On Harmonic Forms for Generic Metrics

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Abstract

Let M be a closed, oriented n-manifold. We first prove that the generic harmonic i-form for $i = 1$ and 2 enjoy the standard transversality properties with respect to the various strata of $\bigwedge^i(\mathbf{R}^n)^*$ under the action of $SO(n)$. We then go on to study two examples: the generic self-dual (SD) or anti-self-dual (ASD) harmonic 2-forms and the generic non-SD/ASD harmonic 2-forms, both on a 4-manifold.

In the SD case, we prove a generalization of Moser's theorem for harmonic forms which are symplectic away from a disjoint union of circles. When $M^4 = N^3 \times S^1$, and g is a product metric, we are able to say more about $\omega = *_3\mu + d\theta \wedge \mu$, with μ a generic harmonic 1-form on N. Using Calabi's characterization of intrinsically harmonic 1-forms, we prove a result on deforming a closed 1-form into a harmonic 1-form and hence a lower bound on the critical points for a Morse harmonic 1-form.

In the non-SD/ASD case, we prove that all closed 2-forms close to a generic harmonic 2-form ω are intrinsically harmonic, subject to a condition on the codimension 1 submanifold on which ω has rank 2.

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Introduction

This work is a first step in the study of generic harmonic forms on compact manifolds, where by 'generic' we mean generic in the space of metrics. Our goal is ambitious: to lay the foundations of what it means for a harmonic form to be generic, and to use generic harmonic forms in the study of singular symplectic geometry. The considerations here were largely motivated by the 'Holy Grail' in symplectic geometry: Characterize when a closed, $2n$-dimensional manifold has a symplectic structure; if it is not possible for a symplectic form to exist, could you still make the closed 2-form symplectic away from a small subset? It turns out that, in dimension 4, the $*$-operator makes life intriguing by sifting harmonic 2-forms into two different classes with very different behavior. If the harmonic 2-form is self-dual (SD) or anti-self-dual (ASD), then generically it is symplectic away from a union of circles, where the form vanishes. On the other hand, if the harmonic 2-form is neither SD nor ASD, then generically it does not vanish, but has a codimension 1 submanifold on which it has rank 2.

The thesis is organized into four chapters. In the first, we prove the genericity results for harmonic 1-forms and harmonic 2-forms on closed manifolds. In the second chapter, we study the harmonic SD/ASD 2-forms, and prove a Moser-type result. The third chapter is, in some sense, an outgrowth of the second, because harmonic 1-forms on a 3-manifold N and harmonic SD 2-forms on a 4-manifold $N \times S^1$ are inseparable. We will discuss Calabi's intrinsic characterization of harmonic 1-forms, and use it to prove a Morse-theoretic conjecture about harmonic 1-forms. Here, a closed form is *intrinsically harmonic* if there exists a metric with respect to which the closed form is harmonic. We will also describe the intrinsic characterization of harmonic $(n-1)$-forms on an n-dimensional manifold, i.e. the dual situation to Calabi's theorem, which is surprisingly different from the 1-form case. Finally, in the last chapter, we delve into a local characterization of non-SD/ASD harmonic 2-forms, primarily because an intrinsic characterization seems much more difficult. We will show that whether a small exact perturbation of a harmonic form is intrinsically harmonic depends on the singularities of the harmonic form; there is also a relationship with the cohomology of a singular differential ideal.

Chapter 1

Harmonic forms for generic metrics

1.1 Introduction

Let us begin our investigations by asking the following question:

Question: Suppose we have a family of Laplacians which are dependent on the metric. Then for a generic metric, would the solution of Laplace's equation have sufficiently generic behavior? In particular, could one force the zeros of the harmonic functions/forms to be regular, i.e. to be submanifolds?

More precisely, we have the two situations:

(A) The Dirichlet problem. Let Δ_g be the Laplacian corresponding to the metric g on a domain in $\mathbf{R}^n$ - for simplicity let this domain be the closed ball D^n. Consider the solutions to the equation $\Delta_g u = 0$, with $u|_{\partial D^n} = f$ fixed. Then for generic metric perturbations of the Dirichlet problem, are the zeros of u regular?

(B) Harmonic forms. Let Δ_g be the Laplacian (Laplace-Beltrami operator) on a closed, oriented n-manifold for the metric g. For generic harmonic forms ω, can we conclude that the zeros of ω are regular? Even better, for a generic harmonic i-form, are the zeros of ω^i regular as well?

It is the goal of this chapter to prove affirmative results for (A) and certain interesting cases in (B). In Section 1.2 we will study generic harmonic forms on compact manifolds, and in Section 1.3 we will treat the Dirichlet Problem.

1.2 Harmonic Forms on Compact Manifolds

In what follows, M will always be a closed, oriented n-manifold and $\mathrm{Met}^k(M)$ the space of C^k-metrics on M, for sufficiently large $k \in \mathbf{R}^+$ (i.e. our metrics are C^k-Hölder metrics). View $([\omega], g)$ in $Q = H^i(M;\mathbf{R}) \times \mathrm{Met}^k(M)$ as a harmonic form ω_g in the class $[\omega] \in H^i(M;\mathbf{R})$ for the metric $g \in \mathrm{Met}^k(M)$. Q, when viewed as a collection (ω, g) with $\Delta_g(\omega) = 0$, is a Banach manifold. (The reader is referred to Section 4.2.2 for some justification of this.) We would like to prove that there is a dense G_δ-set in Q for which the harmonic form ω_g in $H^i(M;\mathbf{R}) \times \mathrm{Met}^k(M)$ has regular zeros.

We will prove genericity for 1-forms on any manifold and 2-forms for even dimensional manifolds (and dually for $(n-1)$-forms on any manifold and $(n-2)$-forms for even-dimensional ones). As we shall see, 4-manifolds exhibit unusual behavior: Both the self-dual (or anti-self-dual) 2-forms and the non-SD (and non-ASD) 2-forms are very intriguing. It appears at the moment that a proof for i-forms for $2 < i < n-2$ requires substantially more work.

Let us describe the general setup. We start with the following evaluation map:

$$ev : Q \times M \to \bigwedge^i(T^*M)$$

$$[(\omega, g), x] \mapsto \omega(x).$$

We want to show that ev is regular, that is, it is transverse to the zero section of $\bigwedge^i(T^*M)$. This means that, for $[(\omega, g), x]$ fixed,

$$(ev)_* : T_{(\omega,g)}Q \times T_xM \to \bigwedge^i(T^*M)_x$$

is surjective whenever $\omega(x) = 0$. In view of the following proposition, this would be sufficient.

Proposition 1 *Let X be a Banach manifold, M, N finite-dimensional manifolds, and $f : X \times M \to N$ be a C^k-map for k sufficiently large. Suppose f is transverse to a submanifold Z of N. Then for a dense G_δ in X, $f_x : M \to N$ is transverse to Z. ($f_x(m) = f(x, m)$.)*

In our present situation, it suffices to show that

$$(ev)_* : T_{(\omega,g)}Q \to \bigwedge^i(T^*M)_x$$

is surjective whenever $\omega(x) = 0$, that is, we don't have to let x vary in M. In fact, $(ev)_*(T_{(\omega,g)}Q)$ is exactly the same as $(ev)_*(T_{(\omega,g)}Q \times T_xM)$. If $\tilde{v} \in T_xM$, then extend $\tilde{v}$ to a vector field V on M. If ϕ_t is the 1-parameter family of diffeomorphisms generated by V, then

$$(ev)_*(\omega, g, x)(0, \tilde{v}) = \mathcal{L}_V\omega(x) = \frac{d}{dt}\phi_t^*\omega(x)\Big|_{t=0},$$

where $\mathcal{L}_V$ is the Lie derivative in the direction of V. Finally we can observe that $\phi_t^*\omega$ is harmonic for the metric ϕ_t^*g.

Since the elements of Q satisfy $\Delta_g \omega = 0$, by differentiating the family ω_t of solutions to Δ_{g_t} with respect to t, we see that $(v,h) \in T_{(\omega,g)}Q$ if and only if

$$\Delta_g v + \frac{d}{dt}(\Delta_{g+th})\Big|_{t=0} \omega = 0. \tag{1.1}$$

For $(v,h) \in T_{(\omega,g)}Q$, $(ev)_* : (v,h) \mapsto v(x)$.

1.2.1 Green's Functions

In order to write the above differential equation in integral form, we make use of the Green's function $G(x,y)$.

Let us first collect some facts about the Green's function that we need. Let $L : \Gamma(E) \to \Gamma(E)$ be a self-adjoint elliptic operator, where $E \to M$ is a vector bundle. Also let π_i be the projections of $M \times M$ onto M, and let $\Delta \subset M \times M$ be the diagonal.

Proposition 2 *There exists a section $G(x,y)$ of $\pi_1^*(E) \otimes \pi_2^*(E)$, called the* Green's function, *with the following properties:*

1. *If L has C^k-coefficients, with k large, then G is C^k on $M \times M - \Delta$.*
2. *$G(x,y) = G(y,x)$.*
3. *$\int_M \langle G(x,y), Ls(y)\rangle dy = s(x)$, if $s \in C^k$ is L^2-orthogonal to $ker(L)$. Here $\langle , \rangle$ is the fiber metric on E.*
4. *$\int_M \langle G(x,y), u(y)\rangle dy = 0$, if $u \in ker(L)$.*

Then we get from Equation 1.1 that, as long as v is orthogonal to $\mathcal{H}_g^i$, the space of harmonic i-forms for the metric g,

$$v(x) = \int_M \langle G(x,y), \frac{d}{dt}(\Delta_{g+th})\Big|_{t=0} \omega(y)\rangle_g dv_g,$$

where dv_g is the volume form with respect to the metric g, and $\langle , \rangle_g$ is the fiberwise inner product induced from g.

Let us now compute $v(x)$:

$$\begin{aligned} v(x) &= \int \frac{d}{dt} \langle G_g(x,y), \Delta_{g+th}\omega(y)\rangle_g dv_g \\ &= \int \frac{d}{dt}\{\langle dG_g(x,y), d\omega(y)\rangle_g + \langle d^*_{g+th} G_g(x,y), d^*_{g+th}\omega(y)\rangle_g\} dv_g \end{aligned}$$

$$
\begin{aligned}
&= \int \frac{d}{dt} \langle *_{g+th} d *_{g+th} G_g(x,y), *_{g+th} d *_{g+th} \omega(y) \rangle_g dv_g \\
&= \int \langle *_g d *_g G_g(x,y), *_g d \,\dot{*}_{g+th} \omega(y) \rangle_g dv_g \\
&= \pm \int \langle *d*d*G_g(x,y), \dot{*}_{g+th}\omega(y) \rangle_g dv_g,
\end{aligned}
$$

keeping in mind that $d\omega = 0$, $d_g^* \omega = 0$.

Thus,

$$v(x) = \pm \int \langle dd^* G_g(x,y), * \dot{*}_{g+th} \omega(y) \rangle_g dv_g.$$

Although we do not have a good grasp of $G_g(x,y)$ in general, we can still take advantage of the asymptotics of $G_g(x,y)$ near the diagonal Δ. This is because the perturbations of the metric g we need are the ones supported arbitrarily close to x. Pick local geodesic coordinates (U, ϕ) around x where $\phi : x \mapsto 0 \in \mathbf{R}^n$.

For g a flat metric,

$$G_g(x,y) = \begin{cases} \sum_{i_1 < \ldots < i_k} \frac{1}{|x-y|^{n-2}} dx_{i_1} \ldots dx_{i_k} \otimes dy_{i_1} \ldots dy_{i_k} & \text{if } n > 2 \\ \sum_{i_1 < \ldots < i_k} \log|x-y| dx_{i_1} \ldots dx_{i_k} \otimes dy_{i_1} \ldots dy_{i_k} & \text{if } n = 2. \end{cases}$$

Here the dy_i terms get paired with respect to $\langle , \rangle_g$, and the dx_i terms are left untouched. Of course, when $n = 2$, the only interesting case is $k = 1$.

Write $F(y) = G_g(0, y)$ for g flat. Then we have the following:

Proposition 3 *$G_g(0,y)$ is asymptotic to $F(y)$ as $y \to 0$. That is, as $y \to 0$, the ratio $[G_g(0,y) - F(y) : F(y)] \to 0$. Moreover, the same is true for $\partial^{i_1 \dots i_l} F(y)$ and $\partial^{i_1 \dots i_l} G_g(0,y)$.*

1.2.2 Computation of $dd^*F(y)$ for g flat

Assume $n > 2$. Let us first compute $d{*}d{*}\frac{1}{|y|^{n-2}} dy_{i_1} \dots dy_{i_k}$. Fix $i_1, \dots, i_n$, which is a permutation of $1, \dots, n$.

$$
\begin{aligned}
*\frac{1}{|y|^{n-2}} dy_{i_1} \dots dy_{i_k} &= \operatorname{sgn}(i_1 \dots i_k i_{k+1} \dots i_n) \frac{1}{|y|^{n-2}} dy_{i_{k+1}} \dots dy_{i_n} \\
d * \frac{1}{|y|^{n-2}} dy_{i_1} \dots dy_{i_k} &= \operatorname{sgn}(i_1 \dots i_n)(2-n) \frac{1}{|y|^n} \sum_{j=1}^{k} y_{i_j} dy_{i_j} dy_{i_{k+1}} \dots dy_{i_n} \\
* d * \frac{1}{|y|^{n-2}} dy_{i_1} \dots dy_{i_k} &= \operatorname{sgn}(i_1 \dots i_k i_{k+1} \dots i_n) \operatorname{sgn}(i_j i_{k+1} \dots i_n i_1 \dots \hat{i_j} \dots i_k) \\
&\quad \times (2-n) \frac{1}{|y|^n} \sum_{j=1}^{k} y_{i_j} dy_{i_1} \dots \widehat{dy_{i_j}} \dots dy_{i_k} \\
&= (-1)^{(n-k)(k-1)+(j-1)} (2-n) \frac{1}{|y|^n} \sum_{j=1}^{k} y_{i_j} dy_{i_1} \dots \widehat{dy_{i_j}} \dots dy_{i_k} \\
d * d * \frac{1}{|y|^{n-2}} dy_{i_1} \dots dy_{i_k} &= (-1)^{(n-k)(k-1)} (2-n) \frac{1}{|y|^n} \sum_{j=1}^{k} dy_{i_1} \dots dy_{i_j} \dots dy_{i_k} \\
&\quad + (-1)^{(n-k)(k-1)+(j+k)} (2-n)(-n) \cdot \\
&\qquad \frac{1}{|y|^{n+2}} \sum_{\substack{j=1,\dots,k \\ l>k \text{ or } l=j}} y_{i_j} y_{i_l} dy_{i_1} \dots \widehat{dy_{i_j}} \dots dy_{i_k} dy_{i_l}.
\end{aligned}
$$

Hence, near $y = 0$ we obtain:

$$
\begin{aligned}
dd^* F(y) &= \sum_{i_1 < \dots < i_k} dx_{i_1} \dots dx_{i_k} \otimes dd^* \left(\frac{1}{|y|^{n-2}} dy_{i_1} \dots dy_{i_k} \right) \\
&= \pm(2-n) \sum_{i_1 < \dots < i_k} dx_{i_1} \dots dx_{i_k} \\
&\quad \otimes \left\{ \left(\frac{k}{|y|^n} - \frac{n}{|y|^{n+2}} (y_{i_1}^2 + \dots + y_{i_k}^2) \right) dy_{i_1} \dots dy_{i_k} \right. \\
&\qquad \left. + \sum_{\substack{j=1,\dots,k \\ i \neq i_1,\dots,i_k}} (-1)^{j+k} \left(-\frac{n}{|y|^{n+2}} y_{i_j} y_i \right) dy_{i_1} \dots \widehat{dy_{i_j}} \dots dy_{i_k} dy_i \right\}.
\end{aligned}
$$

Remark: Consider the following action of $SO(n)$ on $F(y)$: View $A \in SO(n)$ as a diffeomorphism $D^n \xrightarrow{\sim} D^n$, where D^n is a disk centered about the origin. Define $\tilde{A}$ by:

$$\tilde{A} : \bigwedge^i(T^*D^n) \otimes \bigwedge^i(T^*D^n) \to \bigwedge^i(T^*D^n) \otimes \bigwedge^i(T^*D^n),$$
$$\eta(x) \otimes \omega(y) \mapsto A^*\eta(x) \otimes A^*\omega(y).$$

That is, $\tilde{A}$ acts on both the x's and y's simultaneously by pullback. Note that this is the natural way for the Green's function $F(y)$ to transform under rotation. $F(y)$ is invariant under $\tilde{A}$, and hence so is $dd^*F(y)$ (where d, d^* act only on the y variables), since A^* commutes with d and $*$.

Case n>2, k=2: Specializing to $k = 2$,

$$\begin{aligned} dd^*F(y) &= \frac{C}{|y|^{n+2}} \sum_{i,j} dx_i dx_j \otimes \Big\{ (2(y_1^2 + ... + y_n^2) - n(y_i^2 + y_j^2)) dy_i dy_j \\ &\qquad - n \sum_{k \neq i,j} (y_i y_k dy_k dy_j + y_k y_j dy_i dy_k) \Big\}. \end{aligned}$$

In view of the above remark, without loss of generality we can pick $y = (y_1, 0, 0, ...)$ so that

$$dd^*F(y_1, 0, 0, ...) = \frac{C}{|y_1|^n} \sum_{i,j} \xi(i,j) dx_i dx_j \otimes dy_i dy_j,$$

where

$$\xi(i,j) = \begin{cases} 1 - \frac{n}{2} & \text{if one of } i \text{ or } j = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Now let $R_y : \mathbf{R}^n \to \mathbf{R}^n$ be given by the following linear transformation which is almost a reflection along the hyperplane $\langle x, y \rangle = 0$ (and is one for $n = 4$):

$$R_{y=(y_1,0,0,...)} : \begin{cases} e_1 \mapsto (1 - \frac{n}{2}) e_1 & \\ e_i \mapsto e_i, & i > 1. \end{cases}$$

If we extend R_y to $\Lambda^*(\mathbf{R}^n)^*$, then one sees that

Proposition 4 *$R_y = C(y) \cdot \langle dd^*F(y), \cdot \rangle$, as operators $\bigwedge^2(\mathbf{R}^n)^* \xrightarrow{\sim} \bigwedge^2(\mathbf{R}^n)^*$, where $C : \mathbf{R}^n \to \mathbf{R}$ and $C(y) \neq 0$ unless $y = 0$.*

In particular, $dd^*F(y) : \Lambda^2_y \to \Lambda^2_y$ is an isomorphism.

Case n>2, k≥2: In general, for $y = (y_1, 0, ..., 0)$,

$$dd^*F(y_1, 0, ..., 0) = \frac{C}{|y|^n} \sum_{i_1 < ... < i_k} \xi(i_1, ..., i_k) dx_{i_1} ... dx_{i_k} \otimes dy_{i_1} ... dy_{i_k},$$

where

$$\xi(i_1, ..., i_k) = \begin{cases} 1 - \frac{n}{k} & \text{if } i_1 = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Thus, $R_y = C|y|^n \langle dd^* F(y), \cdot \rangle$ as operators $\bigwedge^k (\mathbf{R}^n)^* \xrightarrow{\sim} \bigwedge^k (\mathbf{R}^n)^*$, where

$$R_y : \begin{cases} \frac{y}{|y|} \mapsto \left(1 - \frac{n}{k}\right) \frac{y}{|y|} \\ v \mapsto v, \ \text{for } v \perp y. \end{cases}$$

Case n=2, k=1: We can compute likewise that

$$dd^* \log|y| dy_{i_1} = \left(\frac{1}{|y|^2} - \frac{2}{|y|^4} y_{i_1}^2 \right) dy_{i_1} - \frac{2}{|y|^4} y_{i_1} y_{i_2} dy_{i_2}.$$

Hence, we obtain:

$$\begin{aligned} dd^* F(y) &= \sum_i dx_i \otimes dd^*(\log|y| dy_i) \\ &= C \cdot \sum_i dx_i \otimes \left\{ \left(\frac{1}{|y|^2} - \frac{2}{|y|^4} y_i^2 \right) dy_i - \sum_{j \neq i} \frac{2}{|y|^4} y_i y_j dy_j \right\}. \end{aligned}$$

Once again, pick $y = (y_1, 0)$, and we have

$$\begin{aligned} dd^* F(y_1, 0) &= \frac{C}{|y_1|^4} (-|y_1|^2 dx_1 \otimes dy_1 + |y_1|^2 dx_2 \otimes dy_2) \\ &= \frac{C}{|y_1|^2} (-dx_1 \otimes dy_1 + dx_2 \otimes dy_2). \end{aligned}$$

Proposition 5 *$R_y = C(y) \cdot \langle dd^* F(y), \cdot \rangle$, where $C(y) \neq 0$ when $y \neq 0$.*

1.2.3 An Alternative Approach

Instead of letting our family of harmonic forms $\omega(t) = \omega + tv$ simply satisfy $\Delta_{g+th}(\omega + tv) = 0$, we can require that $v = d\eta$, i.e. that v be *exact*. This would mean letting $Q = [\omega] \times \mathrm{Met}^k(M)$ instead, and all the results we can prove this way have the advantage that they would be valid for a *fixed* cohomology class $[\omega] \in H^i(M; \mathbf{R})$. (This was pointed out to me by Taubes [5].)

Let $v = d\eta$. Using the Hodge decomposition, we can additionally assume that $d^* \eta = 0$. Now, since our family ω_t satisfies $\Delta_t \omega_t = 0$, we have $d\omega_t = 0$ and $d *_t \omega_t = 0$.

Differentiating $d *_t \omega_t$, we get

$$\begin{aligned}\frac{d}{dt}(d *_t \omega_t)|_{t=0} &= d(\dot{*}_t|_{t=0}\omega) + d * \dot{\omega}|_{t=0} \\ &= d(\dot{*}_t)\omega + d * d\eta = 0.\end{aligned}$$

Together with $d * \eta = 0$, we obtain

$$\Delta\eta = \pm * d(\dot{*}\omega).$$

Inverting using Green's functions,

$$\begin{aligned}\eta(x) &= \pm \int \langle G(x,y), *d\dot{*}\omega(y)\rangle dv_g \\ &= \pm \int \langle d_y G(x,y), *\dot{*}\omega(y)\rangle dv_g.\end{aligned}$$

Hence,

$$v(x) = \pm \int \langle d_x d_y G(x,y), *\dot{*}\omega(y)\rangle dv_g.$$

Notice that we are writing d_y to distinguish it from d_x, which is d with respect to the x-variables, while holding the y-variables constant. At this point our $d_x d_y G(x,y)$ does not seem to closely resemble $dd^* G(x,y)$, which appeared in our previous expression for $v(x)$. However, it is quite surprising to find that, upon computing $d_x d_y G(x,y)$ and evaluating it at $x = 0$, these two expressions are identical near the diagonal, as far as the lowest order term is concerned.

