Tannakian Categories and their relationship to algebraic groups

A tensor category over a field k is a symmetric monoidal abelian category $(\mathcal{A}, \otimes, 1, \lambda, \rho, \alpha, \tau)$ with a specified isomorphism $\mathrm{End}(1) \to k$. A functor $F : \mathcal{A} \to \mathcal{B}$ between two tensor categories preserving the tensor structure is called a tensor functor. A tensor functor is actually a functor with additional structure (F, u, m) of an isomorphism $u_F : F(1_\mathcal{A}) \to 1_\mathcal{B}$ and a natural isomorphism of bifunctors $m_F : F(- \otimes_\mathcal{A} -) \to F(-) \otimes_\mathcal{B} F(-)$. An object X in a tensor category is dualizable if there exists an object $X^\vee$ and morphisms morphisms $\delta_X : 1 \to X \otimes X^\vee$ and $\epsilon_V : X^\vee \otimes X \to 1$ satisfying the triangle identities:

$$
\begin{array}{ccccc}
X & \xrightarrow[\cong]{\lambda_X^{-1}} & 1 \otimes X & \xrightarrow{\delta \otimes 1} & X \otimes X^\vee \otimes X \\
& \searrow^{1_X} & & & \downarrow {\scriptstyle 1 \otimes \epsilon} \\
& & X & \xleftarrow[\cong]{\rho_X} & X \otimes 1
\end{array}
\qquad
\begin{array}{ccccc}
X^\vee & \xrightarrow[\cong]{\rho_{X^\vee}^{-1}} & X^\vee \otimes 1 & \xrightarrow{1 \otimes \delta} & X^\vee \otimes X \otimes X^\vee \\
& \searrow^{1_{X^\vee}} & & & \downarrow {\scriptstyle \epsilon \otimes 1} \\
& & X & \xleftarrow[\cong]{\lambda_{X^\vee}} & 1 \otimes X^\vee
\end{array}
$$

Tensor functors evidently preserve duals – if X and $X^\vee$ are duals and $F : \mathcal{A} \to \mathcal{B}$ is a tensor functor, then $F(X)$ and $F(X^\vee)$ are duals using the structure maps $F(\delta)$ and $F(\epsilon)$ combined with the tensor functor structure maps m_F and u_F as appropriate. We call a tensor category rigid if all objects are dualizable.

Given an affine group scheme G over a field k, the category $\mathrm{Rep}_k(G)$ of finite dimensional k-representations of G is an example of a rigid k-linear tensor category. It comes equipped with a forgetful tensor functor $\mathrm{Rep}_k(G) \to \mathrm{Vect}_k$. In general, we will call a rigid k-tensor category $\mathcal{T}$ Tannakian if for some nonempty scheme S over k there exists a right exact tensor functor $\omega : \mathcal{T} \to \mathrm{qcoh}(S)$. Such a functor we will call a fiber functor. A fiber functor is automatically exact and faithful. Since tensor functors preserve dualizability, any such functor lands in the full subcategory $\mathrm{Coh}_p(S)$ of projective coherent sheaves over S (that is, the sheaves which are dualizable). Because of this, if $\mathcal{A}$ is a not-necessarily rigid tensor category, we will call a tensor functor $\mathcal{A} \to \mathrm{Coh}_p(S)$ a fiber functor on $\mathcal{A}$. A Tannakian category over k is called "neutral" if it admits a fiber functor to Vect_k and "algebraic" if it admits a fiber functor to $\mathrm{Vect}_{k'}$ for k' a finite extension of k. Our goal is to show that rigid tensor categories serve as a reasonable generalization of group schemes over G. Under sufficient hypotheses, this generalization turns out to be very mild.

The first theorem in this direction is as follows:

Theorem (Main theorem for neutral Tannakian categories, [3] 2.19)**.** *Let $\mathcal{T}$ be a neutral Tannakian category, and let $\omega : \mathcal{T} \to \mathrm{Vect}_k$ be a fiber functor. Let $\underline{\mathrm{Aut}}_k^\otimes(\omega) : \mathrm{Alg}_\mathrm{k}^\mathrm{op} \to \mathrm{Set}$ be the functor that maps a k-algebra A to the set of tensor automorphisms of $\omega(-) \otimes_k A : \mathcal{T} \to A - mod$. This functor is represented by an affine group scheme G and the functor $\mathcal{C} \to \mathrm{Rep}_k(G)$ defined by ω is an equivalence of tensor categories.*

Thus, neutral Tannakian categories over a field k are the same as affine group schemes. The advantage of this theorem is that it implies that properties of a group scheme G can be stated in terms of $(\mathrm{Rep}_k(G), \omega)$. Any property of a group scheme G that turns out not to involve ω immediately generalizes to a property of rigid tensor categories. For instance:

Theorem ([3] 2.20)**.** *Let G be a group scheme over k.*

(a) G is finite if and only if there exists $X \in \mathrm{Rep}_k(G)$ such that every object of $\mathrm{Rep}_k(G)$ is isomorphic to a subquotient of $X^{\otimes n}$ for some $n \geq 0$.

