument, specifically on good names as in the proof of Claim 2.30 and Remark 2.31, show that $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}' = \dot{\mathbb{Q}}$, and whenever $p \Vdash \sigma \in \dot{\mathbb{Q}}$, there is $\tau \in \dot{\mathbb{Q}}'$ so that $p \Vdash \tau = \sigma$. (A first approximation to τ is the name $\{\langle \rho, q \rangle \in W \times \mathbb{P} \mid q \Vdash_{\mathbb{P}} \rho \in \sigma\}$. τ itself is a subset of this name, thinning it so that for each ρ, $\{q \mid \langle \rho, q \rangle \in \tau\}$ is a maximal antichain among $\{q \mid q \Vdash_{\mathbb{P}} \rho \in \sigma\}$.) □

The name $\dot{\mathbb{Q}}'$ defined in the proof of Claim 4.27 is called the *saturation* of $\dot{\mathbb{Q}}$, denoted $\mathrm{sat}(\dot{\mathbb{Q}})$

To prove preservation of countable closure in countable support iterations, we will need fullness for the poset-names being iterated. There are two approaches for ensuring this. One is to restrict Lemma 4.30 below to situations where each $\dot{\mathbb{Q}}_\xi$ is forced in $\mathbb{P}_\xi$ to be full. Another is to modify the definitions of composition an iterations, liberalizing the requirement that $\tau \in \dot{\mathbb{Q}}$ to the possibly weaker requirement that $\tau \in \mathrm{sat}(\dot{\mathbb{Q}})$. The second is the standard approach, and the one we take here.

Definition 4.28. In Definition 4.2, change the requirement $\tau \in \dot{\mathbb{Q}}$ to $\tau \in \mathrm{sat}(\dot{\mathbb{Q}})$. Similarly in Definition 4.11 (successor case), change the requirement $p(\xi) \in \dot{\mathbb{Q}}_\xi$ to $p(\xi) \in \mathrm{sat}(\dot{\mathbb{Q}}_\xi)$.

Remark 4.29. With this revision for the definition of compositions and iterations, the size of the poset $\mathbb{P} * \dot{\mathbb{Q}}$ is no longer equal to $|\mathbb{P} \times \mathrm{dom}(\dot{\mathbb{Q}})|$. Rather it is equal to $|\mathbb{P} \times \mathrm{dom}(\mathrm{sat}(\dot{\mathbb{Q}}))|$ which may well be larger. This affects the computation in the proof of the consistency of $sfbMA(\omega_1)$, specifically the computation in the proof of Claim 4.19. Assuming that $(\forall \xi < \delta)\xi^\omega \leq \max(\xi, \omega_1)$, the claim continues to hold with the revised definitions, but the proof is a bit harder. (The assumption that $(\forall \xi < \delta)\xi^\omega \leq \max(\xi, \omega_1)$ is satisfied in the situation of Corollary 4.24 on the consistency of MA.)

Lemma 4.30. *Let $\langle \mathbb{P}_\xi \mid \xi \leq \alpha \rangle$, $\langle \dot{\mathbb{Q}}_\xi \mid \xi < \alpha \rangle$ be a countable support iteration. Suppose that for each $\xi < \alpha$, $\Vdash_{\mathbb{P}_\xi}$ "$\dot{\mathbb{Q}}_\xi$ is countable closed." Then $\mathbb{P}_\alpha$ is countably closed. In particular $\mathbb{P}_\alpha$ does not add reals and does not collapse ω_1.*

Proof. Let $p^n \in \mathbb{P}_\alpha$, with $p^{n+1} \leq p^n$. We define a condition p extending all p^n.

Let $Z = \bigcup_{n<\omega} \mathrm{suprt}(p^n)$. Then $Z \subseteq \alpha$ is a countable union of countable sets, hence countable. Our definition of the condition p is by induction. As we define the condition, we will maintain that $(\forall n) p \restriction \xi \leq p^n \restriction \xi$.

Base case: $p \restriction 0$ is equal to $\{0\}$, as is $p^n \restriction 0$ for each n.

Successor case: Suppose we have already defined $p \restriction \xi$, and know that $p \restriction \xi \leq p^n \restriction \xi$ for all n. If $\xi \notin Z$, set $p(\xi) = \mathbb{1}_\xi$. Since in this case $p^n(\xi) = \mathbb{1}_\xi$ for all n, we have $p \restriction \xi + 1 \leq p^n \restriction \xi + 1$. If $\xi \in Z$, we have: (1) $\Vdash_{\mathbb{P}_\xi}$ "$\dot{\mathbb{Q}}_\xi$ is countably closed;" and (2) $p^n \restriction \xi \Vdash_{\mathbb{P}_\xi} p^n(\xi) \in \dot{\mathbb{Q}}_\xi$. Since $p \restriction \xi \leq p^n \restriction \xi$ it follows from (2) that $p \restriction \xi \Vdash_{\mathbb{P}_\xi} p^n(\xi) \in \dot{\mathbb{Q}}_\xi$. From this and (1) it follows that there is a name σ so that $p \restriction \xi \Vdash_{\mathbb{P}_\xi} \sigma \in \dot{\mathbb{Q}}_\xi \wedge (\forall n < \check{\omega}) \sigma \leq p^n(\xi)$. Let $\tau \in \mathrm{dom}(\mathrm{sat}(\dot{\mathbb{Q}}))$ be such that $p \restriction \xi \Vdash_{\mathbb{P}_\xi} \tau = \sigma$. Such τ can be found because $\mathrm{sat}(\dot{\mathbb{Q}}_\xi)$ is *full*. Set $p(\xi) = \tau$. It is each to check that $p \restriction \xi + 1 \leq p^n \restriction \xi + 1$ for each n. We note only that the fact that $p \restriction \xi + 1 \in \mathbb{P}_{\xi+1}$ uses the revised definition of iteration, as τ is only known to belong to $\mathrm{dom}(\mathrm{sat}(\dot{\mathbb{Q}}))$.

Limit case: Let $\gamma \leq \alpha$ be a limit ordinal. Suppose we already defined $p \restriction \gamma$, and know that $p \restriction \xi \leq p^n \restriction \xi$ for each n and each $\xi < \gamma$. By definition, the support of $p \restriction \gamma$ is contained in Z, and in particular it is countable. This and the fact that $p \restriction \xi \in \mathbb{P}_\xi$ for each $\xi < \gamma$, imply that $p \restriction \gamma \in \mathbb{P}_\gamma$. The fact that $p \restriction \xi \leq p^n \restriction \xi$ for each $\xi < \gamma$ implies that $p \restriction \gamma \leq p^n \restriction \gamma$. □

We end the section with a simple example of a countably closed poset and some words on the model obtained by iterating it.

Definition 4.31. The *poset for adding a fast club* is defined as follows. Conditions are pairs $\langle s, A \rangle$ where $s \subseteq \omega_1$ is countable and closed, and $A \subseteq \{\text{clubs in } \omega_1\}$ is countable. s is called the *stem* of the condition, an A is called the *promise*. The ordering on the poset is given by $\langle s^*, A^* \rangle \leq \langle s, A \rangle$ iff s^* is an end extension of s, and $(\forall C \in A) s^* - s \subseteq A$.

Claim 4.32. *Let $\mathbb{P}$ be the poset for adding a fast club. then:*

(1) *$\mathbb{P}$ is countable closed.*
(2) *If the* CH *holds, then $\mathbb{P}$ is ω_2-c.c.*

Proof. Let $\langle s_n, A_n \rangle$ be descending in $\mathbb{P}$. Let $r = (\bigcup_{n<\omega} s_n)$ and set $s = r \cup \sup(r)$. Set $A = \bigcup_{n<\omega} A_n$. It is easy to check that $\langle s, A \rangle$ is a condition extending all s_n. This establishes countable closure.