Let us compute $d_x d_y G(x,y)$ near the diagonal. We may assume without loss of generality that g is flat and

$$G(x,y) = \sum_{i_1<\ldots<i_{k-1}} \frac{1}{|x-y|^{n-2}} dx_{i_1}\ldots dx_{i_{k-1}} \otimes dy_{i_1}\ldots dy_{i_{k-1}}.$$

Then we have,

$$\begin{aligned}d_y G(x,y) &= \pm(2-n) \sum_{i_1<\ldots<i_{k-1}, i} \frac{1}{|x-y|^{n}} dx_{i_1}\ldots dx_{i_{k-1}} \otimes (x_i - y_i) dy_i dy_{i_1}\ldots dy_{i_{k-1}}, \\ d_x d_y G(x,y) &= \pm \Bigg\{ (2-n)(-n) \sum_{\substack{i_1<\ldots<i_{k-1} \\ i,j}} \frac{1}{|x-y|^{n+2}} (x_j - y_j)(x_i - y_i) \cdot \\ &\qquad dx_j dx_{i_1}\ldots dx_{i_{k-1}} \otimes dy_i dy_{i_1}\ldots dy_{i_{k-1}} \\ &\qquad + (2-n) \sum_{i_1<\ldots<i_{k-1}, i} \frac{1}{|x-y|^{n}} dx_i dx_{i_1}\ldots dx_{i_{k-1}} \otimes dy_i \ldots dy_{i_{k-1}} \Bigg\}.\end{aligned}$$

Setting $x = 0$,

$$\begin{aligned}
d_x d_y G(0,y) &= \pm \Bigg\{ (2-n)(-n)\frac{1}{|y|^{n+2}} \sum_{\substack{i_1<\ldots<i_{k-1}\\ i,j}} y_i y_j dx_i dx_{i_1}\ldots dx_{i_{k-1}} \otimes dy_j dy_{i_1}\ldots dy_{i_{k-1}} \\
&\qquad + (2-n)\frac{1}{|y|^n} \sum_{i,i_1<\ldots<i_{k-1}} dx_i dx_{i_1}\ldots dx_{i_{k-1}} \otimes dy_i dy_{i_1}\ldots dy_{i_{k-1}} \Bigg\} \\
&= \pm(2-n) \sum_{i_1<\ldots<i_k} dx_{i_1}\ldots dx_{i_k} \\
&\qquad \otimes \Bigg\{ \left(\frac{k}{|y|^n} - \frac{n}{|y|^{n+2}}(y_{i_1}^2 + \ldots + y_{i_k}^2) \right) dy_{i_1}\ldots dy_{i_k} \\
&\qquad + \sum_{j=1,..,k;i} \left(-\frac{n}{|y|^{n+2}} y_{i_j} y_i (-1)^{j+k} \right) dy_{i_1}\ldots\widehat{dy_{i_j}}\ldots dy_{i_k} dy_i \Bigg\}.
\end{aligned}$$

Thus we see that $d_x d_y G(0,y) = \pm dd^* G(0,y)$. This means that our previous setup give us (essentially for free) stronger genericity results where we fix the cohomology class $[\omega] \in H^i(M;\mathbf{R})$.

1.2.4 Computation of $*\dot{*}_{g+th}\omega(y)$

Consider the family of metrics $g+th$. If we let $\{e_i(t)\}$ be an orthonormal basis for T_y^*M with respect to $g+th$, then we write $\omega = \omega_{i_1\ldots i_k}(t)e_{i_1}(t)\ldots e_{i_k}(t)$. We will compute $*\dot{*}_{g+th}\omega(y)$, which is the term on the right-hand side of equation (8).

Let's first prove a useful lemma. Set $e_i = e_i(0)$.

Lemma 1 *Let $\langle e_i, e_j\rangle_{g+th} = \delta_{ij} + th_{ij}$. Then there exists a basis $\{e_i(t) = e_i - \frac{1}{2}t\sum_j h_{ij}e_j\}$ orthonormal with respect to $g+th$ up to first order in t.*

Proof: Suppose $e_i(t) = e_i + t\sum_j a_{ij}e_j$. Then

$$\begin{aligned}
\delta_{ij} &= \langle e_i(t), e_j(t)\rangle_t \\
&= \langle e_i + t\sum_k a_{ik}e_k, e_j + t\sum_l a_{jl}e_l\rangle_t \\
&= \langle e_i, e_j\rangle_t + t\left\{\langle\sum_k a_{ik}e_k, e_j\rangle_t + \langle e_i, \sum_l a_{jl}e_l\rangle_t\right\} \\
&= \delta_{ij} + th_{ij} + t(a_{ij} + a_{ji}).
\end{aligned}$$

Thus, $h_{ij} = -(a_{ij}+a_{ji})$. Letting $a_{ij} = -\frac{1}{2}h_{ij}$, we obtain the desired result. □

From now on, we use $e_i(t)$ as in the lemma.

Proposition 6 *If $\omega = \sum_{i=1}^n \omega_i(t)e_i(t)$ is a 1-form, then*

$$*\dot{*}_{g+th}\omega = C\cdot\left\{\sum_{i,j} h_{ij}\omega_j e_i - \frac{1}{2}tr(h)\omega\right\}.$$

Proof: Since $*_t\omega = \sum_i(-1)^{i-1}\omega_i(t)e_1(t)\ldots\widehat{e_i(t)}\ldots e_n(t)$, we have

$$\begin{aligned}
\dot{*}_t\omega &= \frac{d}{dt}(*_t\omega) - *\dot{\omega} \\
&= \sum_i(-1)^{i-1}\frac{d}{dt}\left\{\omega_i(t)e_1(t)\ldots\widehat{e_i(t)}\ldots e_n(t)\right\} - *\dot{\omega} \\
&= \sum_i(-1)^{i-1}\dot{\omega}_i e_1\ldots\hat{e}_i\ldots e_n + \sum_{i,j\neq i}(-1)^{i-1}\omega_i e_1\ldots\dot{e}_j\ldots\hat{e}_i\ldots e_n \\
&\qquad - *\sum_i(\dot{\omega}_i e_i + \omega_i\dot{e}_i) \\
&= -*\sum_i\omega_i\dot{e}_i + \sum_{i,j\neq i}(-1)^i\frac{1}{2}\omega_i\cdot \\
&\qquad \left\{h_{jj}e_1\ldots\hat{e}_i\ldots e_n + (-1)^{i-j-1}h_{ji}e_1\ldots\hat{e}_j\ldots e_n\right\},
\end{aligned}$$

using $\dot{e}_i(t) = -\frac{1}{2}\sum_{j=1}^{n} h_{ij}e_j$ from the previous lemma. Then,

$$\begin{aligned} \dot{**}\omega &= (-1)^{n-1}\frac{1}{2}\sum_{ij}\omega_i h_{ij}e_j \\ &\quad + \sum_{i,j\neq i}(-1)^i\frac{1}{2}\omega_i\left\{(-1)^{n-i}h_{jj}e_i + (-1)^{n-i-1}h_{ji}e_j\right\} \\ &= (-1)^{n-1}\sum_{i,j\neq i}\frac{1}{2}\left(\omega_i h_{ji}e_j - \omega_i h_{jj}e_i\right) + (-1)^{n-1}\frac{1}{2}\sum_{i,j}\omega_i h_{ij}e_j \\ &= (-1)^{n-1}\left\{\sum_{i,j}h_{ij}\omega_j e_i - \frac{1}{2}\sum_i(\mathrm{tr}h)\omega_i e_i\right\} \end{aligned}$$

□

Similarly we compute:

Proposition 7 *If* $\omega = \sum_{i_1,\ldots,i_k}\omega_{i_1\ldots i_k}(t)e_{i_1}(t)\ldots e_{i_k}(t)$, *then*

$$\dot{**}_t\omega = C\left\{\sum_{i_1,\ldots,i_k,j}\left(h_{i_1j}\omega_{ji_2\ldots i_k} + \ldots + h_{i_kj}\omega_{i_1\ldots i_{k-1}j}\right)e_{i_1}\ldots e_{i_k} - \frac{1}{2}trh\cdot\omega\right\}.$$

In particular,

Corollary 1 *If* $\omega = \sum_{ij}\omega_{ij}(t)e_i(t)e_j(t)$, *then*

$$\dot{**}_t\omega = C\left\{\sum_{i,j,k}\left(h_{ik}\omega_{kj} + \omega_{ik}h_{kj}\right)e_ie_j - \frac{1}{2}trh\cdot\omega\right\}.$$

If $\omega = \sum_{i,j}\omega_{ij}e_ie_j$, then the corresponding skew-symmetric matrix is $A = \frac{1}{2}(\omega_{ij} - \omega_{ji})$. The variation in the metric, h_{ij}, is a symmetric matrix $H = (h_{ij})$. Then, by Corollary 1, $\dot{**}_t\omega$ corresponds to

$$\{H, A\} - \frac{1}{2}\mathrm{tr}H\cdot A,$$

where $\{H, A\}$ is the anticommutator $HA + AH$.

Since we will need this later, we will compute the image of the map

$$i_\omega : \mathcal{S} \to \Lambda^k\mathbf{R}^n,$$
$$i_\omega(h) = \dot{**}_{g+th}\omega,$$

where ω is a k-form and $\mathcal{S}$ is the set of symmetric $n\times n$ matrices, for certain values of k and n.

k=1: If $\omega = 0$, then Im $i_\omega = 0$. If $\omega \neq 0$, then Im $i_\omega = \Lambda^1 \mathbf{R}^n = \mathbf{R}^n$.

k=2: When dealing with 2-forms, the computations become easier in matrix form. i_ω then becomes

$$i_A : \mathcal{S} \to \mathcal{A},$$
$$i_A(H) = \{H, A\} - \frac{1}{2}\mathrm{tr} H \cdot A,$$

where $\mathcal{A}$ is the set of skew-symmetric $n \times n$ matrices.

We can make a further simplification when $n \neq 4$: Taking $H = I$, we obtain $A \in$ Im i_A, and hence Im $i_A = \mathrm{Im}\{\cdot, A\}$. The situation for $n = 4$ is quite different ($n = 4$ is the only anomaly), and this is the first indication of the differences between $n = 4$ and $n > 4$.

Observe that if $B = C^{-1}AC$, then $i_B(H) = C^{-1} i_A(CHC^{-1})C$. Moreover, if we let $C \in O(n)$, then $C^{-1} = C^t$, and CHC^{-1} will become symmetric. In view of this, it suffices to compute i_A for each orbit of $\Lambda^2 \mathbf{R}^n$ under the action of $O(n)$.

Fact: A skew-symmetric matrix A can always be put into the form

$$\begin{pmatrix} 0 & -\lambda_1 & & & & & & \\ \lambda_1 & 0 & & & & & & \\ & & 0 & -\lambda_2 & & & & \\ & & \lambda_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & \ddots \end{pmatrix}$$

via an orthonormal change of basis. We call this the *normal form* of A. If we assume that $A \in O(n)$ instead of $SO(n)$, then we may assume that all $\lambda_i \geq 0$.

Definition: Let A be a $2n \times 2n$ skew-symmetric matrix. Put A in normal form as above, and let λ_i be the 'eigenvalues'. If $\lambda_i \neq 0$ and $\lambda_i \neq \pm\lambda_j$ for $i \neq j$, then A is said to be of *generic type.*

k=2, n=4: For a 4×4 matrix A, we have the following possibilities:

(1) $A_1 = 0$.

(2) $A_2 = \begin{pmatrix} 0 & -\lambda & & \\ \lambda & 0 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}, \lambda \neq 0.$

(3) $A_3 = \begin{pmatrix} 0 & -\lambda & & \\ \lambda & 0 & & \\ & & 0 & \pm\lambda \\ & & \mp\lambda & 0 \end{pmatrix}, \lambda \neq 0.$

(4) $A_4 = \begin{pmatrix} 0 & -\lambda_1 & & \\ \lambda_1 & 0 & & \\ & & 0 & -\lambda_2 \\ & & \lambda_2 & 0 \end{pmatrix}, \lambda_1 \neq \pm\lambda_2.$

We can easily compute Im i_A for each of the four cases.

(1) Im $i_{A_1} = 0$.

(2) Im $i_{A_2} = \left\{ \begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix} \middle| \begin{array}{c} A = 2 \times 2 \text{ skew-symmetric matrix,} \\ B = 2 \times 2 \text{ matrix} \end{array} \right\}.$

Hence, dim Im $i_{A_2} = 5$. If $e_1, ..., e_4$ is an orthonormal basis, then $(\text{Im } i_{e_1e_2})^\perp = \mathbf{R}\{e_3e_4\} = \mathbf{R}\{*e_1e_2\}$, where the metric on Λ^2T^*M is the one induced from TM.

(3) Im $i_{A_3} = \left\{ \begin{pmatrix} 0 & -a & b & c \\ a & 0 & -c & b \\ -b & c & 0 & a \\ -c & -b & -a & 0 \end{pmatrix} \middle| a, b, c \in \mathbf{R} \right\}$, if A_3 corresponds to $\omega = e_1e_2 + e_3e_4$, that is, ω is self-dual. Note that Im i_{A_3} is the space of anti-self-dual 2-forms.

(4) Im $i_{A_4} = \left\{ \begin{pmatrix} \lambda_1 A & B \\ -B^T & -\lambda_2 A \end{pmatrix} \middle| \begin{array}{c} A = 2\times2 \text{ skew-symmetric matrix} \\ B = 2\times2 \text{ matrix} \end{array} \right\}$. As with (2), dim Im $i_\omega = 5$ and Im $i_\omega = (*\omega)^\perp$. Observe that i_ω is not surjective even if ω is of generic type - this is in sharp contrast with the cases $n \geq 6$.

Aside: The fact that Im $i_{A_3} = \{$anti-self-dual (ASD) 2-forms$\}$ can be rephrased as follows:

Proposition 8 *Let Λ_g^+, Λ_g^- be the self-dual and anti-self-dual subbundles of Λ^2T^*M, respectively. Then the conformal classes near $[g]$ are in $1-1$ correspondence with an open set of Hom$(\Lambda_g^+, \Lambda_g^-)$ containing $0 : \Lambda^+ \to \Lambda^-$.*

To each conformal class $[g + th]$ near g we can assign Λ_{g+th}^+, the self-dual subspace of Λ^2, with respect to $g + th$. We can then view Λ_{g+th}^+ as the graph of an element of $\text{Hom}(\Lambda_g^+, \Lambda_g^-)$, and what we are asserting is that to any $\phi \in \text{Hom}(\Lambda^+, \Lambda^-)$ near $0 \in \text{Hom}(\Lambda^+, \Lambda^-)$, there is a unique conformal class $[g + th]$ with $\Lambda_{g+th}^+ = \text{Graph}(\phi)$.

k=2, n=6: The following are the possible types of orbits (assume without loss of generality that $\lambda_i \geq 0$ via an $O(2n)$ action):

(1) $A=0$. Im $i_A=0$.

(2) $A = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & \\ & 0 & \\ & & 0 \end{pmatrix}$. Im $i_A = \left\{ \begin{pmatrix} * & * & * \\ * & & \\ * & & \end{pmatrix} \right\}$, where the $*$ means that the 2×2 block can be arbitrary (as long as the skew-symmetricity is satisfied) and a blank entry means the 2×2 block is composed of all zeros.

(3) $A = \begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 0 \\ & & & & 0 & 0 \end{pmatrix}$. Im $i_A = \left\{ \begin{pmatrix} * & X & * \\ -X^t & * & * \\ * & * & 0 \end{pmatrix} \right\}$, where X means that the 2×2 block can be an arbitrary matrix of the form $\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$, with $c, d \in \mathbf{R}$.

(4) $A = \begin{pmatrix} 0 & -\lambda_1 & & & & \\ \lambda_1 & 0 & & & & \\ & & 0 & -\lambda_2 & & \\ & & \lambda_2 & 0 & & \\ & & & & 0 & 0 \\ & & & & 0 & 0 \end{pmatrix}$. Im $i_A = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & 0 \end{pmatrix} \right\}$. Here, $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \neq 0$.

(5) $A = \begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & 0 & -\lambda \\ & & & & \lambda & 0 \end{pmatrix}$, with $\lambda \neq 1$. Im $i_A = \left\{ \begin{pmatrix} * & X & * \\ -X^t & * & * \\ * & * & * \end{pmatrix} \right\}$.

(6) $A = \begin{pmatrix} 0 & -\lambda_1 & & & & \\ \lambda_1 & 0 & & & & \\ & & 0 & -\lambda_2 & & \\ & & \lambda_2 & 0 & & \\ & & & & 0 & -\lambda_3 \\ & & & & \lambda_3 & 0 \end{pmatrix}$, where all the λ_i are distinct. Im $i_A = \mathcal{A}$.

In determining the above, we used the following rule:

Rule: Suppose A is of the form

$$\begin{pmatrix} \ddots & & & & & & \\ & 0 & -\lambda_i & & & & \\ & \lambda_i & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & -\lambda_j & \\ & & & & \lambda_j & 0 & \\ & & & & & & \ddots \end{pmatrix}.$$

Then Im i_A will contain

$$\begin{pmatrix} \ddots & & & & \\ & \lambda_i * & & * & \\ & & \ddots & & \\ & * & & \lambda_j * & \\ & & & & \ddots \end{pmatrix},$$

if $\lambda_i \neq \pm\lambda_j$. The $*$'s are placed in the (i,i)-th, (i,j)-th, (j,i)-th, and (j,j)-th 2×2 blocks. On the other hand, if $\lambda_i = \lambda_j$, then Im i_A will contain

$$\begin{pmatrix} \ddots & & & & \\ & * & & X & \\ & & \ddots & & \\ & -X^t & & * & \\ & & & & \ddots \end{pmatrix}.$$

Using this rule, it is easy to prove the following:

Theorem 1 *If a $2n \times 2n$ skew-symmetric matrix A is of generic type, then Im$\{\cdot, A\}$ surjects onto $\mathcal{A}_{2n}$.*

Remark: It should be noted that the situation for $n > 6$ is similar to that of $n = 6$; hence there is a dichotomy $n = 4$ and $n > 6$.

1.2.5 Harmonic 1-forms

Let us gather together the relevant data. $v(0)$, the perturbation of $\omega(0)$, has the form

$$v(0) = \pm \int_M \langle dd^*F(y) + dd^*f(y), \dot{**}_{g+th}\omega(y)\rangle_y dv_g,$$

where $h(y)$ is in C^k for large k (in particular for $k \geq 2$), $G_g(0, y) \sim F(y)$ asymptotically, as $y \to 0$, and $f(y)$ is the difference $G_g(0, y) - F(y)$.

Recall we are using geodesic normal coordinates around 0. Identify all the T_yM's with T_0M.

We showed that

(1) $\langle dd^*F(y), \cdot\rangle = \frac{C}{|y|^n} R_y$ for g flat,
(2) $\dot{**}_{g+th}\omega(y) : \Lambda_0^1 \to \Lambda_0^1$ is surjective, whenever $\omega(y) \neq 0$.

Given $\eta \in \Lambda_0^1$, let y_0 be a point near 0 and h_0 be a variation of the metric at y_0 such that $\dot{**}_{g(y_0)+th_0}\omega(y_0) = \eta$. By taking a sequence of h's with small support approaching $h_0 \cdot \delta(y_0)$, we get that,

$$\pm ev_*(h) = \int_M \langle dd^*F(y) + dd^*f(y), \dot{**}_{g+th}\omega(y)\rangle_g dv_g \to \langle dd^*F(y_0) + dd^*f(y_0), \eta\rangle.$$

Next, suppose we can pick a sequence of pairs $\{(y_i, h_i)\}$ with $\dot{**}_{g(y_i)+th_i}\omega(y_i) = \eta$ and $y_i \to 0$ such that $\frac{y_i}{|y_i|} \to \alpha$ for some fixed unit vector α. Then we get

$$ev_*(h_i\delta(y_i)) = \langle dd^*F(y_i) + dd^*f(y_i), \eta\rangle.$$

However, as $y_i \to 0$, the ratios of $dd^*F(y_i)$ to $dd^*f(y_i)$ become large, and

$$\begin{aligned} \frac{\langle dd^*F(y_i) + dd^*f(y_i), \eta\rangle}{|\langle dd^*F(y_i) + dd^*f(y_i), \eta\rangle|} &\to \frac{\langle dd^*F(y_i), \eta\rangle}{|\langle dd^*F(y_i), \eta\rangle|} \\ &\to \frac{\langle dd^{*_0}F(y_i), \eta\rangle}{|\langle dd^{*_0}F(y_i), \eta\rangle|} \\ &= \frac{R_\alpha(\eta)}{|R_\alpha(\eta)|}, \end{aligned}$$

where d^{*_0} is the adjoint of d with respect to the flat metric g_0 which agrees with g at 0. Thus we have the following proposition:

Proposition 9 *Suppose there exists a sequence $y_i \to 0$ with $\frac{y_i}{|y_i|} \to \alpha$ such that $\eta \in \Lambda_0^1$ is in $Im\ i_{\omega(y_i)}$. Then there exists a sequence of elements in $Im\ ev_*$ approaching $R_\alpha(\eta)$; in particular, we can conclude that $R_\alpha(\eta) \in Im\ ev_*$.*

The last statement of the proposition requires some justification:

Lemma 2 *Suppose $\{v_i\}_{i=1}^{\infty}$ is a sequence in Im ev_* converging to v. Then $v \in$ Im ev_*.*

Proof: Let $V_i = \mathbf{R}\{v_1, ..., v_i\}$. Then $V_1 \subset V_2 \subset ...$, and this sequence must stabilize after some V_n because Im ev_* is finite dimensional. Either $v \in V_n$, in which case we are done, or $v \notin V_n$ but $v_i \in V_n$ for all i, which is a contradiction. $\square$

In view of the above proposition, the fact that R_α is an isomorphism, and that Im $i_{\omega(y)}$ is surjective whenever $\omega(y) \neq 0$, all we need now is that there exists a dense subset near 0 on which ω is nonzero. This is afforded by the following:

Theorem 2 ('Weak' Unique Continuation Theorem) *Let L be a second order elliptic operator with C^k coefficients, for sufficiently large k. Suppose $Lu = 0$ on a domain D, and $u = 0$ on a nonempty open subset of D. Then $u = 0$ on all of D.*

Hence,

Theorem 3 *There exists a dense G_δ-set Q' in $Q = [\omega] \times Met^k(M)$ for which the harmonic form $(\omega, g) \in Q' \subset H^1(M; \mathbf{R}) \times Met^k(M)$ has regular zeros. This implies that the generic harmonic 1-form ω in a fixed cohomology class $[\omega]$ has a collection of isolated points as its zeros.*

The theorem is of course also true for $Q = H^1(M; \mathbf{R}) \times \mathrm{Met}^k(M)$.