(b) G is algebraic if and only if $\mathrm{Rep}_k(G)$ has a tensor generator X – that is, if and only if every object is a subquotient of a sum of $X^{\otimes a_i} \otimes (X^\vee)^{\otimes b_i}$.

Theorem ([3] 2.21)**.** *Let $f : G \to G'$ be a homomorphism of group schemes over k and let $\mathrm{Rep}(f) : \mathrm{Rep}_k(G') \to \mathrm{Rep}_k(G)$ be the corresponding functor.*

(a) f is faithfully flat if and only if $\mathrm{Rep}(f)$ is fully faithful and has essential image closed under subobjects.

(b) f is a closed immersion if and only if every object of $\mathrm{Rep}(G)$ is a subquotient of an object in the essential image of $\mathrm{Rep}(f)$.

Theorem ([3] 2.22). *Let G be a group scheme over k a field of characteristic 0. Then G is connected if and only if $\langle X \rangle$ is not closed under $\otimes$ for all nontrivial G-representations X.*

Theorem ([3] 2.23). *Let G be a connected group scheme over a field k of characteristic 0. Then $\mathrm{Rep}_k(G)$ is semisimple if and only if G is pro-reductive.*

These propositions motivate corresponding definitions for rigid tensor categories.

We also have the following, considerably more general version of the main theorem:

Theorem (Main theorem, [1] 1.12). *Let $\mathcal{T}$ be a Tannakian category and let $\omega : \mathcal{T} \to \mathrm{qcoh}(S)$ be a fiber functor. Then:*

(a) The functor $\underline{\mathrm{Aut}}_k^{\otimes}(\omega)$ is represented by a groupoid scheme over k with object scheme given by S.

(b) The (morphism scheme of the) groupoid scheme $\underline{\mathrm{Aut}}_k^{\otimes}(\omega)$ is affine and faithfully flat over $S \times S$ (with structure map given by the source and target morphisms).

(c) ω induces an equivalence $\mathcal{T} \to \mathrm{Rep}(\underline{\mathrm{Aut}}_k^{\otimes}(\omega))$.

Conversely, if G is a groupoid scheme over k that acts transitively on S and is affine and faithfully flat over $S \times S$, and ω is the forgetful functor $\mathrm{Rep}(G) \to \mathrm{qcoh}(S)$ then

(d) $G \cong \underline{\mathrm{Aut}}_k^{\otimes}(\omega)$

So a Tannakian category with a fiber functor to a scheme S is equivalent to a transitive groupoid scheme over S that is affine and faithfully flat over $S \times S$. Up to stackification, a transitive nonempty groupoid scheme is known as a gerbe over S.

For any rigid tensor category $\mathcal{A}$ over k there is a stack $\mathrm{Fib}(\mathcal{A})$ over k of fiber functors of $\mathcal{A}$. An S-point of $\mathrm{Fib}(\mathcal{A})$ is the same as a tensor functor $\mathcal{A} \to \mathrm{Coh}_p(S)$. If $\mathrm{Fib}(\mathcal{A})$ is not the empty stack, then it is a gerbe over k. If ω is an S-point of $\mathrm{Fib}(\mathcal{A})$, then there is a natural map $\underline{\mathrm{Aut}}_k^{\otimes}(\omega) \to \mathrm{Fib}(\mathcal{A})_S$ where $\mathrm{Fib}(\mathcal{A})_S$ is the base change of $\mathrm{Fib}(\mathcal{A})$ to S, and this map is an equivalence of gerbes. Thus, $\mathcal{A}$ is neutral if and only if $\mathrm{Fib}(\mathcal{A})$ has a k-point and algebraic if and only if it has a k' point for k' a finite extension of k. We can test whether a Tannakian category is algebraic intrinsically by using Prop 2.20 (a). If this condition is satisfied, we can find a field k' and an algebraic group G over k' such that $\mathrm{Rep}'_k(G)$ is equivalent to $\mathcal{A} \otimes_k k'$. Then there is a Galois descent theory for Tannakian categories that intrinsically determines whether $\mathcal{A}$ an algebraic Tannakian category is neutral.