Any two conditions $\langle s, A \rangle$, $\langle s, B \rangle$ with the same stem s are compatible, as $\langle s, A \cup B \rangle$ extends both. Under CH there are only ω_1 possible stems, and it follows that $\mathbb{P}$ is ω_2-c.c. □

Let G be generic for $\mathbb{P}$ over V. By countable closure, $(\omega_1)^{V[G]} = (\omega_1)^V$. Assuming the CH in V, the forcing also preserves all cardinals $\tau \geq \omega_2$, so that V and $V[G]$ have exactly the same cardinals.

Let $F = \bigcup \{ s \mid (\exists A) \langle s, A \rangle \in G \}$. Then F is a closed subset of ω_1, and a density argument shows that it is unbounded, hence it is a club.

For $A, B \subseteq \omega_1$, we say that A is *almost contained* in B, denoted $A \subseteq^* B$, if there is $\alpha < \omega_1$ so that $A - \alpha \subseteq B$.

Claim 4.33. *F is almost contained in every club $C \subseteq \omega_1$ that belongs to the ground model V.*

Proof. Let $C \subseteq \omega_1$ be a club in V. By genericity, there is a condition $\langle s, A\rangle \in G$ so that $C \in A$. Let $\alpha = \sup(s) + 1$. Since every $\xi \in F$ belongs to the stem of a condition in G that extends $\langle s, A\rangle$, $F - \alpha \subseteq C$. □

Iterations of the poset for adding fast clubs can lead to a model satisfying, for example, "for every family $\mathcal{A}$ of clubs in ω_1 with $|\mathcal{A}| = \aleph_8$, there is a club F which is almost contained in every member of $\mathcal{A}$". We leave the proof of this as an exercise, and note only that the reader should pay attention to ensure the iteration preserves cardinals.

5. Proper forcing

We saw in the previous section how c.c.c. posets, and countably closed posets, can be iterated so that the iteration remains c.c.c. or countably closed, and in particular does not collapse ω_1, a crucial property for applications. But we needed to use different kinds of supports for the two classes of posets; finite supports for c.c.c. posets, and countable supports for countably closed posets. In particular, we could not handle mixed iterations that involve both c.c.c. and countably closed poset. Each poset in the iteration by itself preserves ω_1, but we could not iterate in a way that ensures the iteration too preserves ω_1.

In this section we will consider a class of posets that preserve ω_1, called proper, which includes both the classes of c.c.c. and countably closed posets. We will see how to iterate proper posets in a way that maintains properness.

Before getting to that, we give an example of a forcing construction that uses a mixing of c.c.c. and countably closed posets.

5.1. ω_1-dense subsets of $\mathbb{R}$.

$A \subseteq \mathbb{R}$ is *κ-dense* if its intersection with every open interval of $\mathbb{R}$ has size κ.

Theorem 5.1. *Any two ω-dense subsets of $\mathbb{R}$ are order isomorphic.*

Proof. ω-dense subsets of $\mathbb{R}$ are countable dense linear orders with no endpoints. Any two are order isomorphic by the classic back-and-forth argument. □

It is natural to ask about parallels of Theorem 5.1 for ω_1-dense sets. The following claim shows that one can only hope for a consistency proof, and that a model witnessing the consistency must violate the CH.

Claim 5.2. *Assuming the CH, there are two ω_1-dense subsets of $\mathbb{R}$ which are not order isomorphic. In fact there are ω_1-dense sets $A, B \subseteq \mathbb{R}$ so that neither is embeddable in the other.*

Proof. Note that by density of A and B, any order preserving surjection of A onto B must be continuous, and in fact must be the restriction to A of a continuous embedding on $\mathbb{R}$. Under the CH there are ω_1 continuous embedding from $\mathbb{R}$ into $\mathbb{R}$. Build A and B in ω_1 stages, deciding in each stage on at most countably many reals that go into/out of A and B, ensuring in stage ξ that the ξth continuous function is not an isomorphism.

A similar argument leads to sets so that neither is embeddable in the other. Here the embeddings need not be continuous, but they are unions of countably many continuous embeddings. Under the CH, there are ω_1 such embeddings. □

We begin now to work toward a model where all ω_1-dense sets are order isomorphic. In this subsection we handle the basic step. Let $A, B \subseteq \mathbb{R}$ be ω_1-dense. We will see how to force the existence of an order isomorphism between them, without collapsing ω_1. (In an extension that collapses ω_1, the sets become ω-dense, and are trivially order isomorphic by Theorem 5.1.) Later on we will work on iterating the basic step so as to reach a model where an isomorphism has been added for every pair of ω_1-dense sets.

Let $\mathbb{P}$ be the poset to add a fast club. Let G be generic for $\mathbb{P}$ over V, and let C be the fast club added by G. We will define a c.c.c. poset $\mathbb{Q} \in V[G]$, which adds an order isomorphism between A and B. As $\mathbb{P}$ is countably closed, and $\mathbb{Q}$ is c.c.c., the composition preserves ω_1.

The sets A and B have size ω_1. Let φ_A and φ_B enumerate them in order-type ω_1. Let θ be a regular cardinal, larger than the continuum. Note that the set $Z = \{\alpha < \omega_1 \mid (\exists M \prec V_\theta)|M| = \omega, \mathbb{R}, \varphi_A, \varphi_B \in M, \text{ and } M \cap \omega_1 = \alpha\}$ is club in ω_1, and belong to V. Since C is a fast club, it is almost contained in Z. Dropping an initial segment of C if necessary we may assume it is outright contained in C.

Let $\alpha \in C$ and let α^* be the next point of C above α. Define the *$[\alpha, \alpha^*)$-segment of* A, denoted $\mathrm{seg}_A[\alpha, \alpha^*)$, to be the set $\{\varphi_A(\xi) \mid \alpha \leq \xi < \alpha^*\}$. We also refer to $\mathrm{seg}_A[\alpha, \alpha^*)$ as the *α-segment* of A, as α^* can be determined from α. Define segments of B similarly, using φ_B. Each of these segments is countable, and their union contains all of A (resp. B).

Claim 5.3. *The $[\alpha, \alpha^*)$-segment of A is dense in $\mathbb{R}$. Similarly the $[\alpha, \alpha^*)$-segment of B is dense in $\mathbb{R}$.*

Proof. We prove the claim for segments of A. Fix an open interval (x, y) in $\mathbb{R}$. We may assume that the endpoints x and y are rational. Let $M \prec V_\theta$ witness that $\alpha^* \in Z$, meaning that $|M| = \omega$, $\mathbb{R}, \varphi_A, \varphi_B \in M$, and $M \cap \omega_1 = \alpha^*$. Since x and y are rational, they belong to M. Since $\alpha < \alpha^*$, it too belongs to M.

Since A is ω_1-dense and α is countable, there is $\xi \geq \alpha$ so that $\varphi_A(\xi) \in (x, y)$. This statement holds in V_θ, and by elementarity of M it follows that $M \models (\exists \xi \geq \alpha)\varphi_A(\xi) \in (x, y)$. Let $\xi \in M$ witness this. Then $\xi < \alpha^*$ since $\alpha^* = M \cap \omega_1$. We have now $\xi \in [\alpha, \alpha^*)$ so that $\varphi_A(\xi) \in (x, y)$. In other words the $[\alpha, \alpha^*)$-segment of A has points in the interval (x, y). $\square$

Call two points $\alpha, \beta \in C$ are *neighbors* if $\alpha \neq \beta$ and there are only finitely many points of C between them.