Remark: The number of isolated zeros of ω, counted up to sign, is minus the Euler characteristic of TM.

1.2.6 Harmonic 2-forms on a 4-manifold

We will first prove the following theorem:

Theorem 4 *Let M be a closed, oriented 4-manifold. Given a metric g, view $\mathcal{H}^2_+$, the set of self-dual harmonic 2-forms, as a subset of $H^2(M; \mathbf{R})$. If dim $\mathcal{H}^2_+ > 0$, (dim $\mathcal{H}^2_+ = b_2^+(M)$ is the same for all $g \in Met^k(M)$), consider $Q \subset H^2(M; \mathbf{R}) \times Met^k(M)$, where $Q = \{(\omega, g) : \Delta_g \omega = 0, *_g \omega = \omega\}$. Then there exists a dense G_δ-set $Q' \subset Q$ such that the harmonic forms $(\omega, g) \in Q'$ have regular zeros. This means that the zeros of ω consist of disjoint circles for generic g.*

Observe that in this theorem, we could have substituted anti-self-dual 2-forms for self-dual 2-forms with the same result.

Recall: (1) $\langle dd^*F(y), \cdot\rangle = \frac{C}{|y|^4}R_y$.

(2) For ω self-dual, Im i_ω consists of anti-self-dual 2-forms.

(3) R_y swaps self-dual forms and anti-self-dual forms, i.e. $R_y : \Lambda^{\pm} \xrightarrow{\sim} \Lambda^{\mp}$.

Proof of Theorem 4: We use a slightly different evaluation map

$$ev : Q \times M \to \bigwedge{}^+,$$
$$(\omega, g, x) \mapsto \omega(x),$$

where $\bigwedge^+ \to \mathrm{Met}^k(M) \times M$ is the universal vector bundle with fiber $\bigwedge_g^+(x)$ at the point $(g, x) \in \mathrm{Met}^k(M) \times M$.

In this case, we want to show that ev is transverse to the zero section of $\bigwedge^+$, i.e.

$$ev_* : T_{(\omega,g)}Q \to \bigwedge{}_g^+(x)$$

is surjective, whenever $\omega(x) = 0$.

We have two necessary and sufficient conditions for $(v, h) \in T_{(\omega,g)}Q$:

(1) $\Delta_g(v) + \frac{d}{dt}(\Delta_{g+th})|_{t=0}\omega = 0$,

(2) $*_{g+th}(\omega + tv) = \omega + tv$.

Expanding (2) out, we obtain that $v = *_g v + \frac{d}{dt} *_{g+th} \omega$. When $\omega(x) = 0$, this gives $v = *_g v$. Hence, in order to determine all the possible perturbations of $\omega(x)$ when $\omega(x) = 0$, it suffices to compute the $v(x)$'s as before, and project onto $\bigwedge^+$, if that is necessary (as we shall see it is not).

The proof proceeds in the same fashion as for harmonic 1-forms. If ω is self-dual and $\omega(0) = 0$, then there exists an open dense subset near 0 on which $\omega \neq 0$ by the unique continuation theorem. Since at any point y where $\omega(y) \neq 0$, Im $i_{\omega(y)} = \bigwedge^-$, by the proposition in the previous section, Im $ev_* \supset R_y(\bigwedge^-) = \bigwedge^+$. Thus ev_* is surjective when $\omega(x) = 0$. $\square$

Remark: A self-dual form ω is nondegenerate if $\omega \neq 0$. This is because $\omega^2 = \omega \wedge *\omega > 0$, if $\omega \neq 0$. Hence if $b_2^+(M) > 0$, we can construct a self-dual harmonic form which is nearly symplectic, that is, is nondegenerate away from a collection of disjoint circles.

Let $(\tilde{\omega}_0, \tilde{g}_0)$, $(\tilde{\omega}_1, \tilde{g}_1)$ be regular points in Q, and form

$$P = \{(\omega_t, g_t) \in Q \times M | (\omega_0, g_0) = (\tilde{\omega}_0, \tilde{g}_0); (\omega_1, g_1) = (\tilde{\omega}_1, \tilde{g}_1)\}.$$

Consider

$$\widetilde{ev} : P \times M \times [0,1] \to \bigwedge^+$$
$$(\omega_t, g_t, x, t) \mapsto \omega_t(x).$$

It is evident that $\widetilde{ev}_*$ is surjective on $\mathcal{U} = \{(\omega_t, g_t) \in P | \omega_t \not\equiv 0 \text{ on } [0,1]\}$, since at any $(\omega_{t_0}, g_{t_0}, x_0, t_0)$ at which $\omega_{t_0}(x_0) = 0$, $ev_*(\omega_{t_0}, g_{t_0}, x_0)$ is surjective because $\omega_{t_0} \not\equiv 0$.

Theorem 5 *Given two regular points $(\tilde{\omega}_0, \tilde{g}_0)$ and $(\tilde{\omega}_1, \tilde{g}_1)$ in Q, there exists a dense G_δ-set of paths inside $\mathcal{U} = \{(\omega_t, g_t) \in P | \omega_t \not\equiv 0$ on $[0,1]\}$ for which the zeros of (ω_t, g_t) gives a cobordism inside $M \times [0,1]$ between the zeros of $(\tilde{\omega}_0, \tilde{g}_0)$ and the zeros of $(\tilde{\omega}_1, \tilde{g}_1)$.*

Now if the 4-manifold M had a positive definite (or negative definite) intersection matrix, then all the harmonic forms are automatically self-dual (or anti-self-dual), and their generic zeros are circles. Hence $b_2^+(M) = 0$ or $b_2^-(M) = 0$ must be excluded from the following theorem:

Theorem 6 *Assume additionally that $b_2^\pm(M) > 0$. Then there exists a dense G_δ-set Q' in $Q = H^2(M; \mathbf{R}) \times Met^k(M)$ for which the harmonic form $(\omega, g) \in Q'$ has regular zeros. Thus, the generic harmonic 2-form has no zeros.*

In fact, we can do even better. But first we need to discuss the stratification of $\bigwedge^2 \mathbf{R}^4$ under the action of $SO(4)$. Stratify $\bigwedge^2 \mathbf{R}^4$ by rank: Let $V_i = \{\omega \in \bigwedge^2 \mathbf{R}^4 | \text{rk } \omega = i\}$. In particular, $V_0 = \{0\}$. V_4 has two substrata, namely the SD 2-forms and the ASD 2-forms, which we denote $V_{4,+}$ and $V_{4,-}$, respectively. For convenience, we assemble the relevant data in a chart:

Stratum	Typical element	Dim orbit	Dim stratum
V_0	0	0	0
V_2	$\lambda e_1 e_2$	4	5
$V_{4,\pm}$	$\lambda(e_1 e_2 \pm e_3 e_4)$	2	3
$V_4 - (V_{4,+} \bigcup V_{4,-})$	$\lambda_1 e_1 e_2 + \lambda_2 e_3 e_4$, $\lambda_1 \neq \pm\lambda_2$	4	6

Here, $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis for $\mathbf{R}^4$.

We shall obtain Theorem 6 by showing that there exists a dense open set $\mathcal{U} \in Q$ for which

$$ev : \mathcal{U} \times M \to \bigwedge^2 T^* M$$

is a regular section, i.e. locally, taking $\mathcal{U}'$ containing $(\omega, g) \in \mathcal{U}$ and $U \subset M$ containing $x = 0$, and showing that

$$ev : \mathcal{U}' \times U \to \bigwedge^2 \mathbf{R}^4$$

is *transverse* to V_0.

Assume for the moment that we can prove the following:

Theorem 7 *There exists a dense open $\mathcal{U} \subset Q$ on which ev_* is surjective for all points in $\mathcal{U} \times M$. More precisely, $\mathcal{U} \supset Q' = (Q - D) \bigcap H^2(M; \mathbf{R}) \times Met^\infty(M)$, where $Met^\infty(M)$ is the C^∞-metrics of M and $D = \{(\omega, g) \in Q | *\omega = \pm\omega\}$.*

Then $ev : \mathcal{U}' \times U \to \bigwedge^2 \mathbf{R}^4$ is transverse to V_0. Remove $ev^{-1}(V_0)$ from $\mathcal{U}' \times U$. Then

$$ev : \mathcal{U}' \times U - ev^{-1}(V_0) \to \bigwedge{}^2 \mathbf{R}^4 - V_0$$

is transverse to V_2. Proceeding further,

$$ev : \mathcal{U}' \times U - ev^{-1}(V_0 \bigcup V_2) \to \bigwedge{}^2 \mathbf{R}^4 - V_0 - V_2$$

is transverse to $V_{4,+}$, and so on. Hence, by removing lower strata, ev can be made transverse to any given stratum. Then we obtain

Theorem 8 *Suppose $b_2^\pm > 0$. Then there exists a dense G_δ-set Q'' in $Q = H^2(M; \mathbf{R}) \times Met^k(M)$ for which $(\omega, g) \in Q''$ has no zeros, has full rank (and hence is nondegenerate) away from a submanifold of codimension 1, and is SD/ASD on a union of disjoint circles.*

Remark: Theorem 7 implies an analog of Theorem 5 for non-SD or ASD forms. Provided the 1-parameter family (ω_t, g_t) is in Q', we have surjectivity of $\widetilde{ev}_*$.

Proof of Theorem 7: Assume $(\omega, g) \in Q - D$. Then ω can be written as $\omega = \omega_+ + \omega_-$ with ω_+ self-dual and ω_- anti-self-dual, and $\omega_\pm \not\equiv 0$. The points x where ω is SD are the zeros of ω_-, which is harmonic as well, so, by unique continuation, ω is SD (or ASD) only away from a dense open subset of M. Of course, the locus $\{x | \omega(x) \neq 0\}$ also is a dense open subset of M. In what follows, take geodesic normal coordinates y on a suitably small ball D^n about $y = 0$.

Case 1: Suppose $\omega(0) \in V_4 - V_{4,\pm}$. Then we may assume that for all $y \in D^n$, $\omega(y) \in V_4 - V_{4,\pm}$, and Im $i_{\omega(y)} = (*\omega(y))^\perp$, a 5-dimensional subspace of $\bigwedge^2 T_0^* M$. Now, as $y \to 0$ (more precisely, fixing $y_0 \neq 0$ and sending $t \to 0$ with $y = ty_0$),

$$\begin{aligned} \langle dd^* G(y), \text{Im } i_{\omega(y)} \rangle_g &\to \langle R_y, *_0 \omega(y)^{\perp_0} \rangle_{g_0} \\ &= (*_0 \langle R_y, \omega(y) \rangle_{g_0})^{\perp_0}, \end{aligned}$$

with $*_0$, g_0, $\perp_0$ for the inner product at $y = 0$. But since $\omega(y) \to \omega(0)$ as $y \to 0$, the subspaces $\langle R_y, (*_0 \omega(y))^{\perp_0} \rangle_{g_0}$ do not approach the same 5-dimensional space from the various directions on $S_\varepsilon^3 = \{|y| = \varepsilon\}$. Hence $ev_*(\omega, g, x)$ must be surjective.

Case 2: Suppose $\omega(0) \in V_{4,+}$. Since rank is upper semicontinuous, we may assume that for all $y \in D^n$, $\omega(y) \in V_4$; $\omega(y) \in V_4 - V_{4,\pm}$ on a dense open subset of D^n. The considerations of Case 1 then apply here as well.

Case 3: Assume $\omega(0) \in V_2$. Once again, for all $y \in D^n$, $\omega(y) \in V_2 \bigcup V_4$, Im $i_{\omega(y)} = (*\omega(y))^\perp$, which is still 5-dimensional, and sending $y \to 0$,

$$\langle dd^*G(y), \mathrm{Im}\ i_{\omega(y)}\rangle_g \to (*_0\langle R_y, \omega(y)\rangle_{g_0})^{\perp_0},$$

so $ev_*(\omega, g, x)$ must be surjective.

Case 4: Here it becomes important to assume that the metric g is C^∞, and the corresponding harmonic form ω is also C^∞. Assume $\omega(0) \in V_0$. If $ev_*(0)$ is not surjective, then

$$\langle dd^*G(y), \mathrm{Im}\ i_{\omega(y)}\rangle = \langle dd^*G(y), (*_g\omega(y))^\perp\rangle$$

must be a 5-dimensional subspace of $\bigwedge^2 T_0^*M$, independent of y. Since ω is C^∞, we are able to write

$$\omega(y) = \omega_r(y) + \text{h.o.}$$

where $\omega_r(y) = \sum_{i,j} p_r^{ij}(y)dy_i dy_j$ with $p_r^{ij}(y)$ a homogeneous polynomial of degree r, and 'h.o.' consisting of terms of degree $> r$ in y. Note that there must exist some $r < \infty$ for which $\omega_r(y) \neq 0$; this follows from the 'strong' unique continuation theorem. Similarly, we can write $g = g_0 + \text{h.o.}$, $* = *_0 + \text{h.o.}$, $\perp = \perp_0 + \text{h.o.}$, and d^{*_0} the (formal) adjoint of d with respect to $*_0$. Hence

$$\begin{aligned}\langle |y|^6 dd^*G(y), \mathrm{Im}\ i_{\omega(y)}\rangle_g &= |y|^6\langle dd^{*_0}F(y), (*_0\omega_r)^{\perp_0}\rangle_{g_0} + \text{h.o.} \\ &= \langle |y|^2 R_y, (*_0\omega_r)^{\perp_0}\rangle_{g_0} + \text{h.o.}\end{aligned}$$

Observe that the coefficients of $|y|^2 R_y$ are polynomials of degree 2. Then,

$$|y|^2\langle R_y, *_0\omega_r\rangle_{g_0} = \sum_{i,j} f_{r+2}^{ij}(y)dy_i dy_j,$$

with f_{r+2}^{ij} a homogeneous polynomial of degree $r+2$ in y. Since the degree r term dominates as $y \to 0$, we see that, if $ev_*(0)$ is not surjective, then $|y|^2\langle R_y, *_0\omega_r\rangle_{g_0}$ must satisfy

$$|y|^2\langle R_y, *_0\omega_r\rangle_{g_0} = f_{r+2}(y)\widetilde{\omega}_0,$$

with $f_{r+2}(y)$ homogeneous in y, and $\widetilde{\omega}_0$ constant. Hence

$$*_0\omega_r = \frac{f_{r+2}(y)}{|y|^2}\langle R_y, \widetilde{\omega}_0\rangle_{g_0},$$

and $\omega_r = f_{r+2}|y|^4\langle dd^{*_0}F(y), *_0\widetilde{\omega}_0\rangle_{g_0}$. Finally $d\omega = 0$ and $d * \omega = 0$ imply $d\omega_r = 0$ and $d *_0 \omega_r = 0$. Then,

$$\begin{aligned} d\omega_r = 0 &\Rightarrow d(f_{r+2}|y|^4) \wedge \langle dd^{*_0}F(y), *_0\widetilde{\omega}_0\rangle_{g_0} = 0 \\ d(*_0\omega_r) = 0 &\Rightarrow d(f_{r+2}|y|^4) \wedge \langle *_0 dd^{*_0}F(y), *_0\widetilde{\omega}_0\rangle_{g_0} = 0, \end{aligned}$$

using the fact that $(dd^{*_0} + d^{*_0}d)F(y) = \Delta_{g_0}F(y) = 0$ on $D^n - \{0\}$. Now, if $\xi \in T_0^*M$, $\omega \neq 0$ is in $\bigwedge^2 T_0^*M$, and $\xi \wedge \omega = 0$, then ω must be decomposable and ξ must lie on the 2-plane given by ω. If $\xi \wedge *\omega = 0$ as well, then ξ also lies on the 2-plane orthogonal to ω, and $\xi = 0$. Thus $d(f_{r+2}|y|^4) = 0$ and $f_{r+2} = 0$, contradicting the assumption. □

1.2.7 Harmonic 2-forms on a $2n$-manifold

We will prove the following theorem, which is a slightly weaker version of Theorem 8, for higher dimensions. Let M be a $2n$-manifold, with $2n > 4$.

Theorem 9 *There exists a dense open $\mathcal{U} \subset Q = [\omega] \times Met^k(M) \subset H^2(M;\mathbf{R}) \times Met^k(M)$ on which ev_* is surjective for all points in $\mathcal{U} \times M$.*

Proof: We need to prove that, starting from any (ω, g, x) with $\omega \not\equiv 0$, we can find an arbitrarily small perturbation $\omega + v$ (in stages) such that $(\omega + v)(x)$ has generic type. Once we prove this, we can invoke the following lemma:

Lemma 3 *The set $S = \{(\omega, g) \in Q | \omega(x)$ is of generic type on a dense open in $M\}$ is dense in Q.*

Proof of Lemma 3: We will exhibit an $(\omega, g) \in S$ arbitrarily close to $(\omega_0, g_0) \in Q$. Pick a countable dense subset of M, say $\{x_i\}$. Let $\mathcal{U}_0 \ni (\omega_0, g_0)$ be an open set in Q. We pick $\mathcal{U}_i \subset \mathcal{U}_{i-1}$ and $V_i \ni x_i$ inductively, as follows: Given x_i, there exists a point $(\omega_i, g_i) \in \mathcal{U}_{i-1}$ such that $\omega_i(x_i)$ is of generic type. Then there exists an open set $\mathcal{U}_i \times V_i \ni (\omega_i, g_i, x_i)$, $\mathcal{U}_i \subset \mathcal{U}_{i-1}$, on which (ω, g, x) is of generic type, since the generic type condition is an open condition.

Now let $(\omega, g) \in \bigcap_{i=1}^{\infty} \mathcal{U}_i$, which is nonempty because of completeness. By our construction, $\omega(x)$ is of generic type on an open dense set in M. □

Since there exist points of generic type arbitrarily near any point $x \in M$, for $(\omega, g) \in S$, $ev_*(\omega, g, x)$ is surjective for all $x \in M$ and $(\omega, g) \in S$. Now the surjectivity of ev_* is an open condition in $Q \times M$. Combining this with the compactness of M, we obtain that the condition '$ev_*(\omega, g, x)$ is surjective for all $x \in M$' is an open condition in Q. But S is dense, so hence there is an open dense set $\mathcal{U} \subset Q$ on which $ev_*(\omega, g, x)$ is surjective for all $x \in M$. This would complete the proof of the theorem.

Let us now proceed to show that ω can be perturbed at x so that $(\omega+v)(x)$ has generic type. If $\omega(x)=0$, pick a point y arbitrarily close to x such that $\omega(y)\neq 0$. It exists by unique continuation. If $\omega(x)\neq 0$, pick a point y arbitrarily near x such that rk $\omega(y)\geq$ rk $\omega(x)$. This is possible because the rank is upper semicontinuous. Upon picking suitable orthonormal coordinates around x,

$$\omega(y)=\begin{pmatrix} 0 & -\lambda_1 & & & & \\ \lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & -\lambda_k & \\ & & & \lambda_k & 0 & \\ & & & & & 0 \end{pmatrix},$$

where the $\lambda_i>0$ are not necessarily distinct. Then by the Rule in Section 1.2.4, Im $i_{\omega(y)}$ contains an element v of higher rank than $\omega(y)$, if $\omega(y)$ does not have maximal rank already. For example, Im $i_{\omega(y)}$ contains any element of the form

$$\begin{pmatrix} * & & & & & \\ & * & & & & \\ & & \ddots & & & \\ & & & * & * & \\ & & & * & 0 & 0 \\ & & & & 0 & 0 \end{pmatrix}.$$

Form $\omega(x)+tR_yv$. Recall that, as in the section on 1-forms, $R_yv\in$ Im ev_*, so there exists a perturbation of the metric giving rise to $\omega'(x)=\omega(x)+tR_yv$ (at least up to 1st order in t). Since R_yv preserves the rank of v, rk $R_yv>$ rk $\omega(x)$. For small enough t, rk $\omega'(x)>$ rk $\omega(x)$. This follows from observing that, since rk $R_yv>$ rk $\omega(x)$, R_yv is not zero on $\omega(x)^\perp=\{w\in T_xM|\omega(x)(w,\cdot)=0\}$, and, for small enough t, $\omega'=\omega+tR_yv$ is still nondegenerate on $(\omega(x)^\perp)^\perp$ (the second $\perp$ with respect to a Riemannian metric). Continue this process until we get a $\tilde{\omega}(x)$ whose 'eigenvalues' λ_i are all nonzero.

The next step is to perturb until the λ_i become distinct, while keeping them nonzero. Denote by J_k the matrix

$$\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}$$

of rank $2k$.

By an orthonormal change of basis, we may write

$$\tilde{\omega}(x) = \begin{pmatrix} \lambda_1 J_{k_1} & & & \\ & \lambda_2 J_{k_2} & & \\ & & \ddots & \\ & & & \lambda_r J_{k_r} \end{pmatrix},$$

where $\lambda_i > 0$, $\lambda_i \neq \lambda_j$, and $k_1 \geq ... \geq k_r$. Of course, $\sum_{i=1}^r k_i = n$.

