SL(2, ℂ) FLOER HOMOLOGY FOR SURGERIES ON SOME KNOTS

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Abstract. We establish a relationship between the sheaf-theoretic SL(2, ℂ) Floer cohomology $HP(Y)$, as defined by Abouzaid and Manolescu, for $Y$ a surgery on a small knot in $S^3$, and the SL(2, ℂ) Casson invariant, as defined by Curtis. We use this to compute $HP$ for surgeries on the trefoil and the figure-eight knots. We also compute $HP$ for surgeries on two non-small knots, the granny and square knots.

1. Introduction

In [AM], the authors defined a new invariant of closed, connected, orientable 3 manifolds $Y$ called sheaf-theoretic SL(2, ℂ) Floer cohomology, denoted $HP(Y)$. It is defined as the hypercohomology of the perverse sheaf on the character scheme $\mathcal{X}_{\text{irr}}(Y)$, coming from a description of this space as a complex Lagrangian intersection. In this paper, we compute this invariant for surgeries on the trefoil, figure-eight, granny and square knots.

Let $K$ be a knot in $S^3$ and $S^3_{p/q}(K)$ its $p/q$ Dehn surgery. When $S^3\setminus K$ contains no closed, incompressible surfaces, we say that $K$ is a small knot. The calculation of $HP(S^3_{p/q}(K))$ for $K$ a small knot and generic values of $p/q$ reduces to the SL(2, ℂ) Casson invariant $\lambda_{SL(2, ℂ)}$ as defined by Curtis [Cur01] and explored in her joint work with Boden [BC16]. Specifically, we have

**Theorem 1.1.** Let $K \subset S^3$ be a small knot, and let $Y = S^3_{p/q}(K)$ denote $p/q$ surgery on $K$. Then, for all but finitely many values of $p, q$, we have $HP(Y) \cong \mathbb{Z}^{\lambda_{SL(2, ℂ)}(Y)}$.

**Remark 1.2.** We will often use the notation $A(k)$ to denote a graded abelian group with $A$ in degree $k$. A more common notation for this is $A[-k]$.

For example, when $K$ is the right-handed trefoil, we have the following explicit formula:

**Theorem 1.3.** Let $S^3_{p/q}(3_1)$ denote the 3-manifold obtained from $p/q$ Dehn surgery on the right-handed trefoil in $S^3$. Then we have the following formula for the sheaf-theoretic Floer cohomology:

$$HP(S^3_{p/q}(3_1)) = \begin{cases} \mathbb{Z}_{(0)}^{\lfloor \frac{|p-6q|}{2} \rfloor} & \text{if } p \text{ is odd} \\ \mathbb{Z}_{(0)}^{\lfloor \frac{|p-6q|}{2} \rfloor} & \text{if } p \text{ is even, } 12 \not| p \\ \mathbb{Z}_{(0)}^{\lfloor \frac{|p-6q|}{2} \rfloor} & \text{if } 12 | p \end{cases}$$

Similarly for the figure-eight knot,

**Theorem 1.4.** Let $S^3_{p/q}(4_1)$ denote the 3-manifold obtained from $p/q$ Dehn surgery on the figure-eight knot in $S^3$. Then we have the following formula for the sheaf-theoretic Floer cohomology:

$$HP(S^3_{p/q}(4_1)) = \begin{cases} \mathbb{Z}_{(0)}^{\lfloor \frac{|p-4q|+|p+4q|}{4} \rfloor} & \text{if } p \text{ is odd} \\ \mathbb{Z}_{(0)}^{\lfloor \frac{|p-4q|+|p+4q|}{4} \rfloor} & \text{if } p \text{ is even, } p \neq \pm 4 \\ \mathbb{Z}_{(0)}^{\lfloor \frac{|p-4q|+|p+4q|}{4} \rfloor} & \text{if } p = \pm 4 \end{cases}$$
In [AM], the authors also define a framed version of sheaf-theoretic Floer cohomology denoted $\text{HP}_{\#}(Y)$. It is defined as the hypercohomology of a certain perverse sheaf on the representation scheme of $Y$, $\text{Hom}(\pi_1(Y), \text{SL}(2, \mathbb{C}))$. We would like to compute the framed sheaf-theoretic Floer homology, $\text{HP}_{\#}(S^3_{p/q}(K))$, for surgeries on knots. However, the representation schemes are usually not zero-dimensional. In fact, for non-trivial surgeries on the trefoil, they are never zero-dimensional and moreover they are singular schemes when $p$ is divisible by 12. This complicates the identification of the relevant perverse sheaf. We can still compute $\text{HP}_{\#}$ when $p$ is not a multiple of 12.

**Theorem 1.5.** Let $S^3_{p/q}(3_1)$ denote the 3-manifold obtained from $p/q$ Dehn surgery on the right-handed trefoil in $S^3$. Then for $p$ not a multiple of 12, we have the following formula for the framed sheaf-theoretic Floer cohomology:

$$\text{HP}_{\#}(S^3_{p/q}(3_1)) = H^*(pt)^{\oplus d_1} \oplus H^{*+2}(\mathbb{C}P^1)^{\oplus d_2} \oplus H^{*+3}(\text{PSL}(2, \mathbb{C}))^{\oplus d_3}$$

where the multiplicities are given by

$$(d_1, d_2, d_3) = \begin{cases} (1, \frac{1}{2}(|p| - 1), \frac{1}{2}(6q - p) - \frac{1}{2}), & \text{if } p \text{ is odd} \\ (2, \frac{1}{2}(|p| - 2), \frac{1}{2}(6q - p)), & \text{if } p \text{ is even and not a multiple of } 12 \end{cases}$$

We use this partial calculation to show that there does not exist an exact triangle relating $\text{HP}_{\#}$ for surgeries on the trefoil.

In light of Theorem 1.1, we are interested in computing $\text{HP}(S^3_{p/q}(K))$ when $K$ is not a small knot. The character schemes of such manifolds may have positive dimensional components, in which case the calculation of the $\text{SL}(2, \mathbb{C})$ Casson invariant is insufficient to determine $\text{HP}$. In fact, when $K = K_1 \# K_2$ is a composite knot, we are guaranteed to have positive dimensional components. We provide a calculation of $\text{HP}$ with $F = \mathbb{Z}/2\mathbb{Z}$ coefficients for surgeries on the square and granny knots. Recall that the granny knot is the connected sum of two right-handed trefoils, whereas the square knot is a composite of a trefoil with its mirror.

**Theorem 1.6.** Let $S^3_{p/q}(G)$ denote the 3-manifold obtained from $p/q$ Dehn surgery on the granny knot, $G = 3_1^1 \# 3_1^1$. Then we have the following formula for the sheaf-theoretic Floer cohomology:

$$\text{HP}(S^3_{p/q}(G); F) = \begin{cases} F^6q - p + \frac{1}{2}|12q - p| - \frac{1}{2} \oplus F^{12q - p - \frac{1}{2}}_{(-1)} & \text{if } p \text{ is odd} \\ F^6q - p + \frac{1}{2}|12q - p| - 1 \oplus F^{12q - p - 1}_{(-1)} & \text{if } p \text{ is even, } p \neq 12k \\ F^6q - p + \frac{1}{2}|12q - p| - 5 \oplus F^{12q - p + 1}_{(-1)} & \text{if } p = 12k, p/q \neq 12 \\ F^4_{(1)} \oplus F^4_{(0)} \oplus F_{(-2)} & \text{if } p/q = 12 \end{cases}$$

**Theorem 1.7.** Let $S^3_{p/q}(S)$ denote the 3-manifold obtained from $p/q$ Dehn surgery on the square knot, $S = 3_1^1 \# 3_1^1$. Then we have the following formula for the sheaf-theoretic Floer cohomology:

$$\text{HP}(S^3_{p/q}(S); F) = \begin{cases} F^6q - p + \frac{1}{2}|6q + p| + \frac{1}{2}|p| - \frac{1}{2} \oplus F^{\frac{1}{2}|p| - \frac{1}{2}}_{(-1)} & \text{if } p \text{ is odd} \\ F^6q - p + \frac{1}{2}|6q + p| + \frac{1}{2}|p| - 1 \oplus F^{\frac{1}{2}|p| - 1}_{(-1)} & \text{if } p \text{ is even, } p \neq 12k \\ F^6q - p + \frac{1}{2}|6q + p| + \frac{1}{2}|p| - 5 \oplus F^{\frac{1}{2}|p| + 3}_{(-1)} & \text{if } p = 12k, p \neq 0 \\ F^4_{(1)} \oplus F^4_{(0)} \oplus F_{(-2)} & \text{if } p = 0 \end{cases}$$

The organization of this paper is as follows. In Section 2 we provide some background on character varieties and the invariants $\text{HP}$, $\text{HP}_{\#}$, and $\lambda_{\text{SL}(2, \mathbb{C})}$. In Section 3, we prove Theorems 1.1, 1.3 and 1.4. In Section 4, we compute the representation varieties of surgeries on the trefoil and prove Theorem 1.5. In Section 5, we determine the character variety of the composite knot $3_1 \# 3_1$, allowing us to
compute the A-polynomials of the square and granny knots in Section 6. In Section 7, we consider surgeries on composite knots and establish Theorems 1.6 and 1.7. In Section 8, we apply Theorem 1.5 to demonstrate the non-existence of a surgery exact triangle for the trefoil.

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2. Background

For a topological space $X$, let $\mathcal{R}(X)$ denote the SL(2, $\mathbb{C}$) representation scheme of $\pi_1(X)$, defined as

$$\mathcal{R}(X) = \text{Hom}(\pi_1(X), \text{SL}(2, \mathbb{C}))$$

Assuming $\pi_1(X)$ is finitely generated, this set is naturally identified as the $\mathbb{C}$ points of an affine scheme. The character scheme $\mathcal{X}(X)$ is the GIT quotient of $\mathcal{R}(X)$ by the conjugation action of SL(2, $\mathbb{C}$).

A representation $\rho \in \mathcal{R}(X)$ is irreducible if the image of $\rho$ is not contained in any proper Borel subgroup. The irreducible representations comprise the stable locus for the GIT action. Let $\mathcal{R}_{\text{irr}}(X) \subset \mathcal{R}(X)$ denote the open subscheme corresponding to irreducible representations, and similarly $\mathcal{X}_{\text{irr}}(X) \subset \mathcal{X}(X)$. When $X$ is a closed surface of genus $g > 1$, $\mathcal{X}_{\text{irr}}(X)$ is a holomorphic symplectic manifold of dimension $6g - 6$ [Gol04].

To investigate character schemes of 3-manifolds, we take the perspective of [AM] using Heegaard splittings. Let $Y = U_0 \cup_\Sigma U_1$ be a Heegaard splitting of a closed, orientable, 3-manifold $Y$ into two handlebodies $U_0$ and $U_1$ with Heegaard surface $\Sigma$. Then $\mathcal{X}_{\text{irr}}(U_i)$ is a complex Lagrangian in $\mathcal{X}_{\text{irr}}(\Sigma)$ and $\mathcal{X}_{\text{irr}}(Y) = \mathcal{X}_{\text{irr}}(U_0) \cap \mathcal{X}_{\text{irr}}(U_1)$ is a Lagrangian intersection [AM].

In [Bus], the author applies the work of [Joy15] to define a perverse sheaf of vanishing cycles associated to any Lagrangian intersection in a holomorphic symplectic manifold. In [AM], the authors use Bussi’s construction to associate a perverse sheaf with a Heegaard splitting of a 3-manifold. Moreover, they show that the perverse sheaf is independent of the Heegaard splitting. This gives an invariant of the 3-manifold, $P(Y) \in \text{Perv}(\mathcal{X}_{\text{irr}}(Y))$ and its hypercohomology, $HP(Y)$, the sheaf-theoretic SL(2, $\mathbb{C}$) Floer cohomology of $Y$. They also define an invariant using the representation scheme that takes into account the reducibles, called the framed sheaf-theoretic SL(2, $\mathbb{C}$) Floer cohomology of $Y$, $HP_{\#}(Y)$. It is the hypercohomology of a perverse sheaf $P_{\#}(Y) \in \text{Perv}(\mathcal{X}_{\text{irr}}(Y))$.