Working in $V[G]$, let $\mathbb{Q}$ be the following poset. Conditions are finite sets of pairs $\langle a_i, b_i \rangle \in A \times B$ so that:

(1) The map $a_i \mapsto b_i$ is order preserving from A into B.
(2) Let $[\alpha, \alpha^*)$ be such that $a_i \in \mathrm{seg}_A[\alpha, \alpha^*)$ and let $[\beta, \beta^*)$ be such that $b_i \in [\beta, \beta^*)$. Then α and β are neighbors.
(3) For every segment $[\gamma, \gamma^*)$, there is *at most one* i such that either
   - $a_i \in \mathrm{seg}_A[\gamma, \gamma^*)$ and b_i belongs to an earlier segment of B, or
   - $b_i \in \mathrm{seg}_B[\gamma, \gamma^*)$ and a_i belongs to an earlier segment of A.

The ordering on $\mathbb{Q}$ is reverse inclusion, i.e., $p \leq q$ if $p \supseteq q$.

Condition (1) ensures that a filter for $\mathbb{Q}$ produces an order preserving embedding from (a subset of) A into B.

Remember that we want to the poset $\mathbb{Q}$ to be c.c.c. in $V[G]$. Without some restriction on the embeddings we allow, there could be trivial antichains of size

ω_1. For example, given some fixed $a \in A$, the partial embeddings $\{\langle a, b\rangle\}$, $b \in B$ are pairwise incompatible, and since $|B| = \omega_1$, there are ω_1 of them. Condition (2) cuts down on the number of options for b (and symmetrically, for a given b). Given a, the only bs that a can be paired with in a condition are bs that belong to segments that neighbor the segment of a. There are only ω such segments, and each is countable, so altogether there are only countably many options for b.

Condition (3) states that in each segment there is at most one point which is paired with an element of a smaller segment. (All other points in the segments must be paired with points in higher neighboring segments.) We will see later on how this condition is used in the proof that $\mathbb{Q}$ is c.c.c.

Elements of $\mathbb{Q}$ are sets of pairs. When enumerating the elements of $p \in \mathbb{Q}$ as $\langle a_0, b_0\rangle, \ldots \langle a_{n-1}, b_{n-1}\rangle$ we always use an increasing enumeration, meaning that $a_0 < a_1 < \ldots b_{n-1}$ and (since the map $a_i \mapsto b_i$ is order preserving) $b_0 < b_1 < \ldots b_{n-1}$. Given a condition p and a pair $\langle a, b\rangle$, we write $p \cup \langle a, b\rangle$ to mean $p \cup \{\langle a, b\rangle\}$ (a larger condition if $\langle a, b\rangle \notin p$). We write $p - \langle a, b\rangle$ to mean $p - \{\langle a, b\rangle\}$ (a smaller condition if $\langle a, b\rangle \in p$).

Claim 5.4. *For each $a \in A$ the set $D = \{p \in \mathbb{Q} \mid a \in \mathrm{dom}(p)\}$ is dense in $\mathbb{Q}$. Similarly for each $b \in B$, the set $D = \{p \in \mathbb{Q} \mid b \in \mathrm{range}(p)\}$ is dense in $\mathbb{Q}$.*

Proof. We prove the first part; the second is similar. Fix $a \in A$. Fix $p \in \mathbb{Q}$. We find $q \leq p$ with $q \in D$.

Let $[\alpha, \alpha^*)$ be such that $a \in \mathrm{seg}_A[\alpha, \alpha^*)$. Let $\alpha_0 = \alpha$, and let α_{n+1} be the next point of C above α_n. The α_n, $n > 1$, are all neighbors of α. Since p is finite, only finitely many of the segments $[\alpha_n, \alpha_{n+1})$ can have points that appear in p. Fix $n \geq 1$ large enough so that on points in p belong to the segment $[\alpha_n, \alpha_{n+1})$.

By Claim 5.3, the $[\alpha_n, \alpha_{n+1})$-segment of B is dense. It follows that there is $b \in \mathrm{seg}_B[\alpha, \alpha_{n+1})$ so that $p \cup \langle a, b\rangle$ is order preserving. (If $p = \{\langle a_i, b_i\rangle \mid i < n\}$, and i is such that $a_i < a < a_{i+1}$, b, any $b \in seg_B[\alpha, \alpha^*)$ chosen between b_i and b_{i+1} will do. Similarly if $a < a_0$ or $a > a_{n-1}$.) $p \cup \langle a, b\rangle$ then satisfies condition (1) in the definition of $\mathbb{Q}$. By choice of n, a and b belong to neighboring segments, so that condition (2) is satisfied. Condition (3) is satisfied because of the two added points, a is paired with a point b in a higher segment, and b is the only point in its segment paired with a point in a lower segment (since it is outright the only point in its segment). So $q = p \cup \langle a, b\rangle$ is a condition. We have $q \leq p$, and $q \in D$. □

Let H be generic for $\mathbb{Q}$ over $V[G]$. By the last claim, $f = \bigcup\{p \mid p \in H\}$ is total on A, and surjective onto B. f is order preserving by definition of $\mathbb{Q}$. So, in $V[G][H]$, A and B are order isomorphic.

It remains to prove that $(\omega_1)^{V[G][H]} = (\omega_1)^V$. $\mathbb{P}$ is countably closed, so $(\omega_1)^{V[G]} = (\omega_1)^V$. We will show that $\mathbb{Q}$ is c.c.c. in $V[G]$, so that the addition of H preserves ω_1.

Suppose that $T = \{p_\xi \mid \xi < \omega_1\}$ is an antichain of $\mathbb{Q}$ in $V[G]$. We work to derive a contradiction. Shrinking T if necessary we may assume that all conditions in T have the same size n. Pick T so as to minimize n, meaning that there are no antichains of size ω_1 with conditions all of size less than n. Shrinking further, we may assume that $\{\mathrm{dom}(p_\xi) \mid \xi < \omega_1\}$ forms a Δ-system, with root r say.

Claim 5.5. $r = \emptyset$.

Proof. Suppose $a \in r$. Let $[\alpha, \alpha^*)$ be the segment of a, meaning $a \in \mathrm{seg}_A[\alpha, \alpha^*)$. For each $p_\xi \in T$, $p_\xi(a)$ has to belong to a neighboring segment of B. There are only ω neighboring segments, and each segment has only countably many points. So there are only countably many options for $p_\xi(a)$. Shrinking T if needed, we may assume that $p_\xi(a)$ is fixed, i.e., $(\exists b)(\forall \xi)p_\xi(a) = b$.

By minimality of n, the conditions $p_\xi - \langle a, b\rangle$, $\xi < \omega_1$ do *not* form an antichain. So we may fix $\xi_1 \neq \xi_2$ so that $p_{\xi_1} - \langle a, b\rangle$ and $p_{\xi_2} - \langle a, b\rangle$ are compatible. Then $p_{\xi_1} \cup p_{\xi_2} - \langle a, b\rangle$ is a condition, and using the fact that $\langle a, b\rangle$ belongs to both p_{ξ_1} and p_{ξ_2} it is easy to check that this implies that $p_{\xi_1} \cup p_{\xi_2}$ is a condition. But then T is not an antichain, contradiction. □

By the *maximum segment* of $\mathrm{dom}(p)$ we mean the largest α so that $\mathrm{dom}(p)$ has points in the $[\alpha, \alpha^*)$-segment of A. The *minimum segment* is defined similarly, as are the maximum and minimum segments for $\mathrm{range}(p)$ using segments of B. The maximum segment of p is the largest α so that either $\mathrm{dom}(p)$ has points in the $[\alpha, \alpha^*)$-segment of A, or $\mathrm{range}(p)$ has points in the $[\alpha, \alpha^*)$ segment of B. The minimum is defined similarly.