Let

$$\bigwedge\nolimits_{k_1,\dots,k_r} = \left\{ BAB^{-1} \,\middle|\, A = \begin{pmatrix} \lambda_1 J_{k_1} & & \\ & \ddots & \\ & & \lambda_r J_{k_r} \end{pmatrix}, \begin{array}{c} \lambda_i \neq \lambda_j, \lambda_i > 0, B \in O(2n), \\ k_1 \geq ... \geq k_r, k_1 + ... + k_r = n \end{array} \right\}.$$

$\bigwedge_{k_1,\dots,k_r}$ is the stratum consisting of orbits of skew-symmetric matrices of the form

$$\begin{pmatrix} \lambda_1 J_{k_1} & & \\ & \ddots & \\ & & \lambda_r J_{k_r} \end{pmatrix}.$$

Lemma 4 *dim* $\bigwedge_{k_1,\dots,k_r} = \textit{dim}\ O(2n) - 2\sum_{i=1}^r k_i^2 + r.$

Proof: We first compute the dimension of the orbit of $A = \begin{pmatrix} \lambda_1 J_{k_1} & & \\ & \ddots & \\ & & \lambda_r J_{k_r} \end{pmatrix}$. dim $O(2n) \cdot A = \dim O(2n) - \dim \text{Stabilizer} = \dim O(2n) - \dim \ker \text{ad}(A)$, where $\text{ad}(A)(B) = [A, B] = AB - BA$. Writing $B = (B_{ij})$, where B_{ij} is a $2k_i \times 2k_j$ block, $[A, B] = 0$ gives

$$\lambda_i J_{k_i} B_{ij} = \lambda_j B_{ij} J_{k_j}.$$

If $i = j$, then $B_{ii} \in \text{Gl}(k_i, \mathbf{C})$. If $i \neq j$, then $B_{ij} = 0$. Thus,

$$\begin{aligned} \ker\ (\text{ad}(A)) &\simeq [Gl(k_1, \mathbf{C}) \oplus \cdots \oplus Gl(k_r, \mathbf{C})] \bigcap O(2n) \\ &= U(k_1) \oplus ... \oplus U(k_r), \end{aligned}$$

and

$$\dim \ker\ = k_1^2 + ... + k_r^2.$$

Finally, □

$$\dim \bigwedge\nolimits_{k_1,\dots,k_r} = \dim O(2n) \cdot A + r = \dim O(2n) - \sum_{i=1}^r k_i^2 + r.$$

Lemma 5 *dim Im* $i_A = \textit{dim}\ O(2n) - \sum_{i=1}^r k_i^2 + n.$

Proof: Using the Rule in Section 1.2.4, one computes that Im $i_A = \begin{pmatrix} A_1 & * & * & * \\ * & A_2 & * & * \\ * & * & \ddots & * \\ * & * & * & A_r \end{pmatrix}$, with $A_i = \begin{pmatrix} * & X & X & X \\ X & * & X & X \\ X & X & \ddots & X \\ X & X & X & * \end{pmatrix}$, where $*$ means the block is an arbitrary matrix of the correct size and X consists of 2×2 blocks of the form $\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$.

Thus,
$$\begin{aligned} \dim \text{Im } i_A &= \dim O(2n) - \sum_i \frac{k_i(k_i-1)}{2} \\ &= \dim O(2n) - \sum_i k_i^2 + n. \end{aligned}$$ □

We now have that dim Im $i_A >$ dim $\bigwedge_{k_1,\ldots,k_r}$, where $A \in \bigwedge_{k_1,\ldots,k_r}$.

Claim: There exists a perturbation v such that $\tilde{\omega}(x) + tv \in \Lambda_{l_1,\ldots,l_s}$ with $\sum_{i=1}^s l_i^2 < \sum_i^r k_i^2$, as long as not all $k_i = 1$, i.e. $\tilde{\omega}(x)$ is not already generic.

Proof: Pick y arbitrarily close to x such that $\tilde{\omega}(y) \in \Lambda_{m_1,\ldots,m_p}$ with $\sum m_i^2 \leq \sum k_i^2$. Then dim Im $i_{\tilde{\omega}(y)} =$ dim $R_y(\text{Im } i_{\tilde{\omega}(y)}) >$ dim $\bigwedge_{k_1,\ldots,k_r}$. Now for small t, $\tilde{\omega}(x) + tR_y v$, with $v \in$ Im $i_{\tilde{\omega}(y)}$, is still nondegenerate. Hence this means that there exists a v such that $\tilde{\omega}(x) + tR_y v$ exits $\bigwedge_{k_1,\ldots,k_r}$ (as well as avoids other $\Lambda_{k_1',\ldots,k_r'}$ with $\sum k_i'^2 \geq \sum k_i^2$) by dimension count. □

Thus we can perturb in stages until we finally obtain an $(\tilde{\omega}, \tilde{g})$ close to (ω, g) with $\tilde{\omega}(x)$ of generic type. This concludes the proof of the theorem. □

A consequence of the theorem is the general principle that a harmonic 2-form, as regards generic transversality issues, behaves just like an ordinary closed 2-form, which, in turn, behaves like an ordinary 2-form with no differential condition. (Cf. Martinet for a study of generic *closed* forms, which, in the end, turns out to closely resemble our situation of harmonic 2-forms.)

Corollary 2 *There exists a dense G_δ-set of $Q = [\omega] \times Met^k(M) \subset H^2(M; \mathbf{R}) \times Met^k(M)$ on which the harmonic 2-form (ω, g) has no zeros.*

Corollary 3 *Let M be a 6-manifold. There exists a dense G_δ-set of $Q = [\omega] \times Met^k(M) \subset H^2(M; \mathbf{R}) \times Met^k(M)$, on which the harmonic 2-form has no zeros, has isolated points where* *it has rank* 2*, and, away from the rank* 2 *points, has rank* 4 *on a submanifold of codimension* 1.

Question: Is is possible to use results of this kind to construct symplectic forms on M^{2n}?

1.3 The Dirichlet Problem

Using a setup similar to that for harmonic forms on a compact manifold, one can prove an analogous theorem for solutions to the Dirichlet problem.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a C^∞-boundary $\partial\Omega$. For such domains Ω, we have the following:

Fact: There exists a unique solution u to the Dirichlet problem $\Delta_g u = 0$, $u|_{\partial\Omega} = f$, where Δ_g is the Laplacian with respect to the C^k-metric g (with k large), and f is a fixed C^k-function on the boundary.

(Since our goal is not to prove our genericity results in the greatest possible generality, we shall make some simplifications which may turn out to be unnecessary.)

Consider $\mathrm{Met}^k(\overline{\Omega})$, the space of C^k-metrics on $\overline{\Omega}$, i.e. defined and C^k on some open set containing $\overline{\Omega}$. This is a Banach manifold, which we also view as $\{(u,g)|\Delta_g u = 0, u|_{\partial\Omega} = f\}$. We shall prove the following theorem:

Theorem 10 *There is a dense G_δ-set in $Q = Met^k(\overline{\Omega})$ for which the solution to the Dirichlet problem $\Delta_g u = 0$, $u|_{\partial\Omega} = f$, has regular zeros inside Ω, provided $f \not\equiv 0$.*

Remark: Note that no claims are being made about the behavior of zeros as we approach $\partial\Omega$.

As before, we start with the following evaluation map:

$$ev : Q \times \Omega \to \mathbf{R}$$
$$((u,g),x) \mapsto u(x),$$

and we show ev is regular, that is, $ev_*(u,g,x)$ is surjective (i.e. nonzero), whenever $u(x) = 0$.

Computing $ev_*(u,g,x)$ is equivalent to differentiating the conditions $\Delta_{g_t} u_t = 0$, $u_t|_{\partial\Omega} = f$, where $g_t = g + th$ and $u_t = u + tv$. Differentiating, we get

$$\frac{d}{dt}(\Delta_{g_t} u_t)\bigg|_{t=0} = \Delta_g v + \frac{d}{dt}(\Delta_{g+th})\bigg|_{t=0} u = 0,$$

and $v|_{\partial\Omega} = 0$.

At this point, we convert the above differential equation into an integral involving the Green's function.

Fact: If Ω is a bounded domain with C^∞-boundary $\partial\Omega$, then the Green's function $G : \overline{\Omega} \times \overline{\Omega} \to \mathbf{R}$ exists, where $G(\cdot, \partial\Omega) = G(\partial\Omega, \cdot) = 0$.

Hence, we can write

$$
\begin{aligned}
v(x) &= -\int_\Omega G(x,y)\, \frac{d}{dt}(\Delta_{g+th})\Big|_{t=0} u(y) dv_g \\
&= \pm \int_\Omega G(x,y)\, \frac{d}{dt}(*_t d *_t)\Big|_{t=0} du(y) dv_g \\
&= \pm \int_\Omega G(x,y)(\dot{*} d * + * \, d\dot{*}) du(y) dv_g \\
&= \pm \int_\Omega G(x,y)(* d\dot{*}) du(y) dv_g,
\end{aligned}
$$

using $d * du = 0$.

Pick local geodesic coordinates on an open set U around $x = 0$. Since we shall only use perturbations of g which are supported arbitrarily close to x, we have

$$v(0) = \pm \int_U G(0,y)(* d\dot{*}) du(y) dv_g.$$

If we let

$$F(y) = \begin{cases} \frac{1}{|y|^{n-2}} & \text{if } n > 2 \\ \log |y| & \text{if } n = 2, \end{cases}$$

then,

Fact: $G(0,y)$ is asymptotic to $F(y)$ as $y \to 0$. That is, as $y \to 0$, the ratio $[G(0,y) - F(y) : F(y)] \to 0$. The same is true for $\partial^{i_1 \cdots i_l} G(0,y)$ and $\partial^{i_1 \cdots i_l} F(y)$.

This means that $G(0,y) \sim F(y)$ as well as $\frac{\partial}{\partial y_i} G(0,y) \sim C \cdot \frac{y_i}{|y|^n}$.

Now set $\omega(y) = du(y)$, and write $\omega = \sum_i \omega_i(t) e_i(t)$. We can choose an orthonormal basis $e_i(t) = e_i - \frac{1}{2} t \sum_j h_{ij} e_j$ on U for $g_t = g + th$, where $\{e_i\}$ is an orthonormal frame with respect to g.

Then,

$$
\begin{aligned}
v(0) &= \pm \int_U G(0,y) * d(\dot{*} du(y)) dv_g \\
&= \pm \int_U G(0,y) d(\dot{*} du(y)) \\
&= \pm \left\{ \int_U d(G(0,y)\dot{*} du(y)) - \int_U dG(0,y) \wedge \dot{*} du(y) \right\} \\
&= \pm \int_{\partial U} G(0,y)\dot{*} du(y) \pm \int_U \langle dG(0,y), *\dot{*} du(y) \rangle_g dv_g.
\end{aligned}
$$

Since h is supported near x, the first term on the right-hand side vanishes, and we are left with

$$v(0) = \pm \int_U \langle dG(0,y), *\dot{*} du(y) \rangle_g dv_g.$$

If du is identically 0 near y, then u is constant near y, and u must be constant on all of Ω by unique continuation. For u constant, Theorem 10 is trivially true. So assume u is not constant. Then there exist points y arbitrarily near 0 such that $du(y) \neq 0$. When $du(y) \neq 0$, $i_{du(y)}$ is surjective, and, just as in the case of harmonic 1-forms, we see that $ev_*(u, g, x)$ is surjective. This concludes the proof of Theorem 10.

Chapter 2

Moser argument for self-dual harmonic 2-forms on a 4-manifold

2.1 Self-dual harmonic 2-forms and almost complex structures

Let M^4 be a compact, closed, oriented 4-manifold. Assume $b_2^+ > 0$. Then, according to Theorem 4, for a pair (ω, g) consisting of a generic metric g and a self-dual harmonic 2-form ω with respect to g, (ω, g) represents a section of $\bigwedge_g^+ \to M$, which is transverse to the zero section. Here $\bigwedge_g^+$ is the subbundle of $\bigwedge^2 TM \to M$ whose fiber over a point $p \in M$ is $\bigwedge_g^+(p) = \{\omega | *_g \omega = \omega\}$. In particular, the zeros of ω are disjoint embedded circles.

Since $\omega \wedge \omega = \omega \wedge *\omega$, ω is nondegenerate at p if and only if $\omega(p) \neq 0$. That is, ω is closed and symplectic away from the union of circles C, and is identically 0 on C.

This is what enables us to define an almost complex structure J on $M - C$.

Proposition 10 *If ω is a self-dual harmonic 2-form which is nondegenerate on a connected set $M - C$, then there exists a unique almost complex structure J compatible with ω and $\tilde{g}$ on $M - C$, where $\tilde{g}$ is conformally equivalent to g.*

Proof: Any 2-form ω can be written, with respect to the metric g, as

$$\omega = \lambda_1 e_1 e_2 + \lambda_2 e_3 e_4,$$

with $e_1, ..., e_4$ orthonormal and positively oriented at a point $p \in M - C$.

For ω to be self-dual, $\lambda_1 = \lambda_2$. Hence,

$$\omega = \lambda(e_1 e_2 + e_3 e_4).$$

This λ is well-defined up to sign: Simply consider $\frac{1}{2}\omega \wedge \omega = \lambda^2 e_1 ... e_4 = \lambda^2 dv_g$, with dv_g the volume form for g. Since λ^2 is only dependent on ω and g, we can determine λ up to sign. However, taking advantage of $M - C$ being connected, we may fix λ on all of $M - S$ so that $\lambda > 0$.

We then set $J : e_1 \mapsto e_2, e_2 \mapsto -e_1, e_3 \mapsto e_4, e_4 \mapsto -e_3$. This definition is equivalent to the following: Let $\tilde{g} = \lambda g$, and define J such that $\tilde{g}(x, y) = \omega(Jx, y)$. Hence we see that if there is a J compatible with ω and $\tilde{g}$, it must be unique. Thus J is compatible with ω and $\tilde{g} = \lambda g$ on $M - C$. □

Observe that ω is defined on all of M and is zero on C, $\tilde{g}$ can be defined on all of M and is zero on C, but is not smooth on C, while J is defined only on $M - C$.

We also need a relative version of the previous discussion. Recall Theorem 5, which we present here in slightly different form.

Theorem 5A *Let (ω_0, g_0) and (ω_1, g_1) be generic harmonic forms. If there exists a path (ω_t, g_t) of harmonic forms ω_t with respect to g_t such that $\omega_t \neq 0$ for all $t \in [0,1]$, then there exists a G_δ-set of perturbations $\{(\tilde{\omega}_t, \tilde{g}_t)\}$ of this path, fixing endpoints, such that $\{(\tilde{\omega}_t, \tilde{g}_t)\}$ has regular zeros in $M \times [0,1]$.*

Note: The conditions for the theorem are minimal. The space $\{(\omega, g) | g \in \mathrm{Met}^k(M), *\omega = \omega, \Delta_g \omega = 0\}$ is diffeomorphic to $\mathbf{R}^{b_2^+} \times \mathrm{Met}^k(M)$, where $\mathrm{Met}^k(M)$ is the space of C^k-metrics on M, and, as long as $b_2^+ > 1$, we can always connect (ω_0, g_0) to (ω_1, g_1) via a cobordism such that $\omega_t \neq 0$ for all $t \in [0,1]$. In the case $b_2^+ = 1$, as long as (ω_0, g_0) and (ω_1, g_1) lie on the same side of the real line, there exists a cobordism.

Let $\{(\omega_t, g_t)\}$ be a regular cobordism. As in the previous proposition, we can define λ_t uniformly over $\bigcup_{t \in [0,1]} (M \times \{t\} - C_t)$ and get a family $\{(\omega_t, \tilde{g}_t, J_t)\}$, which is compatible where defined.

2.2 Moser argument for self-dual harmonic 2-forms

Consider M^4 as above. Let $\{\omega_t\}$ be a generic family of self-dual harmonic 2-forms such that

(i) $[\omega_t] \in H^2(M; \mathbf{R})$ is constant.

(ii) The sets $C_t = \{x \in M | \omega_t(x) = 0\}$ are all S^1's; hence via a diffeomorphism, we may assume that $C = C_t$ is a fixed S^1.

(iii) Let Ω be an oriented surface with $\partial\Omega = C$. (We are assuming here that C is contractible.) Then $\int_\Omega \omega_t$ does not vary with t.

Then we have the following:

Theorem 11 *There exists a 1-parameter family of C^1-diffeomorphisms of M, which is smooth away from C, and takes $(M - C, \omega_0) \xrightarrow{\sim} (M - C, \omega_1)$ symplectically.*

This generalizes the classical

Theorem 12 (Moser) *Let $\{\omega_t\}$ be a family of symplectic forms on a closed manifold M. Provided $[\omega_t] \in H^2(M; \mathbf{R})$ is fixed, there is a 1-parameter family of diffeomorphisms ϕ_t such that $\phi_t^* \omega_t = \omega_0$.*

Proof: (Moser) Let η_t be a 1-parameter family of 1-forms such that $\frac{d\omega_t}{dt} = d\eta_t$. Thus, if we define X_t such that $i_{X_t}\omega_t = \eta_t$, then $\mathcal{L}_{X_t}\omega_t = (i_{X_t} \circ d + d \circ i_{X_t})\omega = d\eta_t$, which, integrated, gives a 1-parameter family ϕ_t such that $\phi_t^* \omega_t = \omega_0$. □

Proof: **(Theorem 11)** The point here is to find a suitable η_t such that $\frac{d\omega_t}{dt} = d\eta_t$ and $\eta_t|_C = 0$. Fix some $\tilde{\eta}_t$ such that $\frac{d\omega_t}{dt} = d\tilde{\eta}_t$. We shall find a function f_t on M such that $\tilde{\eta}_t = df_t$ "up to first order" near C.

Condition (iii) implies that there exists an f_t on C such that $i^*\tilde{\eta}_t = df_t$, where $i : C \to M$ is the inclusion, i.e. $i^*\tilde{\eta}_t$ is exact. This is because

$$\int_C i^*\tilde{\eta}_t = \int_{\partial\Omega} i^*\tilde{\eta}_t = \int_\Omega d\tilde{\eta}_t = \int_\Omega \frac{d\omega_t}{dt} = 0.$$

In order to extend f_t to a neighborhood $N(C)$ of C, first observe that there is only one orientable rank 3 bundle over S^1 ($\pi_1(BSO(3)) = 0$ implies $S^1 \to BSO(3)$ is homotopically trivial) and hence $N(C) \simeq C \times D^3$. Choose coordinates (θ, x_1, x_2, x_3) such that $d\theta, dx_1, dx_2, dx_3$ at $(\theta, 0)$ are orthonormal.

Setting

$$f_t(\theta, x_1, x_2, x_3) = f_t(\theta, 0) + \sum_i \tilde{\eta}_i(\theta, 0)x_i + \frac{1}{2}\sum_{i,j} \frac{\partial\tilde{\eta}_i}{\partial x_j}(\theta, 0)x_i x_j$$

on $N(C)$, where $\tilde{\eta}_t = \sum_i \tilde{\eta}_i dx_i + \tilde{\eta}_\theta d\theta$, we have

$$\begin{aligned} df_t(\theta, x_1, x_2, x_3) &= \frac{\partial f_t}{\partial\theta}(\theta, 0)d\theta + \sum_i \frac{\partial\tilde{\eta}_i}{\partial\theta}(\theta, 0)x_i d\theta \\ &+ \sum_i \tilde{\eta}_i(\theta, 0)dx_i + \frac{1}{2}\sum_{i,j} \frac{\partial\tilde{\eta}_i}{\partial x_j}(\theta, 0)(x_i dx_j + x_j dx_i) \end{aligned}$$

up to first order in the x_i's. Now observing that

$$(1) \quad \frac{\partial f}{\partial\theta}(\theta, 0) = \tilde{\eta}_\theta(\theta, 0),$$

$$(2) \quad d\tilde{\eta}_t(\theta, 0) = 0,$$

and that (2) gives

$$\begin{aligned} \frac{\partial\tilde{\eta}_\theta}{\partial x_i}(\theta, 0) &= \frac{\partial\tilde{\eta}_i}{\partial\theta}(\theta, 0), \\ \frac{\partial\tilde{\eta}_i}{\partial x_j}(\theta, 0) &= \frac{\partial\tilde{\eta}_j}{\partial x_i}(\theta, 0), \end{aligned}$$

we obtain

$$\begin{aligned} df_t(\theta, x) &= \left(\tilde{\eta}_\theta(\theta, 0) + \sum_i \frac{\partial\tilde{\eta}_\theta}{\partial x_i}(\theta, 0)x_i\right) d\theta \\ &+ \sum_i \left(\tilde{\eta}_i(\theta, 0) + \sum_j \frac{\partial\tilde{\eta}_i}{\partial x_j}(\theta, 0)x_j\right) dx_i \\ &= \tilde{\eta}_\theta(\theta, x)d\theta + \sum_i \tilde{\eta}_i(\theta, x)dx_i \end{aligned}$$

up to first order in x.

Damping f_t out to 0 outside $N(C)$, we arrive at $\eta_t = \tilde{\eta}_t - df_t$. Finally, we obtain the vector field X_t such that $i_{X_t}\omega_t = \eta_t$. X_t will then give rise to a 1-parameter family of symplectomorphisms, away from C, once we establish that $X_t \to 0$ rapidly enough as $p \to C$ ($p \in M$).

On $N(C)$,

$$\begin{aligned}
\omega_t &= L_1(\theta, x)(dx_2 dx_3 + dx_1 d\theta) \\
&+ L_2(\theta, x)(dx_3 dx_1 + dx_2 d\theta) \\
&+ L_3(\theta, x)(dx_1 dx_2 + dx_3 d\theta) \\
&+ Q,
\end{aligned}$$

where $L_i(\theta, x) = \sum_j L_{ij}(\theta) x_j$ and Q consists of forms in dx_i, $d\theta$, whose coefficients are quadratic or higher in the x_i. In terms of matrices, ω_t corresponds to

$$A = \begin{pmatrix} 0 & L_1 & L_2 & L_3 \\ -L_1 & 0 & L_3 & -L_2 \\ -L_2 & -L_3 & 0 & L_1 \\ -L_3 & L_2 & -L_1 & 0 \end{pmatrix} + \tilde{Q},$$

where $\tilde{Q}$ has quadratic or higher terms in the x_i and the matrix is with respect to basis $\{dx_1, dx_2, dx_3, d\theta\}$. $i_{X_t}\omega_t = \eta_t$ then becomes

$$(a_1\ a_2\ a_3\ a_\theta)A = (\eta_1\ \eta_2\ \eta_3\ \eta_\theta)$$

with $X_t = \sum_i a_i dx_i + a_\theta d\theta$. Thus,

$$\begin{aligned}
(a_1\ a_2\ a_3\ a_\theta) &= (\eta_1\ \eta_2\ \eta_3\ \eta_\theta)A^{-1} \\
&= \frac{(\eta_1\ \eta_2\ \eta_3\ \eta_\theta)}{L_1^2 + L_2^2 + L_3^2} \begin{pmatrix} 0 & -L_1 & -L_2 & -L_3 \\ L_1 & 0 & -L_3 & L_2 \\ L_2 & L_3 & 0 & -L_1 \\ L_3 & -L_2 & L_1 & 0 \end{pmatrix}
\end{aligned}$$

up to first order in x. This means that $|X_t| < k|x|$ near C; hence, as $x \to 0$, $|\phi_1(\theta, x) - \phi_0(\theta, x)| \to 0$, where ϕ_t is the flow such that $\frac{d\phi_t}{dt} = X_t$. This concludes our proof. $\square$

Chapter 3

Intrinsic Characterization of Harmonic Forms

3.1 Harmonic 1-forms

Let M be a closed, oriented n-manifold. Calabi, in [C], gave an intrinsic characterization for a closed 1-form to be harmonic, which we will describe presently.