To compute these invariants, we can use the following tool

**Proposition 2.1.** Let $X \subset \mathcal{X}_{\text{irr}}(Y)$ (resp. $X \subset \mathcal{X}_{\text{irr}}(Y)$) be a smooth topological component of the character scheme (resp. representation scheme) of complex dimension $d$. Then the restriction of the perverse sheaf $P(Y)$ (resp. $P_{\#}(Y)$) to $X$ is a local system with stalks isomorphic to $\mathbb{Z}[d]$. In particular, if $X$ is simply connected, then $HP_{\#}(Y)$ (resp. $HP(Y)$) contains $H^*(X; \mathbb{Z}[d])$ as a direct summand.

Furthermore, if $[\rho]$ is an isolated irreducible character and $X \cong \text{PSL}(2, \mathbb{C})$ is the orbit of $[\rho]$ in the representation scheme, then the local system $P_{\#}(Y)|_X$ is trivial.

**Proof.** The first part is Proposition 6.2 in [AM]. The second part is Lemma 8.3 of [AM]. □

When $X$ is smooth but not simply connected, then there is some ambiguity over the local system $P(Y)|_X$. This can be circumvented by using $\mathbb{Z}/2\mathbb{Z}$ coefficients.

**Corollary 2.2.** Assume $\mathcal{X}_{\text{irr}}(M)$ is smooth with topological components $X_i$ of complex dimensions $d_i$. Then $HP(Y; \mathbb{Z}/2\mathbb{Z}) = \bigoplus_i H^*(X_i; \mathbb{Z}/2\mathbb{Z})[d_i]$. 

**Proof.** This follows from the fact that all local systems with $\mathbb{Z}/2\mathbb{Z}$ coefficients are trivial, since $\text{Aut}(\mathbb{Z}/2\mathbb{Z})$ is trivial. □
Morally, $HP(Y)$ should be a version of instanton Floer homology using the gauge group $SL(2, \mathbb{C})$ instead of $SU(2)$. Pursuing this analogy, the Euler characteristic of $HP(Y)$, denoted $\lambda^P(Y)$, should be a type of Casson invariant, just as the Euler characteristic of instanton Floer homology is related to the original Casson invariant, which is a count of irreducible $SU(2)$ characters. There is another invariant called the $SL(2, \mathbb{C})$ Casson invariant defined in [Cur01] that counts isolated, irreducible $SL(2, \mathbb{C})$ characters. To distinguish it from this invariant, $\lambda^P(Y)$ is called the full Casson invariant since it takes into account the positive dimensional components of the character scheme. When $\mathcal{X}_{\text{irr}}(Y)$ is zero-dimensional, $\lambda^P$ and $\lambda_{SL(2, \mathbb{C})}$ agree. In fact, we have

**Theorem 2.3.** Let $Y$ be a 3-manifold such that $\mathcal{X}_{\text{irr}}(Y)$ is zero-dimensional. Then $HP(Y) \cong \mathbb{Z}_\lambda$, where $\lambda = \lambda_{SL(2, \mathbb{C})}(Y)$ is the $SL(2, \mathbb{C})$ Casson invariant as defined in [Cur01].

**Proof.** The definition of $HP(Y)$ uses the characterization of $\mathcal{X}_{\text{irr}}(Y)$ as a complex Lagrangian intersection $L_0 \cap L_1$ in the character scheme of a Heegaard surface for $Y$. The stalk of the perverse sheaf $P^\bullet(Y)$ at a point $p \in \mathcal{X}_{\text{irr}}(Y)$ is the degree-shifted cohomology of the Milnor fiber of some function $f : U \to \mathbb{C}$, for $U$ an open neighborhood in one of the Lagrangians, such that the graph of $\Gamma_f \subset T^*U$ is identified with $L_1$ in an appropriate polarization of the symplectic manifold near $p$. Since $\mathcal{X}_{\text{irr}}(Y)$ is zero-dimensional, we know that $f$ has an isolated singularity at $p$. Thus, the Milnor fiber has the homotopy type of a bouquet of spheres. The number of spheres in the bouquet is the Milnor number, denoted $\mu_p$. Then, the stalk is given by $(P^\bullet(Y))_p \cong \mathbb{Z}_{\mu_p}$. The hypercohomology is $HP(Y) \cong \mathbb{Z}\sum_{(0)} \mu_p$, where the sum is over all components of $\mathcal{X}_{\text{irr}}(Y)$. The definition of the Casson invariant in terms of intersection cycles given in [Cur01] is $\lambda_{SL(2, \mathbb{C})}(Y) = \sum_p n_p$, where the sum is over all zero-dimensional components of $\mathcal{X}_{\text{irr}}(Y)$, and $n_p$ is the intersection multiplicity of $L_0$ with $L_1$. But the Milnor number $\mu_p$ is equal to the intersection multiplicity of $\Gamma_f$ with $L_0$, hence the result follows. $\square$

### 3. Surgeries on Small Knots and the $\lambda_{SL(2, \mathbb{C})}$ Casson Invariant

3.1. **Surgeries on small knots.** By applying Theorem 2.3, we can establish the connection between $HP(Y)$ for $Y$ a surgery on a small knot in $S^3$ and the $SL(2, \mathbb{C})$ Casson invariant, $\lambda_{SL(2, \mathbb{C})}(Y)$ as given in Theorem 1.1.

**Proof of Theorem 1.1.** The group $\pi_1(Y)$ is a quotient of $\pi_1(S^3 \setminus K)$ by the subgroup normally generated by the class of the peripheral curve $m^p \ell^q$, where $m$ is the meridian and $\ell$ the longitude. Thus, $\mathcal{X}_{\text{irr}}(Y)$ is a closed subscheme of $\mathcal{X}_{\text{irr}}(S^3 \setminus K)$. However, dim $\mathcal{X}_{\text{irr}}(S^3 \setminus K) = 1$ when $K$ is a small knot [CCG+94]. Thus, if dim $\mathcal{X}_{\text{irr}}(Y) > 0$, then we must have that the reduced scheme $\mathcal{X}_{\text{irr}}(Y)_{\text{red}}$ appears as one of the irreducible components of $\mathcal{X}_{\text{irr}}(S^3 \setminus K)_{\text{red}}$. Observe that the $\mathcal{X}_{\text{irr}}(S^3 \setminus K)_{\text{red}}$ are disjoint for different values of $p/q$, since if $m^p \ell^q = m'p' \ell'^q$ then we have $m = m'$ and the representation would be trivial because $m$ normally generates the fundamental group. Then, as $\mathcal{X}_{\text{irr}}(S^3 \setminus K)_{\text{red}}$ has only finitely many components, we see that dim $\mathcal{X}_{\text{irr}}(S^3 \setminus K)_{\text{red}} = 0$ for all but finitely many $p/q$. The result then follows from Theorem 2.3. $\square$

The invariant $\lambda_{SL(2, \mathbb{C})}$ has been computed for a range of 3-manifolds, including surgeries on many families of knots [Cur01],[BC16],[BC06]. We provide a few examples of how these results yield formulae for the sheaf-theoretic Floer homology of surgeries on knots.

3.2. **Large surgeries on small knots.** We review the results of [Cur01]. Let $M = S^3 \setminus N(K)$ be a knot exterior. Let $i : \partial M \to M$ denote the inclusion and $r : \mathcal{X}(M) \to \mathcal{X}(\partial M)$ denote the restriction map.

**Definition 3.1.** A slope $\gamma \in \partial M$ is **irregular** if there exists an irreducible representation $\rho$ of $\pi_1(M)$ such that:

(i) the character $[\rho]$ is in a one dimensional component $\mathcal{X}_i$ of $\mathcal{X}_{\text{irr}}(M)$ such that $r(\mathcal{X}_i)$ is also one-dimensional;

(ii) $\text{tr}(\rho(\alpha)) = \pm 2$ for all $\alpha \in \partial M$;
(iii) \( \ker(\rho \circ i_*) \) is cyclic, generated by \([\gamma]\).

**Definition 3.2.** A slope \( p/q \) is admissible if:

(i) It is regular and not a strict boundary slope;

(ii) No \( p' \)-th root of unity is a root of the Alexander polynomial of \( K \), where \( p' = p \) for \( p \) odd and \( p' = \frac{p}{2} \) for \( p \) even.

With these definitions, we can state Theorem 4.8 of [Cur01]:

**Theorem 3.3 ([Cur01]).** Let \( K \) be a small knot in \( S^3 \) with complement \( M \). Let \( \{X_i\} \) be the collection of one-dimensional components of \( X(M) \) such that \( r(X_i) \) is one-dimensional and such that \( X_i \) contains an irreducible representation. Then there exist integral weights \( m_i > 0 \) depending only on \( X_i \) and non-negative \( E_0, E_1 \in \frac{1}{2}\mathbb{Z} \) depending only on \( K \) such that for every admissible \( \frac{p}{q} \) we have

\[
\lambda_{\text{SL}(2,\mathbb{C})}(S^3_{p/q}(K)) = \frac{1}{2} \sum_i m_i |p\mathcal{L} + q\mathcal{Z}|_i - E_0 + E_1
\]

where \( \sigma(p) \in \{0, 1\} \) is the parity, and \( || - ||_i \) is the Culler-Shalen seminorm associated to \( X_i \).

There are only finitely many inadmissible slopes and only finitely many strict boundary slopes [Cur01]. Provided \( p \) is not chosen so that some \( p' \)-th root of unity is a root of the Alexander polynomial, where \( p' \) is as in Definition 3.2(ii), the above theorem only excludes finitely many slopes \( p/q \). Thus, by combining Theorem 2.3 with Theorem 3.3 we obtain a formula for the sheaf-theoretic Floer homology for most surgeries on small knots.

### 3.3. \( HP \) for surgeries on the trefoil.

The character schemes of all surgeries on the trefoil are zero-dimensional. They have been explicitly computed and the \( \text{SL}(2, \mathbb{C}) \) Casson invariant determined in [BC06]. From their computation and Theorem 2.3 we obtain Theorem 1.3.

**Proof of Theorem 1.3.** This follows from the calculation of \( \lambda_{\text{SL}(2,\mathbb{C})}(Y) \) in Theorem 5.9 of [BC06]. \( \square \)

### 3.4. \( HP \) for surgeries on the figure-eight knot.

Since the figure-eight knot is small, we can apply Theorem 1.1. By the results of [BC12], the \( \text{SL}(2, \mathbb{C}) \) Casson invariant of surgeries on the figure-eight knot is

\[
\lambda_{\text{SL}(2,\mathbb{C})}(S^3_{p/q}(4_1)) = \frac{1}{2}(|p - 4q| + |p + 4q|) - E_0
\]

for all admissible slopes \( p/q \), where \( E_0 = 0 \) and \( E_1 = 1 \). The inadmissible slopes are the strict boundary slopes \( \pm 4 \). Thus, it suffices to compute the Casson invariant for \( \pm 4 \) surgery. By [CL98], the \( \lambda \)-polynomial of the figure-eight knot is

\[
A(L, M) = -2 + M^4 + M^{-4} - M^2 - M^{-2} - L - L^{-1}
\]

The surgery curve is given by the equation \( L = M^4 \). These two equations reduce to \((M + M^{-1})^2 = 0\). This yields the solutions \( M = \pm i \), which correspond to the same character. It must correspond to an irreducible character because there are no non-abelian reducibles for the figure-eight knot (the Alexander polynomial has no roots that are roots of unity). From the equation, we observe that this point has multiplicity 2. Thus, \( \lambda_{\text{SL}(2,\mathbb{C})}(S^3_{\pm 4}(4_1)) = 2 \). This calculation provides the proof of Theorem 1.4.

