Goodwillie Differentials and Hopf Invariants

Hood Chatham

October 11, 2017

1 Recollections

Recall that the Goodwillie spectral sequence has E_1 page given by the homotopy of the spectra $L(k)_n$ and converges to the unstable homotopy of S^n:

$$\mathrm{GSS}(S^n)\colon \pi_* D_{2^k} S^n = \pi_{*+n-k} L(k)_n \Rightarrow \pi_* S^n$$

The spectrum $L(k)_n$ has a cell structure in terms of $L(k-1)_m$ for varying m, so there is an AHSS calculating $\pi_* L(k)_n$ from $\pi_* L(k)_m$. Putting all of these Atiyah Hirzebruch spectral sequences together, we get a transfinite AHSS computing the homotopy of $L(k)_n$ from the stable homotopy of the sphere. The homology of $L(k)_n$ has basis given by completely inadmissable (CU) sequences of length k and excess at least n, so that $\mathrm{TAHSS}(L(k)_n)$ has signature

$$\mathrm{TAHSS}(L(k)_n)\colon \bigoplus_{JCU,\mathrm{exc}(J)\geq n,|J|=k} \pi_* S^{||J||} \Rightarrow \pi_* L(k)_n$$

Putting together the TAHSS and the GSS, we get the TGSS:

$$\mathrm{TGSS}(S^n)\colon \bigoplus_{JCU,\mathrm{exc}(J)\geq n} \pi_* S^{||J||} \Rightarrow \pi_* S^n$$

where the CU sequences of length k are contributing to the homotopy of $L(k)_n$, which is the k-line of the GSS.

Also recall that there are 2-primary fibrations:

$$\Omega^2 S^{m+1} \xrightarrow{H} \Omega^2 S^{2m+1} \xrightarrow{P} S^m \xrightarrow{E} \Omega S^{m+1}.$$

These fibrations restrict to fibrations of Goodwillie towers and so give maps of TGSS. On the E_1 page, E is given by:

$$\begin{array}{ccc} D_{2^k}S^m & \longrightarrow & D_{2^k}\Omega S^{m+1} \\ \| & & \| \\ \Sigma^{m-k}L(k)_m & \longrightarrow & \Sigma^{m-k}L(k)_{m+1} \\ & & \\ \beta[J] & \longmapsto & \begin{cases} \beta[J] & \text{if exc } J \geq m+1 \\ 0 & \text{if exc } J = m \end{cases} \end{array}$$

and P, which shifts Goodwillie filtration by one, is given by:

$$\begin{array}{ccc} D_{2^{k-1}}\Omega^2 S^{2m+1} & \longrightarrow & D_{2^k}\Omega S^m \\ \| & & \| \\ \Sigma^{2m-k}L(k-1)_{2m+1} & \longrightarrow & \Sigma^{m-k}L(k)_m \\ & & \\ \beta[J] & \longmapsto & \beta[J,m] \end{array}$$

If $[J]$ is CU and $\mathrm{exc}(J) \geq 2m+1$ then $[J,m]$ is CU and $\mathrm{exc}(J) = m$.

2 The stable Hopf invariant and Goodwillie d_1 on the zero line

We would like to calculate differentials in the Transfinite Goodwillie spectral sequence (TGSS) of the identity functor applied to a sphere. Our starting point for computing differentials in the TGSS is the following fact, which identifies the Goodwillie d_1 on elements in the zero line. For any space X there is a map $\mathrm{JH}\colon QX \to QX^{\wedge 2}_{h\Sigma_2}$ adjoint to a projection map in the Snaith splitting. This is called the James-Hopf map. Note that $Q(S^n)^{\wedge 2}_{h\Sigma_2} = \mathbb{RP}^\infty_n$.

Theorem 1. *There is a fiber sequence:*

$$\begin{array}{ccccc} D_2S^n & \longrightarrow & P_2S^n & & \\ & & \downarrow & & \\ & & P_1S^n & \xrightarrow{\mathrm{JH}} & Q\mathbb{RP}^\infty_n = \Omega^\infty L(1)_n \end{array}$$

Since JH is the going around map and $D_1S^n = P_1S^n$, this identifies d_1 in the GSS as JH. So in the TGSS, $d(\alpha)$ is given by a detecting element for $\mathrm{JH}(\alpha)$ in the E_1 page of the AHSS for $\mathbb{RP}^\infty_n$. We give the following definition:

Definition 2. *Given an element $\alpha \in \pi^s_*$, the stable Hopf invariant* $\mathrm{SHI}(\alpha)$ *is the coset of the E_1-page of the* AHSS *detecting* $\mathrm{JH}(\alpha)$. *Explicitly, lift* $\mathrm{JH}(\alpha) \in \pi_*Q\mathbb{RP}^\infty_n$ *to an element of* $\widetilde{\mathrm{JH}(\alpha)} \in$

$\pi_* Q\mathbb{RP}^m_n$ for m minimal and then map it via the compression map to $\pi_* QS^m$, call the result $\beta[m]$. Then $\beta[m] \in \mathrm{SHI}(\alpha)$.