Since $\{\mathrm{dom}(p_\xi) \mid \xi < \omega_1\}$ is a Δ-system with empty root, the domains $\mathrm{dom}(p_\xi)$ are disjoint. In particular, for any $\alpha < \omega_1$, at most countably many conditions p_ξ can have some points of their domains in segments of A before α. Shrinking the antichain if needed, we may therefore assume that for $\xi < \xi'$, the minimum segment of $\mathrm{dom}(p_{\xi'})$ is greater than the maximum segment of $\mathrm{dom}(p_\xi)$, and in fact it is large enough that the two are not even neighbors. This, by condition (2) in the definition of $\mathbb{Q}$ implies that the minimum segment of $\mathrm{range}(p_{\xi'})$ is greater than the maximum segment of $\mathrm{range}(p_\xi)$, and in fact the minimum segment of $p_{\xi'}$ is greater than the maximum segment of p_ξ.

We have so far made cosmetic modifications to the antichain T, ensuring that:

- All conditions in T are of the same length n.
- n is minimal, meaning that there are no uncountable antichains with conditions all of length less than n.
- The elements of T are arranged in order, so that they use increasing segments of A and B as ξ increases, and spaced, so that for $\xi < \xi'$, the maximum segment of p_ξ is smaller than the minimum segment of $p_{\xi'}$.

Claim 5.6. *Call conditions p, q* order compatible *if $p \cup q$ is order preserving, and order incompatible otherwise. Then for any $p \neq q$ both in T, p and q are order* in*compatible.*

Proof. Fix $p, q \in T$, say $p = p_\xi$, $q = p_{\xi'}$, with $\xi < \xi'$. Consider $p \cup q$. Note that it is a function, since $\mathrm{dom}(p)$ and $\mathrm{dom}(q)$ are disjoint (as the maximum segment used by $\mathrm{dom}(p)$ is smaller than the minimum segment used by $\mathrm{dom}(q)$). It satisfies condition (2) in the definition of $\mathbb{Q}$, since each of p and q satisfies this condition. It satisfies condition (3), since each of p and q satisfies this condition and since the segments used by p and q are disjoint (again because the maximum segment used by p is smaller than the minimum segment used by q). But $p, q \in T$, and hence p and q are *not* compatible in $\mathbb{Q}$. It must therefore be that $p \cup q$ fails to satisfy condition (1) in the definition of $\mathbb{Q}$, meaning that p and q are order incompatible. □

A *good neighborhood* U of length n is a finite set of squares $(x_i^{Left}, x_i^{Right}) \times (y_i^{Left}, y_i^{Right})$, $i < n$, so that:

(1) The end points x_i^{Left}, x_i^{Right}, y_i^{Left}, y_i^{Right} are all rational.
(2) The intervals $(x_i^{Left}, x_i^{Right})$ are disjoint, and so are the intervals $(y_i^{Left}, y_i^{Right})$.

(3) The map $(x_i^{Left}, x_i^{Right}) \mapsto (y_i^{Left}, y_i^{Right})$ is order preserving.

A good neighborhood is an approximation to a condition, using open intervals with rational end-points instead of actual elements of A and B.

A condition $p = \{\langle a_i, b_i\rangle \mid i < n\}$ *belongs* to U if $(\forall i)$, $\langle a_i, b_i\rangle \in (x_i^{Left}, x_i^{Right}) \times (y_i^{Left}, y_i^{Right})$.

Claim 5.7. *Let $p = \{\langle a_i, b_i\rangle \mid i < n\}$ and $p^* = \{\langle a_i^*, b_i^*\rangle \mid i < n\}$ both belong to the same good neighborhood U. Suppose that p and p^* are order incompatible. Then there is $i < n$ so that $\langle a_i, b_i\rangle$ and $\langle a_i^*, b_i^*\rangle$ are order incompatible.*

Proof. The facts that $(x_i^{Left}, x_i^{Right}) \mapsto (y_i^{Left}, y_i^{Right})$ is order preserving, that $a_i, a_i^* \in (x_i^{Left}, x_i^{Right})$, and that $b_i, b_i^* \in (y_i^{Left}, y_i^{Right})$, together imply that for any $i \neq j$, the pairs $\langle a_i, b_i\rangle$ and $\langle a_j^*, b_j^*\rangle$ are order compatible. It follows that any order incompatibility between p and p^* must be witnessed by pairs $\langle a_i, b_i\rangle$ and $\langle a_j^*, b_j^*\rangle$ with $i = j$. $\square$

For a condition $p_\xi = \{\langle a_i^\xi, b_i^\xi\rangle \mid i < n\} \in T$, let $p_\xi - i$ denote the condition $p_\xi - \langle a_i^\xi, b_i^\xi\rangle$, namely the length $n-1$ condition obtained from p_ξ by dropping its ith pair. (Recall that the pairs are indexed in order, that is $a_0^\xi < a_1^\xi \cdots < a_{n-1}^\xi$ and $b_0^\xi < \dots b_{n-1}^\xi$.)

For each good neighborhood U, let $T_U = \{p_\xi \in T \mid p_\xi$ belongs to $U\}$. Let $T_{U,i} = \{p_\xi - i \mid p_\xi \in T_u\}$. Let $B_{U,i} \subseteq \omega_1$ be such that $\{p_\xi \mid \xi \in B_{U,i}\} \subseteq T_U$, and $\{p_\xi - i \mid \xi \in B_{U,i}\}$ is a maximal antichain in $T_{U,i}$. Note that $T_{U,i}$ is a set of conditions of lengths $n-1$. By minimality of n, any antichain contained in $T_{U,i}$ must be countable. So $\sup B_{U,i} < \omega_1$. Let $\delta_{U,i} = \sup B_{U,i}$, and let $\delta = \sup\{\delta_{U,i} \mid U$ is a good neighborhood and $i < n\}$. Since there are only countably many good neighborhoods (recall that a good neighborhood is determined by finitely many rational end-points), $\delta < \omega_1$. Let $T\restriction\delta$ denote $\{p_\xi \mid \xi < \delta\}$. The antichain T was taken in $V[G]$, but the restriction $T\restriction\delta$, being a countable sequence of elements of V, belongs to V, since the forcing $\mathbb{P}$ leading to $V[G]$ is countably closed. Similarly each of $B_{U,i}$ belongs to V, and indeed so does the (countable) sequence $\{B_{U,i} \mid U$ is a good neighborhood and $i < n\}$. Hence the following set belongs to V: $Z' = \{\alpha < \omega_1 \mid (\exists M \prec V_\theta)|M| = \omega$, A, B, φ_A, φ_B, $T\restriction\delta$, and $\{B_{U,i}\}_{U,i}$ all belong to M, and $M \cap \omega_1 = \alpha\}$. The set contains a club, and since C is almost contained in each club of V, it follows that there is α_0 so that $C - \alpha_0 \subseteq Z'$.

We are ready now to derive our contradiction from the antichain T. Let $\xi < \omega_1$ be large enough that the minimum segment used by $p_\xi \in T$ is above α_0. Say $p_\xi = \{\langle a_i, b_i\rangle \mid i < n\}$. Let $[\alpha, \alpha^*)$ be the *maximum* segment used by p_ξ. By condition (3) in the definition of $\mathbb{Q}$ and since points of p_ξ in the segment $[\alpha, \alpha^*)$ can only be paired with points in lower segments (as there are no higher segments used by the condition), there can be only one point of p_ξ in the segment $[\alpha, \alpha^*)$. Suppose without loss of generality that this point is b_0. (The cases where this point is b_i for $i > 0$, or a_i for any i, are similar.)