### 4. \( HP \# \) for surgeries on the trefoil

In this section, we prove Theorem 1.5. First, we will compute the representation schemes for surgeries on the trefoil. Then, we can apply Proposition 2.1 in the cases when the representation scheme is smooth and simply connected (except perhaps for some copies of \( \text{PSL}(2, \mathbb{C}) \) coming from orbits of irreducibles) to determine \( HP \# \).
4.1. Calculation of $\mathcal{R}(S_{p/q}^3(3_1))$. We now determine the structure of the representation scheme and identify some components as smooth.

The Wirtinger presentation for the trefoil knot group is

$$\pi_1(S^3\backslash 3_1) \cong \langle r, s \mid rsr = srs \rangle$$

A meridian is given by $m = r$, and a (0-framed) longitude for the right-handed trefoil is $\ell = r^{-4}sr^2s$. For the left-handed trefoil, a longitude is given by $T = r^4s^{-1}r^{-2}s^{-1}$. Applying the change of variables $r = ab^{-1}, s = b^2a^{-1}$, we obtain the following presentation for the fundamental group of $p/q$ surgery.

$$\pi_1(S^3\backslash 3_1) \cong \langle a, b \mid a^2 = b^3, m^p\ell^q = 1, m = ab^{-1}, \ell = (ba^{-1})^6a^2 \rangle$$

The representation scheme of the knot complement consists of two algebraic components, one corresponding to abelians, $\mathcal{R}_{ab}(S^3\backslash 3_1)$ and one for the closure of the irreducibles: $\mathcal{R}_{irr}(S^3\backslash 3_1)$. We can give an explicit description of the representation scheme as a subscheme of $\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$, with matrix-valued coordinates $(A, B) = (\rho(a), \rho(b))$. We record the value of the meridian $M = \rho(m)$ as well. The two irreducible components are

$$\mathcal{R}_{ab}(S^3\backslash 3_1) = \{A, B, M \in \text{SL}(2, \mathbb{C}) | A = M^3, B = M^2 \} \cong \text{SL}(2, \mathbb{C})$$

$$\mathcal{R}_{irr}(S^3\backslash 3_1) = \{A, B, M \in \text{SL}(2, \mathbb{C}) | \text{tr}(A) = 0, \text{tr}(B) = 1, M = AB^{-1} \} \cong T\mathbb{CP}^1 \times T\mathbb{CP}^1$$

We can stratify this space with a total of four strata: three strata corresponding to irreducible, abelian, and non-abelian reducible (unstable) representations and one strata for abelian representations with the same character as non-abelian reducibles, $\mathcal{R}_{ab}$. The strata are given by

$$\mathcal{R}_{irr} = \{A, B, M \in \text{SL}(2, \mathbb{C}) | \text{tr}(A) = 0, \text{tr}(B) = 1, M = AB^{-1}, \text{tr}(M) \neq \pm \sqrt{3} \}$$

$$\mathcal{R}_{ab} = \{A, B, M \in \text{SL}(2, \mathbb{C}) | A = M^3, B = M^2, \text{tr}(M) \neq \pm \sqrt{3} \}$$

$$\mathcal{R}_{nar} = \{A, B, M \in \text{SL}(2, \mathbb{C}) | \text{tr}(A) = 0, \text{tr}(B) = 1, \text{tr}(M) = \pm \sqrt{3}, M = AB^{-1}, A \neq M^3, B \neq M^2 \}$$

$$\mathcal{R}_{ab}^* = \{A, B, M \in \text{SL}(2, \mathbb{C}) | A = M^3, B = M^2, \text{tr}(M) = \pm \sqrt{3} \}$$

The subcheme $\mathcal{R}_{nar}$ consists of the non-abelian reducible representations. The boundaries of strata, $\partial S = S \backslash S$, are:

$$\partial \mathcal{R}_{irr} = \mathcal{R}_{nar} \cup \mathcal{R}_{ab}$$

$$\partial \mathcal{R}_{ab} = \mathcal{R}_{ab}^*$$

$$\partial \mathcal{R}_{nar} = \mathcal{R}_{ab}$$

$$\partial \mathcal{R}_{ab} = \emptyset$$

See, for instance, [HP15] for details on these representations. The longitude $L = \rho(\ell)$ satisfies $L = I$ on $\mathcal{R}_{ab}$ and $L = -M^{-6}$ on $\mathcal{R}_{irr}$. Hence, the surgery equation $M^pL^q = I$ reduces to $M^p = I$ on $\mathcal{R}_{ab}$ and $M^{-6q} = (-I)^q$ on $\mathcal{R}_{irr}$. Since the scheme consists of two smooth irreducible components, the singular locus is the intersection of these components, which we see is equal to $\mathcal{R}_{nar} \cup \mathcal{R}_{ab}^*$. For representations $\rho \in \mathcal{R}_{nar} \cup \mathcal{R}_{ab}^*$, we have $\text{tr}(M) = \pm \sqrt{3}$, so that $e^{\pm \pi/6}$ is an eigenvalue of $M$. Thus, if $M^p = I$, we must have that $p$ is a multiple of 12. So, the representation variety of the surgery will be smooth provided $12 \mid p$. We proceed to describe $\mathcal{R}(S_{p/q}^3(3_1))$ for $p$ odd and $p$ even but not a multiple of 12.

4.1.1. $\mathcal{R}(S_{p/q}^3(3_1))$ for $p$ odd. Applying the descriptions from above, we find:

$$\mathcal{R}_{ab}(S_{p/q}^3(3_1)) = \{A, B, M \in \text{SL}(2, \mathbb{C}) | A = M^3, B = M^2, M^p = I \}$$

The solutions for the matrix $M$ are $M = I$ or $M \sim \text{diag}(e^{2\pi ik/p}, e^{-2\pi ik/p}) \neq I$, with $\frac{1}{2}(|p| - 1)$ non-conjugate choices for $k$. For each choice of $k$, the variety of representations conjugate to this diagonal representation is isomorphic to $\text{SL}(2, \mathbb{C})/\mathbb{C}^\times \cong T\mathbb{CP}^1$. Overall, the abelian representations consist of one point and $\frac{1}{2}(|p| - 1)$ copies of $T\mathbb{CP}^1$.

The irreducible are given by

$$\mathcal{R}_{irr}(S_{p/q}^3(3_1)) = \{A, B, M \in \text{SL}(2, \mathbb{C}) | \text{tr}(A) = 0, \text{tr}(B) = 1, M = AB^{-1}, M^{p-6q} = (-I)^q \}$$
If \( \mu \) is an eigenvalue of \( M \), we have \( \mu^{p-6q} = (-1)^q \). We obtain \( |p - 6q| \) roots for \( \mu \), but for \( q \) odd we must exclude the case \( \mu = -1 \) and for \( q \) even we exclude \( \mu = 1 \). Indeed, for \( \mu \neq \pm 1 \), \( M \) is diagonalizable, so that \( \mu^{p-6q} = (-1)^q \). But for \( \mu = \pm 1 \), if we had \( M^{p-6q} = (-I)^q \), then in fact \( M = \pm I \) and \( A = \pm B \), which contradicts irreducibility. That is, the irreducible representations with \( \text{tr}(M) = \pm 2 \) cannot have \( M \) diagonalizable, so they do no contribute to any surgeries. This leaves \( |6q - p| - 1 \) roots. The conjugation action identifies \( \mu \) with \( \mu^{-1} \), so we count roots mod this inversion.

This yields the count \( \frac{1}{2}(6q - p) - \frac{1}{2} = \lambda_{\text{SL}(2, \mathbb{C})}(S^3_{p/q}(3\mathfrak{l})) \) classes of irreducible representations, each of which contributes a copy of a scheme isomorphic to \( \text{SL}(2, \mathbb{C})/\{ \pm 1 \} = \text{PSL}(2, \mathbb{C}) \).

### 4.1.2. \( \mathcal{R}(S^3_{p/q}(3\mathfrak{l})) \) for \( p \) even, \( 12 \nmid p \)

To determine the abelian representations, we once again consider Equation 4.1. We find that the solutions for \( M = \pm I \) or \( M \sim \text{diag}(e^{2\pi ik/p}, e^{-2\pi ik/p}) \neq \pm I \), with \( \frac{1}{2}(|p| - 2) \) non-conjugate choices for \( k \). This gives two points and \( \frac{1}{2}(|p| - 2) \) copies of \( T \mathbb{C} \mathbb{P}^1 \).

The irreducibles are given by Equation 4.2. Since \( p \) is even, \( q \) must be odd. So, letting \( \mu \) be an eigenvalue of \( M \), we have \( \mu^{p-6q} = -1 \). This gives \( \frac{1}{2}(6q - p) \) choices for \( \mu \) modulo conjugation (none of these roots are at \( \mu = \pm 1 \)), and as many copies of \( \text{PSL}(2, \mathbb{C}) \).

**Proof of Theorem 1.5.** We have calculated the representation varieties \( \mathcal{R}_{\text{irr}}(S^3_{p/q}(3\mathfrak{l})) \) in 4.1.1 – 4.1.3 for \( p \) not a multiple of 12. They are all smooth, so that we can apply Proposition 2.1 to compute \( HP_{\#} \) in these cases.

Next, we show that \( \mathcal{R}(S^3_{p/q}(3\mathfrak{l})) \) is singular for \( 12 \mid p \). To calculate \( HP_{\#}(S^3_{12k/q}(3\mathfrak{l})) \), one would need to determine the perverse sheaf on these singular spaces. We will not pursue this here, but content ourselves with providing a description of the singular schemes.

### 4.1.3. \( \mathcal{R}(S^3_{p/q}(3\mathfrak{l})) \) for \( p \neq 0 \) a multiple of 12

First, we consider the abelian representations, characterized by the equation \( M^p = I \). As before, there are two central representations, \( M = \pm I \), each contributing one point. We can also take \( M \sim \text{diag}(e^{2\pi ik/p}, e^{-2\pi ik/p}) \neq I \) with \( \frac{1}{2}(|p| - 2) \) non-conjugate choices for \( k \), but since \( p \) is a multiple of 12, if we take \( k = \pm p/12 \) or \( k = \pm 5p/12 \), then \( \text{tr}(M) = \pm \sqrt{3} \).

In that case, the abelian representation is in the closure of the irreducibles, so we will need to count this component separately. Excluding these two copies of \( T \mathbb{C} \mathbb{P}^1 \), we obtain \( \frac{1}{2}(|p| - 2) - 2 \) components isomorphic to \( T \mathbb{C} \mathbb{P}^1 \).

By similar reasoning, the previous count of \( \frac{1}{2}(6q - p) \) irreducibles now requires a correction because two of these classes of representations are actually non-abelian reducibles, not irreducibles. This yields \( \frac{1}{2}(6q - p) - 2 \) copies of \( \text{PSL}(2, \mathbb{C}) \).