When is a rigid tensor category Tannakian

We have seen that we can intrinsically determine when a Tannakian category is algebraic and when an algebraic Tannakian category is neutral. A natural question to ask is when a rigid symmetric monoidal category is Tannakian. In characteristic zero, this also has a good answer under extremely general conditions:

Theorem ([1] 7.1). *Let $\mathcal{T}$ be a rigid tensor category over a field k of characteristic 0. Then the following are equivalent:*

(i) $\mathcal{T}$ is Tannakian.

(ii) For all $X \in \mathcal{T}$, the Euler characteristic $\chi(X)$ is a nonnegative integer.

(iii) For all $X \in \mathcal{T}$, there exists a nonnegative integer n so that $\wedge^n X = 0$.

Condition (iii) is called positivity. In finite characteristic, such a theorem only holds if the following stronger condition is met:

Definition ([4] 1.8.1 and 1.8.6). A tensor category $\mathcal{C}$ is locally finite if:

(i) $\mathcal{C}$ has finite dimensional morphism spaces

(ii) every object of $\mathcal{C}$ has finite length

A tensor category $\mathcal{C}$ is *finite* if

(i) It is locally finite

(ii) $\mathcal{C}$ has enough projectives, that is, every object of $\mathcal{C}$ has a projective cover

(iii) there are finitely many isomorphism classes of simple objects

The only truly restrictive criterion in the definition of finiteness for our purposes is condition (iii) – if $\mathcal{T}$ is Tannakian, then $\mathcal{T}$ must be locally finite and have enough projectives. The following theorem holds even in finite characteristic:

Theorem ([4]. *9.9.30] Let $\mathcal{T}$ be a finite rigid tensor category over a field k. The following are equivalent:*

(i) $\mathcal{T}$ is neutral Tannakian.

(ii) For all $X \in \mathcal{T}$ there exists a nonnegative integer n so that $\wedge^n X = 0$.

(iii) There exists a finite group G such that $\mathcal{T}$ is equivalent to $\mathrm{Rep}_k(G)$ as tensor categories over k.

Examples

All the normal group schemes give rise to Tannakian categories. Two counterexamples are as follows:

Example ([3] 1.26)**.** There is a rigid tensor category GL_t over the field $\mathbb{Q}(t)$ that is initial among rigid tensor categories over a field of characteristic zero with an object T such that $\chi(T) = t$. The construction goes roughly as follows: Given V a free module over a ring k, let $T^{a,b} = V^{\otimes a} \otimes (V^{\vee})^{\otimes b}$. If $l = a = b$, then there is an element $id^{\otimes l} \in T^{l,l}$. Letting S_l act on the contravariant piece of this tensor product gives an injection $S_l \to T^{l,l}$. These maps give rise to adjoint maps $T^{a,b} \to T^{c,d}$ whenever $a + d = b + c$. It turns out that there is a universal formula for composing such maps, so we can define a category $\mathcal{A}_0$ whose objects are of the form $T^{a,b}$ and whose morphism spaces $\mathcal{A}_0(T^{a,b}, T^{c,d}) = \mathbb{Z}[t][S_{a+d}]$ if $a + d = b + c$, and zero otherwise. Because there is a universal formula for the composition of these maps, this gives rise to a composition on $\mathcal{A}_0$. Let $\mathcal{A}$ be the category obtained from $\mathcal{A}_0$ by formally adjoining direct sums. Then $\mathcal{A}$ is the initial rigid symmetric monoidal abelian category with an object $T = T^{1,0}$ whose dimension is given by an indeterminate $\xi(T) = t \in \mathbb{Z}[t]$. Then $\mathrm{GL}_t = \mathcal{A} \otimes_{\mathbb{Z}[t]} \mathbb{Q}(t)$. This category fails positivity, because $\xi(T) = t$ is not a nonnegative integer.

Example ([1] 2.19)**.** For any integer n there is a similar construction that gives a category GL_{t-n} freely generated by a single object T such that $\xi(T) = t - n \in \mathbb{Q}(t)$. The universal property of GL_{t-n} implies that there is a functor $\mathrm{GL}_{t-n} \to \mathrm{GL}_{t-n-1}$ sending $T \mapsto T \oplus 1$. The composite map $\mathrm{GL}_t \to \mathrm{GL}_{t-n}$ sends $T \mapsto T \oplus 1^n$. Passing to the colimit $\mathcal{T} = \mathrm{colim}_n\, \mathrm{GL}_{t-n}$ we get a tensor category over $\mathbb{Q}(t)$ freely generated by one object X_t with given decompositions $X_t = X_{t-n} \oplus 1^n$ for all n. This category is a rigid tensor category that is not locally finite – X_t is of infinite length, and $\mathrm{End}(X_t)$ consists of all matrices of the form $\bigcup_n \mathrm{Mat}_n(\mathbb{Q}(t)) \oplus \mathbb{Q}(t)Id$ so the morphism spaces are not finite dimensional.

