Qualifying Exam LOGIC Fall 03

All questions have equal value.

1. For this problem, work in ZF (ZFC minus the Axiom of Choice). If κ is a cardinal number and X is a set, then $\mathcal{P}_\kappa(X)$ is the set of all subsets of X of size $< \kappa$. Suppose that $f : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \mathbb{R}$ is one-to-one. Prove that there exists a sequence of ω_1 distinct reals.

2. A subset X of a limit ordinal α is *stationary* in α if X meets every closed, unbounded subset of α. Let κ be a regular cardinal and let $X \subseteq \kappa$ be stationary in κ. Let M be a transitive class model of ZFC such that $X \in M$. Prove that X is stationary in κ in M.

3. Assume $V = L$. Define $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ as follows. Let A_α be the $<_L$-least $A \subseteq \alpha$ such that $(\forall \beta < \alpha)\, A \cap \beta \neq A_\beta$ if such an A exists and let $A_\alpha = \emptyset$ otherwise. Prove that for all $A \subseteq \omega_1$ there exists an $\alpha < \omega_1$ such that $A \cap \alpha = A_\alpha$.

4. As with problem 1, work in ZF. Let $\mathrm{AC}^{\mathrm{fin}}$ be the restriction of the Axiom of Choice to collections of *finite* sets. Prove that the Compactness Theorem of model theory implies $\mathrm{AC}^{\mathrm{fin}}$.

5. Let $S(n) = n + 1$ for $n \in \omega$. Prove that the theory of (ω, S) is not finitely axiomatizable.

6. Let $\kappa = \omega_1$ and let $T = \mathrm{Th}(V_\kappa, \in)$. Prove that there is no saturated countable model of T.

7. Let A be an infinite recursively enumerable set. Show that $\{e \mid W_e = A\}$ is many-one complete for Π_2. (W_e here is the eth r.e. set in some standard enumeration.)

8. Let $\mathrm{Prov}(v_1, v_2)$ represent in Peano Arithmetic (PA) the set of all pairs (a, b) such that a is the Gödel number of a sentence τ and b is the Gödel number of a proof of τ from the axioms of PA. Let σ be gotten from the Fixed Point Lemma applied to $\forall v_2 \neg \mathrm{Prov}(v_1, v_2)$. In other words, let σ be a sentence such that $\mathrm{PA} \vdash (\sigma \leftrightarrow \forall v_2 \neg \mathrm{Prov}(\mathbf{k}, v_2))$, where k is the Gödel number of σ. Let T be the theory gotten from PA by adding $\neg\sigma$ as an axiom. Show that T is ω-*in*consistent: that there is a formula $\psi(v_1)$ such that $T \vdash \exists v_1 \psi(v_1)$ and $T \vdash \neg\psi(\mathbf{n})$ for each numeral n.