For the intersection of \( \mathcal{R}(S^3_{p/q}(3\mathfrak{l})) \) with the closure of the locus of non-abelian reducibles, \( \overline{\mathcal{R}}_{\text{nar}}(S^3 \setminus 3\mathfrak{l}) \), we obtain the description

\[
\overline{\mathcal{R}}_{\text{nar}}(S^3_{12k/q}(3\mathfrak{l})) = \{ A, B, M \in \text{SL}(2, \mathbb{C}) | \text{tr}(A) = 0, \text{tr}(B) = 1, \text{tr}(M) = \pm \sqrt{3}, M = AB^{-1}, M^p = I, M^{-6} = -I \} = \overline{\mathcal{R}}_{\text{nar}}(S^3 \setminus 3\mathfrak{l})
\]

After a linear change of coordinates, this affine scheme is seen to be isomorphic to (two disjoint copies of) the 3-dimensional subscheme of \( \mathbb{C}^6 \) specified by the equations

\[
\{ \vec{v}, \vec{w} \in \mathbb{C}^3 | \vec{v} \cdot \vec{v} = \vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{w} = 1 \}
\]

When \( \vec{v} = \vec{w} \), the Jacobian only has rank 2, indicating the scheme is singular at these points. In fact, this scheme consists of two irreducible components intersecting along the stratum \( \mathcal{R}_{\text{ab}} \).
4.1.4. \( \mathcal{R}(S^3_{p/0}(3_1)) \) for \( p = 0 \). The abelians are specified by the trivial equation \( M^0 = I \). This means that all of \( \mathcal{F}_{ab} \) is included.

For the irreducibles, we have
\[
\mathcal{R}_{irr}(S^3_{3}(3_1)) = \{A,B,M \in \text{SL}(2, \mathbb{C}) \mid \text{tr}(A) = 0, \text{tr}(B) = 1, M = AB^{-1}, M^{-6} = -I \}
\]
The three conjugacy classes of solutions for \( M \) have \( \text{tr}(M) = \pm \sqrt{3} \) or \( \text{tr}(M) = 0 \). The case \( \text{tr}(M) = 0 \) contributes an irreducible representation. The cases \( \text{tr}(M) = \pm \sqrt{3} \) occur in the singular locus.

Let \( \mathcal{R}_{red}(S^3_{3}(3_1)) = \mathcal{F}_{ab}(S^3_{3}(3_1)) \cup \mathcal{F}_{nar}(S^3_{3}(3_1)) \) denote the locus of all reducible representations. Then in fact \( \mathcal{R}_{red}(S^3_{3}(3_1)) \) appears as a topological component of \( \mathcal{R}(S^3_{3}(3_1)) \). This scheme manifestly has (at least) two separate algebraic components: \( \mathcal{F}_{ab} \) and \( \mathcal{F}_{nar} \). They intersect non-trivially in \( \mathcal{F}_{irr}^* \), so that their union must be singular.

5. The character variety of \( S^3_{3}(3_1 \# 3_1) \)

The knot group of the trefoil has the presentations
\[
\pi_1(S^3_{3}(3_1)) = \langle a, b | a^3 = b^2 \rangle \\
\simeq \langle rsr = srs \rangle
\]
The character scheme is
\[
\mathcal{X}(S^3_{3}(3_1)) \cong \{(y-2)(x^2-y-1) = 0 \} \subset \mathbb{C}^2
\]
where \( x = \text{tr} \rho(r) \) and \( y = \text{tr}(rs^{-1}) \). The line \( \{y = 2\} \) is \( \mathcal{X}_{\text{red}} \) and \( \{x^2 - y = 1, y \neq 2\} \) is \( \mathcal{X}_{\text{irr}} \).

The fundamental group of the complement of the knot \( 3_1 \# 3_1 \) has the presentation
\[
\Gamma = \langle a, b, c, d | a^3 = b^2, c^1 = d^2, d = ba^{-1}c^2 \rangle
\]
where the subgroup \( \Gamma_0 \) generated by \( a \) and \( b \) corresponds to a copy of \( \pi_1(S^3_{3}(3_1)) \) and similarly the subgroup \( \Gamma_1 \) generated by \( c, d \) corresponds to the knot group of the other \( 3_1 \) summand. The relation \( a^2b^{-1} = c^2d^{-1} \) comes from setting the meridian in \( \Gamma_0 \) equal to the meridian in \( \Gamma_1 \). Consider the following closed subsets of \( \mathcal{X}(\Gamma) \),
\[
\mathcal{X}_{\text{red}} = \{ [\rho] \mid \rho \text{ is abelian} \}
\]
\[
\mathcal{X}_i = \{ [\rho] \mid \rho|_{\Gamma_{1-i}} \text{ is abelian} \}
\]
where clearly \( \mathcal{X}_{\text{red}} \subset \mathcal{X}_i \). Since the abelianization of the knot group is generated by the meridian, we have that \( \mathcal{X}_{\text{red}} = \mathcal{X}(\mathbb{Z}) \cong \mathbb{C} \), where the meridional trace is a coordinate for \( \mathbb{C} \).

**Lemma 5.1.** Let \( \mathcal{X}(\Gamma) \xrightarrow{\tau} \mathcal{X}(\Gamma_1) \) denote the natural restriction map. Then the composite \( \mathcal{X}_i \hookrightarrow \mathcal{X}(\Gamma) \xrightarrow{\tau} \mathcal{X}(\Gamma_1) \) is an isomorphism \( \mathcal{X}_i \cong \mathcal{X}(\Gamma_1) \).

**Proof.** If \( \rho|_{\Gamma_{1-i}} \) is abelian, then it is determined by its value on the meridian. But the value of \( \rho \) on the meridian is determined by its restriction to \( \Gamma_i \), since the meridian lies in the intersection \( \Gamma_0 \cap \Gamma_1 \), establishing injectivity.

For surjectivity, we observe that for any representation \( \rho \in \mathcal{X}(\Gamma_1) \), there exists an extension of \( \rho \) to a representation of \( \Gamma \) given by setting \( \rho|_{\Gamma_{1-i}} \) to be the abelian representation of \( \Gamma_{1-i} \) with the required meridional value. This lies in \( \mathcal{X}_i \) by construction. \( \square \)

Recall the following fact:

**Lemma 5.2.** [CCG+94] Let \( \rho \) be a representation of \( \pi_1(S^3\setminus K) \) with \( [\rho] \in \mathcal{X}_{\text{red}} \cap \overline{\mathcal{X}_{\text{irr}}} \). Then the following equivalent conditions hold:

- \( \Delta(\mu^2) = 0 \), where \( \Delta \) is the Alexander polynomial of \( K \) and \( \mu \) is an eigenvalue of \( \rho(m) \), for \( m \) the meridian of the knot.
- There exists a non-abelian reducible representation \( \rho' \) with the same character as \( \rho \).
The Alexander polynomial of the trefoil is the sixth cyclotomic polynomial, \( \Delta_3(t) = t^2 - t + 1 \). Thus, the above lemma guarantees non-abelian reducibles at meridional trace \( \pm \sqrt{3} \). The same holds for \( 3_1 \# 3_1 \) since \( \Delta_3(\pm \sqrt{3}) = (\Delta_3)^2 \). This allows us to establish the following proposition:

**Proposition 5.3.** Let \( X_{i,irr} = X_{i,irr} \) and let \( S = X(\Gamma) \setminus (X_0 \cup X_1) \). Then the four irreducible components of \( X(\Gamma) \) are \( X_{red}, X_{0,irr}, X_{i,irr} \) and \( S \). Moreover, these four components pairwise intersect in the same two points, corresponding to characters of non-abelian reducibles.

**Proof.** That none of the four closed sets share any irreducible components follows from the description of the intersections. If \([\rho] \in S \cap X_{0,irr} \), then by restricting to \( X(\Gamma_1) \), we see that \([\rho]|_{\Gamma_1} \in X_{red}(\Gamma_1) \cap X_{irr}(\Gamma_1) \). Thus, Lemma 5.2 implies that \([\rho] \) is one of two points in \( X_{red} \) corresponding to non-abelian reducibles. The other intersections follow similarly.

It only remains to check that each of the four pieces is in fact irreducible. From the coordinate description of \( X(S^3 \setminus S_1) \), we see that \( X_{red} = \{ y = 2 \} \) and the \( X_{i,irr} \) are equal to \( \{ x^2 - y = 1 \} \). In either case, they are isomorphic to \( \mathbb{C} \). The irreducibility of \( S \) follows from Proposition 5.4 below. \( \square \)

**Proposition 5.4.** \( S \) is an affine cubic surface with precisely two \( A_1 \) singularities at the points \( S_{sing} = \mathbb{C} \setminus S \), the two characters of non-abelian reducible representations.

**Proof.** Let \( A = \rho(a), B = \rho(b), C = \rho(c) \). If \([\rho] \in S \), then \([\rho]|_{\Gamma_1} \) is non-abelian. But since \( a^3 = b^3 = 1 \) is a central element of \( \Gamma_0 \), we must have \( A^3 = B^3 = \pm I \). However, if \( B^2 = I \), then \( B = \pm I \) and \( \rho|_{\Gamma_1} \) would be abelian. Thus, we must have \( A^3 = B^2 = -I \), and similarly \( C^3 = B^2 = -I \) and \( A, C \neq -I \). These equations are equivalent to \( tr(A) = tr(C) = 1 \) and \( tr(B) = tr(D) = 0 \). Now, since \( d = ba^-2c^2 \), we see that \( D = BAC^{-1} \). Thus, we have the inclusion

\[
S \subset \mathcal{S} = \{ [\rho] \in \mathcal{X}(F_3) \mid tr(A) = tr(C) = 1, tr(B) = tr(BAC^{-1}) = 0 \}
\]

where \( F_3 \) is the free group generated by \( a, b, c \). Also, any representation of \( F_3 \) that lies in \( \mathcal{S} \) is a representation of \( \Gamma \), so that \( \mathcal{S} \subset \mathcal{X}(\Gamma) \). Since \( S \) is open in \( \mathcal{X}(\Gamma) \), \( S \) is the union of the irreducible components meeting \( S \). Thus, \( S = \mathcal{S} \) provided \( \mathcal{S} \) is irreducible.

So, we now turn to describing the algebraic set \( \mathcal{S} \). Regarding \( \mathcal{X}(F_3) \) as the character variety of the four-holed sphere, we see that \( \mathcal{S} \) is a relative character variety; \( \mathcal{S} \) is the locus of characters of \( \pi_1(S^2 - \{ p_0, p_2, p_3, p_4 \}) \) with fixed traces along the four boundary circles. This relative character variety can be computed [FK65] to be the affine cubic hypersurface in \( \mathbb{C}^3 \) given by the equation

\[
f = x^2 + y^2 + z^2 + xyz - z - 2 = 0
\]

where \( x = tr(AB), y = tr(B^{-1}C) \) and \( z = tr(A^{-1}C) \). Furthermore, the reducible representations, which are the points in \( S \setminus S \), correspond to \( (x, y, z) = (\pm \sqrt{3}, \pm \sqrt{3}, 2) \). These are precisely the singular points of the affine cubic surface \( S \). Since the Tjurina number, \( dim \mathcal{O}_{x,y,z}/(f, \partial_x f, \partial_y f, \partial_z f) \), is equal to 1 at the singularities, they are \( A_1 \) singularities. \( \square \)

We record a calculation of the singular cohomology groups of \( S \) for use in Section 7.