I don't know of any examples that meet the positivity criterion but fail to be locally finite. If this can happen, it must be over an imperfect field of finite characteristic, so presumably such an example would have to use an inseparable extension in some way.

Example ([3] 1.25)**.** Let $\mathcal{C}$ be the category whose objects are given by $\mathbb{Z}/2$-graded vector spaces, with the symmetric structure given by $\sigma_{V,W} : v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$. Then $\xi(V) = \dim V^+ - \dim V^-$, and so positivity fails.

Super-Tannakian categories

There is a good notion of a super-Tannakian category. This is a rigid tensor category over k equipped with a fiber functor to $\mathrm{sVect}_{k'}$ for some extension k' of k. We have a main theorem in this case too, but its statement is a little bit more subtle. First we need the following definitions:

Definition. A ring in $\mathcal{T}$ is a ring diagram in $\mathrm{Ind}(\mathcal{T})$ the ind-completion of $\mathcal{T}$. That is, a ring in $\mathcal{T}$ is an object $R \in \mathrm{Ind}(\mathcal{T})$ combined with a multiplication map $R \otimes R \to R$ and a unit map $1 \to R$ satisfying appropriate diagrams. An affine $\mathcal{T}$-scheme is a representable functor $\mathrm{Ring}(\mathcal{T})^{op} \to \mathrm{Set}$. An affine group $\mathcal{T}$-scheme is a representable functor $\mathrm{Ring}(\mathcal{T})^{op} \to \mathrm{Group}$.

Definition. The *fundamental group* of a rigid tensor category is given by $\pi(\mathcal{T}) := \underline{\mathrm{Aut}}_k^{\otimes}(Id)$.

Proposition ([1] 8.11)**.** *If $\mathcal{T}$ is a rigid tensor category, then $\pi(\mathcal{T})$ is an affine group $\mathcal{T}$-scheme.*

Theorem (Main theorem for super-Tannakian categories, [1] 8.19)**.** *If $\mathcal{T}$ is a rigid tensor category and $\omega : \mathcal{T} \to \mathrm{sVect}_{k'}$ is a fiber functor then $T \to \mathrm{Rep}(\omega(\pi(\mathcal{T})))$ is an equivalence of categories.*

Both this and the vanilla main theorem are special cases of the following most general statement, where both the domain and the codomain of our tensor functor are allowed to be any locally finite tensor category:

Theorem ([1] 8.17)**.** *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two locally finite tensor categories over a perfect field k and let η be an exact tensor functor $\mathcal{T}_1 \to \mathcal{T}_2$. Then $\eta(\pi(\mathcal{T}_1))$ is an affine group $\mathcal{T}_2$-scheme and η lifts to an equivalence $\mathcal{T}_1 \to \mathrm{Rep}(\eta(\pi(T_1)))$.*

If we take $\mathcal{T}_2$ to be Vect_k and $\eta = \omega$, then $\omega(\pi(T)) = \underline{\mathrm{Aut}}_k^{\otimes}(\omega)$ and we obtain the vanilla main theorem.

As in the case of vanilla Tannakian categories, there is a notion of affine super-gerbes, and any rigid tensor category $\mathcal{T}$ has an affine super stack $\mathrm{Fib}(\mathcal{T})$ of super fiber functors. In this language, $\mathcal{T}$ is super Tannakian if and only if $\mathrm{Fib}(\mathcal{T})$ is a super gerbe. This is elaborated in [2] 3.9 and 3.10.

There is also a set of equivalent conditions for a rigid tensor categories over a characteristic zero field to be super Tannakian:

Theorem ([2] 2.1)**.** *Let $\mathcal{T}$ be a rigid tensor category over k a characteristic 0 field. The following are equivalent:*

(i) For all $X \in \mathcal{T}$ there exists a Schur functor that annihilates X.

(ii) $\mathcal{T}$ is locally finite and for all $X \in \mathcal{T}$, there exists an N such that $\mathrm{length}(X^{\otimes n}) \leq N^n$ ($\mathcal{T}$ is then said to be of exponential growth).

(iii) $\mathcal{T}$ is super Tannakian

There is also an arbitrary characteristic version, supposing that we require the stronger finiteness condition:

Theorem ([4] 9.9.26)**.** *Let $\mathcal{T}$ be a finite rigid tensor category over a field k. Then $\mathcal{T}$ is neutral super Tannakian.*

Bibliography

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