

Strickland's Theorem

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1 Introduction

Let E be an E_∞ ring. Given a space X , the total power operation is a map $P_m : E^*(X) \rightarrow E^*(X \times B\Sigma_m)$. We are interested in the case of Morava E theory, for which we will show $E^*(B\Sigma_m)$ is free and even over E^* , and so there is a Künneth isomorphism $E^*(X \times B\Sigma_m) \cong E^*(X) \widehat{\otimes}_{E^*} E^*(B\Sigma_m)$. The map $P_m : E^*(X) \rightarrow E^*(X) \widehat{\otimes}_{E^*} E^*(B\Sigma_m)$ is multiplicative, but not additive – in fact $P_m(x+y) = \sum_{i+j=l} \text{Tr}_{\Sigma_i \times \Sigma_j}^{\Sigma_m} (P_i(x) \times P_j(y))$. We would like to force P_m to be additive by quotienting out by all of the cross terms. The set of elements of $E^*(B\Sigma_m)$ which are transfers of elements of $E^*(B\Sigma_i \times B\Sigma_j)$ is an ideal of $E^*(B\Sigma_m)$. If m is not a power of p , say $m = p^k b$ then the p -Sylow subgroup of Σ_m is contained in the product $\Sigma_{p^k} \times \Sigma_{m-p^k}$, and the transfer map $E^*(B\Sigma_{p^k} \times B\Sigma_{m-p^k}) \rightarrow E^*(B\Sigma_m)$ is surjective. Thus, we’re really interested in the case that $m = p^k$. We let $R_k = E^*(B\Sigma_{p^k})$, let $I_k \subseteq R_k$ be the transfer ideal, and let $\overline{R}_k = R_k/I_k$. Then the total additive power operation is a map $\overline{P}_{p^k} : E^*(X) \rightarrow E^*(X) \widehat{\otimes}_{E^*} \overline{R}_k$. If we want to understand this map, a good first step is to understand the ring \overline{R}_k . Strickland’s theorem identifies $Y_k = \text{spf } \overline{R}_k$:

Theorem 1.1. *Let $\text{Sub}_n(\mathbb{G})$ be the formal scheme of finite subgroups of \mathbb{G} of degree n . Then:*

$$\text{spf } \overline{R}_k \cong \text{Sub}_{p^k}(\mathbb{G})$$

E^* -algebra	Name	Name of spf
E^*		X
$E^*(BS^1)$	$\mathcal{O}_{\mathbb{G}}$	\mathbb{G}
$E^*(BU(n))$		$\text{Div}_n(\mathbb{G})$
$E^*(BA)$		$\text{Hom}(A^*, \mathbb{G})$
	$D(A)$	$\text{Level}(A^*, \mathbb{G})$
$E^*(B\Sigma_{p^k})$	R_k	
$E^*(B\Sigma_{p^k})/\text{transfers}$	\overline{R}_k	$Y_k \cong \text{Sub}_{p^k}(\mathbb{G})$
	$E(j, l)$	$Y_k(j, l)$

2 Background

Let E be an E_∞ cohomology theory. We will implicitly treat spaces as suspension spectra for convenience. The total Σ_k power operation on the E -cohomology of a space X is a map $P_{\Sigma_k} : E^*(X) \rightarrow E^*(X \times \Sigma_k)$ constructed as follows: an element of $x \in E^i(X)$ is the same as a map:

$$X \xrightarrow{f} \Sigma^i E$$

We take the n th smash power of this map and form the following diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{\Delta} & X^{\wedge k} & \xrightarrow{f^{\wedge k}} & \Sigma^{ki} E^{\wedge k} & \xrightarrow{\mu} & \Sigma^{ki} E \\
\downarrow & & \downarrow & & \downarrow & \nearrow \mu_{\Sigma_k} & \\
X \wedge B\Sigma_k & \xrightarrow{\cong} & X_{h\Sigma_k} & \xrightarrow{\Delta} & X_{h\Sigma_k}^{\wedge k} & \xrightarrow{f_{h\Sigma_k}^{\wedge k}} & (\Sigma^{ki} E^{\wedge k})_{h\Sigma_k}
\end{array}$$

The top row of maps are all Σ_k equivariant, where Σ_k acts on $X^{\wedge k}$ and $(\Sigma^i E)^{\wedge k}$ by permuting the coordinates, and trivially on X . The action on $\Sigma^{ki} E = (S^i)^{\wedge k} \wedge E$ is trivial on E and by permutations on $(S^i)^{\wedge k}$. Transposing two factors in S^k is a reflection, which is of degree (-1) , and transposing two adjacent S^i 's in $(S^i)^{\wedge k}$ is the composition of i transpositions of pairs of dimensions, each of which is of degree (-1) so the overall degree is $(-1)^i$. Thus, if i is odd, the action of Σ_k on $\Sigma^{ki} E$ is the sign action, if i is even, the action of Σ_k on $\Sigma^{ki} E$ is the trivial action.

The vertical maps are homotopy quotient maps. Since the top row of maps are Σ_k equivariant, we can take homotopy orbits of all of the maps to make the diagram commute. The composition of the top row is an element of $E^{ki}(X)$, this element is just x^k which you can see by the definition of the ring structure on $E^*(X)$. The bottom composite gives an element of $E^{ki}(X \wedge B\Sigma_k)$, which when pulled back to $E^{ki}(X)$ along the homotopy orbits map restricts to x^k . This element of $E^*(X \wedge B\Sigma_k)$ is called $P_k(x)$ or $P_{\Sigma_k}(x)$.

We're going to be thinking about the case where $E = E(n)$ is Morava E-theory, and in this case $E^*(B\Sigma_k)$ is even and free, and this implies that for any space X there is a Künneth isomorphism $E^*(B\Sigma_k \times X) \cong E^*(B\Sigma_k) \hat{\otimes}_{E^*} E^*(X)$, so we get a map $P_k : E^*(X) \rightarrow E^*(X) \hat{\otimes} E^*(B\Sigma_k) = E^*(X) \hat{\otimes} R_k$ where we define $R_k = E^*(B\Sigma_k)$.

Clearly this map $P_k : E^*(X) \rightarrow E^{k*}(X \wedge B\Sigma_k)$ isn't additive: the restriction map $E^{k*}(X \wedge B\Sigma_k) \rightarrow E^{n*}(X)$ is additive and the composite is the k th power map, so P_k can't be additive unless the k th power map is additive (it won't be additive even then though). So how badly does the power operation fail to be additive?

Since we are assuming that X is a suspension spectrum, there is a pinch map $X \rightarrow X \vee X$ which represents the addition map $E^*(X) \oplus E^*(X) \cong E^*(X \vee X) \rightarrow E^*(X)$. The "external sum" of two cohomology classes of X is the corresponding cohomology class on $E^*(X \vee X)$, and we can deduce the

$$\begin{array}{ccccc}
D_n X & \longrightarrow & D_n(X \vee X) & \longrightarrow & E \\
\Delta \uparrow & & \Delta \uparrow & & \\
(D_n S^0) \wedge X & \longrightarrow & D_n(S^0 \vee S^0) \wedge X & &
\end{array}$$

We have

$$(X \vee X)^{\wedge k} = \bigvee_{S \sqcup T = [k]} X^{\wedge S} \wedge X^{\wedge T}$$

so as Σ_k -spaces, we have

$$(X \vee X)^{\wedge k} \simeq \bigvee_{i+j=k} E\Sigma_k \times_{\Sigma_i \times \Sigma_j} (X^{\wedge i} \wedge X^{\wedge j})$$

and $(S^0 \vee S^0)_{h\Sigma_k}^{\wedge k} = \bigvee_{i+j=k} B\Sigma_i \times B\Sigma_j$.

So if we started with a map $X \vee X \rightarrow E$ given by $x \vee y$, so the map $D_n(S^0 \vee S^0) \wedge X \rightarrow E$ is the external sum: $\bigvee P_i x \times P_j y$. Pulling back to $D_n(S^0) \wedge X$ we get

$$P_k(x + y) = \sum_{i+j=k} \text{Tr}_{\Sigma_i \times \Sigma_j}^{\Sigma_k} (P_i x \times P_j y)$$

The transfer map $E^*(BH) \rightarrow E^*(BG)$ is an $E^*(BG)$ module map, where $E^*(BG)$ acts on $E^*(BH)$ by restriction. This implies that the image is an ideal. We can quotient out by the ideal I_k of all transfers $E^*(B\Sigma_i \times B\Sigma_j) \rightarrow E^*(B\Sigma_k)$ where $i+j = k$ and $i, j > 0$, and this kills all of the terms in the sum above except for $P_k(x)$ and $P_k(y)$ so we deduce that the composite map $E^*(X) \rightarrow E^*(X) \widehat{\otimes}_{E^*} R_k \rightarrow E^*(X) \widehat{\otimes}_{E^*} \overline{R}_k$ is additive. Our goal in this paper is to understand $\text{spf } \overline{R}_k$.

3 The Main result

We want:

Theorem 3.1. *Let $\text{Sub}_n(\mathbb{G})$ be the formal scheme of finite subgroups of \mathbb{G} of degree n . Then:*

$$\text{spf } \overline{R}_k \cong \text{Sub}_{p^k}(\mathbb{G})$$

There is a natural map $\text{Sub}_{p^k}(\mathbb{G}) \rightarrow \text{Div}_{p^k}(\mathbb{G})$ because $\text{Sub}_{p^k}(\mathbb{G})$ is the scheme of order p^k subgroups of \mathbb{G} , and forgetting the group structure turns any such group into a divisor of order p^k . Thus, the first step of proving that Y_k is the scheme of subgroups should be producing a map $Y_k = \text{spf } \overline{R} \rightarrow \text{Div}_{p^k}(\mathbb{G})$. This will come from the restriction of a map $\text{spf } R_k \rightarrow \text{Div}_{p^k}(\mathbb{G})$. Conveniently, $\text{Div}_{p^k}(\mathbb{G}) = \text{spf } E^*(BU(p^k))$ so any map from $B\Sigma_{p^k} \rightarrow BU(p^k)$ will give us a map $\text{spf } R_k \rightarrow \text{Div}_{p^k}(\mathbb{G})$. Since maps $B\Sigma_{p^k} \rightarrow BU(p^k)$ are exactly complex G -representations of dimension p^k , it is clear what we should do: the regular representation gives us the map we want.