**Proposition 5.5.** The singular cohomology groups of \( S \) are

\[
H^i(S; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}^2 & i = 2 \\ \mathbb{Z}^4 & i = 3 \\ 0 & i \geq 4 \end{cases}
\]

**Proof.** Let \( Q \) denote the projective closure of \( S \) inside of \( \mathbb{P}^3 \). One can check that \( Q \) is smooth at infinity, meaning that \( Q_{sm} \), the smooth locus of \( Q \), is the complement of the two singularities at \( S \).
By Theorem 4.3 in [Dim92], the homology groups of $Q$ are

$$H_*(Q; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & i = 0 \\
0 & i = 1 \\
\mathbb{Z}^5 & i = 2 \\
0 & i = 3 \\
\mathbb{Z} & i = 4 
\end{cases}$$

By Poincaré duality,

$$H_n(Q_{sm}; \mathbb{Z}) \cong H^{4-n}_c(Q_{sm}; \mathbb{Z})$$

And we can equate the compactly supported cohomology with a relative cohomology group,

$$H^*_c(Q_{sm}; \mathbb{Z}) \cong H^*(Q, Q_{sing}; \mathbb{Z})$$

which can be determined from the long exact sequence

$$\cdots \to H^n(Q, Q_{sing}; \mathbb{Z}) \to H^n(Q; \mathbb{Z}) \to H^n(Q_{sing}; \mathbb{Z}) \to \cdots$$

In particular, since $Q_{sing}$ is zero-dimensional, we see that $H_n(Q_{sm}; \mathbb{Z}) \cong H^{4-n}(Q; \mathbb{Z})$ for $n \leq 2$. And

$$\text{rk } H_3(Q_{sm}; \mathbb{Z}) = \text{rk } H^1(Q, Q_{sing}; \mathbb{Z})$$

$$= \text{rk } H^1(Q; \mathbb{Z}) + |Q_{sing}| - 1$$

So, the homology groups of $Q_{sm}$ are

$$H_*(Q_{sm}; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & i = 0 \\
0 & i = 1 \\
\mathbb{Z}^5 & i = 2 \\
\mathbb{Z} & i = 3 \\
0 & i \geq 4 
\end{cases}$$

Let $Q_\infty = Q \setminus S$. Then $S = Q_{sm} \setminus Q_\infty$. We have $Q_\infty = \{xyz = 0\} \subset \mathbb{P}^2$, which is a triangular arrangement of three lines. The normal bundle of each of these three copies of $\mathbb{P}^1$ has degree $-1$. So, a neighborhood of each sphere inside of $S$ is diffeomorphic to the $D^2$ bundle over $S^2$ with Euler number $-1$. The boundary of this neighborhood is diffeomorphic to $S^3$. Hence, the boundary of a neighborhood of $Q_\infty$, $\partial N(Q_\infty)$, is a necklace of three copies of $S^3$. We can then apply the Mayer-Vietoris sequence

$$\cdots \to H_*(\partial N(Q_\infty)) \to H_*(Q_\infty) \oplus H_*(S) \to H_*(Q_{sm}) \to \cdots$$

to compute the stated cohomology groups.

\[ \square \]

6. The A-polynomials of the square and granny knots

We wish to describe the image of the natural map $r : \mathcal{X}(\Gamma) \to \mathcal{X}(\partial(S^3 \setminus (3_1\#3_1)))$ given by restriction to the boundary torus. Coordinates on $\mathcal{X}(\partial(S^3 \setminus (3_1\#3_1))) = \mathcal{X}(T^2)$ are given by the traces of the meridian and longitude. One may consider the double branched cover $d : C^\times \times C^\times \to \mathcal{X}(T^2)$ where the coordinates on the cover are given by the eigenvalues of the meridian and longitude, $M$ and $L$. The defining polynomial for the closure of the pull-back of the image of $r$ to $C^\times \times C^\times$ is called the A-polynomial [CCG+94].

For the right-handed trefoil, the A-polynomial is $M^{-6} + L = 0$, whereas for the left-handed trefoil it is $M^6 + L = 0$ [CCG+94]. These equations define the image under $r$ of the components $\mathcal{X}_{i,irr}, \mathcal{X}_{red}$ is mapped to the line $L = 1$.

Lemma 6.1. Let $S$ be as in Proposition 5.3. Then the defining equation of the algebraic set $d^{-1}(r(S))$ in eigenvalue coordinates is $L - M^{-12} = 0$ for the granny knot (the composite of two right-handed trefoils) and $L = 1$ for the square knot (the composite of oppositely oriented trefoils).
Proof. Let \( \ell_i \) denote the longitude of the \( i \text{th} \) summand of \( 3_1 \# 3_1 \). Then, the longitude of \( 3_1 \# 3_1 \) is \( \ell = \ell_0 \ell_1 \). Also, each of the \( \ell_i \) commutes with the meridian \( \mu \) in \( \Gamma \). Since \( \rho(\mu) \) is non-central, this means that \( \rho(\ell_0) \) and \( \rho(\ell_1) \) must commute with each other. In fact, for the irreducible representations of the right-handed trefoil, we have \( \rho(\ell_i) = -\rho(m)^{-6} \) and similarly \( \rho(\ell_i) = -\rho(m)^6 \) for the left-handed trefoil.

For \( \rho \in S \), we have that \( \rho \) restricted to either summand is irreducible. So for the granny knot, we then have \( \rho(\ell) = (-\rho(m)^{-6})^2 = \rho(m)^{-12} \) and for the square knot we obtain \( \rho(\ell) = 1 \). These matrix equations give the desired eigenvalue equations. \( \square \)

Proposition 6.2. The A-polynomial of the granny knot, \( 3_1^r \# 3_1^r \), is

\[
A_{3_1^r \# 3_1^r} = (L - 1)(L + M^{-6})(L - M^{-12})
\]

The A-polynomial of the square knot, \( 3_1^l \# 3_1^l \), is

\[
A_{3_1^l \# 3_1^l} = (L - 1)(L + M^{-6})(L + M^6)
\]

Proof. The A-polynomial is a product (omitting repeated factors) of the the defining polynomials for the images of the four components of \( \mathcal{X}(\Gamma) \). Two of the components are copies of \( \mathcal{X}(3_1) \), and therefore contribute factors corresponding to the A-polynomial of right or left-handed trefoil. The reducibles give the factor of \( L - 1 \). The factor coming from the two-dimensional component \( \tilde{\mathcal{S}} \) was determined in Lemma 6.1. \( \square \)

7. Surgeries on the Granny and Square knots

In this section, we prove Theorems 1.6 and 1.7. We proceed by calculating the relevant character schemes, showing they are smooth, and then computing their singular cohomology groups so that we can apply Corollary 2.2 to write \( HP \) as the (degree shifted) singular cohomology of the character scheme.

7.1. Character scheme of a composite knot. First, we establish a general procedure for computing the (set-theoretic) characters of the exterior of a composite knot. Although the character variety of \( 3_1 \# 3_1 \) was computed in Section 5, the description given here will be particularly amenable for computing the character varieties of the surgeries. The description from Section 5 will also be useful.

Let \( K_1 \) and \( K_2 \) be two knots in \( S^3 \) and set \( K = K_1 \# K_2, M_i = S^3 \setminus K_i, M = S^3 \setminus K \). We have the following pushout diagram of spaces:

\[
\begin{array}{ccc}
M & \longrightarrow & M_1 \\
\uparrow & & \uparrow \iota_1 \\
M_2 & \leftarrow & S^1 \\
\end{array}
\]

where \( i_j(S^1) = m_j, \) a meridian for \( K_j, j = 1, 2 \). By the Van Kampen theorem, we have the pushout diagram of groups:

\[
\begin{array}{ccc}
\pi_1(M) & \longrightarrow & \pi_1(M_1) \\
\uparrow & & \uparrow \\
\pi_1(M_2) & \leftarrow & \pi_1(S^1) \\
\end{array}
\]

That is, \( \pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)/(m_1 = m_2) \). We have a pullback diagram of representation spaces:

\[
\begin{array}{ccc}
\mathcal{R}(M) & \longrightarrow & \mathcal{R}(M_1) \\
\downarrow & & \downarrow \iota_1 \\
\mathcal{R}(M_2) & \rightarrow & \mathcal{R}(S^1) \\
\end{array}
\]
To analyze $\mathcal{X}(M) = \mathcal{R}(M)/G$, we can compare it to a simpler object: the fiber product of the character schemes $\mathcal{X}(M_1) \times_{\mathcal{X}(S^1)} \mathcal{X}(M_2)$. We have the diagram

$$
\begin{array}{ccc}
\mathcal{X}(M) & \xrightarrow{\varphi} & \mathcal{X}(M_1) \\
\downarrow & & \downarrow \\
\mathcal{X}(M_1) \times_{\mathcal{X}(S^1)} \mathcal{X}(M_2) & \xrightarrow{\tau_1} & \mathcal{X}(M_1) \\
\downarrow & & \downarrow \\
\mathcal{X}(M_2) & \xrightarrow{\tau_2} & \mathcal{X}(S^1)
\end{array}
$$

where $\tau_1([\rho_1]) = \text{tr}(\rho_1(m_1))$.

### 7.1.1 Pullbacks and quotients
In order to understand the character scheme of $M$ from the fiber product of the character schemes of $M_1$ and $M_2$, we must determine the pre-images of points under $\varphi$. We establish the following lemma:

**Lemma 7.1.** Let $\varphi : \mathcal{X}(M) \rightarrow \mathcal{X}(M_1) \times_{\mathcal{X}(S^1)} \mathcal{X}(M_2)$ denote the natural map as above. Then for any $p = ([\rho_1], [\rho_2]) \in \mathcal{X}(M_1) \times_{\mathcal{X}(S^1)} \mathcal{X}(M_2)$, we have

$$\varphi^{-1}(p) \cong \text{Stab}(m)/(\text{Stab}(\rho_1), \text{Stab}(\rho_2))$$

where $m = r_1(\rho_1) = r_2(\rho_2)$.

**Proof.** The pre-image of $p$ in $\mathcal{R}(M_1) \times \mathcal{R}(M_2)$ is $\text{Orb}(\rho_1) \times \text{Orb}(\rho_2)$. The pair $(\rho_1, \rho_2)$ is a point here that is also in $\mathcal{R}(M_1) \times_{\mathcal{R}(S^1)} \mathcal{R}(M_2)$. All other such points can be obtained by using the action of $\text{Stab}(m)$ on each factor, or else using the diagonal action of $G$. This gives the set

$$\left(\mathcal{R}(M_1) \times_{\mathcal{R}(S^1)} \mathcal{R}(M_2)\right) \cap \left(\text{Orb}(\rho_1) \times \text{Orb}(\rho_2)\right) = G \cdot (\text{Stab}(m) \cdot \rho_1 \times \text{Stab}(m) \cdot \rho_2)$$

Reducing modulo the diagonal action of $G$,

$$G \cdot (\text{Stab}(m) \cdot \rho_1 \times \rho_2)/G = \text{Stab}(m)/(\text{Stab}(\rho_1), \text{Stab}(\rho_2))$$

Thus, $\varphi^{-1}(p) \cong \text{Stab}(m)/(\text{Stab}(\rho_1), \text{Stab}(\rho_2))$. \qed

### 7.2 Irreducible representations in the character scheme of a composite knot
To determine the locus of irreducible representations $\mathcal{X}_{\text{irr}}(M)$, we first describe $\mathcal{X}(M_1) \times_{\mathcal{X}(S^1)} \mathcal{X}(M_2)$ and then use Lemma 7.1 to understand the fibers of $\varphi$ over the various components.

Recall that $\mathcal{X}(M)$ has a stratification $\mathcal{X}_{\text{nar}} \subset \mathcal{X}_{\text{red}} \subset \mathcal{X}$, where $\mathcal{X}_{\text{nar}}$ is the locus of characters of non-abelian reducible representations. The complement $\mathcal{X}_{\text{irr}} = \mathcal{X} \setminus \mathcal{X}_{\text{red}}$ is the locus of irreducibles. The scheme $\mathcal{X}_{\text{nar}}$ can be identified from Lemma 4.2. The characters of non-abelian reducibles are also the characters of abelian reducibles. That is, every reducible character has an associated orbit of abelian representations, but for those characters in $\mathcal{X}_{\text{nar}}$, there is an additional orbit corresponding to non-abelian reducible representations.

Taking the product stratification on $\mathcal{X}(M_1) \times_{\mathcal{X}(S^1)} \mathcal{X}(M_2)$ gives nine different strata of six essentially different types. The following proposition states which strata intersect the image $\varphi(\mathcal{X}_{\text{irr}}(M))$ and also identifies the set of irreducible representations in the fiber of $\varphi$ over a point in a given stratum.