Call a closed 1-form ω *generic*, if ω, as a section of T^*M, is transverse to the zero section. This is equivalent to the critical points of f being Morse, where f is any local function with $df = \omega$. Using a Sard argument, it can be shown that the 'generic' closed 1-form is transverse to the zero section. We also showed in Chapter 1 that it is true even for harmonic 1-forms. We may then talk about the *index* of each zero of ω - this is the Morse index of any local function f satisfying $df = \omega$.

Theorem 13 (Calabi) *Let ω be a generic closed 1-form. Then the following are equivalent:*

(A) There exists a metric g with respect to which ω is harmonic.

(B) (i) ω does not have any zeros of index 0 or n, and (ii) given any two points p, $q \in M$ which are not zeros of ω, there exists a path $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$, such that $\omega(\gamma(t))(\dot\gamma(t)) > 0$ for all $t \in [0,1]$.

(C) (i) ω does not have any zeros of index 0 or n, and (ii') through every point $p \in M$ which is not a zero of ω, there exists a closed curve $\gamma : S^1 \to M$ with $\gamma(0) = p$, such that $\omega(\gamma(\theta))(\dot\gamma(\theta)) > 0$ for all $\theta \in S^1$.

We shall call such a closed 1-form an *intrinsically harmonic* 1-form and a positive path as in (ii) an *ω-positive path.*

Let us first consider the local picture near a zero $p \in M$ of a generic closed 1-form ω. Taking a local coordinate chart U with p mapping to 0, $\omega(x) = df(x)$, with $x = 0$ a Morse critical point of f. Using the Morse lemma, we may assume $f(x) = \frac{1}{2}\{(n-k)\cdot(x_1^2 + ... + x_k^2) - k\cdot(x_{k+1}^2 + ... + x_n^2)\}$ and $\omega(x) = (n-k)\sum_{i=1}^k x_i dx_i - k\sum_{i>k} x_i dx_i$. Hence, setting $g = \sum_{i=1}^n dx_i \otimes dx_i$, we obtain $*_g\omega = (n-k)\sum_{i=1}^k x_i dx_{(i)} - k\sum_{i>k} x_i dx_{(i)}$, and $d *_g \omega = 0$.

Observe that if ω is harmonic with respect to g, and $\omega = df$ locally, then f is harmonic with respect to g, and, by the maximum principle, ω cannot have any zeros of index 0 or n. We summarize the above local considerations in the following proposition:

Proposition 11 *If ω is a generic closed 1-form without any zeros of index 0 or n, then near every zero there exists a metric g for which $*_g\omega$ is closed.*

Proof (Theorem): (A)$\Rightarrow$(B). Assume a generic closed 1-form ω is harmonic. Then (i) is satisfied because of the maximum principle. In order to prove (ii), define the 'upland' U_p (resp. 'lowland' L_p) as follows: U_p (resp. L_p) $= \{q \in M|$ There exists an ω-positive path from p to q (resp. from q to p)$\}$. Also note that ω gives rise to a codimension 1 foliation on M away from the zeros, consisting of integral submanifolds of ω. In this section, if we refer to a leaf of ω, we mean a leaf of $\omega|_{M-\{p_i\}_{i=1}^s}$, where $\{p_1, ..., p_s\}$ is the set of zeros of ω. The upland U_p is a union of leaves of ω: If there is an ω-positive path from p to q, then given $q' \in L_q$ (the leaf through q), we can adjoin the ω-positive path from p to q and the path within L_q from q to q', and perturb it to make it ω-positive from p to q'. The boundary of U_p must also be a union of leaves of ω, which necessarily are closed in $M - \{p_i\}_{i=1}^s$. We now obtain a contradiction if $\partial U_p \neq \emptyset$ because $\int_{\partial U_p} *\omega \neq 0$, whereas $[\partial U_p] = 0 \in H_{n-1}(M; \mathbf{Z})$. Thus, $U_p = L_p = M - \{p_i\}_{i=1}^s$.

(B)$\Rightarrow$(C). Assume a closed 1-form ω is generic, and it satisfies (i) and (ii). (ii) immediately implies (ii'): Given $p \in M - \{p_i\}_{i=1}^s$, take any $q \in M - \{p_i\}_{i=1}^s$. Then there exists ω-positive paths from p to q and from q to p. Now, simply adjoin them and smooth the endpoints. We may also assume that the closed transversal has no self-intersections.

(C)$\Rightarrow$(A). Let ω be a generic closed 1-form satisfying (i) and (ii'), and let $\{p_1, ..., p_s\}$ be the set of zeros of ω. On very small, non-overlapping disks $D(p_i)$ about p_i, there exist metrics g_i such that $*_{g_i}\omega$ is closed on $D(p_i)$. Since $*_{g_i}\omega = d\xi_i$ locally, we can damp ξ_i quickly outside $D(p_i)$. Set $\xi = \sum \xi_i$. Then $d\xi|_{D(p_i)} = *_{g_i}\omega$ on $D(p_i)$ and $\omega \wedge d\xi \geq 0$ on $\bigcup_i D(p_i)$, with strict inequality away from the p_i.

Next, through each point $q \neq p_i$, there exists an embedded $\gamma_q : S^1 \to M$ transverse to the foliation. γ_q can be extended to a foliated embedding $\Gamma_q : S^1 \times D^{n-1} \to M$, where the foliation of $S^1 \times D^{n-1}$ is given by $\pi_1^*(d\theta)$, with π_i the projection onto the i-th factor and $d\theta$ the standard 1-form on S^1. Here, we assume Γ_q misses $\{p_i\}_{i=1}^s$. Now define a Poincaré dual to $\gamma_q(S^1)$: Let μ be a nonnegative $(n-1)$-form on D^{n-1} which is positive at $0 \in D^{n-1}$ and has support on the interior of D^{n-1}, and take $\eta_q = (\Gamma_q)_*(\pi_2^*\mu)$. η_q has the property that $\omega \wedge \eta_q > 0$ on $\gamma_q(S^1)$ and $\omega \wedge \eta_q \geq 0$ on M. By compactness, we need only finitely many q's (say q_1,...,q_m) such that

$$\omega \wedge (d\xi + \eta_{q_1} + \ldots + \eta_{q_m}) > 0$$

on $M - \{p_i\}_{i=1}^s$.

Set $\widetilde{\omega} = d\xi + \sum_i \eta_{q_i}$. In a neighborhood of each critical point p_i there exists a g_i such that $\widetilde{\omega} = *_{g_i}\omega$. We now extend $g|_{D(p_i)} = g_i$ to g defined on all of M such that $\widetilde{\omega} = *_g\omega$. By linear algebra, if $\omega(p) \neq 0$, then $\omega(p) \wedge \widetilde{\omega}(p) > 0$ implies that there is a $g(p)$ such that $\widetilde{\omega}(p) = *_{g(p)}\omega(p)$. Note that, since $\omega \wedge \widetilde{\omega}(p) > 0$ for all $p \in \widetilde{M} = M - \bigcup D(p_i)$, ω gives rise to a nonzero section (line field) of $\bigwedge^1 T^*\widetilde{M}$, and $\widetilde{\omega}$ gives rise to an $(n-1)$-plane field of $\bigwedge^1 T^*\widetilde{M}$ transverse to the line field. If we want $\widetilde{\omega} = *_g\omega$, we must require the line field to be orthogonal to the $(n-1)$-plane field, i.e. there must be a splitting with respect to the metric. Since g has this property on $\bigcup D(p_i)$, extend $g|_{\partial[\bigcup D(p_i)]}$ to all of $\widetilde{M}$, using the standard partition of unity argument for constructing metrics, but making sure the splitting is preserved. $\square$

3.2 Morse theory of harmonic 1-forms

In this section we prove the general form of the theorem conjectured by Farber, Katz, and Levine in [FKL] regarding allowable Morse singularities of harmonic 1-forms. Let M be a closed, oriented n-manifold.

We prove the following theorem:

Theorem 14 *Let ω_0 be a generic closed 1-form with no zeros of index 0 or n. Then there exists a family of generic closed 1-forms ω_t, $t \in [0,1]$, with $[\omega_t] \in H^1(M;\mathbf{R})$ fixed, such that ω_1 is intrinsically harmonic and each ω_t has the same number of zeros of each index.*

This theorem tells us that, in studying the Morse theory of closed 1-forms as in [N], the assumption of harmonicity does not give rise to additional constraints regarding critical point structure.

Observe that if ω is harmonic with respect to g, given any local function f with $df = \omega$, f is harmonic and, by the maximum principle, ω cannot have any zeros of index 0 or n.

Proof: First note that the closed form $\omega = \omega_0$ gives rise to an $(n-1)$-dimensional foliation consisting of integral submanifolds of ω, away from the zeros of ω. For a closed 1-form ω with zeros, define a *leaf* L_p through $p \in M$ by $L_p = \{x \in M|$ There exists a smooth path $\gamma : [0,1] \to M$ from p to x with $\omega(\gamma(t))(\dot{\gamma}(t)) = 0$ for all $t\}$. If a leaf does not pass through a zero of ω, it is called a *nonsingular leaf*; otherwise, it's a *singular leaf*. If L is a singular leaf, then let the *components* of L be the closures in L of the connected components of L restricted to $M - \{x|\omega(x) = 0\}$.

The proof then breaks up into the following components:

(1) The compact leaf case. Here we assume all the leaves of ω are compact, and reduce the problem to a problem in graph theory.

(2) Decompose M into two components M_c and M_∞, consisting of (roughly speaking) the compact leaves and the noncompact leaves, respectively.

(3) The general case. The noncompact leaf case is treated in [FKL]. Using this, and the same methods from (1), we obtain the general result.

(1) **Compact leaf case:**

Let us first assume that all the leaves of ω are compact. If we introduce the equivalence relation $\sim$, where $p \sim q$ if and only if $L_p = L_q$, then $\Gamma = M/\sim$ is a graph. Critical points of index i, $1 < i < n-1$ do not give rise to true vertices of the graph Γ, since the surgeries corresponding to passing such critical points do not change the connectivity of the leaves. Near a critical point of index 1 we have the situation as in Figure 1, whereas near a critical point of index $n-1$, we have the situation as in Figure 2, provided (i) $n \geq 3$, (ii) ω does not

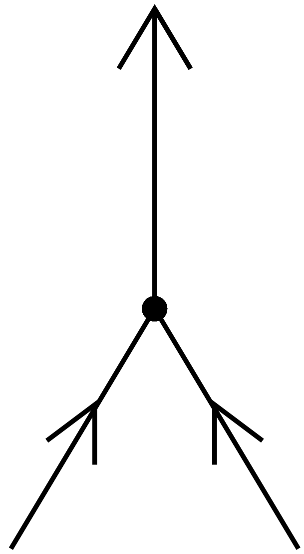

Figure 1

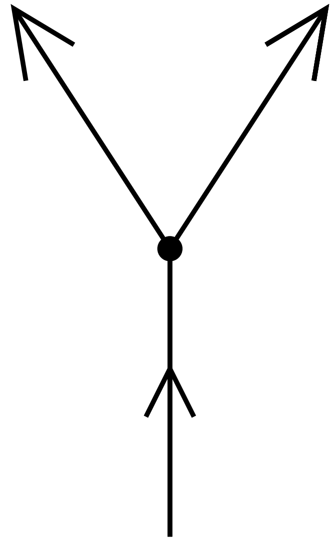

Figure 2

have more than 1 critical point on each leaf, and (iii) the leaves which locally come together or split off weren't parts of the same global leaf. When $n = 2$, at a point of index 1, both Figure 1 and Figure 2 are possible. The arrows represent the direction of increase for the local Morse function (f such that $df = \omega$).

Perturb ω slightly in its cohomology class, so that each leaf contains at most one zero of ω, while keeping all the leaves of ω still compact. Then, $\Gamma = M/\sim$ can be viewed as a trivalent (= each vertex has exactly 3 edges), directed graph, and we can assign weights to each edge: If γ is an edge from p to q, then its weight is the ω-length of any path from $\pi^{-1}(p)$ to $\pi^{-1}(q)$ sitting inside $\pi^{-1}(\gamma)$, where $\pi : M \to \Gamma = M/\sim$. These weights represent $\pi_*\omega$ on $\Gamma = M/\sim$, i.e. they give a cohomology class $[\pi_*\omega]$ on Γ. Thus Γ is a collection $(V(\Gamma), E(\Gamma), W)$ of vertices, directed edges, and weights $W : E(\Gamma) \to \mathbf{R}^+$.

Note also that the vertices have either (i) two incoming edges and one outgoing, or (ii) one incoming and two outgoing, from our previous discussion, i.e. no vertices as in Figure 3.

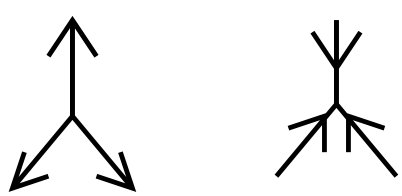

Figure 3

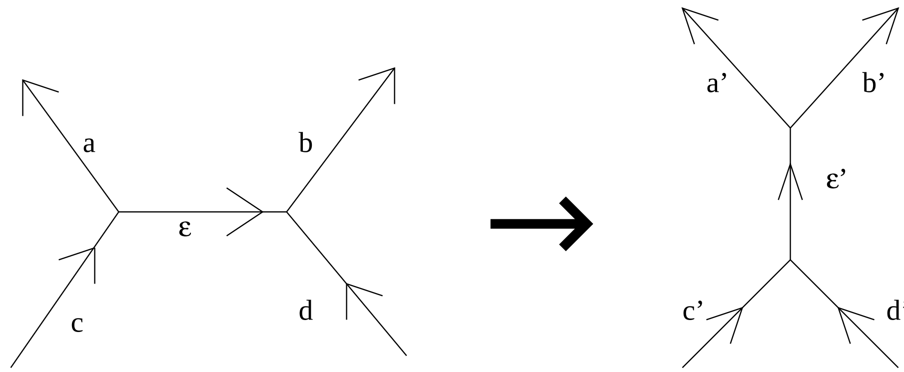

Figure 4

We introduce the two kinds of operations we shall be using:

(A_1). Graph modification to increase connectivity. Whenever we have a subgraph of the type pictured on the left-hand side of Figure 4, we may replace it with one on the right-hand side, provided that $\epsilon < d$.

This modification corresponding to the self-indexing procedure for Morse functions, clearly does not alter the homotopy type of the graph. In order to preserve the cohomology class, we need:

(i) $a + c = a' + \epsilon' + c'$
(ii) $b + d = b' + \epsilon' + d'$
(iii) $a - \epsilon + d = a' + \epsilon' + d'$
(iv) $b + \epsilon + c = b' + \epsilon' + c'$
(v) $d - \epsilon - c = d' - c'$
(vi) $b + \epsilon - a = b' - a'$

Upon a moment's consideration, these six equations are linear combinations of three of them, say (i), (iv), (v).

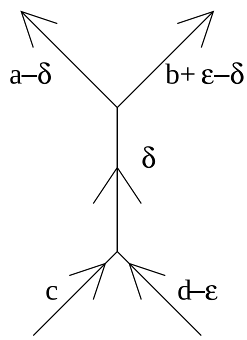

Figure 5

A possible solution is given in Figure 5, provided ϵ is small, i.e. $< d$. Here δ can be any positive number $\leq \min(a, b+\epsilon)$. Of course, the key point in solving for $a' \sim d'$, ϵ' was that they were *positive* numbers.

In terms of Morse theory, if f is a local function with $\omega = df$ on this subgraph, and c_1 and c_2 are the two critical points of index $n-1$ and 1 respectively, then we may reduce $f(c_2)$ so that $f(c_2) - f(c_1) < a$, so long as $d > \epsilon$. We can then raise $f(c_1)$ using the same self-indexing procedure and obtain $f(c_2) < f(c_1)$. See Figure 6.

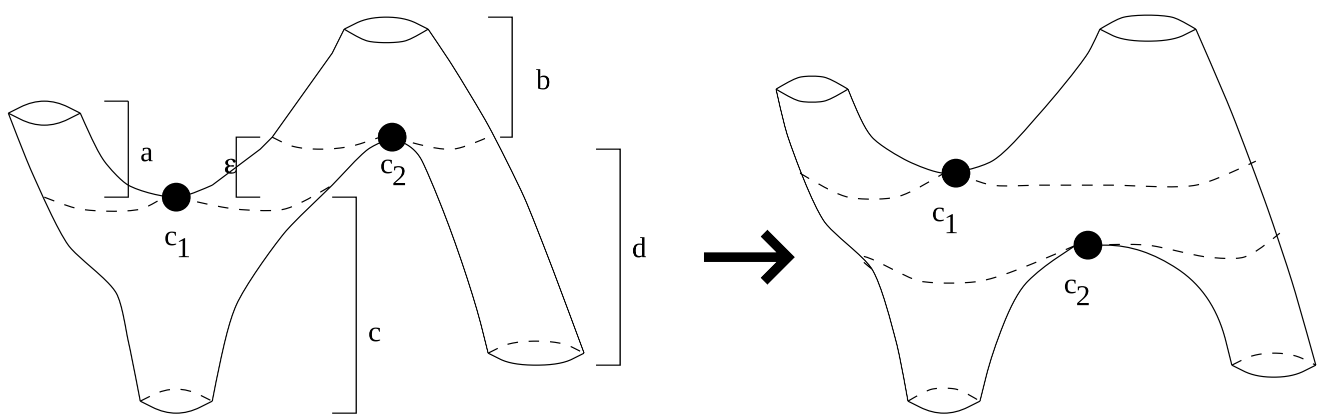

Figure 6

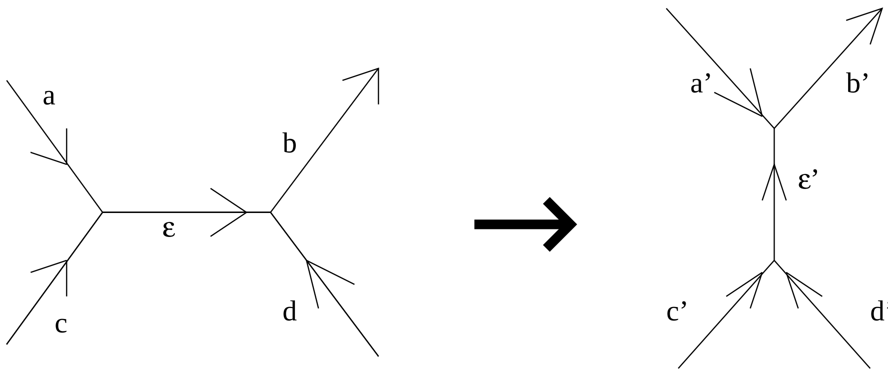

Figure 7

(A$_2$). Graph modification as in Figure 7. Morse theoretically, this corresponds to reversing the heights of the two critical points of index 1. $d > \epsilon$ is sufficient for the modification to be valid. See Figure 8 for a Morse-theoretic interpretation.

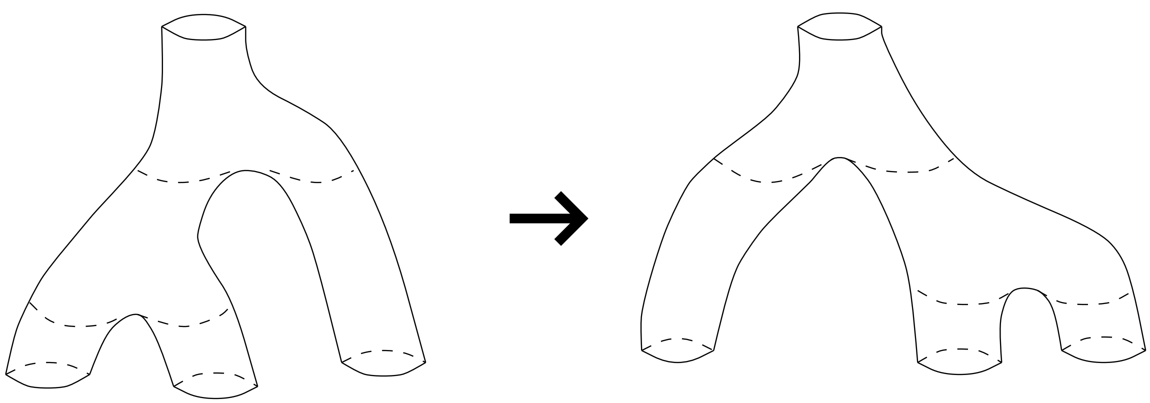

Figure 8

(B). We may alter the weights of the edges, provided the weights remain positive, and the total weight around each closed path remains the same (hence preserving the cohomology class). This corresponds to shortening a Morse function $f : f^{-1}([a,b]) \to [a,b]$ by excising $f^{-1}([r,r']) \subset f^{-1}([a,b])$ and regluing, as long as there are no critical values in $[r,r']$ (or its reverse process).