Then we get the following theorem:

Theorem 3. *Suppose $\alpha \in \pi_*^s$ considered as the zero line of $\mathrm{TGSS}(S^n)$. Then:*

$$d(\alpha) = \begin{cases} \beta[m] & \beta[m] \in \mathrm{SHI}(\alpha) \text{ and } m \geq n \\ 0 & m < n \end{cases}$$

In the first case, the differential is nontrivial.

Proof. The d_1 differential in the $\mathrm{GSS}(S^1)$ on the zero line is given by $\mathrm{JH}\colon QS^1 \to Q\mathbb{RP}^\infty$, and $\mathrm{SHI}(\alpha)$ is defined to be the set of elements detecting $\mathrm{JH}(\alpha)$, it's clear that $d_1(\alpha) = \beta[m]$. Since $m \geq 1$ no matter what, we have to show that this is a nontrivial differential, i.e., that $\beta[m] \neq 0 \in \pi_* Q\mathbb{RP}^\infty$. This is due to the Kahn-Priddy theorem, which says that JH is injective on π_*.

Applying E repeatedly, we deduce that d_1 has the desired form in $\mathrm{TGSS}(S^n)$. If $m \geq n$, then the differential is nontrivial because $\beta[m]$ is a nonzero element of $\pi_* Q\mathbb{RP}^\infty_n$ – no differential hits $\beta[m]$ in $\mathrm{AHSS}(\mathbb{RP}^\infty)$ by Kahn-Priddy, and the AHSS for $\mathbb{RP}^\infty_n$ is a truncation, so no differentail hits $\beta[m]$ there either.

3 The Hopf invariant and longer Goodwillie differentials on the zero line

Definition 4. *Given an element $\alpha \in \pi_t^s$, the Hopf invariant $\mathrm{HI}(\alpha)$ is the coset detecting α in the E_1 page of the EHPSS. Explicitly, lift α to an element $\widetilde{\alpha} \in \pi_{t+n+1} S^{n+1}$ for n as small as possible (we say α is born on S^{n+1}). Then apply $H : \Omega^{n+1} S^{n+1} \to \Omega^{n+1} S^{2n+1}$. Then $H(\widetilde{\alpha}) \in \mathrm{HI}(\alpha)$. The generalized Hopf invariant $\mathrm{GHI}(\alpha)$ is the coset detecting $\mathrm{HI}(\alpha)$ in the E_1 page of $\mathrm{TGSS}(\Omega^{n+1} S^{2n+1})$.*

To relate the Hopf invariant to the stable Hopf invariant, we use the map of spectral sequences $\mathrm{EHPSS} \to \mathrm{AHSS}(Q\mathbb{RP}^\infty)$. This map is given by maps of fibrations:

$$\begin{array}{ccccc} \Omega^n S^n & \xrightarrow{E} & \Omega^{n+1} S^{n+1} & \xrightarrow{H} & \Omega^{n+1} S^{2n+1} \\ \downarrow{\scriptstyle \mathrm{JH}_n} & & \downarrow{\scriptstyle \mathrm{JH}_{n+1}} & & \downarrow{\scriptstyle E^\infty} \\ Q\mathbb{RP}^{n-1} & \longrightarrow & Q\mathbb{RP}^n & \longrightarrow & QS^n \end{array}$$

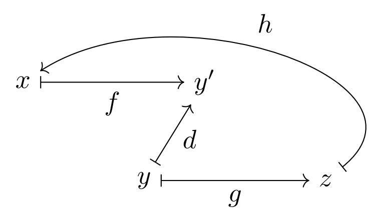
so if $\beta \in \mathrm{HI}(\alpha)$ is stable (meaning $E^\infty(\beta) \neq 0$) then $\mathrm{SHI}(\alpha) = E^\infty(\mathrm{HI}(\alpha))$. If β is unstable, then $E^\infty(\mathrm{HI}(\alpha)) = 0$ and $\mathrm{JH}(\alpha)$ lifts into lower filtration in the Atiyah Hirzebruch spectral sequence than α does in the EHP spectral sequence.