**Proposition 7.2.** Using the previously established notation, $\varphi(\mathcal{X}_{\text{irr}}(M))$ consists of the following pieces

- $\mathcal{X}_{\text{irr}}(M_1) \times_{\mathcal{X}(S^1)} \mathcal{X}_{\text{irr}}(M_2)$
- $\mathcal{X}_{\text{irr}}(M_1)$
- $\mathcal{X}_{\text{nar}}(M_1) \times_{\mathcal{X}(S^1)} \mathcal{X}_{\text{nar}}(M_2)$
The fibers of \( \varphi \) are copies of:

- \( \mathbb{C}^\times \) over points in \( \mathcal{X}_{irr}(M_1) \times \mathcal{X}(S^1) \mathcal{X}_{irr}(M_2) \) with meridional eigenvalue \( \mu \neq \pm 1 \).
- \( \mathbb{C} \) over points in \( \mathcal{X}_{irr}(M_1) \times \mathcal{X}(S^1) \mathcal{X}_{irr}(M_2) \) with meridional eigenvalue \( \mu = \pm 1 \).
- A single point over points in \( \mathcal{X}_{irr}(M_1) \) with \( \Delta(\mu^2) \neq 0 \).
- \( \mathbb{C} \) over points in \( \mathcal{X}_{irr}(M_1) \) with \( \Delta(\mu^2) = 0 \).
- \( \mathbb{C}^\times \) over points in \( \mathcal{X}_{nar}(M_1) \times \mathcal{X}(S^1) \mathcal{X}_{nar}(M_2) \).

**Proof.** First, we identify the copy of \( \mathcal{X}_{irr}(M_1) \) that appears in \( \mathcal{X}(M_1) \times \mathcal{X}(S^1) \mathcal{X}(M_2) \). A reducible character is the character of an abelian representation, and the meridian generates the abelianization of the knot group. Thus, the isomorphism \( H_1(M_1) \cong \pi_1(S^1) \), where \( S^1 \) a meridional circle, yields an isomorphism \( \mathcal{X}_{red}(M_1) \cong \mathcal{X}(S^1) \). And taking fiber products, \( \mathcal{X}_{irr}(M_1) \times \mathcal{X}(S^1) \mathcal{X}_{red}(M_2) \cong \mathcal{X}_{irr}(M_1) \).

Now, we show that image of \( \varphi \) consists of the stated pieces. Indeed, the only strata not included in the list are contained in \( \mathcal{X}_{red}(M_1) \times \mathcal{X}(S^1) \mathcal{X}_{red}(M_2) \) \( \setminus (\mathcal{X}_{nar}(M_1) \times \mathcal{X}(S^1) \mathcal{X}_{nar}(M_2)) \). These correspond to representations of the form \( \rho_1 \ast \rho_2 \) where \( \rho_1 \) is abelian and \( \rho_2 \) is reducible. However, for an abelian representation, \( \text{im}(\rho_1) = \text{im}(\rho_1|_{M_1}) \) since the meridian \( m_1 \) generates the abelianization of \( \pi_1(M_1) \). Thus, since the \( \rho_1 \) agree on \( M_1 \), we see that \( \text{im}(\rho_1 \ast \rho_2) = \text{im}(\rho_2) \), so that the composite representation is also reducible. Thus, none of these pairings provide irreducible representations.

For \( p = ([\rho_1], [\rho_2]) \in \mathcal{X}(M_1) \times \mathcal{X}(S^1) \mathcal{X}(M_2) \), if both \( [\rho_1], [\rho_2] \in \mathcal{X}_{irr} \), then \( \text{Stab}(\rho_1) = \{ \pm 1 \} \). Furthermore, \( r_1(\rho_1) \) is an abelian, non-central representation (if \( \rho_1(m) = \pm 1 \), then the entire representation is central because \( M_1 \) normally generates \( \pi_1(M_1) \)). Thus, \( \text{Stab}(r_1(\rho_1)) \cong \mathbb{C}^\times \) for meridional trace not \( \pm 2 \), and \( \text{Stab}(r_1(\rho_1)) \cong \mathbb{C} \times \mathbb{Z}/2 \) otherwise. So, \( \varphi^{-1}(p) \cong \mathbb{C}^\times /\{ \pm 1 \} \cong \mathbb{C}^\times \) or \( \varphi^{-1}(p) \cong \mathbb{C} \) by Lemma 7.1.

If \([\rho_1]\) is irreducible but \([\rho_2]\) is reducible, then we can find an abelian lift \( \rho_2 \), so that \( \text{Stab}(\rho_2) = \text{Stab}(r_2(\rho_1)) \), and the fiber \( \varphi^{-1}(p) \) is a point. For a non-abelian lift of \( \rho_2 \), we have \( \text{Stab}(\rho_2) \) is trivial, and the stabilizer of the meridian must be \( \mathbb{C}^\times \) (it cannot be \( \mathbb{C} \) because the trace of the meridian cannot be \( \pm 1 \) for a non-abelian reducible, since \( \Delta(\pm 1) \neq 0 \)). The abelian lies in the closure of the orbit of non-abelian reducibles, so that \( \varphi^{-1}(p) = \mathbb{C} \) for such a point.

If both are reducible and at least one is abelian, then the overall representation is reducible. If both are non-abelian reducibles, then the stabilizers of each representation are trivial and the stabilizer of the meridian is \( \mathbb{C}^\times \), giving that the fiber of \( \varphi \) is \( \mathbb{C}^\times \). However, not all of these representations are irreducible. We have that \( \text{im}(\rho_1) \subset B_1 \), for \( B_1, B_2 \) Borel subgroups. For some \( d \in \text{Stab}(r_1(\rho_1)) \), the composite representation corresponding to \( d \) has image generated by \( \text{im}(\rho_1), d^{-1} \text{im}(\rho_2) d \). If this image were contained in some Borel subgroup \( B \), then \( \text{im}(\rho_1) \) would be contained in two Borel subgroups, so either it is contained in a diagonal subgroup (but then \( \rho_1 \) is abelian), or else \( B = B_1 \). Then, we have \( d^{-1} \text{im}(\rho_2) d \subset B_1 \), and so by the same argument we conclude \( B_1 = d^{-1}B_2d \). Thus, \( d \in \text{Stab}(B_2) \), which is trivial in \( G^{ad} \). Hence, precisely one point in \( \text{Stab}(r_1(\rho_1)) \) corresponds to a reducible — the rest are irreducible. So, the irreducibles in \( \varphi^{-1}(p) \) form a copy of \( \mathbb{C}^\times \setminus \{ 1 \} \).

### 7.3. Character scheme of a connected sum of two trefoils

We now focus on the case when \( K_1 = K_2 = 3_1^r \). The character scheme of the trefoil can be described as a plane curve:

\[
\mathcal{X}(3_1^r) \cong \{(y-2)(x^2-y-1) = 0\} \subset \mathbb{C}^2
\]

where \( x \) is the trace of the meridian. In terms of the Wirtinger presentation, we have \( x = \text{tr}(\rho(r)) = \text{tr}(\rho(s)) \) and \( y = \text{tr}(rs^{-1}) \). The line \( \{ y=2 \} \) is \( \mathcal{X}_{red} \) and \( \{ x^2 - y = 1, y \neq 2 \} \) is \( \mathcal{X}_{irr} \). The map \( r_1 \) is projection onto the \( x \) coordinate. The longitude for \( 3_1^r \) is \( \ell = sr^2sr^{-4} \), and its trace in the \( x, y \) coordinates is given by the polynomial

\[
L(x,y) = x^6y - 2x^6 - x^4y^2 - 2x^4y + 8x^4 + 2x^2y^2 + x^2y - 10x^2 + 2
\]

The restriction of \( L(x, y) \) to \( y = 2 \) is the constant function \( 2 \), as expected. On this component, \( \rho(\ell) = 1 \).

The restriction to of \( L(x, y) \) to \( y = x^2 - 1 \) is \( -x^6 + 6x^4 - 9x^2 + 2 \), which can be deduced from the fact that for the irreducible representations, we have \( \rho(\ell) = -\rho(m)^{-6} \).
The Alexander polynomial has roots that are primitive 6th roots of unity. So, non-abelian reducibles occur at the points $(\pm\sqrt{3}, 2) \in \mathcal{X}_{\text{red}}$. Observe that this is precisely $\mathcal{F}_{\text{irr}} - \mathcal{X}_{\text{irr}}$.

The fiber product of the character varieties over the meridional trace map is

$$\mathcal{X}(3_1) \times_C \mathcal{X}(3_1) \cong \{(y - 2)(x^2 - y - 1) = 0, (z - 2)(x^2 - z - 1) = 0\} \subset \mathbb{C}^3$$

Applying Proposition 7.2, we have the following explicit descriptions of the fibers of $\varphi$ over various points in $\mathcal{X}_{\text{irr}}(K_1 \# K_2)$:

- $\mathcal{X}_{\text{irr}} \times_C \mathcal{X}_{\text{irr}} = \{x^2 - y - 1 = 0, x^2 - z - 1 = 0, y \neq 2, z \neq 2\}$. The fibers of $\varphi$ are $\mathbb{C}^\times$ unless $x = \pm 2$, in which case they are $\mathbb{C}$.
- $\mathcal{X}_{\text{irr}}(M_1) = \{z = 2, x^2 - y - 1 = 0, y \neq 2\}$. Note that since $y \neq 2$, we have $x \neq \pm \sqrt{3}$ and $\Delta(m^2) \neq 0$. So, the fibers of $\varphi$ are just points. The same holds for $\mathcal{X}_{\text{irr}}(M_2) = \{y = 2, x^2 - z - 1 = 0, z \neq 2\}$.
- $\mathcal{X}_{\text{nar}} \times_C \mathcal{X}_{\text{nar}} = \{(\pm \sqrt{3}, 2, 2)\}$. The fibers of $\varphi$ are $\mathbb{C}^\times - \{1\}$.

Remark 7.3. To compare this description with that of Proposition 5.3, we see that

- $\varphi^{-1}(\mathcal{X}_{\text{irr}} \times_C \mathcal{X}_{\text{irr}} \cup \mathcal{X}_{\text{nar}} \times_C \mathcal{X}_{\text{nar}}) = S$
- $\mathcal{X}_{\text{irr}}(M_i) = \mathcal{X}_{\text{i, irr}}$

Since $\pi_1(3_1^* \# 3_1^*)$ and $\pi_1(3_1^* \# 3_1^*)$ are isomorphic, the same description applies to $\mathcal{X}_{\text{irr}}(3_1^* \# 3_1^*)$.

7.4. Character scheme for granny knot surgeries. Let $G$ denote the connected sum of two right-handed trefoils, and $S^3_{p/q}(G)$ the $p/q$ surgery. We have the following description of $\mathcal{X}_{\text{irr}}(S^3_{p/q}(G))$.

**Proposition 7.4.** $\mathcal{X}_{\text{irr}}(S^3_{p/q}(G))$ consists of $2\lambda_{\text{SL}(2, \mathbb{C})}(S^3_{p/q}(3_1))$ points and

- $\lambda_{\text{SL}(2, \mathbb{C})}(S^3_{p/q}(3_1))$ copies of $\mathbb{C}^\times$ when $p$ is odd
- $\lambda_{\text{SL}(2, \mathbb{C})}(S^3_{p/q}(3_1))$ copies of $\mathbb{C}^\times$ when $p$ is even, $p \neq 12k$
- $\lambda_{\text{SL}(2, \mathbb{C})}(S^3_{p/q}(3_1)) - 3$ copies of $\mathbb{C}^\times$ when $p = 12k, p/q \neq 12$
- $2$ copies of $\mathbb{C}^\times - \{1\}$ when $p = 12k, p/q \neq 12$
- $S = \varphi^{-1}(\mathcal{X}_{\text{nar}} \times_C \mathcal{X}_{\text{nar}} \cup \mathcal{X}_{\text{irr}} \times_C \mathcal{X}_{\text{irr}})$ when $p/q = 12$

We will describe $\mathcal{X}_{\text{irr}}(S^3_{p/q}(G))$ as a closed subscheme of $\mathcal{X}_{\text{irr}}(S^3\backslash G)$. First, we have the following lemma.