By definition of $\mathbb{Q}$, α belongs to the fast club C. By choice of ξ, $\alpha > \alpha_0$. It follows that $\alpha \in Z'$. Let M witness this. Then $M \prec V_\theta$, $M \cap \omega_1 = \alpha$, and A, B, φ_A, φ_B, $T\restriction\delta$, and $\{B_{U,i}\}_{U,i}$ all belong to M.

Let $p = p_\xi$ and let $\bar{p} = p - 0 = p - \langle a_0, b_0\rangle$. Then a_0 and all points in $\bar{p}$ belong to segments earlier than $[\alpha, \alpha_0)$, and hence they belong to M. On the other hand, b_0 does not belong to M.

Let X be the set of $b \in B$ so that:

(I) $\bar{p} \cup \langle a_0, b\rangle$ is order incompatible with every condition in $T \upharpoonright \delta$.
(II) For every good neighborhood U so that $\bar{p} \cup \langle a_0, b\rangle$ belongs to U, there is $\zeta \in B_{U,0}$ so that $\bar{p}$ is order compatible with $p_\zeta - 0$.

Note that all parameters in the definition of X belong to M, and that the definition can be done in V_θ. (The parameters are B, $\bar{p}$, a_0, $T \upharpoonright \delta$, and $B_{U,0}$.) By elementarity it follows that X belongs to M. We will show that X has size *at most two*. This means that the elements of X are themselves definable and therefore belong to M. But we will show also that b_0 belongs to X, a contradiction as $b_0 \notin M$.

Claim 5.8. *b_0 belongs to X.*

Proof. $\bar{p} \cup \langle a_0, b_0\rangle$ is equal to p_ξ. Since T is an antichain, p_ξ is incompatible with every condition in $T \upharpoonright \delta$. By Claim 5.6 it is in fact order incompatible with every condition in $T \upharpoonright \delta$, so condition (I) holds. Fix a good neighborhood U so that p_ξ belongs to U. Then p_ξ belongs to T_U, and hence $\bar{p} = p_\xi - 0$ belongs to $T_{U,0}$. By definition, $\{p_\zeta - 0 \mid \zeta \in B_{U,0}\}$ is a maximal antichain in $T_{U,0}$, so there is $\zeta \in B_{U,0}$ so that $\bar{p}$ is compatible, and hence in particular order compatible, with $p_\zeta - 0$. This proves condition (II). □

Claim 5.9. *X has at most two elements.*

Proof. Suppose for contradiction that X has three elements, $b_l < b_m < b_h$ say. (The letters "l", "m", and "h" stand for low, middle, and high.) Let $p_l = \bar{p} \cup \langle a_0, b_l\rangle$, and define p_m and p_h similarly. Let U_m be a good neighborhood of p_m, and pick it small enough that it does not contain p_l and p_h. (Since p_l and p_h are the same as p_m except on the first coordinate, it is the points $\langle a_0, b_l\rangle$ and $\langle a_0, b_h\rangle$ of p_l and p_h which are outside U_m.) Let U^* be a larger good neighborhood of p_m, that contains also p_l and p_h.

By condition (II) for p_m, there is $\zeta \in B_{U_m,0}$ so that $p_\zeta - 0$ is order compatible with $p_m - 0 = \bar{p}$. Let q denote p_ζ, and say $q = \{\langle c_i, d_i\rangle \mid i < n\}$. Then q belongs to U_m, and $q - 0$ is order compatible with $\bar{p}$. Since δ is larger than $\sup B_{U_m,0}$, we also know that q belongs to $T \upharpoonright \delta$.

By condition (I) for p_h, q is order incompatible with p_h. Since $U_m \subseteq U^*$, q belongs to U^*. p_h also belongs to U^*, and it follows by Claim 5.7 that there must be an $i < n$ so that the ith coordinate of q is order incompatible with the ith coordinate of p_h. Since $q - 0$ is order compatible with $\bar{p} = p_h - 0$, it must be that $i = 0$, in other words $\langle c_0, d_0\rangle$ is order incompatible with $\langle a_0, b_h\rangle$.

A similar argument using condition (I) for p_l shows that $\langle c_0, d_0\rangle$ is also order incompatible with $\langle a_0, b_l\rangle$.

But now remember that $\langle c_0, d_0\rangle$ is within a very small neighborhood of $\langle a_0, b_m\rangle$, since q belongs to U_m. This neighborhood was chosen small enough to not overlap with $\langle a_0, b_l\rangle$ at the low end, and not overlap with $\langle a_0, b_h\rangle$ at the high end. It follows that d_0 is between b_l and b_h. In other words, $d_0 < b_h$ and $d_0 > b_l$.

c_0 can either be smaller than a_0, or larger. (Equality is impossible since the maximum segment used by $q = p_\zeta$ is earlier than the minimum segment used by $p = p_\xi$.) If $c_0 < a_0$, then since $d_0 < b_h$ it follows that $\langle c_0, d_0\rangle$ is order compatible with $\langle a_0, b_h\rangle$. If $c_0 > a_0$ then since $d_0 > b_l$ it follows that $\langle c_0, d_0\rangle$ is order compatible with $\langle a_0, b_l\rangle$. But using condition (I) for p_h and p_l we saw that $\langle c_0, d_0\rangle$ is order compatible with neither of these pairs, contradiction. □

We noted above that X is definable over V_θ from parameters that belong to M, and therefore by elementarity, X belongs to M. Since X has at most two elements, the elements are themselves definable from X (as the smallest and largest respectively), so by elementarity, they belong to M. We saw also that b_0 is one of the elements of X. So b_0 belongs to M. But this is a contradiction, since the elements of B that belongs to M are the ones enumerated by φ_B before $\omega_1 \cap M = \alpha$, while b_0 belongs to the $[\alpha, \alpha^*)$-segment of B, meaning that it is enumerated between α and α^*. This contradiction completes the proof that $\mathbb{Q}$ is c.c.c. in $V[G]$.

5.2. **Proper forcing.** Our work in the previous subsection yields the following theorem:

Theorem 5.10. *Let $A, B \subseteq \mathbb{R}$ be ω_1-dense. Then there is a poset $\mathbb{P}$, and a name $\dot{\mathbb{Q}}$ so that:*

- *$\mathbb{P}$ is countably closed.*
- *$\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is c.c.c."*
- *Forcing with $\mathbb{P} * \dot{\mathbb{Q}}$ adds an order isomorphism from A to B.*

Note that ω_1-dense subsets of $\mathbb{R}$ have size ω_1, and can be coded by subsets of ω_1. If we could iterate applications of Theorem 5.10 without collapsing ω_1, then with a bookkeeping argument similar to the one used in our consistency proof for $\mathsf{bMA}(\omega_1)$, we would be able to predict all names for ω_1-dense sets in the end extension, and ensure that the iteration adds an order isomorphism between any two of them. This would lead to an end extension where all ω_1-dense subsets of $\mathbb{R}$ are order isomorphic.

Unfortunately our current preservation theorems show that a countable support iteration preserves countable closure, while a finite support iteration preserves c.c.c. We do not have any preservation results for iterations that mix both countably closed and c.c.c. posets.

Our goal in this subsection is to develop such a result. We will define a class of posets, called *proper*, that subsumes both the class of countably closed posets and the class of c.c.c. posets. We will show that proper posets do not collapse ω_1. And we will show that the countable support iteration of proper posets is proper.

Definition 5.11. Let θ be a limit. $C \subseteq \mathcal{P}_\omega(V_\theta)$ is a *club* if there is a function $\varphi \colon V_\theta^{<\omega} \to V_\theta$ so that C consists precisely of all countable $X \subseteq V_\theta$ which are closed under φ (meaning that $a_0, \ldots, a_{n_1} \in X \to \varphi(a_0, \ldots, a_{n-1}) \in X$).