So now we want to use the Hopf invariant to understand longer differentials on the zero line in the TGSS. The idea is to study how the GSS interacts with the EHP fibration. Suppose that $\alpha \in \pi_t^s$ is born on S^{n+1}. This means that α supports a Goodwillie differential in GSS(S^n) but is a permanent cycle in GSS(S^{n+1}). We want to relate the target of the Goodwillie differential on α in GSS(S^n) to the image of $\alpha \in \pi_* \Omega S^{n+1}$ under the going around map $H : \Omega S^{n+1} \to \Omega S^{2n+1}$. To do this we use the geometric boundary theorem:

Theorem 5 (Geometric Boundary Theorem (part 5)). *Suppose that*

$$\Omega Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a fibration of filtered objects (so that the maps give maps of spectral sequences). Suppose that $d_i(y) = y'$ in $E_i(Y)$ and $g(y)$ is a permanent cycle. Then $y' = f(h(z))$.



Also, suppose that z detects a nonzero element in $\overline{z} \in \pi_* Z$. Then either $h(\overline{z})$ is detected by x or x is hit by a differential longer than d and $h(\overline{z})$ is detected in higher filtration.

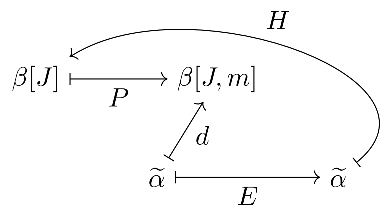
Now we use this theorem to calculate Goodwillie differentials:

Theorem 6. *Suppose that α is born on S^{m+1}. Then there is a nontrivial differential $d(\alpha) = \beta[J, m]$ in* TGSS(S^m) *where $\beta[J] \in$* GHI(α)*. In* TGSS(S^n) *for $n < m$, either $d(\alpha) = \beta[J, m]$ is a nontrivial differential or α supports a shorter differential.*

Proof. Write $\widetilde{\alpha}$ for the element of $E^1_{0,*}$(TGSS) and write α for the element $\widetilde{\alpha}$ detects in homotopy. Because α is born on S^{m+1}, $\widetilde{\alpha}$ isn't a permanent cycle on S^m, so it must support a differential in TGSS(S^m). On the other hand, $E(\widetilde{\alpha})$ is a permanent cycle in TGSS(S^{m+1}) detecting α. So we can apply the geometric boundary theorem to the fibration:

$$\Omega^2 S^{m+1} \xrightarrow{H} \Omega^2 S^{2m+1} \xrightarrow{P} S^m \xrightarrow{E} \Omega S^{m+1}$$

so f is P, g is E, and h is H and we have:



Suppose $H(\widetilde{\alpha}) = \beta[J]$. By our expression for the P map, $P(\beta[J]) = \beta[J, m]$. Now $\widetilde{\alpha}$ detects α in $\mathrm{TGSS}(S^{m+1})$ so either $\beta[J] = H(\widetilde{\alpha})$ detects $H(\alpha)$ or $\beta[J]$ is hit by a longer differential than $\beta[J, m]$ and $H(\alpha)$ is detected in higher filtration. But $\beta[J, m]$ is hit by α which is in the zero line, so there are no longer differentials. Thus $\beta[J]$ detects $H(\alpha)$ and so $\beta[J] \in \mathrm{GHI}(\alpha)$.

4 Propagating Differentials

Now we know the behavior of differentials that start on the zero line, at least insofar as we can calculate Hopf invariants. We would like to convert this into information about differentials starting in higher filtration. Conveniently, the P map shifts filtration, and this is the tool we need. We have the following theorem:

Theorem 7. *Suppose that* $m \leq 2n + 1$, $n \geq m'$ *and* $d(\alpha[J, a]) = \beta[J', a']$ *in* $\mathrm{TGSS}(S^m)$ *for* $a, a' \geq 2n + 1$. *Then either:*

(1) $d(\alpha[J, a, n]) = \beta[J', a', n])$ *is a nontrivial differential in* $\mathrm{TGSS}(S^{m'})$

(2) $\beta[J', a', n]$ *is the target of a shorter differential in* $\mathrm{TGSS}(S^{2n+1})$

(3) $\alpha[J, n]$ *supports a shorter differential in* $\mathrm{TGSS}(S^{m'})$

5 Exotic Differentials

Differentials that don't come from propogating Hopf invariants are called "exotic". There are two types of these that occur in the Toda range:

- Geometric boundary effect differentials which come from applying the geometric boundary theorem for an EHP fibration to nonexotic TGSS differentials.
- "Bizarre" differentials, which are needed to make $\mathrm{GSS}(S^1)$ correctly calculate the known unstable homotopy of S^1.