The final steps of Strickland's theorem are to check that the restriction of V_k to \overline{R}_k is a subgroup divisor, so that we get a factorization of the map $Y_k \rightarrow \text{Div}_{p^k}(\mathbb{G})$ through a map $Y_k \rightarrow \text{Sub}_{p^k}(\mathbb{G})$, and then to check that this is an isomorphism.

$$\begin{array}{ccc} \text{spf } \overline{R}_k & \longrightarrow & \text{spf } E^*(B\Sigma_k) \\ \downarrow \text{sl} & \lrcorner & \downarrow V_k \\ \text{Sub}_{p^k}(\mathbb{G}) & \longrightarrow & \text{Div}_{p^k}(\mathbb{G}) = \text{spf}(E^*(BU(p^k))) \end{array}$$

Most of the work of Strickland's theorem is getting enough algebraic data about $\text{Sub}_{p^k}(\mathbb{G})$ and about \overline{R}_k to do these last two steps. I'm going to assume all the facts I need here and give the proof of these two final steps, then explain how to prove the facts I am assuming in later sections.

3.1 The factorization

The goal of this section is to prove:

Theorem 3.2 ([7, Proposition 9.1]). *There is a factorization as follows:*

$$\begin{array}{ccc} & & \text{Sub}_{p^k}(\mathbb{G}) \\ & \nearrow \text{dashed} & \downarrow \\ \text{spf } \overline{R}_k & \xrightarrow{V_k} & \text{Div}_{p^k}(\mathbb{G}) \end{array}$$

The moral of character theory tells us that to understand $\text{spf } E^*(BG)$, it suffices to test BG by mapping in abelian p -subgroups of G . For abelian groups A , we have that $\text{spf } E^*(BA) = \text{Hom}(A^*, \mathbb{G})$. We have a natural map $\text{Hom}(A^*, \mathbb{G}) \rightarrow \text{Div}_{|A|}(\mathbb{G})$ given by $\phi \mapsto [\phi(A^*)]$, that is by sending the homomorphism to its image. If ϕ is injective, then $[\phi(A^*)]$ is a subgroup, but the image of the map $\text{Sub}_{p^k}(\mathbb{G}) \rightarrow \text{Div}_{p^k}(\mathbb{G})$ consists of divisors that contain each point with multiplicity at most one, so if ϕ is not injective, its associated divisor does not lie in $\text{Sub}_{p^k}(\mathbb{G})$. Thus, we are naturally interested in restricting from $\text{Hom}(A^*, \mathbb{G})$ to injective maps. It turns out that there is no scheme of injections, but there's a scheme called $\text{Level}(A^*, \mathbb{G})$ that does it's best to be the scheme of injections $A^* \rightarrow \mathbb{G}$. We'll need the following facts about $\text{Level}(A, G)$:

Fact 3.3.

1. $\text{Level}(A, G)(R) = \text{Mon}(A, \mathbb{G}(R))$ if R is a domain of characteristic 0 [6, Proposition 7.6].
2. $\text{Level}(A, G)$ is smooth and connected [6, Theorem 7.3].
3. There are maps $\text{Level}(A, G) \rightarrow \text{Hom}(A, G)$ and $\text{Level}(A, G) \rightarrow \text{Sub}_{|A|}(\mathbb{G})$ making the following square a pullback:

$$\begin{array}{ccc} \text{Level}(A, G) & \longrightarrow & \text{Sub}_{|A|}(\mathbb{G}) \\ \downarrow & \lrcorner & \downarrow \\ \text{Hom}(A, G) & \longrightarrow & \text{Div}_{|A|}(\mathbb{G}) \end{array}$$

An extremely convenient consequence of character theory is as follows:

Proposition 3.4 ([1, Theorem 3.7]). *For any finite group G , there is a surjection*

$$\bigsqcup_{A \subseteq G} \text{Level}(A^*, \mathbb{G}) \longrightarrow \text{spf } E^*(BG)$$

where the union is over all abelian p -subgroups of G and the map $\text{Level}(A^*, \mathbb{G}) \rightarrow \text{spf}(E^*BG)$ factors as the composite $\text{Level}(A^*, \mathbb{G}) \rightarrow \text{Hom}(A^*, \mathbb{G}) = \text{spf}(E^*BA) \rightarrow \text{spf}(E^*BG)$, this last map the one induced by $A \rightarrow G$. In fact, this comes from an injection $E^*(BG) \rightarrow \prod_A D(A)$, which is actually what I'll need.

We want to check that the image of the map $\text{spf } \overline{R}_k \rightarrow \text{Div}_{p^k}(\mathbb{G})$ lies inside the closed subscheme $\text{Sub}_{p^k}(\mathbb{G})$. To do so, it would suffice to check that the image of $\text{spf } \overline{R}_k$ lies inside the union of the images of the maps $\text{Level}(A^*, \mathbb{G}) \rightarrow \text{Div}_{p^k}(\mathbb{G})$ for certain abelian subgroups of Σ_{p^k} .

We need to assume one more difficult fact, which is the main result from the first three quarters of [7]. Let $c_k = c_{p^k-1}(V_k)$ be the top Chern class of V_k (V_k contains a trivial summand, so $c_{p^k}(V_k) = 0$).

Fact 3.5 ([7, Theorem 8.6]). $\overline{R}_k = R / \text{ann}(c_k)$

Now I can give the proof of the theorem. First a couple lemmas:

Lemma 3.6. *The following is a pullback:*

$$\begin{array}{ccc} \bigsqcup_{A \text{ transitive}} \text{Level}(A^*, \mathbb{G}) & \longrightarrow & \text{spf } \overline{R}_k \\ \downarrow & \lrcorner & \downarrow \\ \bigsqcup_{A \subseteq \Sigma_{p^k}} \text{Level}(A^*, \mathbb{G}) & \longrightarrow & \text{spf } R_k \end{array}$$

where the bottom left hand corner union is over all abelian p -subgroups of Σ_{p^k} , and the top left hand corner union is over all transitive abelian p -subgroups of Σ_{p^k} .

Proof. For A an abelian p -subgroup of Σ_{p^k} and let $i_A : A \rightarrow \Sigma_{p^k}$ be the inclusion map. Let $D(A) = \mathcal{O}_{\text{Level}(A^*, \mathbb{G})}$ and let $f : \text{spf}(E^* B \Sigma_{p^k}) \rightarrow D(A)$ be the map $\text{Level}(A^*, \mathbb{G}) \rightarrow \text{spf} R_k$ on coordinate rings. Since $\bar{R}_k = R_k / \text{ann}(c_k)$, the pullback is $\text{spec}(\prod_A D(A) / \text{ann}(f_A(c_k)))$, so we need to determine the image of we need to quotient by the image of the annihilator of c_k in each $D(A)$. Note that the map $\text{Level}(A^*, \mathbb{G}) \rightarrow \text{spf} R_k$ factors through $\text{Hom}(A^*, \mathbb{G}) = \text{spf}(E^* B A)$. Let's denote the map $\text{spf}(E^* B A) \rightarrow D(A)$ by p so that $f = pi^*$.

First suppose that A is not transitive. Then $i^* V_k$ has two trivial summands, given by averaging over the orbits of the action of A , so we see that $i^*(c_k) = i^*(c_{p^k-1}(V_k)) = c_{p^k-1}(i^* V_k) = 0$ since the top two Chern classes are both zero when a vector bundle with has two trivial summands. We deduce that $f(c_k) = 0$ and hence $D(A) / \text{ann}(f(c_k)) = 0$.

Now suppose that A is transitive. The pullback $i^* V_k$ is the regular representation of A which has the decomposition $i^* V_k = \bigoplus_{L \in A^*} L$. Each L gives a map $[L] : \text{spf}(E^* B A) = \text{Hom}(A^*, \mathbb{G}) \xrightarrow{L} \mathbb{G} = \text{Div}_1(\mathbb{G})$ and thus, $\mathbb{D}(i^* V_k) = \sum_{L \in A^*} [L]$. If $\phi : A^* \rightarrow \mathbb{G}$ is a level structure, then ϕ is injective (this follows from Fact 3.3 part (1) because E^* is an integral domain where $p \neq 0$). We deduce that $f(c_k) = \prod_{L \neq 1} e(L) \neq 0$ in $D(A)$, and because $D(A)$ is smooth and connected, c_k is a nonzero divisor, that $\text{ann}(f(c_k)) = (0)$. \square

In the process, we also deduced:

Lemma 3.7. *If A is transitive, the following diagram commutes:*

$$\begin{array}{ccc} \text{Hom}(A^*, \mathbb{G}) = \text{spf}(E^0 B A) & \xrightarrow{i} & \text{spf}(E^0 B \Sigma_{p^k}) \\ & \searrow \phi \mapsto [\phi(A^*)] & \downarrow V_k \\ & & \text{Div}_{p^k}(\mathbb{G}) \end{array}$$

Proof. We checked that $\mathbb{D}(i^* V_k) = \sum_{L \in A^*} [L]$ in the proof of the last lemma, but this is exactly $[\phi(A^*)]$. \square

Proof of Theorem 3.2. We have shown that there is a commutative diagram as follows:

$$\begin{array}{ccc} \bigsqcup_{A \text{ transitive}} \text{Level}(A^*, \mathbb{G}) & \longrightarrow & \text{Sub}_{p^k}(\mathbb{G}) \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \text{spf} \bar{R}_k & \longrightarrow & \text{Div}_{p^k}(\mathbb{G}) \end{array}$$

The desired factorization is the dashed map. Because $\bar{R}_k \rightarrow \prod_A D(A)$ is an injection on the level of rings, it is immediate that there is a lift. \square

3.2 $\text{spf } \overline{R}_k \rightarrow \text{Sub}_{p^k}(\mathbb{G})$ is an isomorphism

The goal of this section is to prove:

Theorem 3.8 ([7, Theorem 9.2]). *The map $\text{spf } \overline{R}_k \rightarrow \text{Sub}_{p^k}(\mathbb{G})$ is an isomorphism*

Again, we're going to defer the actual work of proving the theorem until later sections. We need the following facts:

Fact 3.9 ([7, Theorem 8.6] and [6, Theorem 10.1]). *\overline{R}_k and $\mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G})}$ are free E^0 modules of the same rank.*

Thus it suffices to check that the map $\mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G})} \rightarrow \overline{R}_k$ is an injection. Since they are free, we can check it after restricting to the special fiber – that is, we want to show that $\mathbb{F}_{p^n} \otimes_{E^0} \mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G})} \rightarrow \mathbb{F}_{p^n} \otimes_{E^0} \overline{R}_k$ is an injection. Writing \mathbb{G}_0 for the restriction of \mathbb{G} to the special fiber, we have $\mathbb{F}_{p^n} \otimes_{E^0} \mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G})} = \mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G}_0)}$ because basechange commutes with Sub.