**Lemma 7.5.** Let $\varphi : \mathcal{X}_{\text{irr}}(S^3\backslash G) \rightarrow \mathcal{X}(S^3\backslash 3_1) \times_C \mathcal{X}(S^3\backslash 3_1)$ denote the map to the fiber product over the meridional trace. Then

$$\mathcal{X}_{\text{irr}}(S^3_{p/q}(G)) = \varphi^{-1}(\varphi(\mathcal{X}_{\text{irr}}(S^3_{p/q}(G))))$$

**Proof.** A character $[\rho] = [(\rho_1, \rho_2)] \in \mathcal{X}_{\text{irr}}(S^3\backslash G)$ is in the character scheme for the $p/q$ surgery if the surgery equation $\rho(m^p \ell^q) = I$ is satisfied. For a composite knot, the longitude $\ell$ is the product of the two longitudes for the constituent knots. Thus, the surgery equation is

$$\rho_1(m)p^p (\rho_1(\ell_1)\rho_2(\ell_2))^q = I$$

If $[\rho'] \in \varphi^{-1}(\varphi([\rho]))$, then it is of the form $[\rho'] = [(\rho_1, g^{-1}\rho_2g)]$ for some $g \in \text{Stab}(\rho(m))$. For an irreducible representation, we cannot have $\rho(m) = \pm I$. Thus, $\text{Stab}(\rho(m))$ is one-dimensional. Furthermore, since $\ell_2$ and $m$ commute, we must have $\text{Stab}(\rho(m)) \subset \text{Stab}(\rho(\ell_2))$. Therefore, $g^{-1}\rho_2(\ell_2)g = \rho_2(\ell_2)$, verifying the surgery equation for $[\rho']$. \qed

Thanks to this lemma, it suffices to describe $\varphi(\mathcal{X}_{\text{irr}}(S^3_{p/q}(G)))$. We consider each of the three different types of points in $\mathcal{X}_{\text{irr}}(3_1) \times_C \mathcal{X}_{\text{irr}}(3_1)$ separately.

**Lemma 7.6.** The locus of characters of $\pi_1(S^3_{p/q}(G))$ that restrict to an irreducible in $\pi_1(S^3\backslash K_1)$ and an abelian in $\pi_1(S^3\backslash K_2)$ is $\mathcal{X}_{\text{irr}}(S^3_{p/q}(G)) \cap \varphi^{-1}(\mathcal{X}_{\text{irr}}(M_i))$. This space consists of $\lambda_{\text{SL}(2, \mathbb{C})}(S^3_{p/q}(3_1))$ points.
Lemma 7.7. The set of characters that restrict to an irreducible representation on both factors is given by
\[ X = \varphi(\mathcal{X}^r(S^3_{p/q}(G))) \cap \mathcal{X}^r(M_1) = \lambda_{\text{SL}(2,\mathbb{C})}(S^3_{p/q}(3)) \]
Since the fibers of \( \varphi \) over these types of characters are just points, we obtain the result. \( \square \)

Lemma 7.8. The set of irreducible representations formed from a composite of non-abelian reducible representations is
\[ \mathcal{X}^r(S^3_{p/q}(G)) \cap \varphi^{-1}(\mathcal{X}^r \times \mathcal{X}^r) = \begin{cases} 2 \text{ copies of } \mathbb{C}^\times - \{1\} & \text{if } p = 12k \\ \emptyset & \text{else} \end{cases} \]

Proof. For irreducible representations of \( \pi_1(S^3 \setminus 3) \), \( \rho(\ell) \) is determined by \( \rho(m) \). In fact, we have \( \rho(\ell) = -\rho(m)^{-6} \). For a point \( \varphi([\rho]) = ([\rho_1], [\rho_2]) \in \mathcal{X}^r \times \mathcal{X}^r \), we have \( \rho_1(m) = \rho_2(m) \), so that \( \rho_1(\ell_1) = \rho_2(\ell_2) \). Thus,
\[ \rho(m^p\ell^q) = \rho_1(m^p\ell^q) \]
For \( p \) odd, the equation \( \rho_1(m^p\ell^{2q}) = I \) is just the defining equation for \( p/2q \) surgery on the trefoil. Thus, we obtain \( \lambda_{\text{SL}(2,\mathbb{C})}(S^3_{p/2q}(3)) \) points. None of these occur at meridional trace \( \pm 2 \), so that the fiber of \( \varphi \) is a copy of \( \mathbb{C}^\times \) for all of these points.

For \( p \) even, \( p \neq 12k \), the surgery equation
\[ \rho(m)^{p-12q} = I \]
has an even exponent. Thus, we obtain
\[ \frac{1}{2}(12q - p) \]
distinct characters, where the \(-2\) term serves to discount the roots at \( \rho(m) = \pm 1 \). For \( p = 12k, p/q \neq 12 \), two of the characters in this count occur at meridional trace \( \pm \sqrt{3} \), so we subtract 2 in this case. Again, all of the fibers of \( \varphi \) are \( \mathbb{C}^\times \).

For \( p/q = 12 \), the surgery equation is trivial, so that every representation of this form provides a representation of the surgery. \( \square \)

Remark 7.9. 12 surgery on the granny knot yields a Seifert fiber space fibered over the orbifold base \( S^2(2,2,3,3) \) [KT90]. Thus,
\[ \pi_1(S^3_{12}(G)) \cong \langle a, b, c | a^3 = b^3 = c^2 = (abc)^{-2} \rangle \]
7.5. Character scheme for square knot surgeries. Let $S$ denote the square knot, a connected sum of two mirror trefoils, and $S_{p/q}^3(S)$ the $p/q$ surgery. We have the following description of $\mathcal{X}_{Irr}(S_{p/q}^3(S))$.

**Proposition 7.10.** The character scheme $\mathcal{X}_{Irr}(S_{p/q}^3(S))$ consists of $\lambda_{SL(2,\mathbb{C})}(S_{p/q}^3(3^1)) + \lambda_{SL(2,\mathbb{C})}(S_{p/q}^3(3^1))$ points and

- $\frac{1}{2}|p| - \frac{1}{2}$ copies of $\mathbb{C}^\times$ when $p$ is odd
- $\frac{1}{2}|p| - 1$ copies of $\mathbb{C}^\times$ when $p$ is even, $p \neq 12k$
- $\frac{1}{2}|p| - 3$ copies of $\mathbb{C}^\times$ when $p = 12k 
eq 0$
- 2 copies of $\mathbb{C}^\times - \{1\}$ when $p = 12k 
eq 0$
- $S = \varphi^{-1}(\mathcal{X}_{narrow} \times \mathcal{X}_{narrow} \cup \mathcal{X}_{irr} \times \mathcal{X}_{irr})$ when $p = 0$

**Proof.** The proof is analogous to that of Proposition 7.4. The essential difference is that we also need to consider the representations of the left-handed trefoil. Since $S_{p/q}^3(3^1) \cong S_{-p/q}^3(3^1)$, we can relate the Casson invariants by

$$\lambda_{SL(2,\mathbb{C})}(S_{p/q}^3(3^1)) = \lambda_{SL(2,\mathbb{C})}(S_{-p/q}^3(3^1))$$

Thus, the intersection of $\mathcal{X}_{Irr}(S_{p/q}^3(S))$ with the two copies of $\mathcal{X}_{Irr}(3^1)$ give contributions of $\lambda_{SL(2,\mathbb{C})}(S_{p/q}^3(3^1))$ and $\lambda_{SL(2,\mathbb{C})}(S_{-p/q}^3(3^1))$ points, depending on whether the copy of $\mathcal{X}_{Irr}(3^1)$ corresponds to the right or left-handed trefoil.

For irreducible representations of the right-handed trefoil, we have $\rho_1(\ell_1) = -\rho_1(m)^{-6}$, whereas for the left-handed trefoil we have $\rho_2(\ell_2) = -\rho_2(m)^6$. So, for a representation of the composite that restricts to irreducibles on either factor, we find that $\rho(\ell) = \rho(\ell_1\ell_2) = I$. The equation for $p/q$ surgery reduces to

$$\rho(m)^p = I$$

Throwing away the solutions $\rho(m) = \pm I$ and counting solutions up to conjugacy (i.e. dividing by the equivalence $\rho(m) \sim \rho(m)^{-1}$), we find $\frac{1}{2}|p| - \frac{1}{2}$ solutions for $p$ odd, and $\frac{1}{2}|p| - 1$ solutions for $p$ even, $p \neq 12k$. For $p = 12k \neq 0$, we omit the two solutions with $\text{tr}(\rho(m)) = \pm \sqrt{3}$, as these correspond to non-abelian reducible representations rather than irreducibles. The case of irreducibles formed from the composite of non-abelian reducible representations, which only occurs when $p = 12k$, is the same as in Lemma 7.8. When $p = 0$, the surgery equation is trivial, and we have the same situation as for $p = 12$ for the granny knot. \hfill $\square$

**Remark 7.11.** 0 surgery on the square knot yields a Seifert fiber space fibered over the orbifold base $S^2(-2,2,3,3)$ [KT90]. Thus,

$$\pi_1(S^3_0(S)) \cong \langle a, b, c | a^3 = b^3 = c^2 = (abc)^2 \rangle$$


**Proposition 7.12.** Let $G$ and $S$ denote the granny and square knots, respectively. The schemes $\mathcal{X}_{Irr}(S_{p/q}^3(G))$ and $\mathcal{X}_{Irr}(S_{p/q}^3(S))$ are smooth schemes for all $p$ and $q$.

**Proof.** The sets of complex points of these schemes were computed in the previous section. They consisted of components of dimensions zero, one, and, in the cases of $S^3_0(G)$ and $S^3_0(S)$, two. To establish the smoothness of the character scheme near some irreducible representation $\rho$, we must show that the local dimension of the set of complex points at $\rho$ equals the dimension of the tangent space to the scheme at $\rho$. Recall that for an irreducible representation $\rho$ the tangent space is computed by $T_\rho \mathcal{X}_{Irr}(\Gamma) = H^1(\Gamma, \text{ad} \rho)$. Thus, the claim follows from the calculation of these $H^1$ groups in Lemma 7.14 below. \hfill $\square$
Lemma 7.13. Let $\rho$ be an irreducible representation of $\pi_1(S^3 \setminus (3_1 \# 3_1))$ (where $3_1 \# 3_1$ is either the square or granny knot, which have isomorphic fundamental groups). Let $\rho_1$ and $\rho_2$ be the restrictions of $\rho$ to each of the two copies of $\pi_1(S^3 \setminus 3_1)$. Then,

$$\dim H^1(\pi_1(S^3 \setminus (3_1 \# 3_1)), \text{ad } \rho) = \begin{cases} 
2 & \text{if neither of the } \rho_i \text{ are abelian} \\
1 & \text{if either of the } \rho_i \text{ are abelian}
\end{cases}$$

Proof. We can compute $H^1(\pi_1(S^3 \setminus (3_1 \# 3_1)), \text{ad } \rho)$ (we will suppress the $\pi_1$ from this notation without confusion, as all spaces in consideration are aspherical) from the following portion of the Mayer-Vietoris sequence:

$$0 \to H^0(S^3 \setminus 3_1, \text{ad } \rho_1) \oplus H^0(S^3 \setminus 3_1, \text{ad } \rho_2) \to H^0(S^1, \text{ad } \rho) \to H^1(S^3 \setminus (3_1 \# 3_1), \text{ad } \rho) \to \cdots \label{7.1}$$

The $\rho_i$ are the restrictions of $\rho$ to the two copies of $S^3 \setminus 3_1$, and the $S^1$ refers to the meridional annulus along which the connected sum operation is performed. Technically, $\rho$ restricts to the complement of the meridional annulus inside of $S^3 \setminus 3_1$, but since removing a subset of the boundary of a manifold does not change its homotopy type, this is homotopy equivalent to $S^3 \setminus 3_1$ so we ignore the distinction.

