Algebraic structures in equivariant homotopy theory

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G-equivariant abelian groups

Lukas told us last time that a weak homotopy equivalence in the category of G-spectra is a map $f : X \to Y$ inducing isomorphisms $\pi_*^H(X) \to \pi_*^H(Y)$ for all subgroups $H \subseteq G$. In standard homotopy theory, $\pi_*(X)$ is a graded abelian group. In equivariant homotopy theory, we should replace the category of abelian groups with something that tracks the structure of systems like $\underline{\pi_0}(X) := \{\pi_0^H(X) \,|\, H \subseteq G\}$. We would like to put as rich of a structure on these $\underline{\pi_0}(X)$ as we can. So what structure does $\underline{\pi_0}$ have? Clearly we have restriction maps $\pi_0^H \to \pi_0^K$ for any $K \subseteq H$ because any H fixed point is naturally a K fixed point. We also have conjugation maps $c_g : \pi_0^H \to \pi_0^{gHg^{-1}}$.

What else do we have? One useful observation is that

$$\pi_0^H(X) = [\Sigma_+^\infty G/H, X]^G.$$

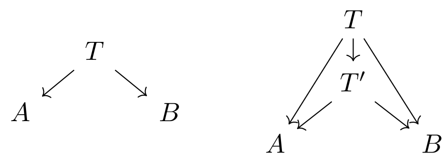
This means that we can make $\underline{\pi_0}(X)$ into a contravariant functor $\Sigma_+^\infty\{\text{finite } G\text{-sets}\} \to \text{Set}$ by setting $\underline{\pi_0}(X)(A) = [\Sigma_+^\infty A, X]^G$ for A a finite G-set, where $\Sigma_+^\infty\{\text{finite } G\text{-sets}\}$ is the category of finite G-sets and stable maps – that is, the full subcategory of G-spectra on the suspension spectra of finite G-sets. The functors we get this way from the $\underline{\pi_0}$ construction take coproducts to products:

$$\begin{aligned}\underline{\pi_0}(X)(A \sqcup B) &= [\Sigma_+^\infty(A \sqcup B), X] \\ &= [\Sigma_+^\infty A \vee \Sigma_+^\infty B, X] \\ &= [\Sigma_+^\infty A, X] \times [\Sigma_+^\infty B, X] \\ &= \underline{\pi_0}(X)(A) \times \underline{\pi_0}(X)(B)\end{aligned}$$

So our equivariant version of abelian groups we will take to be functors $\Sigma_+^\infty\{\text{finite } G\text{-sets}\}^{\text{op}} \to \text{Set}$ which take coproducts to products. Now we have a description of the category $\Sigma_+^\infty\{\text{finite } G\text{-sets}\}$ from the tom Dieck splitting principal. But for motivation, let's first consider the case when $G = \{e\}$.

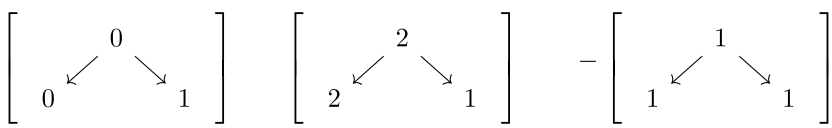
In this case, $\underline{\pi_0}(X)(A) = \pi_0(X)^{\times A}$ because $A = \bigsqcup_A *$. What does $[\Sigma_+^\infty A, \Sigma_+^\infty B]$ look like? We know from Spanier-Whitehead duality that $[A, B] = [S^0, A^\vee \times B] \cong \mathbb{Z}\{A^\vee \times B\}$, where of course $A^\vee \simeq A$, but its convenient to record the domain. We should think about the elements of $[A, B] = \mathbb{Z}\{A^\vee \times B\}$ as A by B matrices $\sum n_{a,b}(a^\vee, b)$. Composition in Σ_+^∞finite sets is then given by composition of matrices. We'd like to take this category and throw in G-actions everywhere, but this is difficult because there isn't an obvious G-equivariant notion of a matrix.