We now have a map between Artinian local rings. To show that the map is injective, we will check that it is injective when restricted to the socle. The socle of a local ring R with maximal ideal \mathfrak{m} is the set $\{r \in R \mid \mathfrak{m}r = 0\}$. The socle is dual to the indecomposables of an augmented ring, and this method of checking that a map is injective is dual to checking that a map is surjective by showing that it hits all of the generators. So to do this, we need to know $\text{soc}(\mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G}_0)})$.

A divisor is of the form $\sum \sigma_i t^{n-i}$ and the coefficients σ_i are functions on the scheme of divisors – in fact, $\mathcal{O}_{\text{Div}_{p^k}(\mathbb{G})} = \mathcal{O}_{\mathbb{G}}[\sigma_1, \dots, \sigma_{p^k}]$. This element σ_{p^k-1} pulls back along the map $\text{Sub}_{p^k}(\mathbb{G}) \rightarrow \text{Div}_{p^k}(\mathbb{G})$ to give an element $a' \in \mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G})}$.

Fact 3.10 ([6, Proposition 10.15]). *$\text{soc}(\mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G}_0)}) = \mathbb{F}_{p^n} \{a'^N\}$ where $N = p + \dots + p^{n-1}$*

We need to show that a'^N maps to a nonzero element in \overline{R}_k . The map $\text{Sub}_{p^k}(\mathbb{G}) \rightarrow \text{Div}_{p^k}(\mathbb{G})$ factors the map $V_k : \text{spf } \overline{R}_k \rightarrow \text{Div}_{p^k}(\mathbb{G})$, so the image of a' in \overline{R}_k is $c_{p^k-1}(V_k) = c_k$. So it remains to check that $c_k^N \neq 0$ in \overline{R}_k .

Fact 3.11 ([7, Proposition 3.4]). *The element $c_k^{N+1} \neq 0$ in $R_k = E^*(B\Sigma_{p^k})$ and $E^*\{c_k^i\}$ is a summand of R_k as an E^* -module for all i .*

Since $c_k^{N+1} \neq 0$, we have that $c_k^N \notin \text{ann } c_k$ so $c_k^N \neq 0$ as an element of \overline{R}_k . Also, since \overline{R}_k is free as an E^* -module too, we deduce that $E^*\{c_k^N\}$ is a summand of \overline{R}_k . Tensoring down to \mathbb{F}_{p^n} , we deduce that $c_k^N \neq 0$ as an element of $\overline{R}_k \otimes_{E^*} \mathbb{F}_{p^n}$ and hence that $\text{soc}(\mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G}_0)})$ is mapped injectively. It follows that $\mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G})} \rightarrow \overline{R}_k$ is an injection.

4 Goodness

Now let's start over at the beginning and work towards proving the six facts needed to prove the main theorem.

A finite group G is good if $K(n)^*(BG)$ is generated by the transfers of Euler classes of complex representations of subgroups H of G for all $n \geq 1$. Facts about good groups:

Proposition 4.1 ([2, Proposition 7.2 and Theorem 7.3]).

1. If G and G' are good, then so is $G \times G'$.
2. If G is abelian, then G is good.
3. If $H \subseteq G$ is a p -Sylow subgroup of G and H is good, then G is good.
4. If G is good, then $C_p \wr G$ is good too.

Proof.

1. The Euler class of an external product of bundles is the external product of the Euler classes. Since Künneth isomorphisms hold for $K(n)$, if G and G' are good, say $K(n)^*(BG)$ and $K(n)^*(BG')$ are generated by the Euler classes of V_1, \dots, V_n and V'_1, \dots, V'_n respectively, then $K(n)^*(B(G \times G'))$ is generated by $\{e(V_i \times V'_j)\}_{i,j}$.
2. By part (1), this reduces to the case that $G = C_m$ is cyclic. The Gysin sequence associated to $BC_m \rightarrow BS^1 \xrightarrow{*m} BS^1$ is $K(n)^*(BS^1) \rightarrow K(n)^*(BS^1) \rightarrow K(n)^*(BC_m)$. Now $K(n)^*(BS^1) = \mathcal{O}_{\mathbb{G}_0} \cong K(n)^*[[x]]$ and the map induced on cohomology by multiplication by m is the m -series of the formal group law. This is nonzero, so the map $K(n)^*(BS^1) \rightarrow K(n)^*(BS^1)$ is injective and $K(n)^*(BC_m) \cong K(n)^*[[x]]/[m](x)$. The map $BC_m \rightarrow BS^1$ is picking out the standard representation of C_m , the class $x \in \mathcal{O}_{\mathbb{G}_0}$ is the Euler class of the canonical line bundle on BS^1 so it pulls back to the Euler class of the standard representation of C_m , which we've also called x and which we see generates $K(n)^*(BC_m)$.
3. If H is a p -Sylow subgroup, then the transfer map $E^*(BH) \rightarrow E^*(BG)$ is onto.
4. See the proof of Proposition 4.3.

□

Because a good group has finitely generated even K theory for all n , and all groups G have finitely generated E -theory, it is easy to check by chasing exact sequences that $E^*(BG)$ is even, free as an E^* module, and generated by the Euler classes of the same representations [7, Proposition 3.5]. With little extra effort, Strickland deduces the following facts:

Proposition 4.2 ([7, Theorem 3.2]). *If G is a group, E^0BG is a Noetherian local ring and a free module over E^0 and:*

$$\begin{aligned} E^\vee BG &= \text{Hom}_{E^0}(E^0BG, E^0) \\ K^0BG &= \mathbb{F}_{p^n} \otimes_{E^0} E^0BG \\ K_0BG &= \text{Hom}_{\mathbb{F}_{p^n}}(K^0BG, \mathbb{F}_{p^n}) \\ E^1BG &= E_1^\vee BG = K^1BG = K_1BG = 0 \end{aligned}$$

Let W_0 be the trivial group and $W_k = C_p \wr W_{k-1}$. The p -Sylow subgroup of a symmetric group is a product of the W_k 's, which are all good, and so all of the symmetric groups are good. We also need to know (Fact 3.11) that $c_k = c_{p^k-1}(V_k)$ raised to the power $N+1 = \frac{p^n-1}{p-1}$ is nonzero in $E^*(B\Sigma_{p^k})$. It suffices to show that the restriction of this element to W_k is nonzero. We deduce this by chasing through the proof of Proposition 4.1 part (4) for the special case of W_k .

Proposition 4.3 ([7, Proposition 3.4]). *W_k is good for all k . The element $b_k = c_k^{(p^n-1)/(p-1)} \neq 0$.*

Proof. Suppose for sake of induction that the proposition holds for W_k . The base case of W_0 is trivial. Choose a basis $\{e_1, \dots, e_d\}$ of $K^0 BW_k$ where $e_1 = b_m$ and the e_i 's are all transferred Euler classes. The exact sequence:

$$1 \longrightarrow W_k^p \longrightarrow W_{k+1} \longrightarrow C_p \longrightarrow 1$$

leads to an Atiyah-Hirzebruch-Serre spectral sequence in K -theory:

$$H^*(BC_p; K(n)^*(BW_k)^{\otimes p}) \Rightarrow K(n)^*(B(W_{k+1}))$$

where we used the Kunnetth isomorphism for $K(n)$. We need to understand $K(n)^*(BW_k)^{\otimes p}$ as a C_p -module. For each i , $K(n)^*(BW_k)^{\otimes p}$ has a trivial C_p -submodule generated by $e'_i = e_1 \otimes \dots \otimes e_i$. For each sequence $I = (i_0, \dots, i_{p-1})$ considered up to cyclic permutation such that $i_j \neq i_k$ for some j, k , there is a summand with free action. Let e_I be the sum of the C_p orbit of $e_{i_0} \otimes \dots \otimes e_{i_{p-1}}$. Then the E_2 term of our spectral sequence is:

$$E(\beta) \otimes P(x) \otimes \mathbb{F}_p e_I, e'_i / (x e_I, \beta e_I)$$

This spectral sequence is a module over the Atiyah-Hirzebruch spectral sequence:

$$E(\beta) \otimes P(x) = H^*(BC_p; K(n)^*) \Rightarrow K(n)^*(BC_p) = P[x]/(x^{p^n})$$

The only possible differential that can kill x^{p^n} is $d_{p^n-1}(\beta) = x^{p^n}$, and after this the spectral sequence is even concentrated so this is the only differential. Using the module structure, we see that in our spectral sequence $d_{p^n-1}(\beta e'_i) = x^{p^n} e'_i$, which also leaves an even concentrated spectral sequence.