It is easy to check that if we were to replace V_θ in the definition with ω_1, every club in the ordinary sense would be a club in the new sense. That is, every club subset of ω_1 is the set of closure points of some function. Conversely, every club in the new sense contains a club in the ordinary sense.

Claim 5.12. *The intersection of countably many clubs in V_θ is a club in V_θ.*

Proof. Let C_n be clubs, and let φ_n witness this. Let X_n be the smallest set closed under φ_n, and let $X = \bigcup_{n<\omega} X_n$. X is non-empty since $\varphi_0(\emptyset) \in X$, and is countable. Each $Y \in C_n$ is closed under φ_n, and hence $X_n \subseteq Y$. It follows that $Y \in \bigcup_{n<\omega} C_n \to X \subseteq Y$.

Suppose for simplicity that $|X| = \omega$. The case that X is finite is slightly harder and is left as an exercise.

Let $\{a_i \mid i < \omega\}$ enumerate X with no repetitions, and define $\varphi\colon V_\theta^{<\omega} \to V_\theta$ by $\varphi(\emptyset) = a_0$, $\varphi(a_i) = a_{i+1}$, $\varphi(a_0, a_n, u_0, \dots, u_{k-1}) = \varphi_n(u_0, \dots, u_{k-1})$, and $\varphi(\dots) = a_0$ in all other cases. It is clear that any Y which is closed under φ is closed under φ_n for each n. Conversely, if Y is closed under φ_n for all n, then $X \subseteq Y$ and from this and the closure of Y under each φ_n it follows that Y is closed under φ. □

Claim 5.13. *The set $\{X \subseteq V_\theta \mid |X| = \omega$ and $X \prec V_\theta\}$ contains a club.*

Proof. For each formula ψ, let $f_\psi\colon V_\theta^{<\omega} \to V_\theta$ be a function that assigns to each tuple of $a_1, \dots, a_k \in V_\theta$ some b so that $V_\theta \models \psi(b, a_1, \dots, a_k)$ if such b exists, and $\emptyset$ if no such b exists. Let C_ψ be the club of closure points of f_ψ, and let $C = \bigcap_\psi C_\psi$. Then C is club by Claim 5.12. If $X \subseteq V_\theta$ belongs to C, then for every formula ψ and any $a_1, \dots, a_k \in X$, $(\exists b \in V_\theta) V_\theta \models \psi(b, a_1, \dots, a_k) \to (\exists b \in X) V_\theta \models \psi(b, a_1, \dots, a_k)$. (This is because X is an element of C_ψ and therefore closed under f_ψ.) It follows that $X \prec V_\theta$. □

Definition 5.14. Let $\mathbb{P} \in V_\theta$. Let $X \prec V_\theta$ with $|X| = \omega$ and $\mathbb{P} \in X$. $q \in \mathbb{P}$ is a *master condition* for X if for every dense $D \subseteq \mathbb{P}$ with $D \in X$, $q \Vdash \dot{G} \cap \check{D} \cap \check{M} \neq \emptyset$.

Note in connection to definition 5.14 that, since D is dense, it is outright forced that $\dot{G} \cap \check{D} \neq \emptyset$. The additional requirement on q is that for $D \in X$, it forces not only that $\check{D}$ meets $\dot{G}$, but that it does so inside $\check{X}$.

Claim 5.15. *q is a master condition for X iff for every maximal antichain A of $\mathbb{P}$ with $A \in x$, $q \Vdash \dot{G} \cap \check{A} \cap \check{X} \neq \emptyset$.*

Proof. This is the standard conversion between meeting dense sets and meeting maximal antichains. We note only that by elementarity of X, the conversion can be done inside X. More precisely, if $D \in X$ is dense, then by elementarity of X one can find a maximal antichain in D that also belongs to X. Similarly, if $A \in X$ is a maximal antichain, then by elementarity the (dense) set $\{r \mid (\exists s \in A) r \leq s\}$ belongs to X. □

Definition 5.16. $\mathbb{P}$ is *proper* if there exists θ large enough that $\mathbb{P} \in V_\theta$, and a club $C \subseteq \mathcal{P}_\omega(V_\theta)$, so that for every $X \in C$, and for every condition $p \in \mathbb{P} \cap X$, there is $q \leq p$ which is a master condition for X.

The definition states that many elementary substructures X have many master conditions in $\mathbb{P}$.

Lemma 5.17. *If $\mathbb{P}$ is c.c.c. then $\mathbb{P}$ is proper.*

Proof. Let $\theta > \omega$ be large enough that $\mathbb{P} \in V_\theta$. We prove that in fact every condition is a master condition for every countable $X \prec V_\theta$ with $\mathbb{P} \in X$. The set of these X contains a club by Claim 5.13.

Fix X and a condition q. Let $A \in X$ be a maximal antichain in $\mathbb{P}$. Since $\mathbb{P}$ is c.c.c., A is countable. From this, the fact that $A \in X$, and the elementarity of X, it follows that $A \subseteq A$. Since $q \Vdash \dot{G} \cap \check{A} \neq \emptyset$, the fact that $A \subseteq X$ immediately implies $q \Vdash \dot{G} \cap \check{A} \cap \check{X} \neq \emptyset$. □

Lemma 5.18. *If $\mathbb{P}$ is countably closed, then $\mathbb{P}$ is proper.*

Proof. Let θ be large enough that $\mathbb{P} \in V_\theta$. Let $X \prec V_\theta$ with $|X| = \omega$ and $\mathbb{P} \in X$. Let $p \in \mathbb{P} \cap X$. We prove that p can be extended to a master condition for X.

Let D_n, $n < \omega$ enumerate all dense subsets of $\mathbb{P}$ that belong to X. By density of D_n, $(\forall r \in \mathbb{P})(\exists s \in D_n) s \leq r$. Since $D_n, \mathbb{P} \in X$, it follows by the elementarity of X that for $r \in X$, one can find *inside* X an $s \in D_n$ which extends r.

Set now $p_0 = p \in X$, and using the observation of the previous paragraph, inductively find $p_{n+1} \in X$, extending p_n, with $p_{n+1} \in D_n$.

Since $\mathbb{P}$ is countably closed, there is $q \in \mathbb{P}$ so that $(\forall n) q \leq p_n$. Then $q \Vdash \check{p}_{n+1} \in \dot{G}$, and since $p_{n+1} \in D_n \cap X$ it follows in particular that $q \Vdash \dot{G} \cap \check{D}_n \cap \check{X} \neq \emptyset$. So $q \leq p_0 = p$ is a master condition for X. □

Our proofs of Lemmas 5.17 and 5.18 are in some sense orthogonal. The master condition in Lemma 5.17 gave no information at all about the the way the generic meets the dense sets (or rather maximal antichains) in X. The master condition in Lemma 5.18 determined specific conditions where the generic meets dense sets in X.

Lemma 5.19. *Let $\mathbb{P}$ be proper. Let θ and C witness this. Let $\theta^* > \theta$ be a limit. Then for every countable $X^* \prec V_{\theta^*}$ with $\mathbb{P} \in X^*$, every condition $p \in X^*$ extends to a master condition q for X^*.*

Proof. Fix θ^* and fix X^*. We prove that every condition $p \in X^*$ extends to a master condition q for X^*.

Note that V_{θ^*} satisfies the statement "$(\exists \theta)(\exists C \subseteq \mathcal{P}(V_\theta)$ club) so that $\mathbb{P} \in V_\theta$ and for every $X \in C$ with $\mathbb{P} \in X$, every condition $p \in X$ extends to a master condition q for X". This is a statement with only one parameter, $\mathbb{P}$. By elementarity of X^* and since $\mathbb{P} \in X^*$, the same statement hold in X^*. We can therefore fix θ' and C' in X^* witnessing this statement. Since C' is club, it is the set of countable closure points of some function $\varphi \colon V_{\theta'}^{<\omega} \to V_{\theta'}$. Again by elementarity, and since $C' \in X^*$, we can fix $\varphi \in x^*$.