Observe that all the above operations can be performed through a 1-parameter family of

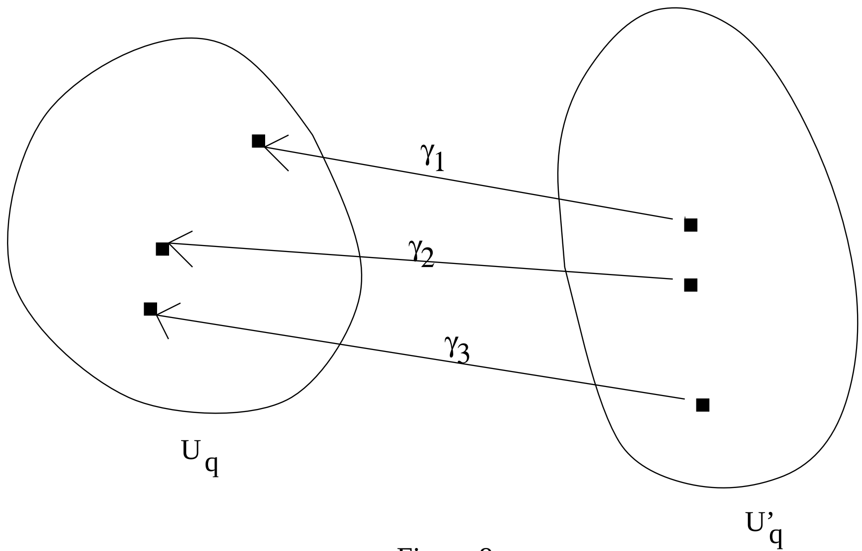

Figure 9

closed 1-forms in the same cohomology class.

We shall modify the graph Γ using the above operations and inductively increasing the 'connectivity'. Pick a vertex p on $\Gamma = M/\sim$. Let U_p be the 'upland' of p, that is, the subgraph consisting of all the vertices and edges that can be reached from p by traveling in the positive direction. Define the 'lowland' L_p in a similar fashion. We stipulate that $p \notin U_p$ unless there is a positive path from p back to itself. (Same for p in L_p.)

It is sufficient to show that there exists a q such that $U_q = L_q = \Gamma$. We find it most convenient to use a q such that $q \in U_q$ (and hence $q \in L_q$). Indeed, start with p and keep traveling in some positive direction until we return to a vertex on the path. Simply take q to be this point.

Operation (B) may not appear practical, until we couple it with the following observation:

Key Observation: If $U_q \neq \Gamma$, then there exist disjoint subgraphs U_q and U'_q, with $V(U_q) \bigcup V(U'_q) = V(\Gamma)$ and $E(U_q) \bigcup E(U'_q) \bigcup \{\text{edges between } U_q \text{ and } U'_q\} = E(\Gamma)$, such that all the edges between U_q and U'_q are directed from U'_q to U_q.

Let γ_1,..., γ_s be the edges from U'_q to U_q. Then, modifying each of the $W(\gamma_i)$ by the same constant (that is, modifying $W(\gamma_i) \mapsto W(\gamma_i) + C$ for some C) will not change the cohomology class represented by ω; this is due to the fact that each closed loop in Γ must traverse the same number of times from U'_q to U_q as from U_q to U'_q.

Hence, we may now apply (B) and assume without loss of generality that $W(\gamma_1) = \epsilon$ where ϵ can be made as small as we want.

There are two possibilities for Γ near γ_1 - see Figure 10.

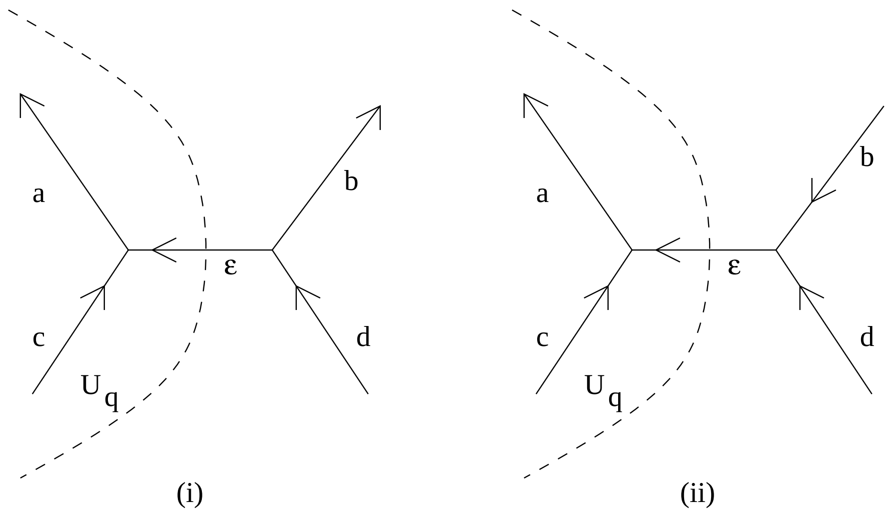

Figure 10

(i) Since ϵ can be made arbitrarily small, we simply make $\epsilon < c$, apply (A_1), and increase the number of vertices in U_q as in Figure 11.

(ii) Apply (A_2) with $\epsilon < c$. See Figure 12.

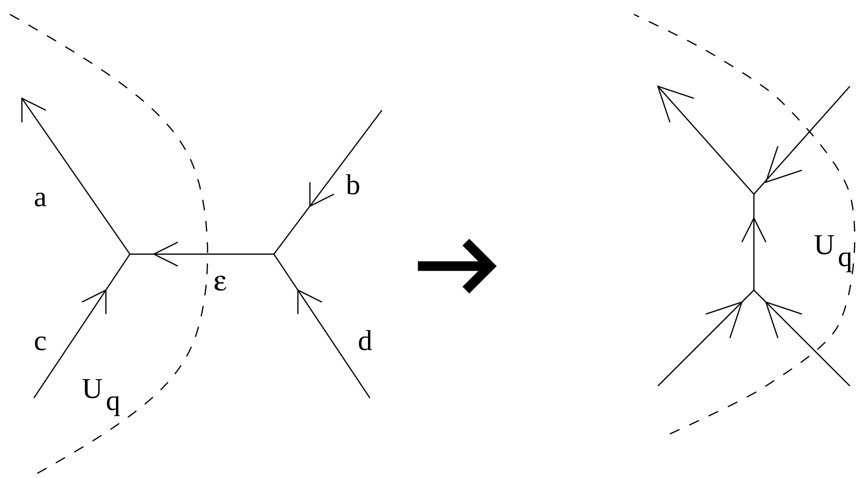

Figure 12

Since the number of vertices in U_q increases after each modification (whereas the total number of vertices of Γ remains the same), by induction, we can find a modified Γ with $U_q = \Gamma$. Next, apply the same procedure for L_q - here we simply note that neither (A_1) nor (A_2) decreases the circulation, and modifying Γ to make L_q larger will not affect $U_q = \Gamma$.

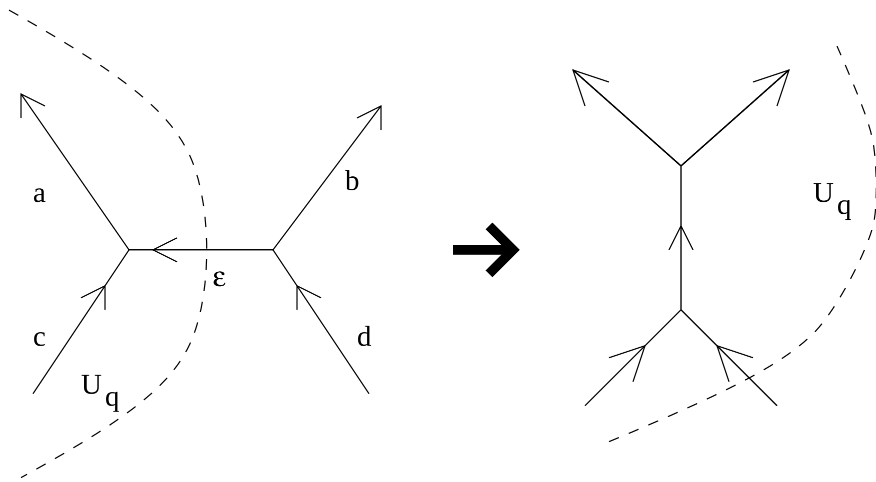

Figure 11

(2) **Compact and noncompact leaves:**

Here we closely follow the discussion in [FKL].

Let ω now be a generic closed 1-form. We describe how M is split into M_c, which is roughly the union of compact leaves, and M_∞, roughly the union of noncompact leaves.

First, if L is a compact leaf, then there exists an open neighborhood of L consisting solely of compact leaves. This is because we can integrate ω to give f with $df = \omega$ near L. Hence, the union of compact leaves in M is open.

Next, the union of all compact leaves (nonsingular or singular), as well as compact components of noncompact singular leaves, is closed in M. This is because all the above leaves and leaf components are closed leaves when restricted to $M - \{x|\omega(x) = 0\}$. We use a result of Haefliger's (c.f. [Ha]): Given a codimension 1 foliation on a (not necessarily closed) manifold N with dim $H^1(M;\mathbf{R})$ finite, the union of closed leaves is closed.

These tell us that $M = M_c \bigcup M_\infty$, where

$$M_c = \overline{\bigcup_{L_\alpha \text{ compact}} L_\alpha}, \quad M_\infty = \overline{\bigcup_{\substack{L_\alpha \text{ nonsingular} \\ \text{noncompact}}} L_\alpha}$$

are both n-dimensional submanifolds with (non-smooth) boundary, and $\partial M_c = \partial M_\infty = M_c \bigcap M_\infty$ is a union of compact components of noncompact leaves.

Lemma 6 *There exists a small perturbation of ω in its cohomology class so that each leaf contains at most one zero of ω.*

Proof: It suffices to modify ω by $\sum_i df_i$, where f_i is a compactly supported function near a zero p_i. Consider $\int \omega : H_1(M;\mathbf{Z}) \to \mathbf{R}$, and let $S = \mathrm{Im}(\int \omega)$. Since S is a countable set in $\mathbf{R}$, there exist f_i, new zeros $\tilde{p}_i$, and paths γ_{ij} from $\tilde{p}_i$ to $\tilde{p}_j$ such that $\omega(\gamma_{ij}) \notin S$. This prevents $\tilde{p}_i$ and $\tilde{p}_j$ from lying on the same leaf of $\omega + \sum_i df_i$. $\square$

Lemma 7 *If all the zeros of ω lie on distinct leaves, then $\partial M_c = \partial M_\infty$ is the union of all compact components of singular noncompact leaves.*

Proof: Consider $\omega = df$ near the singular point of a singular noncompact leaf with a compact component. Without loss of generality assume f has index 1 and $f(x_1, ..., x_n) = -x_1^2 + \sum_{i=2}^n x_i^2$ near the zero of ω. If we integrate (in the negative direction) a gradient flow emanating from the compact component, we see that all the neighboring leaves of ω 'below' the compact component are compact nonsingular leaves. On the other hand, all the leaves 'above' the singular noncompact leaf will be noncompact and all the leaves 'below' the noncompact component of the noncompact leaf are also noncompact. $\square$

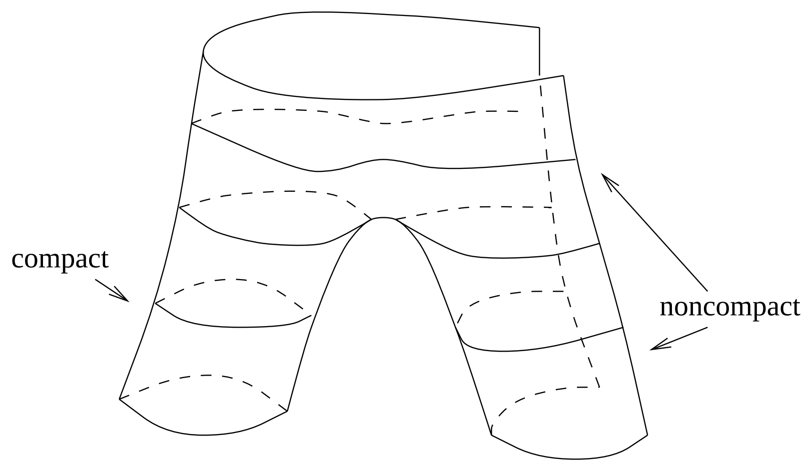

Figure 13

(3) **General result:**

We start with a lemma:

Lemma 8 (FKL) *If L is a nonsingular noncompact leaf of ω, then there is an ω-positive path between any two points of L.*

Proof: We invoke the standard argument used to find a closed transversal to a non-closed leaf L. Let $p \in M$ be a limit point of L. Then there exists a (distinguished) neighborhood $U \subset M$ of p, with respect to which $L \bigcap U$ consists of hyperplanes converging to the hyperplane containing p. Which also means that there exist q_1 and q_2 in L and an ω-positive path from q_1 to q_2. Now take a path on L from any r_1 to q_1, the ω-positive path from q_1 to q_2, and some path on L from q_2 to r_2, and perturb so that the composite path is ω-positive and smooth. $\square$

Let ω be a closed 1-form. We may assume that all the zeros of ω lie on distinct leaves. Since the interior of each connected component of M_∞ is ω-transitive (can get from any point to any other via an ω-positive path), we can graph-theoretically represent each connected component of M_∞ by a single vertex, and take $\Gamma = M/\sim$, where $\sim$ is the equivalence relation: (i) $p \sim q$ if $p, q \in L$, L compact, and (ii) $p \sim q$ if p, q belong to the same connected component of M_∞. Note that the resulting graph is no longer trivalent, and the subgraphs below become possible (where a black dot denotes a component of M_∞):

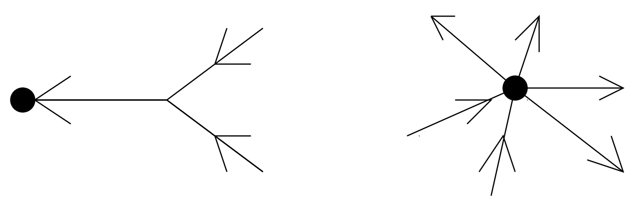

Figure 14

Let us start with $p \in M$ on a nonsingular noncompact leaf L. The same graph-theoretic maneuvers carry over to this case with a few differences. Whenever one of the critical points sits in a noncompact singular leaf (say the higher critical point), we have the following situations:

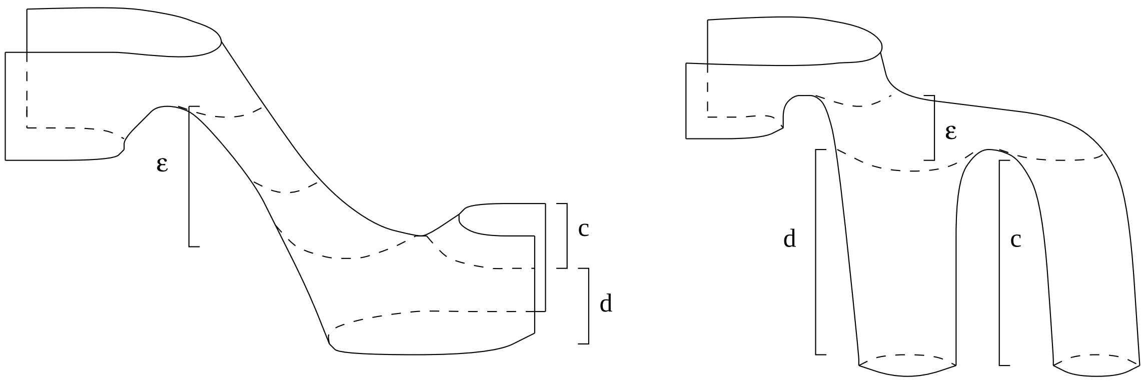

Figure 15

In these cases, as long as ϵ can be made small, moving a critical point of higher index above one of lower index is easily done. This is because one only needs to modify the Morse function in an arbitrarily small neighborhood of the trajectories (for a gradient-like flow into and out of a critical point) in order to change the height of the critical point.

The analogs of (A_1) are depicted in Figure 16. We have two possibilities, depending on whether the lower critical point sits in (i) a compact singular leaf, or (ii) a noncompact singular leaf.

(i)

(ii)

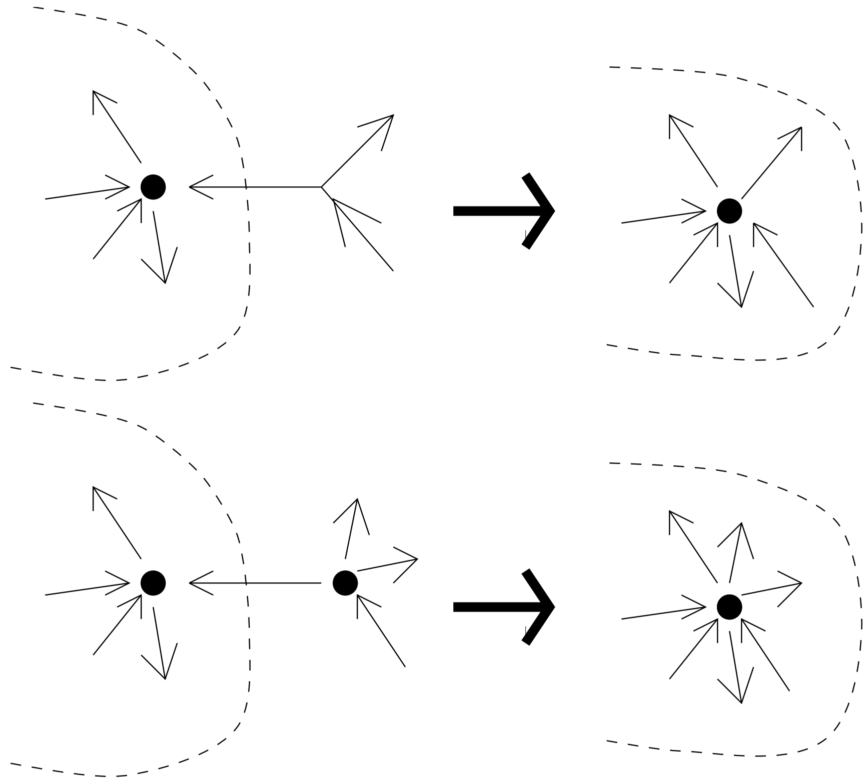

Figure 16

The analog of (A_2) is as in Figure 17.

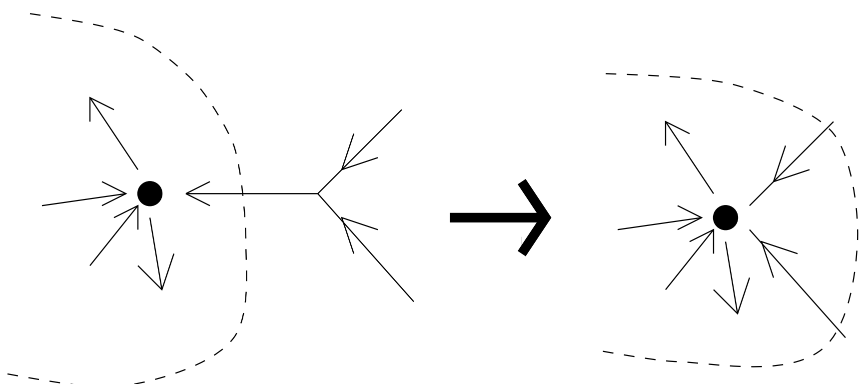

Figure 17

The above modifications decrease the total number of vertices in Γ by 1, while keeping the number of vertices in U_q at least the same. This completes the proof of Theorem 14. $\square$

3.3 Harmonic $(n-1)$-forms

Next, let ω be a closed $(n-1)$-form on an n-dimensional manifold M. Assuming ω has 'generic' zeros, we give an intrinsic characterization for ω to be harmonic. This is the dual situation to Calabi's result - note that however Calabi's result *does not* imply the dual $(n-1)$-form result.

Call a closed $(n-1)$-form *generic* if ω, as a section of $\bigwedge^{n-1} T^*M$, is transverse to the zero section.

Let us begin by analyzing a generic closed ω near a zero p. Taking local coordinates centered around $x=0$,

$$\omega(x) = \sum_{i,j} a_{ij} x_j dx_{(i)} + \text{h.o.},$$

where 'h.o.' refers to terms quadratic or higher in the x_i's. Here, we write $dx_{(i)} = (-1)^{i+1} dx_1 ... \widehat{dx_i} ... dx_n$.

Lemma 9 *If the matrix $A=(a_{ij})$ is diagonalizable, ω can be written as $\omega(x) = \sum_{i=1}^n \lambda_i x_i dx_{(i)} +$ h.o., with $\sum_i \lambda_i = 0$.*

Proof: It suffices to consider just the first order part of ω, i.e. let

$$\omega = \sum_{i,j} a_{ij} x_j dx_{(i)}.$$

If we make coordinate changes $x_i = b_{ij} x'_j$, then,

$$\begin{aligned} \omega &= \sum_{i,j,k,l_1,\dots,l_n} (-1)^{i+1} a_{ij} b_{jk} x'_k \left[b_{1l_1} ... \widehat{b_{il_i}} ... b_{nl_n} dx'_{l_1} ... \widehat{dx'_{l_i}} ... dx'_{l_n} \right] \\ &= \sum_{i,j,k,l} a_{ij} b_{jk} \tilde{b}_{il} x'_k dx_{(l)}, \end{aligned}$$

where $\tilde{b}_{ij}$ is the (i,j)-th minor of $B=(b_{ij})$. Let $\tilde{B} = (\tilde{b}_{ij})$. Hence the coordinate transformation gives us

$$A \mapsto (\tilde{B})^T AB = \det B \cdot (B^{-1}AB).$$

If A is diagonalizable, we can choose B such that the right-hand side expression becomes a diagonal matrix $D = \mathrm{diag}(\lambda_1, ..., \lambda_n)$. Finally, $d\omega = 0$, and hence $\mathrm{tr} D = 0$. □

Observe: The λ_i's are completely determined up to permutation and scaling. Thus there is no unique normal form for ω with a generic zero at $x=0$.

Theorem 15 *A generic closed $(n-1)$-form ω on M is harmonic if and only if (1) the traceless linear transformation associated to every zero $p \in M$ as in the previous lemma is diagonalizable, and (2) there exists an $(n-1)$-dimensional submanifold N_p through every point $p \in M$ which is not a zero of ω, such that $\omega(T_x N_p)(x) \neq 0$ for all $x \in N_p$.*

Proof: ($\Rightarrow$) Assume the generic closed $(n-1)$-form ω is harmonic.

(1) Choose geodesic normal coordinates x_i about the zero p. Then

$$\omega = \sum_{i,j} a_{ij} x_j dx_{(i)} + \text{h.o.},$$

where the $\{dx_i\}$ are orthonormal up to first order near x. Now, $*\omega = \sum_{i,j} a_{ij} x_j dx_i + \text{h.o.}$, and $d * \omega = 0 \Rightarrow A = (a_{ij})$ is symmetric, hence diagonalizable.