Now we exhibit elements \tilde{e}_I and \tilde{e}'_i that hit e_I and e'_i under the edge map $K^*(BW_{k+1}) \rightarrow K^*(BW_k^p)$. Set

$$\tilde{e}_I = \text{Tr}_{W_k^p}^{W_{k+1}}(e_{i_0} \otimes \dots \otimes e_{i_{p-1}})$$

Because W_k^p is normal in W_{k+1} , $\text{Res Tr}(x) = \sum_{g \in C_p} gx$ and this was how we defined e_I in terms of $e_{i_0} \otimes \dots \otimes e_{i_{p-1}}$, so indeed \tilde{e}_I hits e_I under restriction. Now suppose $e_i = \text{Tr}_H^{W_k}(e(V))$ for some complex representation V of a subgroup H of W_k . We get an action of $C_p \wr H$ on $V^{\oplus p}$. Let $e'_i = \text{Tr}_{C_p \wr H}^{W_{k+1}}(e(V^{\oplus p}))$. The restriction of $V^{\oplus p}$ to W_k^p splits as an external sum of p copies of V , so the restriction of e'_i to W_k is $e_i \otimes \dots \otimes e_i = e'_i$. We deduce that

$$K^* BW_{k+1} = K^* \{\tilde{e}_I, x^j \tilde{e}'_i\}_{0 \leq j < p^n}$$

We need to show that $b_{k+1} \neq 0$. I claim that b_{k+1} is a unit multiple of $x^{p^n-1} \tilde{e}'_1$. Let U be the pullback of the regular representation of C_p along the projection $W_{k+1} \rightarrow C_p$. Let V_{k+1} be the regular representation of W_{k+1} . Then $V_{k+1} - U$ restricts to $p \otimes (V_k - 1)$. Because $e_1 = e(V_k - 1)^N$, the fact that $NV_{k+1} - NU$ restricts to $p \otimes (NV_k - N)$ implies by definition of \tilde{e}'_1 that $\tilde{e}'_1 = e(V_{p^{k+1}} - U)^N$. On the other hand, have that $b_{k+1} = e(V_{p^{k+1}} - 1)^N$, so

$$b_{k+1} = e(V_{p^{k+1}} - 1)^N = e(V_{p^{k+1}} - U)^N e(U - 1)^N = \tilde{e}'_1 e(U - 1)^N$$

. It remains to calculate the Euler class of $U - 1$. Since $U \cong \bigoplus_{r=0}^{p-1} L^{\otimes r}$, and $e(L^{\otimes r}) = [r](x)$ is a unit times x for $r \neq 0$, we see that $e(U - 1) = ux^{p-1}$, so $e(U - 1)^N = u^N x^{(p-1)N} = u^N x^{p^n-1}$, and it follows that $b_{m+1} = u^N x^{p^n-1} \tilde{e}'_1$ as desired. \square

We'll need later to follow [3] to deduce that $E^0(B\Sigma_{p^k})$ is power series. As a part of this argument, Kashiwabara makes a similar argument about $BP^* B\Sigma_{p^k}$. First we need the following:

Theorem 4.4 ([5, Theorem 2.2.2]). *Suppose that G is good. Then $BP^*(BG)$ is generated by transferred Euler classes of subgroups. Suppose that $BP^*(BG) \widehat{\otimes}_{BP^*} \mathbb{F}_p \cong \mathbb{F}_p\{b_i\}$. Then:*

$$BP^*(B(C_p \wr G)) \widehat{\otimes}_{BP^*} \mathbb{F}_p \cong \mathbb{F}_p \left\{ \tilde{e}'_i, \tilde{e}_I x^s, x^{s'} \mid s \geq 0, s' > 0, i \text{ and } I \text{ as before} \right\}$$

This is a simple application of the following local to global principal for BP , which is a specialized version the main result of BP from K -theory [5, Theorem 1.21]:

Theorem 4.5. *Let X be a CW complex such that $\lim^1 BP^*(X^m)$ vanishes and suppose that $T \subseteq BP^*(X)$ is a set such that all but finitely many elements of T are in the m th skeletal filtration and that $K(q)^*(X)$ is topologically generated by the image of T as a $K(q)^*$ -module for all $q > 0$. Then $BP^*(X)$ is topologically generated by T as a BP^* -module.*

Proof of Theorem 4.4. Take T to be the subset of $BP^*(BG)$ multiplicative generated by all transferred Euler classes of irreducible subrepresentations. The set of irreducible subrepresentations is finite and each Euler class is of positive skeletal filtration, so the number of products in skeletal filtration less than s for any s is finite. Because G is good the image in $K(n)^*(BG)$ is a topological generating set for all n . Also $\lim^1 BP^*(BG) = 0$ for all groups G . Thus the hypotheses of Theorem 4.5 are satisfied and it follows that T is a topological generating set of $BP^*(BG)$.

Now consider the second statement. We again get an Atiyah-Hirzebruch-Serre spectral sequence in K -theory and the same proof used in Proposition 4.3 shows that

$$H^*(BC_p; K(n)^*(BG^p)) \Rightarrow K(n)^*(B(C_p \wr G))$$

has the same differentials determined by the module structure over $H^*(BC_p; K(n)^*) \Rightarrow K(n)^*(BC_p)$ for any G . This implies that the elements $T = \{\tilde{e}'_i, \tilde{e}_I x^s, x^{s'}\}$ descend to a generating set for $K(n)^*$ homology of $B(C_p \wr G)$. In fact, the subset of these elements where x is taken to a power less than p^n give a basis for the $K(n)^*$ homology of $B(C_p \wr G)$. Since the element x is in positive skeletal filtration, we again see that the set of elements in skeletal filtration less than or equal to s is finite for any s . Thus, the given set is a BP^* basis for $BP^*(B(C_p \wr G))$. \square

Lemma 4.6 ([3, Theorem 2.9]).

$$H\mathbb{F}_p^*(B(C_p \wr G)) \cong H^*(BC_p; H\mathbb{F}_p^*(BG)^{\otimes p})$$

Proposition 4.7 ([3, Proposition 5.1]). *Let G be a good group such that $BP^*(BG) \widehat{\otimes}_{BP^*} \mathbb{F}_p \hookrightarrow H\mathbb{F}_p^*(BG)$. Then $BP^*(B(C_p \wr G)) \widehat{\otimes}_{BP^*} \mathbb{F}_p \hookrightarrow H\mathbb{F}_p^*(B(C_p \wr G))$.*

Proof. By Theorem 4.4 and Lemma 4.6, we see that the map $BP^*(B(C_p \wr G)) \widehat{\otimes}_{BP^*} \mathbb{F}_p \rightarrow H\mathbb{F}_p^*(B(C_p \wr G))$ is the inclusion of the elements of $H\mathbb{F}_p^*(B(C_p \wr G)) \cong H^*(BC_p; H\mathbb{F}_p^*(BG)^{\otimes p})$ with no Bockstein. \square

Because the p -Sylow subgroup of S_m is a product of W_k 's, we get:

Corollary 4.8 ([3, Corollary 5.2]).

$$BP^*(B\Sigma_m) \widehat{\otimes}_{BP^*} \mathbb{F}_p \hookrightarrow H\mathbb{F}_p^*(B\Sigma_m)$$

and

$$BP^*(DS^0) \widehat{\otimes}_{BP^*} \mathbb{F}_p \hookrightarrow H\mathbb{F}_p^*(DS^0)$$

5 Hopf Rings

It turns out that DS^0 is a co-Hopf ring spectrum, so that $E^*(DS^0) = \prod_m E^*(B\Sigma_m)$ is a Hopf ring. We'll eventually learn that $\text{Ind}(E^*(DS^0)) \cong \prod_k \overline{R}_{p^k}$, and leveraging the Hopf ring structure will tell us most of what we need to know about \overline{R}_k .

5.1 What is a Hopf ring?

A bialgebra is a monoid in the category of coalgebras, a Hopf algebra is a group in the category of coalgebras, and a Hopf ring is a ring in the category of coalgebras – so it is a bialgebra and a Hopf algebra with common comultiplication satisfying a distributivity condition.

The simplest example of a bialgebra is a monoid ring, the simplest example of a Hopf algebra is a group ring, and the simplest example of a Hopf ring is a ring ring. So I'll give the example of what all the structure is for a ring ring $R[S]$. We're going

1. An R -module structure.
2. A single coproduct Δ given for a ring ring by $\Delta([a]) = [a] \otimes [a]$. This is the common coproduct of the monoid ring and the group ring.
3. An additive product \star given for a ring ring by $[a] \star [b] = [a + b]$ on the basis. This is the multiplication from the $R[S^+]$ group ring.
4. An additive unit $[0]$.
5. An additive counit $\epsilon([0]) = 1$, $\epsilon([r]) = 0$ if $r \neq 0$.
6. An additive antipode $\xi([r]) = [-r]$.
7. A multiplicative product denoted \circ or by juxtaposition, given by $[a] \circ [b] = [ab]$ on the basis. This is the multiplication from the $R[S^\times]$ monoid ring.
8. A multiplicative unit $[1]$.
9. A multiplicative counit $\epsilon([1]) = 1$, $\epsilon([r]) = 0$ if $r \neq 1$.

with the following properties:

1. $(A, \Delta, \epsilon_\star)$ is a cocommutative coalgebra.
2. $(A, \Delta, \star, [0], \epsilon_\star, \xi)$ is a commutative Hopf algebra
3. $(A, \Delta, \circ, [1], \epsilon_\circ)$ is a bialgebra.
4. $a \circ (b \star c) = \sum (a^{(1)} \circ b) \star (a^{(2)} \circ c)$ where $\Delta(a) = \sum a^{(1)} \otimes a^{(2)}$.

5.2 DS^0 as a co-Hopf ring spectrum

5.3 Rational Morava E -theory of DS^0

6 $E^0 DS^0$ is a formal power series ring under \star

The goal of this section is to prove that $E^0(DS^0)$ is a formal power series ring under the \star product. We do this by a fairly circuitous route: the first step is to show that $K(n)_*(DS^0)$ is polynomial under first note that since \underline{BP}_0 is an E_∞ ring space and since DS^0 is the initial E_∞ ring space, we get a map $DS^0 \rightarrow \underline{BP}_0$.

6.1 Determining the map $BP^*(\underline{BP}_0) \rightarrow BP^*(DS^0)$

The goal of this section is to prove:

Theorem 6.1 ([3, Theorem 1.1]).

$$BP^*(DS^0) \widehat{\otimes}_{BP^*} \mathbb{F}_p \hookrightarrow HF_p^*(DS^0)$$

and the image coincides with the image of the map $HF_p^*(\underline{BP}_0) \rightarrow HF_p^*(DS^0)$. The image is dual to the subalgebra of $(HF_p)_*(DS^0)$ generated under \star by the elements $Q^I[1]$ where Q^I is any Dyer Lashof operation with no Bocksteins. Also the map $BP^*\underline{BP}_0 \rightarrow BP^*DS^0$ is a surjection.

