The positive complete model structure and why we need it

Hood Chatham

Alan told us in his talk about what information we can get about the homotopical structure of $\mathcal{S}^G$ directly. In particular, he built the homotopy category by "bare hands". However, we have a large number of functors that we care about, both on $\mathcal{S}^G$, and importantly on the category of rings comm^G. Most of these functors are not homotopical on the whole category of $\mathcal{S}^G$. In order to get these functors to behave well, they need to be derived. Thus, we want to find some appropriate model structure on $\mathcal{S}^G$ for which the various functors we care about are homotopical on maps between cofibrant objects so that we can use cofibrant replacement to derive them. In particular, we want a monoidal cofibrantly generated model structure on $\mathcal{S}^G$ such that:

(1) The weak equivalences are the stable equivalences

(2) The Kan Transfer Theorem applies to the adjunction $\mathcal{S}^G \underset{U}{\overset{\mathrm{Sym}}{\rightleftarrows}} \mathrm{comm}^G$

(3) If f is a cofibration or trivial cofibration then so is $\Phi^H(f)$ for all $H \subseteq G$.

(4) Many other things we don't explicitly have to worry about: cofibrant replacement should derive all of our favorite functors.

I should note at this point that (2) is the condition that is most important and will require the most work. It has nothing in particular to do with equivariant homotopy theory – this talk would be almost the same even if we were only interested in nonequivariant ring spectra. Condition (3) is the source of a minor modification to the model structure that explicitly relates to equivariantness.

Recall that a model category is a category with specified classes of cofibrations, fibrations, and weak equivalences satisfying the following conditions:

(1) All three classes are closed under composition and retracts.

(2) If two of f, g, and gf are weak equivalences, then so is the remaining.

(3) Call a morphism a trivial cofibration (resp triv. fibration) if it is a cofibration (resp. fibration) and a weak equivalence. Given a class C of morphisms, write $\mathrm{LLP}(C)$ for the set of maps with the left lifting property against elements of C and $\mathrm{RLP}(C)$ for the set of maps with the right lifting property against elements of C. We should have:
   (a) RLP(cofibrations) = trivial fibrations
   (b) RLP(trivial cofibrations) = fibrations
   (c) LLP(fibrations) = trivial cofibrations
   (d) LLP(trivial fibrations) = cofibrations

(4) Every map can be factored either as a trivial cofibration followed by a fibration or as a cofibration followed by a trivial fibration.

Note that if you write down two of the three classes of cofibrations, fibrations, and weak equivalences, there is at most one model structure with these classes as specified: for instance, if we have cofibrations and weak equivalences, the fibrations must be RLP(trivial cofibrations). If cofibrations and fibrations are given, then we know the trivial cofibrations and trivial fibrations too. Any weak equivalence may be factored as a trivial cofibration followed by a fibration, and by two of three the fibration must be a weak equivalence too, so we get the class of weak equivalences by taking compositions of any trivial cofibration followed by any trivial fibration. Note that not any pair of classes satisfying conditions (1) and (2) gives rise to a third collection that makes a model category – condition (3) need not be satisfied.

In general, there can be set theoretic problems in constructing the factorizations for condition (4). The concept of a cofibrantly generated model structure is designed to ensure that our model category can be defined using a set-theoretically small amount of data, which allows us to use the small object argument to show that condition (4) is satisfied, and also allows us to use a large collection of tools that only work on cofibrantly generated model categories, most importantly for this talk, the Kan Transfer theorem.

Recall that a cofibrantly model category is a category with two specified sets, the set $\mathcal{I}$ of generating cofibrations, and the set $\mathcal{J}$ of generating trivial cofibrations. These need to satisfy:

(1) $\mathcal{I}$ and $\mathcal{J}$ admit the small object argument (so there exists some cardinal κ such that all the domains of maps in $\mathcal{I}$ and $\mathcal{J}$ are κ-small)

(2) Write cofib(S) for LLP(RLP(S)). Set cofibrations = cofib($\mathcal{I}$), fibrations = RLP($\mathcal{J}$), trivial cofibrations = cofib($\mathcal{J}$), and trivial fibrations = RLP($\mathcal{I}$). Set the weak equivalences to be compositions of trivial cofibrations with trivial fibrations. These classes must satisfy condition (3) for model categories, and also we must have that trivial (co)fibrations = (co)fibrations $\cap$ weak equivalences.