(2) Observe that a closed form α with $[\alpha] \neq 0$ in $H^1(M; \mathbf{Z})$ gives rise to a map $f : M \to S^1$, where $\alpha = f^{-1} d\theta$. (We use the convention that $\int_{S^1} d\theta = 1$.) Then $f^{-1}(c)$ for generic c is a compact $(n-1)$-dimensional submanifold of M such that $\text{PD}([f^{-1}(c)]) = [d\theta]$.

Lemma 10 *There exists a closed 1-form $\widetilde{\omega} = *\omega + \xi$ such that (1) $\widetilde{\omega} = *\omega$ near the zeros of ω, (2) $\omega \wedge \widetilde{\omega} > 0$ on $M - \bigcup D(p_i)$, where $\{p_i\}$ is the set of zeros and $D(p_i)$ is a small disk around p_i, and (3) $[\widetilde{\omega}] \in H^1(M; \mathbf{Q})$.*

Proof: If $*\omega$ is in a rational cohomology class we would be done. If not, take a basis $\{\alpha_i\}$ for $H^1(M; \mathbf{Z})$. Any closed ω_i representing α_i gives rise to $f_i : M \to S^1$ such that $\omega_i = f_i^{-1} d\theta$, and we can hence find a submanifold N_i avoiding $\{p_i\}$ which is Poincaré dual to α_i. Now, take a neighborhood of N_i of the form $N_i \times I$. Take a compactly supported function g_i on I with $\int_I g_i(x) dx = 1$. Then $\pi_2^*(g_i(x)dx) \in \alpha_i$. For small enough constants c_i, $*\omega + \sum_i c_i \pi_2^*(g_i(x)dx)$ satisfies

$$\omega \wedge \left[*\omega + \sum_i c_i \pi_2^*(g_i(x)dx) \right] > 0$$

on $M - \bigcup D(p_i)$. We can certainly find suitable c_i so that $\widetilde{\omega}$ is rational. $\square$

Now, choose a suitable integral multiple $n\widetilde{\omega}$ of $\widetilde{\omega}$ so that that $[n \cdot \widetilde{\omega}] \in H^1(M; \mathbf{Z})$. Representing $n\widetilde{\omega}$ by a function $f : M \to S^1$, we see that, away from critical levels of f, there exists a submanifold N_p through each point $p \in M$ (a level submanifold) which must satisfy $\omega(T_x N_p)(x) \neq 0$ for all $x \in N_p$. If p happens to sit on one of the critical levels, simply perturb a nearby level surface so that it passes through p.

($\Leftarrow$) Let ω now be a generic closed $(n-1)$-form satisfying (1) and (2) in the theorem. We shall first obtain a local metric g for which $*_g \omega$ is closed. By Lemma 9, ω can be written as

$$\omega(x) = \sum_{i=1}^{n} \lambda_i x_i dx_{(i)} + \text{h.o.},$$

with $\sum_i \lambda_i = 0$ near a zero $x = 0$.

Claim: There exists a metric g on a small disk $D(p)$ around the zero p, for which $*_g\omega = \sum_{i=1}^n \lambda_i x_i dx_i$ (i.e. there are no higher order terms in x).

Proof: We define the metric g by specifying an orthonormal basis $e_i(x) = \sum_j c^{ij}(x)dx_j$, or $dx_i = \sum_j c_{ij}(x)e_j(x)$, where $c_{ij}(0) = I$. We compute

$$\begin{aligned}
*_g dx_i &= \sum_j c_{ij} e_{(j)} \\
&= \sum_{j,k_1,\ldots,k_n} (-1)^{j+1} c_{ij} c^{1k_1} \ldots \widehat{c^{jk_j}} \ldots c^{nk_n} dx_{k_1} \ldots \widehat{dx_{k_j}} \ldots dx_{k_n} \\
&= \sum_{j,k} c_{ij} \tilde{c}^{jk} dx_{(k)} \\
&= \sum_k \left(\frac{C \cdot C^T}{\det C} \right)_{ik} dx_{(k)}.
\end{aligned}$$

Here, $C = (c_{ij})$, and $(\tilde{c}^{jk}) = \frac{C^T}{\det C}$ is the adjoint of C^{-1}. Setting $\widetilde{\omega} = \sum_i \lambda_i x_i dx_i$, we obtain

$$*_g\widetilde{\omega} = \sum_i \lambda_i x_i (*_g dx_i) = \sum_i \lambda_i x_i S_{ik} dx_{(k)},$$

where we have assigned $S(x) = \frac{C \cdot C^T}{\det C}(x)$. If the symmetric matrix $S(x)$ is near I, it must be positive definite and must have a symmetric positive square root. It is C^∞ by the inverse function theorem applied to $F : \mathrm{Sym}^2(\mathbf{R}^n) \to \mathrm{Sym}^2(\mathbf{R}^n)$, with $F(A) = A^2$ near I. Here $\mathrm{Sym}^2(\mathbf{R}^n)$ is the space of symmetric matrices. Thus, we see that given an $S(x)$ with $S(0) = I$, we can find a unique symmetric positive $C(x)$ with $S(x) = \frac{C \cdot C^T}{\det C}(x)$.

Now write $S(x) = I + \sum_j x_j S^j(x)$, with $S^j(x)$ symmetric. Then $*_g\widetilde{\omega} = \sum_i \lambda_i x_i dx_{(i)} + \sum_{i,j,k} \lambda_i x_i x_j S^j_{ik}(x) dx_{(k)}$. On the other hand,

$$\omega = \sum_i \lambda_i x_i dx_{(i)} + \sum_{i,j,k} \lambda_k a^k_{ij}(x) x_i x_j dx_{(k)},$$

where $a^k_{ij}(x)$ can be made symmetric in i, j. Hence, for $*_g\widetilde{\omega} = \omega$,

$$\lambda_i S^j_{ik} + \lambda_j S^i_{jk} = 2\lambda_k a^k_{ij}.$$

Permuting i, j, k and taking linear combinations, we obtain:

$$S^k_{ij} = \frac{1}{\lambda_i \lambda_j} \left[\lambda_i^2 a^i_{jk} + \lambda_j^2 a^j_{ik} - \lambda_k^2 a^k_{ij} \right].$$

This proves the claim. $\square$

Thus, there exists a metric on $\bigcup_i D(p_i)$ such that $*_g\omega$ is closed on $\bigcup D(p_i)$. Let $*_g\omega = d\xi$, and damp ξ out on $M - \bigcup D(p_i)$. Given $p \neq p_i$, there exists an $(n-1)$-dimensional submanifold N_p transverse to the characteristic vector field X with $i_X\omega = 0$. As before, we can construct the Poincaré dual to N_p, supported on $N_p \times I \subset M$. First we extend $N_p \subset M$ by flowing along X, which we may assume is nonzero. Thus we obtain $\phi : N_p \times [-\varepsilon, \varepsilon] \to M$. Now, take a bump function f on $[-\varepsilon, \varepsilon]$ with compact support and integral 1, and form $\eta_p = \pi_2^*(f(x)dx)$, where π_2 is projection $N_p \times [-\varepsilon, \varepsilon] \to [-\varepsilon, \varepsilon]$.

η_p satisfies $\omega \wedge \eta_p > 0$ on N_p, $\omega \wedge \eta_p \geq 0$ on all of M, and $d\eta_p = 0$. Again, by compactness, there exist finitely many $q_1, ..., q_r$ such that

$$\omega \wedge (d\xi + \eta_{q_1} + ... + \eta_{q_r}) > 0$$

on $M - p_i$. Let $\widetilde{\omega} = d\xi + \eta_{q_1} + ... + \eta_{q_r}$. Using the same argument as for 1-forms, by linear algebra, there exists a g such that $\widetilde{\omega} = *_g\omega$. This completes the proof of Theorem 15. □

3.4 Self-dual 2-forms on a 4-manifold

In this section we prove the following proposition:

Proposition 12 *Let ω be a 2-form (not necessarily closed) on M^4 which is nondegenerate away from a union of S^1's, has a metric g defined on a small tubular neighborhood $N(\bigcup S^1)$ for which $*_g\omega = \omega$, and has zeros precisely on $\bigcup S^1$ which are generic as sections of $\bigwedge_+^2 T^*M|_{N(\bigcup S^1)}$. Then there exists a metric $\widetilde{g}$ for which $*_{\widetilde{g}}\omega = \omega$ on all of M.*

Proof: Extend g to all of M. We recall here the standard procedure for finding $\widetilde{g}$ and J compatible with ω. Locally pick an orthonormal frame for g. Then, with respect to this frame $g(x)$ corresponds to I and $\omega(x)$ corresponds to a skew-symmetric matrix $A(x)$. Let $B(x)$ be the unique symmetric positive square root of $-A^2(x)$ (i.e. $B^2(x) = -A^2(x)$, and $B(x)$ is symmetric, with positive eigenvalues). If we make the following correspondences, then $\widetilde{g}$, ω, and J are compatible:

$$\begin{aligned} \widetilde{g} &\leftrightarrow B(x) \\ \omega(x) &\leftrightarrow A(x) \\ J(x) &\leftrightarrow A^{-1}(x)B(x) \end{aligned}$$

One can check that this operation can be globalized.

Note that we need to examine the special case of $N(\bigcup S^1)$. Here,

$$\begin{aligned} \omega &= L_1(x)(e_2e_3 + e_1e_4) \\ &+ L_2(x)(e_3e_1 + e_2e_4) \\ &+ L_3(x)(e_1e_2 + e_3e_4), \end{aligned}$$

where $\{e_i\}$ is an orthonormal frame with respect to g. ω would then correspond to

$$A = \begin{pmatrix} 0 & L_3 & -L_2 & L_1 \\ -L_3 & 0 & L_1 & L_2 \\ L_2 & -L_1 & 0 & L_3 \\ -L_1 & -L_2 & -L_3 & 0 \end{pmatrix},$$

and $-A^2 = (L_1^2 + L_2^2 + L_3^2)I$. Hence $B = \sqrt{L_1^2 + L_2^2 + L_3^2} \cdot I$. Thus, $\tilde{g}$ is conformally equivalent to g on $N(\bigcup S^1)$. Instead of letting $\tilde{g} \to 0$ on $\bigcup S^1$, take a nonzero conformal multiple of g near $\bigcup S^1$. (Note that an almost complex structure J would not exist on $N(\bigcup S^1)$, but our new $\tilde{g}$ would make ω self-dual.) □

Remark: The above proposition did not use the closedness of ω. Of course, if ω is a closed 2-form with the requisite 'self-dual zeros' near $\bigcup S^1$, ω becomes 'degenerate symplectic'.

3.5 Symplectic forms on 2n-manifolds

For the sake of completeness we include the following result, which suggests possible future applications of harmonic forms to symplectic geometry.

Proposition 13 *If ω is a symplectic form on a 2n-manifold, then ω is intrinsically harmonic.*

Proof: Let g, J be compatible with ω. Then with respect to an orthonormal frame $\{e_i\}$ for g, $\omega = \sum_{i=1}^n e_{2i-1} \wedge e_{2i}$, and $*\omega = \sum_{i=1}^n e_{(2i-1,2i)}$. Also note that $\omega^{n-1} = (n-1)! * \omega$. Since $d\omega = 0$ and $d(\omega^{n-1}) = 0$, $d * \omega = 0$ as well. Hence ω is harmonic with respect to a g compatible with ω. □

Chapter 4

Local characterization of harmonic forms

In the previous chapter we gave an intrinsic characterization of harmonic 1-forms and $(n-1)$-forms on an n-manifold M. Here we shall occupy ourselves with the following question:

Question: Let (ω, g) be a harmonic i-form on M. For which small perturbations $\omega_t = \omega + t\eta$ with $\eta = d\xi$ is ω_t harmonic as well? In particular, is the space of harmonic i-forms open in $\Omega^i_\alpha(M) = \{\omega | \omega \in \alpha\} \subset \Omega^i(M)$, where we fix $\alpha \in H^i(M; \mathbf{R})$?

In what follows, fix $\Omega^i(M)$ to be the space of i-forms of class C^k, for large k. Denote by $\mathcal{H}^i_\alpha(M) \subset \Omega^i_\alpha(M)$ the space of intrinsically harmonic i-forms, and let $\widetilde{\mathcal{H}^i_\alpha}(M) \subset \mathcal{H}^i_\alpha(M)$ be the generic (i.e. ev_* surjective at every point of M) harmonic i-forms in the class α. Note that in this chapter, by *generic* we mean that ev_* is surjective at every point of M. If we mean a different kind of genericity, we will say 'generic'.

4.1 Harmonic 1-forms

Let $\omega \in \Omega^1(M)$. Denote the C^k-norm of ω by $|\omega|_k$.

Lemma 11 *Let $\omega \in \mathcal{H}^1_\alpha(M)$. If $\eta = dh$ is supported away from the zeros of ω, then $\omega + \eta \in \mathcal{H}^1_\alpha(M)$, provided $|\eta|_0$ is small.*

Proof: Let (ω, g) be a harmonic 1-form. Then $\omega + \eta = \omega$ and $*_g(\omega+\eta) = *_g\omega$ on $M - \mathrm{Supp}(\eta)$. Provided $|\eta|_0$ is small, we have $(\omega + \eta) \wedge *_g\omega > 0$ on $\mathrm{Supp}(\eta)$. Hence, as in the proof of Calabi's theorem, we can extend $g|_{M-\mathrm{Supp}(\eta)}$ to $\widetilde{g}$ on all of M such that $*_{\widetilde{g}}(\omega + \eta) = *_g\omega$. $\square$

If we restrict ourselves to generic harmonic forms, we can do much better:

Theorem 16 $\widetilde{\mathcal{H}}^1_\alpha \subset \Omega^1_\alpha$ *is open.*

Proof: Let $(\omega, g) \in \widetilde{\mathcal{H}}^1_\alpha$. We shall prove that $\omega + \eta \in \widetilde{\mathcal{H}}^1_\alpha$ for all $|\eta|_1$ small. Let p be a zero of ω. There exists a disk $D(p) = \{\sum x_i^2 \le c_p\}$ centered at p, on which $\omega = df$ and $f = (x_1^2 + ... + x_r^2) - (x_{r+1}^2 + ... + x_n^2)$. Write $\eta = dh$ on $D(p)$, with $h(0) = 0$. The critical points of $f + h$ occur when

$$\frac{\partial}{\partial x_i}(f+h) = \pm 2x_i + \frac{\partial h}{\partial x_i} = 0.$$

Whenever there is a critical point, the Hessian is

$$H(x) = 2\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + \left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right).$$

If $|\eta|_1$ is small, then $|h|_2$ is small, and every critical point $\widetilde{p}$ of $f + h$ inside $D(p)$ must be Morse with index $(n-r)$. Now, if $|\eta|_0$ is small (i.e. $|h|_1$ is small), then no critical points cross $\partial D(p)$, and $\sum_{\widetilde{p} \in D(p) \text{ critical pt.}} (-1)^{\text{index}(\widetilde{p})}$ remains invariant. However, since all the critical points must have the same index, there must still be only one critical point inside $D(p)$, and it's necessarily a saddle of index $(n - r)$.

Claim: $f + h$ is harmonic with respect to some metric on $D(p)$.

Proof of claim: Let $\widetilde{p}$ be the critical point of $f + h$ in $D(p)$. Then near $\widetilde{p}$, using the Morse lemma, there exists a coordinate system $D(\widetilde{p}) \subset D(p)$, on which $f + h = \frac{1}{2}\{(n - r)(y_1^2 + ... + y_r^2) - (r)(y_{r+1}^2 + ... + y_n^2)\}$. As before (c.f. the proof of Calabi's theorem), there exists a metric $\widetilde{g}$ such that $*_{\widetilde{g}}(\omega + \eta)$ is closed on $D(\widetilde{p})$. Next, through each point q of $D(p) - \{\widetilde{p}\}$, we can find a transverse γ_q with endpoints in $\partial D(p)$. Using the same procedure as in the proof of Calabi's theorem, one then obtains a metric $\widetilde{g}$ on $D(p)$ for which $\omega + \eta$ is harmonic. □

Since $*_{\widetilde{g}}(\omega + \eta)$ is exact on $D(p)$, we can write $*_{\widetilde{g}}(\omega + \eta) = d\xi$ on $D(p)$. Damping ξ out away from $D(p)$, we obtain an exact $(n-1)$-form $\widetilde{\omega}_p$, which coincides with $*_{\widetilde{g}}(\omega + \eta)$ on $D(p)$ and has small support outside $D(p)$.

Now, let $\{p_1, ..., p_s\}$ be the set of zeros of ω. Then for each p_j, there exist $D(p_j)$, $\widetilde{g}_j$, and $\widetilde{\omega}_{p_j}$. Given $q \in M - \bigcup_j D(p_j)$, there exists a closed transversal γ_q which avoids $\{p_1, ..., p_s\}$. For each such transversal, we can extend $\gamma_q : S^1 \to M$ to a foliated embedding $\Gamma_q : S^1 \times D^{n-1} \to M$, with the foliation on $S^1 \times D^{n-1}$ given by $\pi_1^* d\theta$ and the one on M given by ω. By compactness there exist $q_1, ..., q_m \in M - \bigcup_j D(p_j)$ and $\Gamma_{q_j} : S^1 \times D^{n-1} \to M$ such that the $\Gamma_{q_j}(S^1 \times D^{n-1}_{1/2})$ cover $M - \bigcup_j D(p_j)$, with $D^{n-1}_{1/2}$ a smaller concentric disk in D^{n-1}.

Now if $|\eta|_0$ is small, the $\Gamma_{q_j}(S^1 \times \{x\})$ would be transverse to the foliation given by $\omega + \eta$ for all $j = 1, ..., m$ and $x \in D^{n-1}$. Therefore we can find $\eta_{q_j} = (\Gamma_{q_j})_*(\pi_2^* \mu)$, Poincaré duals of the γ_{q_j}'s, which are nonzero on $\Gamma_{q_j}(S^1 \times D^{n-1}_{\frac{1}{2}})$, but are supported on $\Gamma_{q_j}(S^1 \times D^{n-1})$. Here, μ is a nonnegative $(n-1)$-form on D^{n-1} which is positive on $D^{n-1}_{\frac{1}{2}}$ and has support on a small neighborhood of $D^{n-1}_{\frac{1}{2}}$. Then, taking suitable positive multiples of η_{q_j},

$$(\omega + \eta) \wedge (\sum_{j=1}^{s} \widetilde{\omega}_{p_j} + \sum_{j=1}^{m} c_j \cdot \eta_{q_j}) > 0$$

on $M - \{\widetilde{p}_1, ..., \widetilde{p}_s\}$, and $\sum_j \widetilde{\omega}_{p_j} + \sum_j c_j \cdot \eta_{q_j} = *_{\widetilde{g}_j}(\omega + \eta)$ near p_j. Thus $\omega + \eta$ is intrinsically harmonic. $\square$

Remark: In the proof of the theorem all we needed to control was the C^1-norm of the perturbation η.

4.2 Infinitesimal harmonic perturbations

In what follows, the most convenient norm to use is the C^k-Hölder norm with $k > 0$ which is *not* an integer. Let $\mathrm{Met}^k(M)$ be the space of C^k-metrics on an n-dimensional manifold M, and let $T_g\mathrm{Met}^k(M) = \Gamma^k(\mathrm{Sym}^2(TM))$ be its tangent space at $g \in \mathrm{Met}^k(M)$, consisting of C^k-sections. Let $C^k(\Omega^i(M))$, $C^k(\Omega^i_\alpha(M))$ be of class C^k, where $\alpha \in H^i(M;\mathbf{R})$. Define

$$\Phi_\alpha : \mathrm{Met}^k(M) \to C^k(\Omega^i_\alpha(M)),$$

which sends the metric g to the i-form ω with $\Delta_g\omega = 0$ and $[\omega] = \alpha$. The map is well-defined because of the following proposition:

Proposition 14 *If $g \in Met^k(M)$, then ω satisfying $\Delta_g\omega = 0$ is in C^k.*

Proof: This follows immediately from the basic estimate for the Hölder norm (c.f. [GT]):

$$|\omega|_l \le C(|\Delta_g\omega|_{l-2} + |\omega|_0),$$

valid for $l \le k$. This is because if $g \in C^k$, then the coefficients of Δ_g are in C^{k-2}. □

The derivative of Φ_α is the *infinitesimal harmonic perturbation map*

$$d\Phi_\alpha(g) : \Gamma^k(\mathrm{Sym}^2(TM)) \to C^k(d\Omega^{i-1}(M)),$$

which we shall now compute.

Consider a 1-parameter family (ω_t, g_t) of harmonic i-forms on M, with $g_0 = g$, $[\omega_t] \in \alpha$, $h = \frac{d}{dt}g_t|_{t=0}$, and $\eta = \frac{d}{dt}\omega_t|_{t=0}$ exact. We differentiate

$$d\omega_t = 0, d^*\omega_t = 0$$

to obtain

$$\begin{array}{ll} (i) & d\eta = 0, \\ (ii) & d^*\eta = \pm d^*(*\dot{*}_{g_t}\omega). \end{array}$$

The Hodge decomposition gives

$$\Omega^i = d\Omega^{i-1} \oplus d^*\Omega^{i+1} \oplus \mathcal{H}^i,$$

so we find that $\eta = \pm\pi_1(*\dot{*}_{g_t}\omega)$, where π_1 is the projection onto the $d\Omega^{i-1}$ factor. Hence, $d\Phi_\alpha(g)$ is the composite map

$$\Gamma^k(\mathrm{Sym}^2(TM)) \xrightarrow{*\dot{*}\omega} C^k(\Omega^i(M)) \xrightarrow{\pi_1} C^k(d\Omega^{i-1}(M))$$

$$h \mapsto *\dot{*}_{g+th}\omega \mapsto \pi_1(*\dot{*}_{g+th}\omega).$$

Question: Compute the image of Φ_α, or, more realistically, the image of $d\Phi_\alpha(g)$ for generic g.

In order to compute the image of $d\Phi_\alpha(g)$, we solve the equation

$$\eta + *\eta' + \mu = *\dot{*}_{g+th}\omega, \tag{4.1}$$

where the exact form η is the given candidate for an infinitesimal harmonic perturbation, and we determine η' exact, μ harmonic, and h, the metric perturbation.

4.2.1 Some words on the Hodge decomposition

We explain here the effect of the Hodge decomposition for forms of class C^k. For this we refer to [Fu] and summarize the relevant results.