The statement that the map is an injection is just Corollary 4.8. Most of the work involved is calculating the image of the map. The last sentence is the key fact for our purposes. The proof is slightly different in case $p = 2$. I'm going to treat the case $p \neq 2$ here, see the appendix of [3] for the case when $p = 2$. Let K be the image of the map $BP^*(DS^0) \rightarrow HF_p^*(DS^0)$ and let A be the subalgebra of $(HF_p)_*(DS^0)$ generated under \star by the elements $Q^I[1]$ where Q^I is any Dyer Lashof operation with no Bocksteins, so that our goal is to prove that $K \cong A^*$.

In order to do this, first we'll prove the lower bound:

Proposition 6.2 ([3, Lemma 4.1]). *The map $HF_p^*(DS^0) \rightarrow \text{Hom}((HF_p)_*(DS^0), \mathbb{F}_p)$ induces a surjection $K \rightarrow A^*$.*

This is an easy corollary of Wilson's Theorem, which we'll assume:

Theorem 6.3 ([3, Theorem 2.1]).

1. $HZ_{(p)}^* \underline{BP}_0$ is torsion free.
2. The kernel of the map $(HF_p)_*(DS^0) \rightarrow (HF_p)_*(\underline{BP}_0)$ is generated by the elements $Q^I[1]$ where I is an admissible sequence containing at least one Bockstein.

Proof of Proposition 6.2. Because $HZ_{(p)} \underline{BP}_0$ is torsion free, the Bockstein spectral sequence for $HF_p^*(\underline{BP}_0)$ collapses, and so the map $BP^*(\underline{BP}_0) \rightarrow HF_p^*(\underline{BP}_0)$ is surjective. Consider the following diagram:

$$\begin{array}{ccc}
BP^*(\underline{BP}_0) & \longrightarrow & BP^*(DS^0) \\
\downarrow & & \downarrow \\
H\mathbb{F}_p^*(\underline{BP}_0) & \longrightarrow & H\mathbb{F}_p^*(DS^0)
\end{array}$$

Then K is the image of $BP^*(DS^0) \rightarrow H\mathbb{F}_p^*(DS^0)$ which contains the image of the composite $BP^*(\underline{BP}_0) \rightarrow H\mathbb{F}_p^*(DS^0)$ which is equal to the image of $H\mathbb{F}_p^*(\underline{BP}_0) \rightarrow H\mathbb{F}_p^*(DS^0)$ because the left map is surjective. By Wilson's theorem, the map $H\mathbb{F}_p^*(\underline{BP}_0) \rightarrow H\mathbb{F}_p^*(DS^0) \rightarrow \text{Hom}((H\mathbb{F}_p)_*(DS^0), \mathbb{F}_p)$ surjects onto A^* . \square

The harder part is establishing the upper bound. We have the following fact from [4]:

Theorem 6.4 ([3, Theorem 3.6]). *Ind $_*(H\mathbb{F}_p)_*(DS^0)$ is generated under \circ by $[1]$, $Q^s[1]$ and $\beta Q^s[1]$. Also, if $I = (\epsilon_1, i_1, \dots, \epsilon_n, i_n)$ then*

$$Q^I[1] = \prod_{j=1}^n \beta^{\epsilon_j} Q^{i_j}[1] \pmod{\text{decomposables}}$$

This essentially tells us that $(H\mathbb{F}_p)_*DS^0$ is completely controlled by the homology of the symmetric group – the elements $Q^s[1]$ and $\beta Q^s[1]$ all lie in the image of the map $(H\mathbb{F}_p)_*(B\Sigma_p) \rightarrow (H\mathbb{F}_p)_*(DS^0)$:

Lemma 6.5 ([3, Proposition 3.7]). *The inclusion $C_p \rightarrow \Sigma_p$ induces an injection on cohomology*

$$H\mathbb{F}_p^*(B\Sigma_p) \hookrightarrow H\mathbb{F}_p^*(BC_p) = E(\beta) \otimes P(x)$$

with image generated x^{p-1} and βx^{p-2} .

Proof. If $p = 2$, there is nothing to check, so suppose $p \neq 2$. Because C_p is a p -Sylow subgroup in Σ_p , the map on cohomology is an injection, with image given by the subalgebra of $H\mathbb{F}_p^*(BC_p)$ invariant under the action of the normalizer of C_p inside of Σ_p . Suppose C_p is the subgroup generated by the cycle $g = (0 \cdots (p-1))$ inside $\Sigma_p = \Sigma_{\mathbb{Z}/p}$. Then the cycle $\sigma = (0, 2, 4, \dots, 2(p-1))$ conjugates g to g^2 . This map generates $\text{Aut}(C_p) \cong C_{p-1}$, so we just need to find the elements that are invariant under all automorphisms of C_p . We have that $H\mathbb{F}_p^1(BC_p)$ is generated by the cocycle $f(g^i) = i$, and the automorphism σ^{-i} takes f to $(\sigma^{-i}f)(g^j) = f((g^j)^i) = ij = if(g)$, so $\sigma^{-i}\beta = i\beta$. Because the Bockstein is equivariant for this action, we deduce also that $\sigma^{-i}x = ix$. Now $i^a = 1$ for all $i \in \mathbb{F}_p$ if and only if $(p-1)$ divides a so $H\mathbb{F}_p^*(B\Sigma_p)$ is the subalgebra of $H\mathbb{F}_p^*(B\Sigma_p)$ generated by x^{p-1} and βx^{p-2} . \square

The element $x^{i(p-1)}$ is dual to an element $e_{2i(p-1)}$ and the element $\beta x^{i(p-1)-1}$ is dual to $e_{2i(p-1)-1}$. Unravelling the definition of the Dyer-Lashof operations gives that $Q^s[1]$ is the image of $e_{2i(p-1)}$ and $\beta Q^s[1]$ is the image of $e_{2i(p-1)-1}$. The goal is to understand the image of $BP^*(DS^0) \rightarrow H\mathbb{F}_p^*(DS^0)$ by figuring out the image of $BP^*(BC_p) \rightarrow H^*(BC_p)$ and combining this with Theorem 6.4. We have:

Lemma 6.6 ([3, Proposition 3.8]).

$$BP^*(BC_p) \widehat{\otimes}_{BP^*} \mathbb{F}_p \hookrightarrow H^*(BC_p) = E(\beta) \otimes P(x)$$

with image given by $P(x)$.

Proof. By the standard Gysin argument, $BP^*(BC_p) \cong BP^*[[x]]/([p](x))$. The element x is the BP euler class of the standard one dimensional representation of C_p and since the map $BP \rightarrow H\mathbb{F}_p$ is a complex oriented map, the euler class is mapped to the euler class, so we deduce that $P(x)$ is the $BP^*(BC_p) \rightarrow H\mathbb{F}_p^*(BC_p)$. The coefficients of $[p](x)$ all lie in the maximal ideal (p, v_1, v_2, \dots) , so $BP^*(BC_p) \widehat{\otimes}_{BP^*} \mathbb{F}_p = H\mathbb{F}_p^*[[x]]$ and hence the map is injective. \square

We can use Theorem 6.4 to build a map from a disjoint union of infinitely many $B(\Sigma_p)^n$'s to DS^0 that detects all of the $H\mathbb{F}_p$ homology of DS^0 :

Proposition 6.7 ([3, Lemma 4.2]). *There is a space X and a map $X \rightarrow DS^0$ such that the map $(H\mathbb{F}_p)_*(X) \rightarrow (H\mathbb{F}_p)_*(DS^0)$ is surjective and such that the image of $BP_*(X) \rightarrow BP_*(DS^0) \rightarrow H\mathbb{F}_{p*}(DS^0)$ has image A .*

Proof. By Theorem 6.4, to get the surjectivity it suffices to hit all expressions of the form

$$(\beta^{\epsilon_{1,1}} Q^{s_{1,1}}[1] \circ \dots \circ \beta^{\epsilon_{1,m_1}} Q^{s_{1,m_1}}[1]) \star \dots \star (\beta^{\epsilon_{n,1}} Q^{s_{n,1}}[1] \circ \dots \circ \beta^{\epsilon_{n,m_n}} Q^{s_{n,m_n}}[1])$$

given by first circling together various elements $Q^s[1]$ and then starring together the results. To get these, we take $X = \bigsqcup_m B\Sigma_p^{\Sigma^{m_i}}$ where m is allowed to vary over all finite length sequences of nonnegative rational numbers. The map $X \rightarrow DS^0$ is given on components by

$$B\Sigma_p^{\Sigma^{m_i}} \rightarrow B\left(\prod \Sigma_{p^{m_i}}\right) \rightarrow B\Sigma_{\prod p^{m_i}} \subseteq DS^0$$

where the first map is given by the multiplication maps $\Sigma_a \times \Sigma_b \rightarrow \Sigma_{ab}$ and the second map by the addition maps $\Sigma_a \times \Sigma_b \rightarrow \Sigma_{a+b}$.