These conditions are sufficient for these collections to be the collections of cofibrations, fibrations, and weak equivalences of a model category. This statement is called the Kan Recognition theorem. Note that in our case, we will be dealing with topological model categories, and all the domains of all of the maps in $\mathcal{I}$ and $\mathcal{J}$ will always be compact, so they will always admit the small object argument.

Given a category and a set of maps L, we call a map $f : X \to Y$ a relative L cell complex if f is a possibly transfinite composition of pushout maps $Z \to Z'$ of the form:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ Z \arrow[r] & Z' \end{tikzcd}$$

for $i \in L$. In fact, cofib(L) is the class of relative L-cell complexes. The two cases we care about are of course $L = \mathcal{I}$ and $L = \mathcal{J}$.

In Top, we obtain a cell complex by coning off maps from a sphere, which suggests that we should define the generating cofibrations and generating trivial cofibrations to be

$$\mathcal{I} = \{S^{n-1} \to D^n\}$$

$$\mathcal{J} = \{I^{n-1} \to I^n\}$$

This definition satisfies the Kan recognition theorem and gives a cofibrantly generated model structure on Top. This model structure has the standard cofibrations (relative cell complexes), fibrations (Serre fibrations), and weak equivalences (weak homotopy equivalences).

Now I will make an analogue for $\mathcal{S}^G$; this will be the *complete level model structure*. We set

$$\mathcal{I} = \{G_+ \wedge_H S^{-V} \wedge (S^{n-1}_+ \to D^n_+) \;:\; H \le G \text{ and } V \text{ an } H\text{-rep}\}$$
$$\mathcal{J} = \{G_+ \wedge_H S^{-V} \wedge (I^{n-1}_+ \to I^n_+) \;:\; H \le G \text{ and } V \text{ an } H\text{-rep}\}$$

Here "complete" means that H is allowed to range over the complete family of all subgroups of G (rather than some smaller family). The word "level" is a synonym for "strict," which was the term used yesterday. It refers to the weak equivalences – it turns out that the weak equivalences we get for this model structure are the levelwise weak equivalences: they are the maps $f : X \to Y$ of spectra such that $f_V : X_V \to Y_V$ is a weak equivalence of G-spaces for all G.

Let's demonstrate that the levelwise equivalences are the class of weak equivalences we get for the cofibrantly generated model structure on $\mathcal{S}^G$ specified by this $\mathcal{I}$ and $\mathcal{J}$. Note that all elements of $\mathcal{I}$ and $\mathcal{J}$ are levelwise weak equivalences. This implies that all relative $\mathcal{J}$ complexes are levelwise weak equivalences. We just need to check that all trivial fibrations are levelwise weak equivalences too.

The trivial fibrations are given by RLP($\mathcal{I}$). Such a trivial fibration needs to admit a lift for all diagrams of the form:

$$\begin{tikzcd} G_+ \wedge_H S^V \wedge S^{n+1}_+ \arrow[r] \arrow[d] & X \arrow[d] \\ G_+ \wedge_H S^{-V} \wedge D^n_+ \arrow[r] & Y \end{tikzcd}$$

which is equivalent by the adjunction between $G_+ \wedge -$ and Res^G_H to admitting a lift for all diagrams of the form:

$$\begin{tikzcd} S^{-V} \wedge S^{n-1}_+ \arrow[r] \arrow[d] & \mathrm{Res}^G_H X \arrow[d] \\ S^{-V} \wedge D^n_+ \arrow[r] & \mathrm{Res}^G_H Y \end{tikzcd}$$