It helps to give the following alternate description of the category $\Sigma_+^\infty\{\text{finite sets}\}$ (of $\mathbb{Z}$-matrices): consider the bicategory of spans of sets. In the category of spans, the morphisms between two objects are spans. A span is a diagram of sets as below, left:

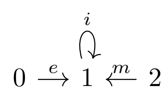


A bijection $T \to T'$ as in the diagram on the right making the diagram commute is a two-morphism. Composition is given by pullbacks. From such a diagram, we can obtain an $A \times B$ matrix of natural numbers by setting $n_{a,b} = |p^{-1}(a) \cap q^{-1}(b)|$. Clearly, if we take the disjoint union of spans, this corresponds to sum of matrices, and it's not hard to see that pullback of spans is composition of matrices. Sets and matrices are a 1-category, so we take isomorphism classes of spans to get a 1-category of spans. However, isomorphism classes of spans only correspond to $\mathbb{N}\{A^\vee \times B\}$, and we wanted $\mathbb{Z}\{A^\vee \times B\}$, so we group complete with respect to disjoint unions. Thus, we define our category $\mathcal{A}$ with objects finite sets and morphisms $\mathcal{A}(A, B) =$ group complete $(\pi_0(\text{spans from } A \text{ to } B))$ with composition given by pullbacks and addition given by disjoint union. This structure makes $\mathcal{A}$ into an Ab-enriched category. In the case that G is trivial, we have that $\underline{\pi_0}$ is valued in functors $\text{Set}^{\mathcal{A}^{op}}$ that take disjoint union (of sets) to products.

What is the category of functors $\text{Set}^{\mathcal{A}^{op}}$ taking disjoint unions to products? Well as we said, such a functor necessarily satisfies $F(S) = F(1)^S$, so it is specified on objects by a single set. In $\mathcal{A}$ we have the following morphisms:



which together give a diagram in $\mathcal{A}$ as follows:



and these arrows satisfy associativity, commutativity, unit, and inverse axioms making $F(1)$ into an abelian group. It should be no surprise that we got back abelian groups here – we weren't expecting to discover any new structure on $\pi_0(X)$ when X is a nonequivariant spectrum.

Our description of $\mathcal{A}$ is given entirely in terms of diagrams in Set, so we can just replace Set with G-Set everywhere and we get a category $\mathcal{A}_G$.

Theorem. $\mathcal{A}_G = \Sigma^\infty_+\{\text{finite } G\text{-sets}\}$

This is just tom Dieck splitting in the case when the input G-space is a finite G-set.

We obtain the following definition:

Definition. A Mackey functor is a functor $\text{Set}^{\mathcal{A}_G^{op}}$ sending disjoint unions to products.

Let $\underline{\text{M}}$ be such a functor. Unpacking the definition, we get back the normal definition of a Mackey functor:

1) Every finite G-Set has a unique orbit decomposition, so it suffices to write down a set $\underline{\text{M}}(G/H)$ for each subgroup $\underline{\text{M}}(G/H)$. Likewise, any span has an orbit decomposition, so we only need to record what $\underline{\text{M}}$ does on spans where all three objects are orbits – the values of $\underline{\text{M}}$ on all other morphisms can be computed from these.

2) Taking the product of G/H with the abelian group structure diagram in $\mathcal{A}$ gives an abelian group structure diagram on G/H in $\mathcal{A}_G$, so we deduce that $\underline{\mathrm{M}}(G/H)$ is an abelian group for each H. This structure is compatible with the action of the other maps in a way that makes all of the morphisms into abelian group homomorphisms.