The image of $BP_*(X) \rightarrow H\mathbb{F}_{p*}(X)$ is the set of elements of the form $Q^s[1]$ with no β , so then taking the image in $H\mathbb{F}_{p*}(DS^0)$ we recover A . \square

Proposition 6.8 ([3, Proposition 4.3]). $K \cong A^*$

Proof. Consider the following diagram, where each Im is the image of the vertical map above and H I'll define in a moment.

$$\begin{array}{ccccc} BP^*(DS^0) & \longrightarrow & BP^*(X) & & \\ \downarrow & & \downarrow & & \\ H\mathbb{F}_p^*(DS^0) & \hookrightarrow & H\mathbb{F}_p^*(X) & & \\ \uparrow & & \uparrow & \searrow & \\ K & \xlongequal{\quad} & Im & \hookrightarrow & Im & \hookrightarrow & H^* \end{array}$$

By the lemmas, we see that the image of $BP^*(B\Sigma_p)$ in $H\mathbb{F}_p^*(B\Sigma_p)$ is contained in the intersection of the image of $BP^*(BC_p)$ in $H\mathbb{F}_p^*(BC_p)$ and the image of $H\mathbb{F}_p^*(B\Sigma_p)$ in $H\mathbb{F}_p^*(BC_p)$, which just contains the elements $x^{s(p-1)}$ dual to $e_{s(p-1)}$ that correspond map to $Q^s[1]$. Letting H be the subspace of $H\mathbb{F}_p^*(X)$ generated by these elements $e_{s_1(p-1)} \otimes \dots \otimes e_{s_k(p-1)}$, we get a dual surjection $H\mathbb{F}_p^*(X) \rightarrow H^*$

Let B be the image of H in $H\mathbb{F}_{p*}(DS^0)$. We have:

$$\begin{array}{ccccc} & & H\mathbb{F}_{p*}(DS^0) & \longleftarrow & H\mathbb{F}_{p*}(X) & & \\ & \swarrow & \uparrow & & \swarrow & \searrow & \\ K & \longleftarrow & B & \longleftarrow & & & H \end{array}$$

We know that H is contained in the Hurewicz image of $BP_*(X)$ in $H\mathbb{F}_{p^*}(X)$, so by the second statement in Proposition 6.7, we deduce that $B \subseteq A$. Dually, we have that:

$$\begin{array}{ccc} B^* & \longleftarrow & A^* \\ \uparrow & \nearrow & \\ K & & \end{array}$$

We deduce that K has the same dimension in each degree as A^* and B^* so all the maps are isomorphisms. \square

Proof of Theorem 6.1. All we have left to check is that the map $BP^*\underline{BP}_0 \rightarrow BP^*DS^0$ is a surjection. Consider again the diagram:

$$\begin{array}{ccc} BP^*(\underline{BP}_0) & \longrightarrow & BP^*(DS^0) \\ & \searrow & \downarrow \\ & & H\mathbb{F}_p^*(DS^0) \end{array}$$

We now know that the images of the diagonal map and the vertical map are equal. Furthermore, $BP^*(DS^0) \widehat{\otimes}_{BP^*} H\mathbb{F}_p^* \hookrightarrow H\mathbb{F}_p^*(DS^0)$, so in order for the images to match, $BP^*(\underline{BP}_0) \widehat{\otimes}_{BP^*} H\mathbb{F}_p^* \rightarrow BP^*(DS^0) \widehat{\otimes}_{BP^*} H\mathbb{F}_p^*$ must be a surjection. Lifting the $H\mathbb{F}_p^*$ -module generators of $BP^*(DS^0) \widehat{\otimes}_{BP^*} H\mathbb{F}_p^*$ gives BP^* -module generators of $BP^*(DS^0)$, so $BP^*(\underline{BP}_0) \rightarrow BP^*(DS^0)$ is also a surjection. \square

Dually we deduce:

Corollary 6.9. *The map $BP_*(DS^0) \rightarrow BP_*(\underline{BP}_0)$ is an injection.*

Corollary 6.10. *The map $K(n)_*(DS^0) \rightarrow K(n)_*(\underline{BP}_0)$ is an injection.*

Proof. Because Σ_m is good, $K(n)_*(B\Sigma_m) = BP_*(B\Sigma_m) \otimes_{BP_*} K(n)_*$, and so $K(n)_*(DS^0) = BP_*(DS^0) \otimes_{BP_*} K(n)_*$. Thus it suffices to the injection $BP_*(DS^0) \hookrightarrow BP_*(\underline{BP}_0)$ up to $K(n)_*$. \square

6.2

7 Finite Subgroups of Formal Groups

In this section, we develop the formal group theory needed for the proof of the main theorem. Suppose that X is a formal scheme and \mathbb{G} is a height n formal group over \mathcal{O}_X . In the first subsection, we define the scheme $\text{Level}(A, \mathbb{G})$ of A level structures on \mathbb{G} for a finite abelian group A as a closed subscheme of $\text{Hom}(A, \mathbb{G})$, and verify Fact 3.3. In the second subsection, we define the scheme $\text{Sub}_{p^k}(\mathbb{G})$ and check part of Fact 3.9 by showing that $\mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G})}$ is free over E^0 and by computing its rank. Then we prove Fact 3.10. This is all from sections 7 and 10 of [6].

Most of these theorems work for arbitrary formal groups over an arbitrary formal scheme X , but occasionally they use assumptions like \mathbb{G} is the universal deformation or that \mathcal{O}_X is an integral domain with $p \neq 0$. The theorems about $\mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G})}$ I'll only prove in the case that \mathcal{O}_X is an integral domain with $p \neq 0$. This

hypothesis is removed using an easy base change argument in [6], but in these notes we only care about the case of $X = \text{spf } E(n)^0$ and $\mathbb{G} = \text{spf } E(n)^0(BU(1))$, so I'm going to leave out that unnecessary extra work.

7.1 The scheme of level structures

First we need to check:

Proposition 7.1. *Let A be a finite abelian p -group. The functor $\text{Hom}(A, G)(R) = \text{Hom}(A, \mathbb{G}(R))$ is a scheme.*

Note that because \mathbb{G} is a formal group, its R -points form a group for every complete local \mathcal{O}_X -algebra R . By $\text{Hom}(A, \mathbb{G}(R))$ I mean the set of group homomorphisms.

Proof. Suppose that $A = \prod_{i=1}^r C_{p^{d_i}}$. Then $\text{Hom}(A, \mathbb{G}) = \prod_{i=1}^r \mathbb{G}(d_i)$ is the product of the p^{d_i} torsion subschemes of \mathbb{G}^r . If x is a coordinate on \mathbb{G} then $\text{Hom}(A, \mathbb{G}) = \text{spf } \mathcal{O}_X[[x_1, \dots, x_r]] / ([p^{d_i}]_{\mathbb{G}}(x_i))$. \square

If G is a finite group, then we have the decomposition (of sets)

$$\text{Hom}(A, G) = \bigsqcup_{B \subseteq A} \text{Mon}(A/B, G).$$

However there is no scheme $\text{Mon}(A, \mathbb{G})$ satisfying $\text{Mon}(A, \mathbb{G})(R) = \text{Mon}(A, \mathbb{G}(R))$ – this isn't even a presheaf, because the map $R \rightarrow 0$ should induce a restriction map $\text{Mon}(A, \mathbb{G}(R)) \rightarrow \text{Mon}(A, \mathbb{G}(0))$, but $\mathbb{G}(0)$ is a single element set and so $\text{Mon}(A, \mathbb{G}(0))$ is empty. The scheme $\text{Level}(A, \mathbb{G})$ is our replacement for $\text{Mon}(A, G)$.

Definition 7.2 ([6, Definition 7.1]). *A level- A structure on a formal group \mathbb{G} over an X -scheme Y is a map $\phi : A \rightarrow \mathbb{G}(Y)$ such that $[\phi(A(1))] \leq \mathbb{G}(1)$ as divisors.*

In order to show that level structures form a scheme, we need the following proposition:

Proposition 7.3 ([6, Proposition 4.6]). *Given two divisors D and D' on \mathbb{G} over X there is a closed subscheme $Y \leq X$ such that Y is universal among maps $a : Z \rightarrow X$ such that $a^*D \leq a^*D'$.*

Proof. This is equivalent to checking that there is a closed subscheme $Y \leq X$ where $D \cap D' \rightarrow D$ is an isomorphism. If $\mathcal{O}_D = \mathcal{O}_X[x]/(f_D(x))$ and $\mathcal{O}_{D'} = \mathcal{O}_X[x]/(f_{D'}(x))$ and $\mathcal{O}_{D \cap D'} = \mathcal{O}_X[x]/f_{D \cap D'}$ then $f_{D \cap D'} = \text{gcd}(f_D, f_{D'})$, so the map $D \cap D' \rightarrow D$ corresponds to the map $\mathcal{O}_X[x]/(f_D(x)) \rightarrow \mathcal{O}_X[x]/(\text{gcd}(f_D, f_{D'}))$. This is an isomorphism if and only if f_D divides $f_{D'}$, which occurs on the closed subscheme cut out by the ideal generated by the coefficients of $f_{D'}(x) \pmod{f_D(x)}$.

This in fact holds more generally: given a diagram

$$\begin{array}{ccc} Z & \longleftrightarrow & W \\ \downarrow & \swarrow & \\ X & & \end{array}$$

where $Z \rightarrow W$ is a closed inclusion, there is a closed subscheme Y of X universal with the property that that the map $Z \times_X Y \rightarrow W \times_X Y$ is an isomorphism. To see this, let $J = \ker \mathcal{O}_W \rightarrow \mathcal{O}_Z$. If $Z \rightarrow W$ is an isomorphism over a point $I \in X$, then this implies that $\mathcal{O}_W/I = \mathcal{O}_Z/(I+J) \rightarrow \mathcal{O}_Z/I$ is a bijection, so that $I+J=I$, so we must have $J \subseteq I$. Now let I be the smallest ideal I of \mathcal{O}_X such that $\mathcal{O}_W I$ contains J , and $Y = V(I)$ is our desired subscheme.

In our case the ideal J is generated by $(f_{D'} \pmod{f_D})$ and in order to get an ideal I of \mathcal{O}_X such that $I\mathcal{O}_D \supseteq (f_{D'} \pmod{f_D})$, we need to put all of the coefficients of $(f_{D'} \pmod{f_D})$ into I . \square

Corollary 7.4 ([?, FiniteSubgroups]). *The functor $\text{Level}(A, G)(R) = \{\text{level-}A \text{ structures on } \mathbb{G} \text{ over } Y\}$ is represented by a scheme.*

Proof. Over $\mathbb{G} \times_X \text{Hom}(A, G)$ there is a universal map $\phi : A \rightarrow \text{Hom}(A, G)(\mathbb{G})$ and so a divisor $[\phi(A(1))]$. Apply the previous proposition to get a closed subscheme of $\text{Hom}(A, G)$ where $[\phi(A(1))] \leq [\mathbb{G}(1) \times_X \text{Hom}(A, G)]$ and define $\text{Level}(A, G)$ to be this subscheme. \square

Lemma 7.5 ([6, Lemma 7.5]). *If $\text{rank}(A) > n$ then $\text{Level}(A, \mathbb{G}) = \emptyset$.*

Proof. In this case, the degree of $[A(1)]$ will be greater than the degree of $[\mathbb{G}(1)]$, so the scheme where $[A(1)] \leq [\mathbb{G}(1)]$ is empty. \square

From now on, assume $\text{rank}(A) \leq n$.