Now using the adjunction between $S^{-V} \wedge : \mathrm{Top}^G_* \to \mathcal{S}^G$ to get S^{-V} to the other side:

$$\begin{tikzcd} S^{n-1}_+ \arrow[r] \arrow[d] & (\mathrm{Res}^G_H X)_V \arrow[d] \\ D^n_+ \arrow[r] & (\mathrm{Res}^G_H Y)_V \end{tikzcd}$$

Now asking for a map $Z \to W$ of pointed spaces to admit all lifts of this form implies that $Z \to W$ is a trivial fibration. We see that $X_V \to Y_V$ is a weak equivalence for all V and hence $X \to Y$ is a levelwise weak equivalence. Since every weak equivalence is the composition of a trivial cofibration with a trivial fibration and both trivial cofibrations and trivial fibrations are levelwise weak equivalences, we deduce that the weak equivalences are exactly the set of levelwise weak equivalences.

So we get the levelwise model structure. This isn't so nice because we care about stable things. If we can make the maps $e_{V,W} : S^{-(V\oplus W)} \wedge S^V \to S^{-W}$ into weak equivalences, this will get us our stable complete model structure. We have the most direct control over generating trivial cofibrations, which all must be weak equivalences, so our approach will be to add the maps $e_{V,W}$ to the weak equivalences by putting them into the set $\mathcal{J}$. Of course $\mathcal{J}$ can only contain cofibrations (that is, elements of $\mathrm{cofib}(\mathcal{I})$). Since we aren't planning on changing $\mathcal{I}$ and $\mathcal{I}$ admits the small object argument, we can use it to factor a morphism as an element of $\mathrm{cofib}(\mathcal{I})$ followed by an element of $\mathrm{RLP}(\mathcal{I})$ – that is as a cofibration followed by a trivial fibration. Let $\widetilde{e}_{V,W}$ be this cofibrant replacement of $e_{V,W}$. Let $\mathcal{K} = \{G_+ \wedge_H S^{-V} \wedge (I_+^{n-1} \to I_+^n) : H \leq G, V \text{ an } H\text{-rep}\}$ be our old $\mathcal{J}$, and let our new $\mathcal{J}$ be the union $\mathcal{J} = \mathcal{K} \cup \{\widetilde{e}_{V,W}\}$. (This isn't actually exactly what we do, because we want a monoidal cofibrantly generated model category, so we have to take $\mathcal{J} = \mathcal{K} \cup (\mathcal{I} \mathbin{\square} \{\widetilde{e}_{V,W}\})$, but I don't feel like defining the $\square$ product.)

This gives us the stable complete model structure. Later, I'll show that the fibrations in this model structure are relative Ω-spectra.

0.1. The positive (stable) complete model structure.

We are interested in studying G-equivariant commutative rings, so naturally we want to get a model structure on them. As is usually the case with categories of algebraic structures in another category, there is a free forgetful adjunction $\mathcal{S}^G \underset{U}{\overset{\mathrm{Sym}}{\rightleftarrows}} \mathrm{comm}^G$, where U is the functor that "forgets" the ring structure from a G-ring to get the underlying G-spectrum, and Sym is the functor $X \mapsto \bigvee_{n\in\mathcal{N}} \mathrm{Sym}^n X$, where $\mathrm{Sym}^n X = S^{\wedge n}/\Sigma_n$ (here the quotient is formed levelwise). Because we're going to frequently use both Sym and the forgetful functor, we want to make sure that the model structure we put on comm^G makes this adjunction to be a Quillen adjunction between model categories.

The Kan transfer theorem solves exactly the problem we're interested in. Given a cofibrantly generated model category $\mathcal{M}$, some other category $\mathcal{N}$ and an adjunction $\mathcal{M} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{N}$ (where $F : \mathcal{M} \to \mathcal{N}$ is the left adjoint), the Kan Transfer theorem gives us conditions that will allow us to "transfer" the model structure from $\mathcal{M}$ to $\mathcal{N}$ to get a cofibrantly model structure on $\mathcal{N}$ with weak equivalences given by maps that become weak equivalences after applying U ("underlying weak equivalences") and making the adjunction into a Quillen adjunction.