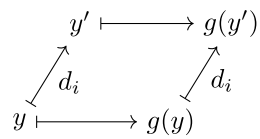
To say a bit more about Geometric boundary effect differentials, here is the full geometric boundary theorem:

Theorem 8 (Geometric boundary theorem). *Suppose*

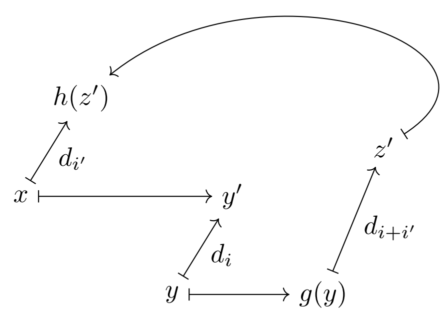
$$\Omega Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a fibration of filtered objects (so that the maps give maps of spectral sequences). Suppose that $d_i(y) = y'$ *in* $E_i(Y)$*. Then one of the following occurs:*

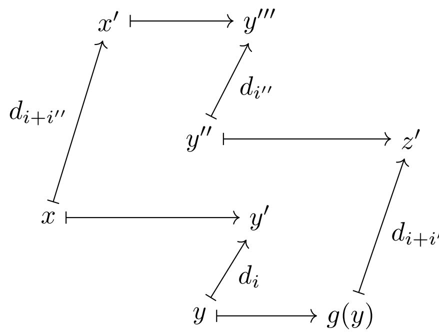
(1) $g(y') \neq 0 \in E_n(Z)$ *and so* $d_n(g(y)) = g(y')$ *is a nontrivial differential:*



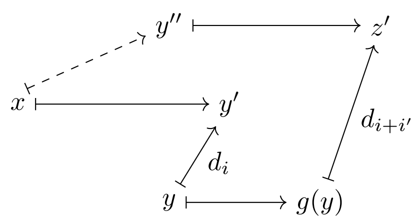
(2) $g(y') = 0$ *and* $g(y)$ *supports a longer differential* $d_{i+i'}(g(y)) = z'$ *and* $h(z') \neq 0$*. Then there is a differential* $d_{i'}(x) = h(z')$ *and* $f(x) = y'$*.*



(3) As in case (2), $g(y') = 0$ *and* $g(y)$ *supports a longer differential* $d_{i+i'}(g(y)) = z'$*. However, now suppose* $h(z') = 0$ *and* $z' = g(y'')$*. Suppose also that* y'' *supports a differential* $d_{i''}(y'') = y'''$*. Then there are is a differential* $d_{i'+i''}(x) = x'$ *as in the following picture:*

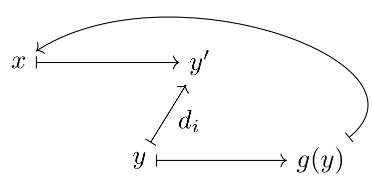


(4) *As in case (3) suppose that* $g(y') = 0$, *that* $g(y)$ *supports a differential* $d_{i+i'}(g(y)) = z'$, *that* $h(z') = 0$, *and that* $z' = g(y'')$. *However, now suppose that* y'' *is a permanent cycle. Then there is an element* $\overline{x} \in \pi_*(X)$ *and elements* x *and* y'' *as in the following picture:*



and either x *detects* $\overline{x}$ *in homotopy or* x *is hit by a differential of length greater than* $i + i'$ *and* $\overline{x}$ *is detected in higher filtration. Also either* y'' *detects* $f(\overline{x})$ *in homotopy or* y'' *is hit by a differential longer than* $i + i'$ *and* $f(\overline{x})$ *is detected in higher filtration.*

(5) *Now suppose* $g(y') = 0$ *and* $g(y)$ *is a permanent cycle. Then* $y' = f(h(z))$. *Also, suppose that* z *detects a nonzero element in* $\overline{z} \in \pi_* Z$. *Then either* $h(\overline{z})$ *is detected by* x *or* x *is hit by a differential longer than* d *and* $h(\overline{z})$ *is detected in higher filtration.*



We've already investigated case (5) and case (1). The "geometric boundary effect" come from case (3).