Proposition 7.6 ([6, Lemma 7.6]). *If \mathcal{O}_Y is an integral domain and $p \neq 0$ in \mathcal{O}_Y then $\phi : A \rightarrow \mathbb{G}(Y)$ is a level structure if and only if ϕ is injective.*

Proof. Suppose that $\phi : A \rightarrow \mathbb{G}(Y)$ is a level structure. Note that for a map of p -groups $A \rightarrow B$ to be injective it suffices to check that the map $A(1) \rightarrow B(1)$ is injective, so we can assume that $A = A(1)$. We need to check that $\phi(a) \neq 0$ for all $a \in A$. Let x be a coordinate on \mathbb{G} and let $x_a = x(\phi(a))$ for $a \in A$. Then $f_{[\phi(A)]}(x) = \prod_{a \in A} (x - x_a) \in \mathcal{O}_{\mathbb{G} \otimes_X Y}$, and we have that $f_{[\phi(A)]}(x)[p](x) = px + \dots$. Since $p \neq 0$ in \mathcal{O}_Y , x^2 does not divide $[p](x)$, and so we must have $x_a \neq 0$ for $a \neq 0$. It follows that the map $A \rightarrow \mathbb{G}(Y)$ is injective.

Conversely, if $A \rightarrow \mathbb{G}(Y)$ is injective, then because \square

Proposition 7.7 ([6, Proposition 7.10]). *If \mathbb{G} is the universal deformation of \mathbb{G}_0 then $\text{Level}(A, \mathbb{G})$ is a smooth scheme of dimension n .*

Proposition 7.8 ([6, Proposition 7.12]). *If ϕ is a level structure then $[\phi(A(k))]$ is a subgroup scheme of $\mathbb{G}(k)$.*

7.2 The Scheme of Subgroups

Given a finite subgroup $K \subseteq \mathbb{G}$ of a formal group, we get an associated divisor on \mathbb{G} by taking the sum of all of the points in K . To make the scheme of subgroups of a given degree, we take the scheme of divisors of that degree, and then ask for those divisors K for which the following map factors:

$$\begin{array}{ccc}
 K \times K & \longrightarrow & \mathbb{G} \times \mathbb{G} \\
 \vdots & & \downarrow \mu \\
 K & \longrightarrow & \mathbb{G}
 \end{array}
 \qquad
 \begin{array}{ccc}
 E^*[[x, y]]/(f_K(x), f_K(y)) & \longleftarrow & E^*[[x, y]] \\
 \uparrow & & \uparrow x+fy \\
 E^*[[x]]/(f_K(x)) & \longleftarrow & E^*[[x]]
 \end{array}$$

The image of f_K in $\mathcal{O}_{K \times K}$ is $f_K(x +_F y)$ and the condition that the map factors is asking for $f_K(x +_F y) = 0 \pmod{f_K(x), f_K(y)}$. To form the scheme of subgroups of degree p^k , we take the universal divisor D over $\text{Div}_{p^k}(\mathbb{G})$ and set $\text{Sub}_{p^k}(\mathbb{G})$ to be the vanishing locus of all the coefficients of $f_D(x +_F y) \pmod{f_D(x), f_D(y)}$. This is evidently the right scheme. It's also clear that $X' \times_X \text{Sub}_{p^k}(\mathbb{G}) = \text{Sub}_{p^k}(X' \times_X \mathbb{G})$.

Let $\Lambda = (\mathbb{Q}_p/\mathbb{Z}_p)^n$. Our goal in this section is to prove:

Theorem 7.9 ([6, Theorem 10.1]). *The projection $\text{Sub}_{p^k}(\mathbb{G}) \rightarrow \text{spf } E^*$ is a finite flat map of degree d the number of subgroups of Λ of order n .*

7.2.1 Upper bound

To get an upper bound on the dimension, we are going to restrict to the special fiber \mathbb{G}_0 of \mathbb{G} and let \mathbb{H} be the base change of \mathbb{G}_0 to $\text{Sub}_{p^k}(\mathbb{G}_0)$. Then \mathbb{H} has a universal subgroup $K \subseteq \mathbb{H}$ and so we get a map $q : \mathbb{H} \rightarrow \mathbb{H}/K$. Then q corresponds in coordinates to the map $x \mapsto f_K(x) = x^{p^m} + \sum_{i=1}^{m-1} a_i x^i$ which has height m and \mathbb{H}/K has the same height n as \mathbb{H} . Suppose $[p]_{\mathbb{H}/K}(x) = x^{p^n} + \sum_{i=p}^{n-1} v_i x^i$ (because we're working at the special fiber $p = 0$ and $[p](x)$ starts $a_p x^p + \dots$). Let $Y = \text{Sub}_{p^k}(\mathbb{G}_0)$ and let $Y_{j,l}$ be the closed subscheme cut out by $(a_1, a_p, \dots, a_{p^{j-1}}, v_p, \dots, v_{p^l})$ – that is, $Y_{j,l}$ is the sublocus of Y where $q(x) = a_{p^l} x^{p^j} + \dots$ and $[p]_{\mathbb{H}/K}(x) = v_{p^l} x^{p^l} + \dots$. Let $E_{j,l} = \mathcal{O}_{Y_{j,l}}$ and let $e(j,l) = \dim E_{j,l}$. Note that since $\text{Sub}_{p^k}(\mathbb{G})$ is a closed subscheme of $\text{Div}_{p^k}(\mathbb{G})$, and the coefficients of the universal divisor generate the maximal ideal of $\mathcal{O}_{\text{Div}_{p^k}(\mathbb{G})}$, the coefficients of q generate the maximal ideal of $\mathcal{O}_{\text{Sub}_{p^k}(\mathbb{G}_0)}$. In particular, all of the maximal ideal is killed in $E(m,l)$, so $e(m,l) \leq 1$ for all l . We also have that $Y(0,1) = Y$, so the plan is to count the dimension of Y by inducting downwards on j and l . In [6], my $E(j,l)$ is called $E'(j,l)$, and my $e(j,l)$ is called $e'(j,l)$. His $e(j,l)$ is my $f(j,l)$ defined below.

First we need to do some counting.

Note that a subgroup of order m in Λ is the same as a lattice of index m in $\Lambda^* = \text{Hom}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$. Every lattice is of the form $M\mathbb{Z}_p$ for some $M \in M_n(\mathbb{Z}_p)$, and two such matrices M and N give the same lattice if there exists $P \in \text{GL}_n(\mathbb{Z}_p)$ such that $PM = N$.

Lemma 7.10 ([6, Lemma 10.2]). *Every matrix $M \in M_n(\mathbb{Z}_p)$ is equivalent to one of the form:*

$$\begin{pmatrix} p^{\alpha_0} & 0 & 0 & 0 \\ a_{01} & p^{\alpha_1} & 0 & 0 \\ a_{02} & a_{12} & p^{\alpha_2} & 0 \\ a_{03} & a_{13} & a_{23} & p^{\alpha_3} \end{pmatrix}$$

where $a_{ij} \leq p^{\alpha_j}$. The index of such a lattice is p^N where $N = \sum_{i=0}^{n-1} \alpha_i$.

Proof. Find the element of minimal valuation in the top row and move it's column to the top left column, then multiply this column by a unit to get p^{α_0} in the top right corner. Then use column operations to clear the rest of the top row. Recurse on the $(n-1) \times (n-1)$ submatrix without the first row and column. At the end, the matrix will have powers of p along the diagonal and zeroes above the diagonal. Now working from top to bottom use column operations to make the a_{ij} 's into the desired range. \square

Notation 7.11. Given a sequence $\alpha = (\alpha_0, \dots, \alpha_{n-1})$ such that $\sum_{i=0}^{n-1} \alpha_i = m$. Define:

$$|\alpha| = \sum_{j>0} \alpha_j$$

$$\|\alpha\| = \sum_j j\alpha_j$$

When writing such a sequence α , we will leave off α_0 since there's a unique value of α_0 making $\sum_i \alpha_i = m$.

Corollary 7.12 ([6, Corollary 10.3]).

$$d(m, n) = \sum_{\substack{\alpha \\ |\alpha| \leq m}} p^{|\alpha\|}$$

Now we also need another arithmetic function that we will eventually show is equal to $e(k, l)$.

Definition 7.13. Fix m and n . Define

$$f(j, l) = \sum_{\alpha \in S(j, l)} p^{|\alpha\|}$$

where $S(j, l) = \{\alpha \mid \alpha = (0, \dots, 0, \alpha_l, \dots, \alpha_{n-1}) \text{ and } |\alpha| \leq m - j\}$

We have that $f(0, 1) = d(m, n)$ and $f(j, n) = 1$ if $j \leq m$, which is promising – we have already seen that $e_{j, n} = 1 = f(j, n)$, and we want to show that $e_{0, 1} = d(m, n)$, for which it will suffice to show that $e(j, l) = f(j, l)$ for all j and l . We need the following recursive characterization of f .