Theorem 0.1 (Kan Transfer Theorem). *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations $\mathcal{I}$ and trivial cofibrations $\mathcal{J}$, let $\mathcal{N}$ be a bicomplete category and let $\mathcal{M} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{N}$ be an adjunction. Let $F\mathcal{I} = \{Fi : i \in \mathcal{I}\}$ and likewise with $F\mathcal{J}$. Then if:*

(1) both $F\mathcal{I}$ and $F\mathcal{J}$ admit the small object argument and

(2) U takes relative $F\mathcal{J}$-cell complexes to weak equivalences

then there is a cofibrantly generated model structure on $\mathcal{N}$ such that $F\mathcal{I}$ and $F\mathcal{J}$ are the generating cofibrations and trivial cofibrations and the weak equivalences are the maps taken to weak equivalences by U. With respect to this model structure on $\mathcal{N}$, the adjunction (F, U) is a Quillen adjunction.

Condition (1) is harmless – it will be satisfied more or less automatically. Condition (2) is clearly necessary because we must have that all maps in $\mathrm{cofib}(F\mathcal{J})$ ("relative $F\mathcal{J}$-cell complexes") are weak equivalences, which means that Uf must be a weak equivalence for $f \in \mathrm{cofib}(F\mathcal{J})$.

From now on, we'll say that a map $f : X \to Y$ of rings is a weak equivalence if Uf is a weak equivalence. We want to apply the Kan transfer theorem with the adjunction $\mathcal{S}^G \underset{U}{\overset{\mathrm{Sym}}{\rightleftarrows}} \mathrm{comm}^G$. Unfortunately, with the current model structure on $\mathcal{S}^G$, condition (2) fails. Note that because weak equivalences are closed under transfinite compositions, it would suffice for the pushout map $X \to Y$ in the following diagram to be a weak equivalence for all $f : A \to B \in \mathcal{J}$ and for any ring X.

$$\begin{tikzcd} \mathrm{Sym}\, A \arrow[r, "\mathrm{Sym}\, f"] \arrow[d] & \mathrm{Sym}\, B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{0.1}$$

As a special case of this, for any trivial cofibration $A \to B$, we must have $\mathrm{Sym}\, A \to \mathrm{Sym}\, B$ a weak equivalence. By Ken Brown's Lemma, this implies that for any weak equivalence between cofibrant objections $A \to B$, we must have $\mathrm{Sym}\, A \to \mathrm{Sym}\, B$ a weak equivalence. I claim that this is not true. This is a purely non-equivariant concern, so let's take $G = \{e\}$ for the moment (the same problem occurs for any G). Consider the weak equivalence $e_{1,1} : S^{-1} \wedge S^1 \to S^0$. In the complete stable model structure, both $S^{-1} \wedge S^1$ and S^0 are cofibrant, so Ken Brown's Lemma tells us that if condition (2) of the KTT holds then $\mathrm{Sym}\, e_{1,1}$ must be a weak equivalence. I claim that $\mathrm{Sym}(e_{1,1})$ is not a weak equivalence. This will contradict condition (2).