Observe that $H^1(S^1, \text{ad } \rho) \cong H^0(S^1, \text{ad } \rho) \cong \mathbb{C}$. The first isomorphism follows from Poincaré duality. The second follows from the fact that since $\rho$ is an irreducible representation of $\pi_1(S^3 \setminus (3_1 \# 3_1))$, it restricts to a non-central abelian representation on the meridian and the invariants of such a representation are 1 dimensional subspace of $\text{ad } \rho$.

The last map in \eqref{7.1} is the sum of two maps, each of the form $H^1(S^3 \setminus 3_1, \text{ad } \rho_1) \to H^1(S^1, \text{ad } \rho)$. When $\rho_1$ is irreducible, this is the derivative at $[\rho_1]$ of the natural map $\mathcal{X}_{irr}(S^3 \setminus 3_1), \text{ad } \rho_1) \to \mathcal{X}(S^1, \text{ad } \rho)$, where $S^1$ refers to the meridional circle. From our description of $\mathcal{X}_{irr}(S^3 \setminus 3_1)$ as a plane curve, we observe that the meridional trace map is non-singular at all points. Thus, the map on tangent spaces is surjective.

We now consider the case when the $\rho_i$ are both irreducible or both non-abelian reducible. In this case, $H^0(S^3 \setminus 3_1, \text{ad } \rho_i) = 0$. When $\rho_i$ is an irreducible representation, we observe that $\dim H^1(S^3 \setminus 3_1, \text{ad } \rho_i) = 1$ because the character scheme is smooth of dimension 1. When $\rho_i$ is a non-abelian reducible, we can compute $\dim H^1(S^3 \setminus 3_1, \text{ad } \rho_i) = 1$ directly, as there are only finitely many non-abelian reducible representations up to conjugacy. From this data, \eqref{7.1} yields $\dim H^1(S^3 \setminus (3_1 \# 3_1), \text{ad } \rho) = 2$.

When $\rho_1$ is abelian and $\rho_2$ is irreducible, $H^0(S^3 \setminus 3_1, \text{ad } \rho_2) = 0$ and the map $H^0(S^3 \setminus 3_1, \text{ad } \rho_1) \to H^0(S^1, \text{ad } \rho)$ at the start of \eqref{7.1} is an isomorphism. For an abelian representation, $\dim H^1(S^3 \setminus 3_1, \text{ad } \rho_1) = 1$. Thus, we compute $\dim H^1(S^3 \setminus (3_1 \# 3_1), \text{ad } \rho) = 1$.

Lemma 7.14. Let $G$ and $S$ denote the square and granny knots (and let $3_1 \# 3_1$ denote either). Let $\rho$ be an irreducible representation of $\pi_1(S^3_{p/q}(3_1 \# 3_1))$. Let $\rho_1$ and $\rho_2$ be the restrictions of $\rho$ to each of the two copies of $\pi_1(S^3_{p/Q}(3_1 \# 3_1))$. Then,

$$\dim H^1(\pi_1(S^3_{p/Q}(3_1 \# 3_1)), \text{ad } \rho) = \begin{cases} 
2 & \text{if both of the } \rho_i \text{ are non-abelian and } p/q=12 \text{ for the granny knot or } p/q=0 \text{ for the square knot} \\
1 & \text{if both of the } \rho_i \text{ are irreducible and we are not in the above case} \\
0 & \text{if either of the } \rho_i \text{ are abelian}
\end{cases}$$

Proof. We can compute $H^1(S^3_{p/Q}(3_1 \# 3_1), \text{ad } \rho)$ from the following Mayer-Vietoris sequence:

$$0 \to H^1(S^3_{p/Q}(3_1 \# 3_1), \text{ad } \rho) \to H^1(S^3 \setminus (3_1 \# 3_1), \text{ad } \rho) \oplus H^1(D^2 \times S^1, \text{ad } \rho) \to H^1(T^2, \text{ad } \rho) \to \cdots \label{7.2}$$

Since $\rho$ must restrict to a non-central abelian representation on the boundary torus, we have $H^2(T^2, \text{ad } \rho) \cong H^0(T^2, \text{ad } \rho) \cong \mathbb{C}$. From the Euler characteristic, we compute $\dim H^1(T^2, \text{ad } \rho) = 2$. Similarly, $\rho$ restricts to a non-central abelian representation on the solid torus (if it sent the core of the solid torus to...
a central element, then in fact $\rho$ would be central on the entire boundary torus, and in particular on the meridian). So, $\dim H^1(D^2 \times S^1, \text{ad}\, \rho) = 1$.

We claim that $r$ has rank 1 when $p/q = 12$ for the granny knot and $p/q = 0$ for the square knot and neither of the $\rho_i$ are abelian representations, and in all other cases, $r$ has rank 2.

Let $[m]$ and $[\ell]$ denote the standard generators of $H^1(T^2, \text{ad}\, \rho)$. The image of the map $H^1(D^2 \times S^1, \text{ad}\, \rho) \to H^1(T^2, \text{ad}\, \rho)$ is the span of $-q[m] + p[\ell]$. Thus, the rank of $r$ is 2 unless the image of $H^1(S^3 \setminus (3_1 \# 3_1), \text{ad}\, \rho) \to H^1(T^2, \text{ad}\, \rho)$ is precisely the span of $-q[m] + p[\ell]$, in which case it is 1.

We can identify the map $H^1(S^3 \setminus (3_1 \# 3_1), \text{ad}\, \rho) \to H^1(T^2, \text{ad}\, \rho)$ with the derivative at $[\rho]$ of the restriction map $\mathcal{X}_{\text{irr}}(S^3 \setminus (3_1 \# 3_1)) \to \mathcal{X}(\partial S^3 \setminus (3_1 \# 3_1))$. The image of this map is essentially the A-curve of the knot. Thus, we see that the rank of $r$ is 2 provided that the A-curve is transverse to the surgery curve $M^p L^q = 1$, where $M$ and $L$ are the meridional/longitudinal eigenvalues.

Recall our calculation of the A-polynomials from Section 6,

$$A_{3_1 \# 3_1}(M, L) = (L-1)(L + M^{-6})(L - M^{-12})$$

$$A_{3_1 \# 3_1}(M, L) = (L-1)(L + M^{-6})(L + M^6)$$

A factor of $L-1$ comes from the reducibles. The factor of $L + M^{-6}$ (which is the A-polynomial of the right-handed trefoil) comes from representations that are irreducible on a $3_1'$ summand and abelian on the other summand. Similarly, $L + M^6$ is the A-polynomial of the left-handed trefoil. The last factors come the composites of two non-abelian representations. For such representations of the the granny knot, we have $L_1 = L_2 = -M^{-6}$ and $L = L_1L_2$, so that $L = M^{-12}$. For the square knot, $L_1 = L_2^{-1}$, so that this component is mapped to the line $L = 1$.

Now we see that the only situation in which the A-curves (ignoring the reducible representations) are not transverse to $M^p L^q = 0$ is when $p = 12, q = 1$ for the granny knot and $p = 0, q = 1$ for the square knot.

From (7.2), we see that

$$\dim H^1(S^3_{p/q}(3_1 \# 3_1), \text{ad}\, \rho) = \dim H^1(S^3_{p/q}(3_1 \# 3_1), \text{ad}\, \rho) + 1 - \text{rank}(r)$$

The result follows from combining the above formula, our computations of the rank of $r$, and Lemma 7.13. 

Theorems 1.6 and 1.7 now follow from applying Corollary 2.2 to the calculation of the respective character varieties in Propositions 7.4 and 7.10 and the determination of the singular cohomology of these character schemes from Proposition 5.5.

Remark 7.15. We use $HP$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients in Theorems 1.6 and 1.7 only to avoid determining the relevant local system. Indeed, the character schemes of surgeries on $3_1 \# 3_1$ include some components isomorphic to $\mathbb{C}^\times$ and $\mathbb{C}^\times - \{1\}$, while the other topological types of components that appear are simply connected. We conjecture that the local systems are in fact trivial on all of the components and that Theorems 1.6 and 1.7 hold over $\mathbb{Z}$.

8. Further Discussion

8.1. Exact triangles. In analogy with other Floer theories [OS04][Sca15][Flo00], one may conjecture the existence of a surgery exact triangle for $HP_\#$. That is, one may hope that there exists a long exact sequence

$$HP_\#(S^3)[1] \to HP_\#(S^3_{p+1}(K)) \to HP_\#(S^3_p(K)) \to HP_\#(S^3)$$

However, since $HP_\#(S^3)$ is supported in degree zero, such a long exact sequence would imply that $HP_\#(S^3_p(K))$ and $HP_\#(S^3_{p+1}(K))$ are isomorphic except possibly in degrees $-1, 0,$ and $1$. Yet the data from Theorem 1.5 shows that this is not the case. For example, $HP_\#(S^3_2(3_1))$ has rank 2 in degree $-3$, whereas $HP_\#(S^3_3(3_1))$ has rank 1 in degree $-3$. 

One can also ask whether a surgery exact triangle exists for $HP$. The data in Theorem 1.3 can be used to show that such a triangle cannot exist for the trefoil. However, one would not even expect such a surgery exact triangle for $HP$ since for other Floer theories such exact triangles are not usually formulated for the versions that exclude reducibles. For example, there is no surgery exact triangle for $HF^{\circ}_{\text{red}}$ in Heegaard Floer homology.

8.2. A conjecture. In [BC16], the authors define an $SL(2, \mathbb{C})$ Casson knot invariant by

$$\lambda_{SL(2,\mathbb{C})}^\prime(K) = \lim_{q \to \infty} \frac{1}{q} \lambda_{SL(2,\mathbb{C})}(S^3_{p/q}(K))$$

where $p$ is fixed and the limit is taken over all $q$ relatively prime to $p$. In particular, this quantity is independent of $p$. We can make the analogous conjecture for $HP$ and $HP\#_n$.

**Conjecture 8.1.** Let $K \subset S^3$ be a knot and $S^3_{p/q}(K)$ its $p/q$ surgery. Then the quantities

$$\lim_{q \to \infty} \frac{1}{q} \text{rk}(HP^n(S^3_{p/q}(K)))$$

and

$$\lim_{q \to \infty} \frac{1}{q} \text{rk}(HP^n_{\#}(S^3_{p/q}(K)))$$

are well-defined invariants of the knot $K$.

For example, by Theorems 1.6 and 1.7 we can verify this conjecture for $HP$ of surgeries on the granny and square knots. We obtain the numerical data

$$\lim_{q \to \infty} \frac{1}{q} \text{rk}(HP^0(S^3_{p/q}(G))) = 12$$

$$\lim_{q \to \infty} \frac{1}{q} \text{rk}(HP^{-1}(S^3_{p/q}(G))) = 6$$

and

$$\lim_{q \to \infty} \frac{1}{q} \text{rk}(HP^0(S^3_{p/q}(S))) = 6$$

$$\lim_{q \to \infty} \frac{1}{q} \text{rk}(HP^{-1}(S^3_{p/q}(S))) = 0$$

**References**


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