3) We can factor any span as follows:

$$\left[\begin{array}{ccccc} & & A & & \\ & \overset{f}{\swarrow} & & \overset{g}{\searrow} & \\ G/H & & & & G/k \end{array}\right] = \left[\begin{array}{ccccccc} & & & A & & & \\ & & /\!\!/ & & \backslash\!\!\backslash & & \\ & A & & & & A & \\ \overset{f}{\swarrow} & & \backslash\!\!\backslash & & /\!\!/ & & \overset{g}{\searrow} \\ G/H & & & A & & & G/K \end{array}\right]$$

so if for any map $h : S \to T$ of finite G-sets, we write R_h for the span $\left[\ S \xleftarrow{h} T = T\ \right]$ and T_h for the span $\left[\ S = S \xrightarrow{h} T\ \right]$, we have a factorization of every map in $\mathcal{A}_G$ as a product $T_g R_f$.

4) Let $g : A \to B$ and $f : C \to B$ be maps. Composing in the other way gives:

$$R_f T_g = \left[\begin{array}{ccccccc} & & & P & & & \\ & & \overset{f'}{\swarrow} & & \overset{g'}{\searrow} & & \\ & A & & & & C & \\ /\!\!/ & & \underset{f}{\searrow} & & \underset{g}{\swarrow} & & \backslash\!\!\backslash \\ A & & & B & & & C \end{array}\right] = \left[\begin{array}{ccccc} & & P & & \\ & \overset{f'}{\swarrow} & & \overset{g'}{\searrow} & \\ A & & & & C \end{array}\right] = T_{g'} R_{f'}$$

This is the usual push-pull identity.

The simplest examples of Mackey functors are representable functors: given any finite G-set S we obtain a Mackey functor by setting $\underline{\mathrm{M}}(T) = \mathcal{A}_G(T, S)$. These satisfy $\mathrm{Mackey}_G(\underline{\mathrm{M}}, \underline{\mathrm{N}}) = \underline{\mathrm{N}}(S)$ by the Yoneda lemma. In particular, in the case that $S = *$, we obtain what is called the Burnside Mackey functor, denoted $\underline{\mathcal{A}_G}$.

How do we know that $\underline{\mathrm{M}}$ has the maximum amount of structure that we can put on $\underline{\pi_0}$? As in the nonequivariant case, there is an Eilenberg Maclane construction $\mathrm{H} : \mathrm{Mackey}_G \to G$-spectra that forms an adjunction with $\underline{\pi_0}$:

$$G\text{-spectra}(X, \mathrm{H}\underline{\mathrm{M}}) \overset{\pi_0}{\underset{\cong}{\longrightarrow}} \mathrm{Mackey}_G(\underline{\pi_0}(X), \underline{\mathrm{M}})$$

In particular, the category of G-spectra with homotopy only in degree zero is equivalent to the category of G Mackey functors, and so we see that Mackey_G is indeed the right category to replace abelian groups in the equivariant setting. As in the nonequivariant case, H is built by setting up cellular cohomology and then using equivariant Brown representability.

G-equivariant rings

Suppose that R is an associative algebra in G-spectra. We get the following extra structure on $\underline{\pi_0}$:

$$\underline{\pi_0}(R)(S) \times \underline{\pi_0}(R)(T) = [\Sigma^\infty_+ S, R] \times [\Sigma^\infty_+ T, R] \xrightarrow{\wedge} [\Sigma^\infty_+(S \times T), R \wedge R] \xrightarrow{m_*} [\Sigma^\infty_+(S \times T), R] = \underline{\pi_0}(R)(S \times T)$$

These maps are bifunctorial in S and T. We define a symmetric monoidal structure on Mackey_G so that makes $\underline{\text{M}} \otimes \underline{\text{N}} \to \underline{\text{P}}$ are the same as bifunctors $\underline{\text{M}}(-) \times \underline{\text{N}}(-) \to \underline{\text{P}}(- \times -)$. We can calculate this as the coend

$$\int^S \underline{\text{M}}(S) \times \underline{\text{N}}(S^\vee \times -)$$

We deduce that $\underline{\pi_0}(R)$ is a monoid object in Mackey_G and that if R is homotopy commutative then $\underline{\pi_0}(R)$ is a commutative monoid.