On the one hand, we have $\mathrm{Sym}^n S^0 = (S^0)^{\wedge n}/\Sigma_n = (S^0)/\Sigma_n = S^0$ (since the only Σ_n action on S^0 is trivial). On the other hand, $(S^{-1} \wedge S^1)^{\wedge n} = (S^{-1})^{\wedge n} \wedge S^n = S^{-n} \wedge S^n$. To form Sym^n from this, we take the quotient, so we just need to figure out what the Σ_n action is. Now Σ_n acts on S^n in the expected way – treating S^n as the one-point compactification of $\mathbb{R}^n$, it permutes the n factors of $\mathbb{R}^n$. What about the action on S^{-n}? Recall that $\mathscr{J}(n, -) = \mathrm{Thom}(O(n, -) \downarrow \text{something})$ and this has a free left $O(n)$-action because $O(n)$ acts freely on $O(n, -)$. The action of Σ_n on $S^{-n} = \mathscr{J}(n, -)$ is via the inclusion $\Sigma_n \subset O(n)$, so that's also free. Since the action is free, $(S^{-n} \wedge S^n)/\Sigma_n$ is the same as the homotopy quotient $(S^{-n} \wedge S^n)_{\mathrm{ho}\,\Sigma_n} \simeq (S^0)_{\mathrm{ho}\,\Sigma_n} := (E\Sigma_n)_+ \wedge_{\Sigma_n} S^0 = \Sigma^\infty (B\Sigma_n)_+$.

So we deduce that $\mathrm{Sym}(S^{-1} \wedge S^1) = \bigvee_{n \in \mathbb{N}} \Sigma^\infty_+ B\Sigma_n$. This is not weak equivalent to $\mathrm{Sym}\, S^0 = S^0$, so the Kan Transfer Theorem doesn't apply. (You should think of $\mathrm{Sym}(S^{-1} \wedge S^1)$ as what $\mathrm{Sym}\, S^0$ "should" be – that is, this is the derived version of $\mathrm{Sym}\, S^0$.)

Note that Sym^n is the composition of the n-fold smash power with the quotient. The n-fold smash power operation is homotopical, but the quotient is not. however, that as long as the action of Σ_n is free, then the actual quotient will be homotopy equivalent to the homotopy quotient, and the homotopy quotient is homotopical. This free action happens quite generally: our proof above that S_n acts freely on $(S^{-V})^{\wedge n}$ works as long as $\dim V > 0$. So now let's change the definition of $\mathcal{I}$ and $\mathcal{J}$, replacing "for all H-representations V" to "for all representations V such that $\dim V^H > 0$":

$$\begin{aligned}
\mathcal{I} &= \{G_+ \wedge_H S^{-V} \wedge (S_+^{n-1} \to D_+^n) \ : \ \ H \le G \text{ and } V \text{ an } H\text{-rep with } \dim V^H > 0\} \\
\mathcal{K} &= \{G_+ \wedge_H S^{-V} \wedge (I_+^{n-1} \to I_+^n) \ : \ H \le G \text{ and } V \text{ an } H\text{-rep with } \dim V^H > 0\} \\
\mathcal{J} &= \mathcal{K} \cup \{\widetilde{e}_{V,W}\} \,\square\, \mathcal{I}
\end{aligned}$$

Requiring $\dim V > 0$ would be sufficient to ensure that Sym condition (2) of the Kan Transfer Theorem is met, but then $\varphi^H(G_+ \wedge_H S^{-V} \wedge (S_+^{n-1} \to D_+^n)) = S^{-V^H} \wedge (S_+^{n-1} \to D_+^n))$ and so if $\dim V^H = 0$ then $\varphi^H(f)$ is not positive cofibrant even though f is. So instead we require the mildly stronger condition that $\dim V^H > 0$.

Remark 0.2. This modification of the model structure means that the sphere spectrum is no longer cofibrant. This was necessary because $e_{1,1} : S^{-1} \wedge S^1 \to S^0$ is a weak equivalence and $\mathrm{Sym}(e_{1,1})$ is not a weak equivalence. Condition (2) of the KTT implies (by Ken Brown's Lemma) that Sym takes weak equivalences between cofibrant objects to weak equivalences. Thus, in a model structure where the KTT applies, either $S^{-1} \wedge S^1$ or S^0 must not be cofibrant. The positivity condition keeps $S^{-1} \wedge S^1$ cofibrant and makes S^0 not cofibrant. This was more or less necessary, since for any reasonable change of the generating set, we will have $S^{-n} \wedge S^n$ cofibrant for some $n \gg 0$, and then the same problem will apply to $e_{n,n}$.