Let $X = X^* \cap V_{\theta'}$. Note that for every $a_0, \dots, a_{k-1} \in X \subseteq X^*$, by elementarity of X^* and since $\varphi \in X^*$, $\varphi(a_0, \dots, a_{k-1}) \in X^*$ and hence $\varphi(a_0, \dots, a_{k-1}) \in X$. So X is closed under φ, and hence $X \in C'$. By choice of θ' and C' it follows that every condition $p \in X$ extends to a master condition q for X.

It remains to convert the conclusion for the previous paragraph from a statement about master conditions for X, to a statement about master conditions for X^*. Fix a condition $p \in X^*$. Now $\mathbb{P} \subseteq V_{\theta'}$, so $\mathbb{P} \subseteq V_{\theta'}$, so $\mathbb{P} \cap X = \mathbb{P} \cap X^*$, and hence p is a condition in X. By the conclusion of the previous paragraph, p extends to a master condition q for X. Since θ' is a limit, every subset of $\mathbb{P}$ belongs to $V_{\theta'}$. It follows that X and X^* have exactly the same dense subsets of $\mathbb{P}$. (In fact they have the same subsets of $\mathbb{P}$, regardless of density.) This implies that every master condition for X is a master condition for X^*, so q is a master condition for X^*. □

We say that θ witnesses the properness of $\mathbb{P}$, if for every countable $X \prec V_\theta$ with $\mathbb{P} \in X$, every condition $p \in X$ extends to a master condition q for X. The previous lemma shows that if $\mathbb{P}$ is proper, then in fact *all* sufficiently large limit θs witness the properness of $\mathbb{P}$.

Lemma 5.20. *Let $\mathbb{P}$ be proper. Then forcing with $\mathbb{P}$ does not collapse ω_1.*

Proof. Let $\dot{f}$ be forced to be a function from $\check{\omega}$ into $\check{\omega}_1$. Let $p \in \mathbb{P}$.

We prove that there is $q \leq p$ and a countable set X so that $q \Vdash \text{range}(\dot{f}) \subseteq \check{X}$. In particular q forces that $\dot{f}$ is not a surjection on $\check{\omega}_1$, as X, being countable and in V, does not contain ω_1.

Let θ witness the properness of $\mathbb{P}$, with θ large enough that $\mathbb{P}, \dot{f}, \omega_1$ all belong to V_θ. Let $X \prec V_\theta$ be countable with $\mathbb{P}, \dot{f}, p \in X$. (By elementarity we have $\omega_1 \in X$ also.) Let $q \leq p$ be a master condition for X. This is possible since θ witnesses the properness of $\mathbb{P}$.

We prove that $q \Vdash \text{range}(\dot{f}) \subseteq \check{X}$. Let G be generic for $\mathbb{P}$ over V with $q \in G$. It is enough to show that $\dot{f}[G](n) \in X$ for each $n < \omega$.

Fix $n < \omega$. Working in V, let $D = \{r \in \mathbb{P} \mid (\exists \alpha < \omega_1) r \Vdash \dot{f}(\check{n}) = \check{\alpha}\}$. Since $\dot{f}$ is forced to be a function from $\check{\omega}$ into $\check{\omega}_1$, the set D is dense. The parameters in the definition of D are $\mathbb{P}$, n, $\dot{f}$, and ω_1. All belong to X, so $D \in X$. Since q is a master condition for X, it follows that $q \Vdash \dot{G} \cap \check{D} \cap \check{X} \neq \emptyset$.

Since $q \in G$, we have then that $G \cap D \cap X \neq \emptyset$. Let $r \in G \cap D \cap X$. By definition of D, there is α so that $r \Vdash_{\mathbb{P}} \dot{f}(\check{n}) = \check{\alpha}$. By elementarity of X, and since $r, \dot{f}, \mathbb{P} \in X$, such α can be found in X. We have now $r \in G$, $\alpha \in X$, and $r \Vdash \dot{f}(\check{n}) = \check{\alpha}$, so $\dot{f}[G](n) = \alpha \in X$. $\square$

Given a generic G for $\mathbb{P}$ over V, let $X[G]$ (for X which is not a name, for example $X \prec V_\theta$) denote $\{\tau[G] \mid \tau \in X, \tau \text{ a name}\}$. The argument used in the proof of Lemma 5.20 to show that a master condition for X forces the range of $\dot{f}$ to be contained in X, shows that if τ is a name for an ordinal and $\tau \in X$, then a master condition for X forces the interpretation of τ to belong to X. A similar argument applies for any name τ for an element of V, not necessarily an ordinal. This gives the following lemma:

Lemma 5.21. *Let $\mathbb{P} \in X \prec V_\theta$ and let q be a master condition for X. Let G be generic for $\mathbb{P}$ over V with $q \in G$. Then $X[G] \cap V = X$.*

The next claim shows that this property, and even the seemingly weaker $X[G] \cap \text{Ord} = X \cap \text{Ord}$, characterizes master conditions:

Claim 5.22. *Let $\mathbb{P} \in X \prec V_\theta$. Suppose that $q \Vdash \check{X}[\dot{G}] \cap \text{Ord} \subseteq \check{X}$. Then q is a master condition for X.*

Proof. Let $D \in X$ be dense. Let G be generic for $\mathbb{P}$ over V, with $q \in G$. We must show that $D \cap G \cap X \neq \emptyset$.

Working in V, let $\alpha = |D|$ and let $f\colon \alpha \to D$ be a bijection. By elementarity, and since $D \in X$, we have $\alpha \in X$ and can pick $f \in X$. Let τ name the least ordinal $\xi < \alpha$ so that $\check{f}(\xi) \in \dot{G}$. (Such ξ is forced to exist since $\dot{G}$ is forced to meet the dense set D.) Again by elementarity, $\tau \in X$.

By assumption of the claim, and since τ is a name for an ordinal, $\tau[G] \in X$. Let $\xi = \tau[G]$ and let $r = f(\xi)$. Since $\xi \in X$ and $f \in X$, by elementarity of X, $r = f(\xi) \in X$. By definition of τ, r belongs to G, and by definition of f, $r \in D$. So $r \in D \cap G \cap X$. $\square$

Claim 5.23. *Let $\mathbb{P} \in X \prec V_\theta$. Let q be a master condition for X. Let G be generic for $\mathbb{P}$ over V with $q \in G$. Then $X[G] \prec V_\theta[G]$.*

Remark 5.24. The assumption that G includes a master condition for X is in fact not necessary. The conclusion $X[G] \prec V_\theta[G]$ holds for any G, but the argument is more involved. In case G does not include a master condition, $X[G] \cap V$ will include elements outside X. In other words there will be names in X for elements of V that do not belong to X. These can be used to show that $X[G]$ is elementary.

Proof of Claim 5.23. It is enough to show that for every formula ψ and every $a_1, \ldots, a_k \in X[G]$, if $V_\theta[G] \models (\exists y)\psi(y, a_1, \ldots, a_k)$ then $(\exists y \in X[G])V_\theta[G] \models \psi(y, a_1, \ldots, a_k)$.

Fix ψ and $a_i = \sigma_i[G]$, with $\sigma_i \in X$. Let $D = \{r \in \mathbb{P} \mid V_\theta \models (\exists \tau) r \Vdash \psi(\tau, \sigma_1, \ldots, \sigma_k)$ or $V_\theta \models r \Vdash \neg(\exists y)\varphi(y, \sigma_1, \ldots, \sigma_k)\}$. Then D is dense, and by elementarity, $D \in X$. Since $q \in G$ is a master condition for X, there is $r \in G \cap D \cap X$.