Definition. A Green functor is a monoid in Mackey_G. A commutative Green functor is a commutative monoid in Mackey_G.

Note that $\underline{\mathcal{A}_G}$ is the tensor unit because it is represented by $*$ so we calculate:

$$\int^S \underline{\text{M}}(S) \times \underline{\mathcal{A}_G}(S^\vee \times -) = \int^S \underline{\text{M}}(S) \times \mathcal{A}_G(-, S) = \underline{\text{M}}(S)$$

where the first equality is by definition of $\underline{\mathcal{A}_G}$ and duality in $\mathcal{A}_G$, and the second equality is the Coyoneda lemma. This is a nice result because $\underline{\mathcal{A}_G} = \underline{\pi_0}(\mathbb{S})$.

Interestingly, an E_∞ ring has richer structure on its $\underline{\pi_0}$ than a commutative Green functor. This is because E_∞ rings have multiplicative norms $\pi_0^K(R) \to \pi_0^H(R)$ for $K \subseteq H$, but this norm is missing in a Green functor. To fix this, let's replicate the procedure we used to get Mackey_G but now with rings. Remember, we calculated $\mathcal{A}(S,T) = [S,T] = \text{Ab}(\mathbb{Z}\{S\}, \mathbb{Z}\{T\})$. Observe that $\mathbb{Z}\{S\}$ represents the functor $U^S : \text{Ab} \to \text{Set}$ that takes an abelian group to functions from S to its underlying set. We deduce that $\mathcal{A}(S,T) = \text{Set}^{\text{Ab}}(U^S, U^T)$. Now in this description, we can replace Ab with Ring to get a definition:

$$\mathcal{U}(S,T) = \text{Set}^{\text{Ring}}(U^S, U^T)$$

Now as before, we want to get a description of $\mathcal{U}$ in terms of diagrams of sets. Then we will replace the sets with G-sets and pray that we get the right thing. So first lets calculate $\mathcal{U}(n, *)$. This is the collection of maps $R^n \to R$ that are natural in R. Such a map is an expression

$$\sum_{\overline{a}} n_{\overline{a}} \cdot \prod_i x_i^{a_i}$$

where the coefficients $n_{\overline{a}} \in \mathbb{Z}$ are zero for all but finitely many $\overline{a}$.

We can express an expression

$$\left(\sum_{\overline{a}} n_{j,\overline{a}} \cdot \prod_i x_i^{a_i} \right)_{j \in T}$$

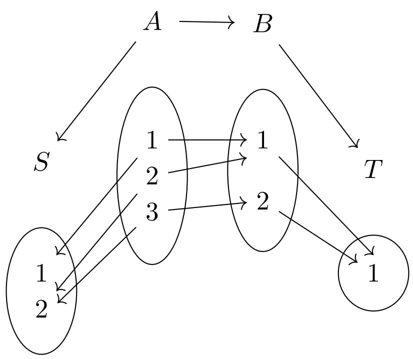
when the coefficients $n_{j,\overline{a}}$ are natural numbers as the collection of isomorphism classes of diagrams:

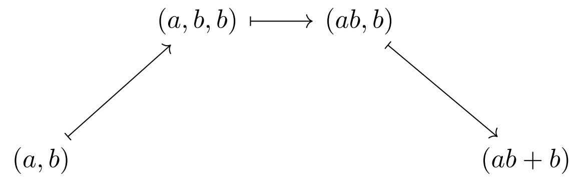
$$\begin{array}{ccccc} & A & \xrightarrow{g} & B & \\ {}^{f}\swarrow & & & & \searrow^{h} \\ S & & & & T \end{array}$$

Such a diagram represents the map $T_h N_g R_f : U^S \to U^T$ where R_f is restriction along f, N_g is taking the product along the fibers of g and T_h is summing along the fibers of f. An isomorphism of such diagrams is a pair of bijections making the following diagram commute:

$$\begin{array}{ccccc} & A & \xrightarrow{g} & B & \\ {}^{f}\swarrow & {\scriptstyle a}\downarrow\cong & & {\scriptstyle b}\downarrow\cong & \searrow^{h} \\ S & & & & T \\ {}_{f'}\nwarrow & A' & \xrightarrow[g']{} & B' & \nearrow_{h'} \end{array}$$