Lemma 7.14 ([6, Lemma 10.6]). $f(j, l)$ satisfies the following recurrence:

$$\begin{aligned} f(j, l) &= 0 && \text{if } j > m \\ f(j, n) &= 1 && \text{if } j \leq m \\ f(j, l) &= f(j, l+1) + p^l f(j+1, l) && \text{if } l < n \end{aligned}$$

Proof. Consider a sequence $\alpha = (0, \dots, 0, \alpha_l, \dots, \alpha_{n-1})$. If $\alpha_l = 0$ then $\alpha \in S(j, l+1)$. If not, then $\alpha' = (0, \dots, 0, \alpha_l - 1, \dots, \alpha_{n-1})$ has $|\alpha'| = |\alpha| - 1 \leq m - j - 1$ so $\alpha' \in S(j+1, l)$. However, $\|\alpha'\| = \|\alpha\| - l$, so we see that

$$\begin{aligned} f(j, l) &= \sum_{\alpha \in S(j, l)} p^{|\alpha\|} \\ &= \sum_{\alpha \in S(j, l+1)} p^{|\alpha\|} + p^l \sum_{\alpha' \in S(j+1, l)} p^{|\alpha'|} \\ &= f(j, l+1) + p^l f(j+1, l) \end{aligned}$$

□

Lemma 7.15 ([6, Lemma 10.7]). Let B be a finite-dimensional algebra over a field k , and suppose $v, a \in B$ such that $va^r = 0$. Then $\dim_k B \leq r \dim_k B/a + \dim_k B/v$

Proof. For each $0 \leq j < r$, the quotient Ba^j/Ba^{j+1} is a cyclic module over B/a so $\dim_k Ba^j/Ba^{j+1} \leq \dim_k B/a$. Also, Ba^r is a cyclic module over B/v so $\dim_k Ba^r \leq \dim_k B/v$. It follows that

$$\dim_k B = \sum_{j=0}^{\infty} \dim_k Ba^j/Ba^{j+1} \leq r \dim_k B/a + \dim_k B/v$$

□

Proposition 7.16 ([6, Lemma 10.8]). *For all j and l , $e(j, l) \leq f(j, l)$. In particular, $\deg(\text{Sub}_m(\mathbb{G}) \rightarrow X) \leq d(m, n)$.*

Proof. The plan is to induct downwards on j and l . As a base case, we know that $e(m, l) = 1 \leq f(m, l) = 1$ for all $l \leq n$. Now we want to show that $e(j, l) \leq e(j, l+1) + p^l e(j+1, l)$ when $l < n$. Over $E(j, l)$, we have:

$$\begin{aligned} [p]_{\mathbb{H}}(x) &= ux^{p^n} + \cdots \quad \text{for some unit } u \\ [p]_{\mathbb{H}/K}(x) &= v_{p^l} x^{p^l} + \cdots \\ q(x) &= a_{p^j} x^{p^j} + \cdots \end{aligned}$$

Calculating the lowest terms in the equation $q([p]_{\mathbb{G}}(x)) = [p]_{\mathbb{H}}(q(x))$ gives

$$a_{p^j} u^{p^j} x^{p^{n+j}} + \cdots = v_{p^l} a_{p^j}^{p^l} x^{p^{j+l}} + \cdots$$

Since $l < n$, the power of x on the right hand side is strictly less than the power on the left hand side and we deduce that $v_{p^l} a_{p^j}^{p^l} = 0$. By Lemma 7.15 it follows that

$$e(j, l) = \dim E(j, l) \leq \dim(E(j, l)/v_{p^l}) + p^l \dim(E(j, l)/a_{p^j}) = e(j, l+1) + p^l e(j+1, l).$$

It remains to check that $e(j, n) = 1$. In this case we need to show that $a_{p^j} = 0$. By the same argument we get:

$$a_{p^j} u^{p^j} x^{p^{n+j}} + \cdots = v_{p^n} a_{p^j}^{p^n} x^{p^{j+n}} + \cdots$$

so $a_{p^j} (u^{p^j} - v_{p^n} a_{p^j}^{p^n-1}) = 0$. Since a_{p^j} is in the maximal ideal and u is a unit, $(u^{p^j} - v_{p^n} a_{p^j}^{p^n-1})$ is a unit, and we deduce that $a_{p^j} = 0$. □

7.2.2 Lower Bound and flatness

For these last two subsections, we assume \mathcal{O}_X is an integral domain where $p \neq 0$. In [6], this hypothesis is removed, but we don't bother because $E(n)^0$ is an integral domain where $p \neq 0$. Fix a subgroup K of \mathbb{G} of degree p^m and a level structure $\phi : \Lambda(m) \rightarrow \mathbb{G}$.

Proposition 7.17 ([6, Proposition 10.9]). *The map $\mathcal{O}_{\text{Sub}_m(\mathbb{G})}[1/p] \rightarrow F(\text{Sub}_m(\Lambda), \mathcal{O}_X[1/p])$ is surjective with nilpotent kernel.*

Lemma 7.18.

Lemma 7.19.

Corollary 7.20 ([6, Corollary 10.12]). *The map $\text{Sub}_m(\mathbb{G}) \rightarrow X$ is flat with degree $d(m, n)$.*

7.2.3 The Socle

Let $a = a_1$. Recall that the socle of a finite dimensional local algebra B over a field with maximal ideal \mathfrak{m} is the set of elements $\{b \in B \mid b\mathfrak{m} = 0\}$.

Proposition 7.21 ([6, Proposition 10.15]). *The ring $\mathcal{O}_{\text{Sub}_m(\mathbb{G}_0)}$ has socle generated by the element a^N where $N = p + \dots + p^{n-1}$.*

This proposition implies that $\text{Sub}_m(\mathbb{G})$ is Gorenstein. We need the following commutative algebra lemma:

Lemma 7.22 ([6, Lemma 10.16]). *If B is a finite dimensional local algebra over a field k with maximal ideal \mathfrak{m} and $v, a \in \mathfrak{m}$ satisfy $va^r = 0$ and $\dim_k(B) = r \dim_k(B/a) + \dim_k(B/v)$ then multiplication by a^k is a monomorphism $B/v \rightarrow B$ whose image is the annihilator of v . Also, this map induces an isomorphism $\text{soc}(B/v) \cong \text{soc}(B)$.*

Proof. As in the proof of Lemma 7.15, we filter B by the quotients Ba^j/Ba^{j+1} for $0 \leq j < r$ and Ba^r . Each Ba^j/Ba^{j+1} is a cyclic module over B/a and Ba^r is cyclic over B/v . In order to get the equality $\dim_k(B) = r \dim_k(B/a) + \dim_k B/v$, each of the resulting inequalities has to be an equality, and so Ba^j/Ba^{j+1} and Ba^r are in fact free modules of rank one, respectively over B/a and B/v . This implies that $*a^r : B/v \rightarrow Ba^r$ is an isomorphism and the corresponding map $B/v \rightarrow B$ is an injection. This map lands inside $\text{ann } v$, and because the sequence

$$0 \longrightarrow \text{ann } v \longrightarrow B \xrightarrow{\cdot v} B \longrightarrow B/v \longrightarrow 0$$

is exact, $\dim(\text{ann } v) = \dim(B/v)$ and so $B/v \cong Ba^r = \text{ann } v$.

Now we check that a^k induces an isomorphism $\text{soc } B/v \rightarrow \text{soc } B$. First note that a^k maps the socle of B/v into the socle of B : if $x\mathfrak{m}_{B/v} = 0$ this implies that $x\mathfrak{m} \subseteq (v)$, so $a^k x\mathfrak{m} = 0$. Since the map $B/v \rightarrow B$ is an injection, and this map is a restriction, the map is injective. Suppose $b \in \text{soc } B$. Then since $v \in \mathfrak{m}$, we see that $b \in \text{ann } v$. Thus, $b = a^r b'$ for some $b' \in B/v$. It remains to check that $b' \in \text{soc } B/v$. We have that multiplication by a^r maps $b'\mathfrak{m}_{B/v}$ to $a^r b'\mathfrak{m} = 0$. Since multiplication by a^r is an injection $B/v \rightarrow B$ we deduce that $b'\mathfrak{m}_{B/v} = 0$ and so that $b' \in \text{soc}(B/v)$. \square

Proof of Proposition 7.21. Recall that $e(j, l) \leq e(j, l+1) + p^l e(j+1, l)$ and that $f(j, l) = f(j, l+1) + p^l f(j+1, l)$. We know also that $e(0, 1) = d(m, n) = f(0, 1)$, so it follows that the inequalities for $e(j, l)$ must all be equalities. Now we derived the inequality $e(j, l) \leq e(j, l+1) + p^l e(j+1, l)$ in the proof of Proposition 7.16 by using that $E(j, l+1) = E(j, l)/v_{p^l}$, $E(j+1, l) = E(j, l)/a_{p^j}$, and $a_{p^j}^{p^l} v_{p^l} = 0$ in $E(j, l)$, so Lemma 7.15 implies the inequality. Because $e(j, l) = e(j, l+1) + p^l e(j+1, l)$ the hypotheses of Lemma 7.22 are satisfied and we deduce that multiplication by $a_{p^j}^{p^l}$ induces an isomorphism $\text{soc}(E(j, l+1)) \rightarrow \text{soc}(E(j, l))$. We're interested in the case $j = 0$. We have that $e(0, n) = 1$ and so the maximal ideal of $E(0, n)$ is zero and $\text{soc}(E(0, n)) = E(0, n)$ is generated by 1. Composing all the isomorphisms $a_1^{p^l} : \text{soc}(E(0, l+1)) \rightarrow \text{soc}(E(0, l))$, we deduce that $\text{soc } E(0, 1)$ is generated by a_1^N where $N = \sum_{l=1}^{n-1} p^l$ as desired. \square

References

- [1] JPC Greenlees and NP Strickland. Varieties and local cohomology for chromatic group cohomology rings. *Topology*, 38(5):1093–1139, 1999.

- [2] Michael Hopkins, Nicholas Kuhn, and Douglas Ravenel. Generalized group characters and complex oriented cohomology theories. *Journal of the American Mathematical Society*, 13(3):553–594, 2000.
- [3] Takuji Kashiwabara. Brown-Peterson Cohomology of QS^{2n} . *The Quarterly Journal of Mathematics*, 49(3):345–362, 1998.
- [4] JP May. E-infinity ring spaces and e-infinity spectra. *Lecture Notes in Mathematics, Springer, Berlin*, 1977.
- [5] Douglas C Ravenel, W Stephen Wilson, and Nobuaki Yagita. Brown–Peterson cohomology from Morava K-theory. *K-theory*, 15(2):147–199, 1998.
- [6] Neil P. Strickland. Finite subgroups of formal groups. *Journal of Pure and Applied Algebra*, 121(2):161 – 208, 1997.
- [7] Neil P. Strickland. Morava E-theory of symmetric groups. *Topology*, 37(4):757 – 779, 1998.
- [8] Neil P. Strickland and Paul R. Turner. Rational Morava E-theory and DS^0 . *Topology*, 36(1):137 – 151, 1997.