0.2. Bonus: Fibrations are relative Ω-spectra. The category $\mathcal{S}^G$ is a topological model category. This means that given $i : A \to B$ a cofibration and $p : X \to Y$ a fibration, one which must be trivial, there is a contractible choice of lifts $A \to X$ for any diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r] & Y
\end{tikzcd} \tag{0.2}$$

In particular, the lifting condition for a topological model category $\mathcal{M}$ is as follows: given (i, p) as above, there is a space of pairs $\mathcal{M}(A, X) \times \mathcal{M}(B, Y)$ of maps $B \to Y$ and $A \to X$. We want to add the condition that the diagram commutes. There is a continuous map $\mathcal{M}(B, Y) \to \mathcal{M}(A, Y)$ given by precomposing with i and a map $\mathcal{M}(A, X) \to \mathcal{M}(A, Y)$ given by postcomposing with p. The condition that a pair of maps $f : A \to X$ and $g : B \to Y$ make the diagram commute is that $pf = gi$. Thus, the space of commuting diagrams is given by

$$\mathcal{M}(A, X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B, Y).$$

Any lift $l : B \to X$ gives a commuting diagram by taking $f = li$ and $g = pl$. Thus there is a map

$$\mathcal{M}(B, X) \to \mathcal{M}(A, X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B, Y).$$

If $\mathcal{M}$ is a topological model category, this map is assumed to be a trivial fibration whenever i is a cofibration, p a fibration, and one of the two is a weak equivalence.

$$\begin{tikzcd}
\mathcal{M}(B,X) \arrow[dr, "\text{triv. fib.}"] \arrow[drr, bend left, "i^*"] \arrow[ddr, bend right, "p_*"'] & & \\
& \mathcal{M}(A,X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B,Y) \arrow[r] \arrow[d] & \mathcal{M}(A,X) \arrow[d, "p_*"] \\
& \mathcal{M}(B,Y) \arrow[r, "i^*"'] & \mathcal{M}(A,Y)
\end{tikzcd}$$

Let's consider what this means for $\mathcal{S}^G$. Consider a map $p : X \to Y$ of spectra. It suffices to check cofibrancy on the generating trivial cofibrations. We showed above that lifting against the maps in $\mathcal{K}$ means that p is a levelwise fibration, except that we checked this before we added the positivity condition. With the extra positivity condition, p_V only has to be a fibration when $\dim V^H > 0$, you can call this a "positive levelwise fibration". Thus, we just need to check what lifting against $e_{V,W} : S^{-(V\oplus W)} \wedge S^V \to S^{-W}$ means. Note that $\mathcal{S}^G(S^{-W}, Z) = Z_W$ and $\mathcal{S}^G(S^{-(V\oplus W)} \wedge S^V, Z) = \Omega^V X_{V\oplus W}$. Instantiating the previous diagram using $i = e_{V,W}$ and using these identities of mapping spaces, we get the following diagram:

$$\begin{tikzcd}
X_W \arrow[dr, "\text{triv. fib.}"] \arrow[drr, bend left] \arrow[ddr, bend right, "p_W"'] & & \\
& \Omega^V X_{V\oplus W} \times_{\Omega^V Y_{V\oplus W}} Y_W \arrow[r] \arrow[d, "i"'] & \Omega^V X_{V\oplus W} \arrow[d, "\Omega^V p_{V\oplus W}"] \\
& Y_W \arrow[r] & \Omega^V Y_{V\oplus W}
\end{tikzcd}$$

So we learn that for all V, W, we have that $X_W \simeq \Omega^V X_{V\oplus W} \times_{\Omega^V Y_{V\oplus W}} Y_W$. When Y is a point, this says that $X_W \simeq \Omega^V X_{V\oplus W}$, which means that X is an Ω-spectrum. Because of this, in the general case we say that $p : X \to Y$ is a relative Ω-spectrum.