For example,





Evidently all expressions with $n_{\overline{a}} \in \mathbb{N}$ come about in this way. We can add them as:

$$\left[\begin{array}{ccc} & A \xrightarrow{g} B & \\ {}^{f}\swarrow & & \searrow^{h} \\ S & & T \end{array}\right] + \left[\begin{array}{ccc} & A' \xrightarrow{g'} B' & \\ {}^{f'}\swarrow & & \searrow^{h'} \\ S & & T \end{array}\right] = \left[\begin{array}{ccc} & A \sqcup A' \xrightarrow{g \sqcup g} B \sqcup B' & \\ \swarrow & & \searrow \\ S & & T \end{array}\right]$$

so we group complete with respect to this operation. All the other structure, including multiplication, composition, 0 and 1 can be written in terms of (complicated) set theoretic operations on these diagrams. So we can again just throw a G action into our description and set $\mathcal{U}_G(S,T)$ to be the resulting set. This category is called $\mathrm{Tambara}_G$. By miracle we get:

Theorem. *There is a functor* $\mathrm{H} : \mathrm{Tambara}_G \to E_\infty$ G-spectra *left adjoint to* $\underline{\pi_0}$*, which makes the category of* E_∞ *rings with homotopy concentrated in degree zero equivalent to* $\mathrm{Tambara}_G$*.*

Proof sketch. By the Coyoneda lemma,

$$\underline{\mathrm{R}} = \int^{\mathcal{U}_G} \mathcal{U}_G(G/K, -) \times \underline{\mathrm{R}}(G/K).$$

Our goal is to lift this coend to the level of rings. The proof goes as follows:

Step 1. (hard part) First we need to build rings with homotopy given by $\mathcal{U}_G(G/K,-)$. There is a free functor $\mathbb{T} : \mathrm{Mackey}_G \to \mathrm{Tambara}_G$. Since it's a left adjoint, it takes representable functors to representables and we have $\mathbb{T}(\mathcal{A}_G(G/K,-)) = \mathcal{U}_G(G/K,-)$. This suggests that $\mathbb{S}[\mathcal{A}_G(G/K,-)]$ is a good candidate for a ring with $\underline{\pi_0}$ equal to $\mathcal{U}_G(G/K,-)$. The heart of Ullman's paper is the proof that indeed $\underline{\pi_0}(\mathbb{S}[\mathcal{A}_G(G/K,-)]) = \mathcal{U}_G(G/K,-)$.

Step 2. Compute $ho \int^{\mathcal{U}_G} \mathbb{S}[\mathcal{A}_G(G/K,-)] \wedge \underline{\mathrm{R}}(G/K)_+$. This gives us an E_∞ ring with $\underline{\pi_0} = \underline{\mathrm{R}}$.

Step 3. Take the 0th Postikov section, which is 0-coconnective and so is Eilenberg Maclane, but still has $\underline{\pi_0}$ equal to $\underline{\mathrm{R}}$.

□

So we have that {commutative Green functors} $\simeq$ EM E_2-rings and {Tambara functors} $\simeq$ EM E_∞-rings. This gives us an opportunity to study the difference purely algebraically.

Proposition. *If* G *is nontrivial, there exists a Green functor which does not admit a Tambara structure.*

Proposition. *If G is nontrivial, there exists Green functors with arbitrarily many distinct Tambara structures.*

Proposition. *If G is nontrivial, there does not exist a functorial choice of Tambara structures on Green functors that admit at least one Tambara structure.*

These results are completely concrete and they give constructions of an E_2 G-spectrum which has no E_∞ structure and one that has arbitrarily many nonequivalent E_∞ structures.