Since $r \in G$ and $V_\theta[G] \models (\exists y)\psi(y, \sigma_1[G], \ldots, \sigma_k[G])$, the second clause in the definition of D fails for r, and the membership of r in D must be through the first clause. Let τ witness this. Since $r \in X$ we may by elementarity of X fix $\tau \in X$. Then setting $y = \tau[G]$ we have $y \in X[G]$ and $V_\theta[G] \models \psi(y, a_1, \ldots, a_k)$. $\square$

We have so far defined the class of proper posets, showed it subsumes both the class of c.c.c. posets and the class of countably closed posets, and showed that proper posets do not collapse ω_1. Our plan next is to show that the class of proper posets is closed under countable support iterations. This will allow us to form combined iterations of c.c.c. and countably closed posets without fear of collapsing ω_1.

Lemma 5.25. *Suppose that $\mathbb{P}$ is proper, and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper". Then $\mathbb{P} * \dot{\mathbb{Q}}$ is proper.*

Proof. Let θ be large enough to witness the properness of $\mathbb{P}$, and be forced by every condition in $\mathbb{P}$ to witness the properness of $\dot{\mathbb{Q}}$. Let $X \prec V_\theta$ be countable with $\mathbb{P}, \dot{\mathbb{Q}} \in X$. We will prove that every $\langle p, \tau \rangle \in \mathbb{P} * \dot{\mathbb{Q}} \cap X$ extends to a master condition for X.

Claim 5.26. *Let $q \in \mathbb{P}$ be a master condition for X. Let $\tau \in X$ and suppose $q \Vdash \tau \in \dot{\mathbb{Q}}$. Then there is τ^* so that $q \Vdash$ "$\tau^* \in \dot{\mathbb{Q}}$, $\tau^* \leq \tau$, and τ^* is a master condition for $\check{X}[\dot{G}]$". (Here as usual $\dot{G}$ is the name for a generic filter for $\mathbb{P}$.)*

Proof. Let G be generic for $\mathbb{P}$ over V, with $q \in G$. Then $X[G] \prec V_\theta[G]$ by claim 5.23, $\dot{\mathbb{Q}}[G] \in X[G]$ since $\dot{\mathbb{Q}} \in X$, and $t = \tau[G]$ is a condition in $\dot{\mathbb{Q}}[G]$ that belongs to $X[G]$. Since θ witnesses the properness of $\dot{\mathbb{Q}}[G]$, it follows that t extends to a master condition t^* for $X[G]$. This is true for all generic G for $\mathbb{P}$ with $q \in G$, so it is forced by q. In other words, $q \Vdash (\exists t^* \in \dot{\mathbb{Q}})(t^* \leq t$ and t^* is a master condition for $\check{X}[\dot{G}]$". It follows that there exists a name τ^* so that $q \Vdash$ "$\tau^* \leq \tau$ and τ^* is a master condition for $\check{X}[\dot{G}]$". $\square$

Fix $\langle p, \tau \rangle \in \mathbb{P} * \dot{\mathbb{Q}} \cap X$. We show how to extend $\langle p, \tau \rangle$ to a master condition for X. Since θ witnesses that $\mathbb{P}$ is proper, there is $p^* \leq p$ so that p^* is a master condition for X in $\mathbb{P}$. By Claim 5.26, there is τ^* so that $p^* \Vdash$ "$\tau^* \in \dot{\mathbb{Q}}$, $\tau^* \leq \tau$, and τ^* is a master condition for $\check{X}[\dot{G}]$ in $\dot{\mathbb{Q}}$". Then $\langle p^*, \tau^* \rangle \leq \langle p, \tau \rangle$ in $\mathbb{P} * \dot{\mathbb{Q}}$. We prove that $\langle p^*, \tau^* \rangle$ is a master condition for X.

Let $D \in X$ be dense in $\mathbb{P} * \dot{\mathbb{Q}}$. Let $G * H$ be generic for $\mathbb{P} * \dot{\mathbb{Q}}$ over V, with $\langle p^*, \tau^* \rangle \in G * H$.

Let $D^* = \{\sigma[G] \mid (\exists s \in G)\langle s, \sigma \rangle \in D\}$. We have seen when analyzing compositions that D^* is dense in $\dot{\mathbb{Q}}[G]$. Since $D \in X$, by elementarity a name for D^* belongs to X, so $D^* \in X[G]$. By Claim 5.23, $X[G] \prec V_\theta[G]$. By choice of X, $\dot{\mathbb{Q}} \in X$, so $\dot{\mathbb{Q}}[G] \in X[G]$. By choice of τ^*, $\tau^*[G] \in H$ is a master condition for $X[G]$. It follows from all this that H meets D^* inside $X[G]$. In other words there is some

$\sigma \in X$ so that $\sigma[G] \in H$ and $\sigma[G] \in D^*$. Since $\sigma[G] \in D^*$, there exists $\langle s, \sigma' \rangle \in D$, with $s \in G$, so that $\sigma[G] = \sigma'[G]$.

The statement that $(\exists \langle s, \sigma' \rangle \in D)(s \in G$ and $\sigma'[G] = \sigma[G])$ is true in $V_\theta[G]$. The parameters of the statement are D, G, and σ. All belong to $X[G]$. By elementarity of $X[G]$ we may therefore pick $\langle s, \sigma' \rangle$ in $X[G]$. Note that $\langle s, \sigma' \rangle \in D \subseteq V$, so by Lemma 5.21, the fact that this pair belongs to $X[G]$ implies that it belongs to X.

We have now $\langle s, \sigma' \rangle \in X$, $\langle s, \sigma' \rangle \in D$, and (since $s \in G$ and $\sigma'[G] = \sigma[G] \in H$) $\langle s, \sigma' \rangle \in G * H$. $\square$

5.3. **Future additions.** We must still show that (infinite) countable support iterations of proper posets are proper. This will allow us to reach an extension where all ω_1-dense subsets of $\mathbb{R}$ are order isomorphic, as discussed earlier in the section.

In fact for this particular application we could manage with just a c.c.c. iteration. A careful examination of our proof in Subsection 5.1 shows that there are only continuum many clubs of the ground model V that our fast club C must be almost contained in. (This is because the clubs Z' we defined in the proof were defined from countable parameters, and there are only continuum many options for the values of these parameters.) If the continuum is ω_1, then we can obtain C without any forcing, simply by taking the diagonal intersection of all these clubs. We then do not need the initial countably closed forcing that was used to introduce C. The remaining forcing, $\mathbb{Q}$, is c.c.c. Thus, under CH, any two ω_1-dense subsets of $\mathbb{R}$ can be made isomorphic by a c.c.c. poset. A finite support iteration of these posets, that starts from a model of the CH, covers all pairs of potential names for ω_1-dense subsets of $\mathbb{R}$ in ω_2 stages, and preserves the CH along the way (though it fails in the end model), then reaches a model where all ω_1-dense subsets of $\mathbb{R}$ are order isomorphic.

However it is *not* the case in general—meaning without the CH—that any two ω_1-dense subsets of $\mathbb{R}$ can be made isomorphic using a c.c.c. forcing. Indeed, $\mathsf{MA}(\omega_1)$ does *not* imply that all ω_1-dense subsets of $\mathbb{R}$ are order isomorphic. Moreover, for many other forcing constructions, a combination of countably closed and c.c.c. posets, or a proper poset that is neither c.c.c. nor countably closed, is necessary.

The preservation of properness under countable support iterations, applications, and a parallel of MA for proper forcing, will be discussed in a current literature seminar in fall 2011.

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