Denote by $L^2_k(\Omega^i)$ the Sobolev L^2_k-sections (k derivatives) of $\Omega^i(M)$ and $C^k(\Omega^i)$ the C^k-sections of Ω^i. Let $\pi : L^2(\Omega^i) \to \mathcal{H}^i$ be the orthogonal projection onto $\mathcal{H}^i = \ker \Delta$, where the metric g is of class C^k and k is not integral.

Proposition 15 *There exists a continuous linear operator $G : L^2(\Omega^i) \to L^2(\Omega^i)$, called the Green's operator, satisfying*

1. $G|_{\mathcal{H}^i} = 0$.
2. *G is a right inverse for Δ on $(\mathcal{H})^\perp$, i.e. $\Delta G u = (I - \pi)u$ for all $u \in L^2(\Omega^i)$.*
3. *$G\Delta u = (I - \pi)u$ for all u in the domain of definition of Δ (i.e. for all $u \in L^2_2(\Omega^i)$).*
4. *$G : C^{l-2}(\Omega^i) \to C^l(\Omega^i)$ is continuous, where l is not integral and $l \le k$.*
5. *$G : L^2_{l-2}(\Omega^i) \to L^2_l(\Omega^i)$ is continuous, where $l \le k$.*
6. *There exists an orthogonal decomposition $C^l(\Omega^i) = C^l(\Delta(\Omega^i)) \oplus \mathcal{H}^i$, for $l \le k$. (This follows from (2) and (4) and observing that $\Delta G = G\Delta$ on the complement of $\mathcal{H}^i$).*

Now, $\pi_1 = dd^*G = G(dd^*)$, so it is a continuous map from C^l to C^l, using (4). Here we assume $l \le k$. Using (6), we obtain an orthogonal decomposition

$$C^l(\Omega^i) = C^l(d(\Omega^{i-1})) \oplus C^l(d^*(\Omega^{i+1})) \oplus \mathcal{H}^i,$$

and hence π_1 is surjective. Finally we claim that $C^l(d(\Omega^{i-1})) \subset C^l(\Omega^i)$ is closed, hence a Banach subspace. Assume $df_i \to g$ in C^l. Then, by (4), $dd^*G(df_i) \to dd^*Gg$ in C^l, but since $dd^*G(df_i) = df_i$, we have $g = dd^*Gg$, which proves the claim. Let us assemble the above facts into the following proposition:

Proposition 16 *$C^l(\Omega^i) \xrightarrow{\pi_1} C^l(d(\Omega^{i-1}))$ is a bounded, surjective map of Banach spaces, for $l \le k$.*

Combining with the fact that $**\omega$ is a continuous map (since it is essentially given by multiplication against ω, which is of class C^k), we have

Proposition 17 *$d\Phi_\alpha(g)$ is a bounded linear map.*

4.2.2 Differentiability of Φ_α

Here we outline the proof of the following proposition:

Proposition 18 *$\Phi_\alpha : Met^k(M) \to C^k(\Omega^i_\alpha(M))$ is a C^∞-differentiable map of Banach manifolds.*

Define

$$\begin{array}{rcl} \Pi : \mathrm{Met}^k(M) \times C^{l-2}(\Omega^i(M)) & \to & C^l(\Omega^i(M)) \\ \Pi : (g, \omega) & \mapsto & \pi_g(\omega), \end{array}$$

where $\pi_g : \Omega^i(M) \to \mathcal{H}_g$ is the g-orthogonal projection onto $\mathcal{H}_g$.

Proposition 19 *Π is a C^∞-differentiable map of Banach manifolds, if $l \leq k$.*

In order to prove Proposition 18, it suffices to prove Proposition 19 and observe that $\Phi_\alpha = \Pi \circ i$, where ω_0 is a fixed element in α, and

$$\begin{array}{rcl} i : \mathrm{Met}^k(M) & \to & \mathrm{Met}^k(M) \times C^{l-2}(\Omega^i(M)) \\ g & \mapsto & (g, \omega_0). \end{array}$$

Proof of Proposition 19: Let us begin with the basic estimate for the Hölder norm (c.f. [GT], [Fu, Theorem 4.3] for families): Assume $l \leq k$ is not integral and the norms $|\cdot|_l$ are with respect to a fixed metric g_0. Write $\Delta_g(\lambda) = \Delta_g - \lambda$ for λ a constant. Given ω,

$$|\omega|_l \leq C_g(|\Delta_g(\lambda)\omega|_{l-2} + |\omega|_0), \tag{4.2}$$

where C_g is dependent only on k, λ, and $|g|_k$. Hence, if we fix g_0 and λ_0, then there exist $\delta > 0$ and $C > 0$ such that $|g - g_0|_l \leq \delta$, $|\lambda - \lambda_0| \leq \delta$ is sufficient for a uniform estimate:

$$|\omega|_l \leq C(|\Delta_g(\lambda)\omega|_{l-2} + |\omega|_0). \tag{4.3}$$

This estimate implies the following lemma ([K, Theorem 7.5], [Fu, Theorem 7.1]):

Lemma 12 *Choose (g_0, λ_0) so that $\Delta_{g_0}(\lambda_0)$ has no kernel. Then there exist $\delta > 0$ and $C_l > 0$ such that for all g, λ with $|g - g_0|_k \leq \delta$, $|\lambda - \lambda_0| \leq \delta$, we have*

$$|\omega|_l \leq C_l |\Delta_g(\lambda)\omega|_{l-2}.$$

Here $l \leq k$.

Proof: Assume otherwise, i.e. there exist $(\omega_i, g_i, \lambda_i)$ with

$$|\omega_i|_l = 1, \ |\Delta_{g_i}(\lambda_i)\omega_i|_{l-2} \to 0.$$

After taking a subsequence, $(\omega_i, g_i, \lambda_i) \to (\omega, g_0, \lambda_0)$ in C^l, and $\Delta_{g_i}(\lambda_i)\omega_i \to \Delta_g(\lambda)\omega$ in C^{l-2} by continuity of $\Delta : C^l \to C^{l-2}$. But then $\Delta_{g_0}(\lambda_0)\omega = 0$, and we conclude that $\omega = 0$, which is a contradiction. □

We also need to invoke some standard results on the spectrum of Δ_g: The Green's operator G_g is a compact operator on L^2, whose spectrum consists of point spectra, and there is only one accumulation point, 0. Hence,

Proposition 20 *(1) Δ_g has eigenvalues $\lambda_1(g) = 0 \leq \lambda_2(g) \leq \ldots$ only on the (nonnegative) real line. (2) Each eigenspace is finite-dimensional. (3) There exists a complete orthonormal basis of eigenfunctions in L^2.*

Now pick a point λ_0 which is not an eigenvalue for g_0. Then $\Delta_{g_0}(\lambda_0) : L^2 \to L^2$ is an isomorphism, and, applying Lemma 12, $|\omega|_l \leq C_l|\Delta_g(\lambda)\omega|_{l-2}$ for all g, λ, ω with $|g - g_0|_k \leq \delta$, $|\lambda - \lambda_0| \leq \delta$, and $l \leq k$. In particular, $\Delta_g(\lambda) : C^l(\Omega^i) \to C^{l-2}(\Omega^i)$, with $l \leq k$, is injective for nearby g, λ. It is also surjective: Take $\eta \in C^{l-2}(\Omega^i)$. We know that there exists an $\omega \in L^2(\Omega^i)$ such that $\Delta_g(\lambda)\omega = \eta$. Using the Sobolev estimates, we can bootstrap ω up to $C^0(\Omega^i)$, provided l is large. Finally, using the Hölder estimates, we can bootstrap ω up to $C^l(\Omega^i)$. We also have the following proposition:

Proposition 21 *Let $W \subset Met^k(M) \times \mathbf{R}$ be the set of (g, λ) for which $\Delta_g(\lambda)$ is injective (and hence isomorphic). Then W is open in $Met^k(M) \times \mathbf{R}$.*

Let $G_g(\lambda) : C^{l-2}(\Omega^i) \to C^l(\Omega^i)$, $l \leq k$, be the Green's operator for $\Delta_g(\lambda)$. Near (g_0, λ_0), $\Delta_g(\lambda)$ is invertible and $G_g(\lambda)$ satisfies $|G_g(\lambda)\eta|_l \leq C_l|\eta|_{l-2}$, with $l \leq k$. Using this we show:

Proposition 22 *$G_g(\lambda) : C^{l-2}(\Omega^i) \to C^l(\Omega^i)$, $l \leq k$, is C^∞-differentiable in g.*

Proof: It is straightforward to check that

$$\Delta(\lambda) : \mathrm{Met}^k(M) \times C^l(\Omega^i) \to \mathrm{Met}^k(M) \times C^{l-2}(\Omega^i)$$

$$(g, \omega) \mapsto (g, \Delta_g(\lambda)\omega)$$

is C^∞-differentiable, provided $l \leq k$. If (g, λ) is near (g_0, λ_0), then $\Delta_g(\lambda)$ is invertible, since $|\omega|_l \leq C_l|\Delta_g(\lambda)\omega|_{l-2}$. This implies that $d\Delta(\lambda)$ is invertible, with a continuous inverse. Using the inverse function theorem, we obtain the C^∞-differentiability of

$$\widetilde{G}(\lambda) : U \times C^{l-2}(\Omega^i) \to U \times C^l(\Omega^i)$$

$$(g, \eta) \mapsto (g, G_g(\lambda)\eta),$$

where U is an open set containing g_0. This, in turn, implies the C^∞-differentiability of

$$G(\lambda) : U \times C^{l-2}(\Omega^i) \to C^l(\Omega^i).$$

□

Following the proof of [K, Theorem 7.6], given $g_0 \in \mathrm{Met}^k(M)$, find a small circle $C \subset \mathbf{C}$ with center 0, so that all the eigenvalues of Δ_{g_0} besides 0 lie outside of C. Also, using the openness condition in Proposition 21, there exists a $\delta > 0$ such that $|g - g_0|_k \leq \delta$ implies $\Delta_g(\lambda)$ is isomorphic for all $\lambda \in C$.

Now, define

$$\pi_g(C)(\eta) = -\frac{1}{2\pi i}\int_C G_g(\lambda)\eta d\lambda.$$

If $\eta = \sum_i a_i e_i(g)$, where $\{e_i(g)\}$ is an orthonormal basis of eigenvectors for $L^2(\Omega^i)$, then $G_g(\lambda)(\eta) = \sum_i \frac{a_i}{\lambda_i(g)-\lambda} e_i(g)$, with $\lambda_i(g)$ eigenvectors corresponding to $e_i(g)$. Hence,

$$\pi_g(C)(\eta) = \sum_{\lambda_i(g) \in \mathrm{Int}(C)} a_i e_i(g),$$

i.e. the projection onto the eigenspaces enclosed by C. Since $G_g(\lambda) : C^{l-2}(\Omega^i) \to C^l(\Omega^i)$ is C^∞-differentiable in g, so is $\pi_g(C)$.

Claim: $\pi_g(C) = \pi_g$.

Proof: We know $\pi_{g_0}(C) = \pi_{g_0}$. If there does not exist $\delta > 0$ such that $|g - g_0|_k \leq \delta$ implies $\pi_g(C) = \pi_g$, then there is a sequence $g_i \to g_0$ and (λ_{g_i}, e_{g_i}) with $\lambda_{g_i} \neq 0$, $\|e_{g_i}\|_{L^2,g} = 1$, and $e_{g_i} \perp \mathcal{H}_{g_i}$. After taking a subsequence, we may assume that $\lambda_{g_i} \to \lambda$. Bootstrapping using the elliptic estimates, we obtain C^{k+2}-bounds on e_{g_i}, and hence there exists a subsequence $e_{g_i} \to e_{g_0}$ in C^{k+2}; however, $\|e_{g_0}\|_{L^2,g_0} = 1$ and $e_{g_0} \perp \mathcal{H}_{g_0}$, giving us a contradiction. □

Thus we have proved that Π is a C^∞-differentiable map of Banach manifolds, for $l \leq k$. □

4.3 Non-self-dual (or anti-self-dual) harmonic 2-forms on a 4-manifold

In contrast to the situation of 1-forms, an intrinsic characterization of harmonic 2-forms appears to be much more complicated. In this section, we shall carry out an *infinitesimal* study of harmonic 2-forms on a 4-manifold and deduce some local facts. Since SD (or ASD) harmonic 2-forms have been considered in Chapter 2, we will focus on harmonic forms which are neither SD nor ASD in this section.

Recall the stratification of $\bigwedge^2 \mathbf{R}^4$ under the action of $SO(4)$, and the corresponding strata for a generic non-SD/ASD harmonic 2-form ω on a 4-manifold:

(i) ω has no zeros.

(ii) The locus where ω is SD/ASD consists of a union of circles $C = \bigcup S^1$.

(iii) The locus where ω has rank 2 is a 3-manifold N (possibly disconnected).

Note that C and N are disjoint.

4.3.1 Infinitesimal computation

Assume in this section that $\Phi_\alpha : \mathrm{Met}^\infty(M) \to C^\infty(\Omega^2_\alpha(M))$. (We shall worry about C^k-differentiability in the next section.) Here we prove the following theorem:

Theorem 17 *Let (ω, g) be a generic non-SD/ASD harmonic 2-form on M^4 in the class $\alpha \in H^2(M;\mathbf{R})$. If $i_N^*(*_g\omega)$ is not zero on each connected component of N, where $i : N \to M$ is the inclusion, then $d\Phi_\alpha(g)$ is surjective, i.e. all the exact 2-forms on M are infinitesimal harmonic perturbations.*

Remark: With our usage of *generic*, a generic harmonic 2-form must necessarily be non-SD/ASD, when both $b_2^+ > 0$ and $b_2^- > 0$.

The primary difficulty with the generic non-SD/ASD harmonic 2-form on a 4-manifold is that Im i_ω is not surjective, even if ω is of generic type. Recall that Im $i_\omega = (*\omega)^\perp$ for ω of generic type.

It is most convenient to rewrite Equation 4.1 as follows: Noting that Im $i_\omega \subset (*\omega)^\perp$ whenever $\omega \neq 0$, and Im $i_\omega = (*\omega)^\perp$ in particular when ω is not SD/ASD, we obtain, after taking $*$,

$$\eta' + *\eta + \mu \perp \omega, \tag{4.4}$$

where $\perp$ is the pointwise inner product with respect to g, and μ is some harmonic 2-form which may not be the same μ as in Equation 4.1. This can be rephrased as

$$(\eta' + *\eta + \mu) \wedge *\omega = 0. \tag{4.5}$$

We will thus compute the image of $d\Phi_\alpha(g)$ in the following fashion: Fix $\eta \in d\Omega^1(M)$, and solve for $\eta' = d\xi$ and μ harmonic in Equation 4.5, where we additionally require on each component S^1 of C that $(\eta' + *\eta + \mu)|_{S^1}$ be ASD whenever $\omega|_{S^1}$ is SD (and vice versa). If there exist such η' and μ, then, by linear algebra, we can find an h solving Equation 4.1. Neighborhoods of C require a little care when solving for h.

Singular differential ideal

We want to compute the image of the following composite map:

$$\Omega^1(M) \xrightarrow{\mathcal{A}} \Omega^3(M) \xrightarrow{d} \Omega^4(M)$$

$$\xi \mapsto \xi \wedge *\omega \mapsto d(\xi \wedge *\omega) = d\xi \wedge *\omega.$$

We shall relate the image of this map to the cohomology $H^4(M, \mathcal{I})$ of a singular differential ideal, and compute it in this section. Let $\mathcal{I} = (*\omega)$ be the differential ideal generated by $*\omega$. The ideal has the following chain complex:

$$0 \to \mathcal{I}^0 = 0 \to \mathcal{I}^1 = 0 \to \mathcal{I}^2 \to \mathcal{I}^3 \to \mathcal{I}^4 \to 0.$$

Observe that $\mathcal{I}^4 = \Omega^4$: As long as $*\omega$ has no zeros, there exists a 2-form ξ such that $\xi \wedge *\omega = F\omega \wedge *\omega$ for given F. Also noting that $\mathcal{I}^3 = \{\xi \wedge *\omega | \xi \in \Omega^1\}$, we have

Lemma 13 *$H^4(M, \mathcal{I}) = \Omega^4(M)/Im\ d \circ \mathcal{A}$.*

Hence, our problem is equivalent to computing $H^4(M, \mathcal{I})$ of a *singular differential ideal.*

Proposition 23 *$H^4(U, \mathcal{I}) = H^4(U, \mathbf{R})$, if $U \subset \{x \in M | \omega^2(x) \neq 0\}$.*

Proof: This follows from observing that if ω is symplectic at x, then $\xi \mapsto \xi \wedge *\omega$ gives an isomorphism $\bigwedge^1(\mathbf{R}^4)^* \simeq \bigwedge^3(\mathbf{R}^4)^*$. □

Corollary 4 *If ω is symplectic, then $H^4(M, \mathcal{I}) = \mathbf{R}$.*

Corollary 5 *If $\omega(x)$ is of generic type for all $x \in M$, then $d\Phi_\alpha(g)$ is surjective.*

Proof: Note that $*\eta \wedge *\omega = \eta \wedge \omega$, with η exact. Hence $[*\eta \wedge *\omega] = 0 \in H^4(M;\mathbf{R})$. That is, we can let $\mu = 0$ and solve for $d\xi \wedge *\omega = *\eta \wedge *\omega$, which has a solution $d\xi$ by the proposition. $\square$

Let us now examine $\mathcal{I} = (*\omega)$ near the rank 2 submanifold N. Let $I \times N$ be a neighborhood of N, with coordinates (t, x). We can write

$$\omega = (\mu_1 + dt \wedge \tilde{\mu}_2) + t(\omega_1 + dt \wedge \tilde{\omega}_2),$$

with μ_1, ω_1 2-forms, and $\tilde{\mu}_2$, $\tilde{\omega}_2$ 1-forms, all without a dt-term. Here μ_1, $\tilde{\mu}_2$ do not depend on t.

On $I \times N$, we can solve for α in $d\alpha = *\eta \wedge *\omega$. Since α must satisfy $\alpha = \xi \wedge *\omega$ for some 1-form ξ, we require $\alpha|_N = 0$. Let us then modify $\alpha \mapsto \alpha - \delta\alpha$ so that $\alpha - \delta\alpha|_N = 0$. We write

$$\begin{aligned} \alpha &= \alpha_1(t, x) + dt \wedge \tilde{\alpha}_2(t, x) && (4.6) \\ &= \alpha_1(0, x) + dt \wedge \tilde{\alpha}_2(0, x) + \text{ h.o. in } t. && (4.7) \end{aligned}$$

Here, α_1 is a 3-form and $\tilde{\alpha}_2$ is a 2-form, both without dt-terms.

If we let $\delta\alpha(t, x) = \alpha_1(0, x) + d(t\tilde{\alpha}_2(0, x))$, then $(\alpha - \delta\alpha)|_N = 0$; since $\delta\alpha$ is closed, we still have $d(\alpha - \delta\alpha) = *\eta \wedge *\omega$. It is not difficult to see that $\alpha - \delta\alpha|_N = 0$ is sufficient to ensure the existence of a ξ such that $\xi \wedge *\omega = \alpha - \delta\alpha$. This follows from the genericity of ω near N. Summarizing,

Proposition 24 $H^4(I \times N, \mathcal{I}) = 0$.

Having taken care of the local aspects, we can pass from local to global. Let $\{N_j\}_{j=1}^r$ be the set of connected components of N. ω is said to be *semi-contact* on N_j if $\omega = \mu_1 + t(\omega_1 + dt \wedge \tilde{\omega}_2)$, with μ_1 nonsingular and closed on N_j, i.e. $i^*_{N_j}(*\omega) = 0$, where $i_{N_j} : N_j \to M$ is the inclusion. Let N' be the union of all the semi-contact N_j. Then we have the following theorem:

Theorem 18 *dim* $H^4(M, \mathcal{I}) = (\#$ *of connected components of* $M - N')$.

Proof: If $[\beta] = 0 \in H^4(M;\mathbf{R})$, then there exists a global α such that $d\alpha = \beta$.

Claim 1: If $i^*_{N_j}[\alpha] = 0 \in H^3(N_j;\mathbf{R})$, then we can modify α so that $\alpha|_{N_j} = 0$.

Proof of Claim 1: Recall Equation 4.6 in the proof of Proposition 24. If $i^*_{N_j}[\alpha] = 0$, then we can write $\alpha_1(0, x) = d_3\gamma_j$ on N_j. Extend γ_j to $I \times N_j$ so that $\gamma_j(t, x) = \gamma_j(0, x)$, and damp $\gamma_j + t\tilde{\alpha}_2(0, x)$ out outside of $I \times N_j$. Finally, modify $\alpha \mapsto \alpha - \delta\alpha$, where $\delta\alpha = d(\gamma_j + t\tilde{\alpha}_2(0, x))$. $\square$

Claim 2: If $M - N'$ is connected, then we can modify $\alpha \mapsto \alpha + \delta\alpha$ with $\delta\alpha \in H^3(M;\mathbf{R})$ so that $i^*_{N_j}[\alpha + \delta\alpha] = 0 \in H^3(N_j;\mathbf{R})$ for all N_j semi-contact.

Proof of Claim 2: Consider the exact sequence

$$H^3(M) \xrightarrow{i} H^3(N') \to H^4(M,N') \to H^4(M) \to 0. \tag{4.8}$$

Since $M - N'$ is connected, $H^4(M,N') \simeq H_0(M-N') \simeq \mathbf{R}$. This implies that i is surjective, and that there exists a $\delta\alpha \in H^3(M;\mathbf{R})$ such that $i^*_{N_j}[\alpha+\delta\alpha] = 0 \in H^3(N_j;\mathbf{R})$ for all N_j semi-contact. $\square$

Claim 3: If $i^*_{N_i}[\alpha] = 0$ for all N_i semi-contact, then there exists an $\alpha = \xi \wedge *\omega$ such that $d\alpha = \beta$.

Proof of Claim 3: Let α satisfy $d\alpha = \beta$, with the additional condition that $i^*_{N_i}[\alpha] = 0$ for all N_i which is semi-contact. By Claim 1, we may also assume that $\alpha|_{N_i} = 0$ for all N_i semi-contact. Now assume N_j is not semi-contact. Then we can write

$$\omega = (\mu_1 + dt \wedge \tilde{\mu}_2) + t(\omega_1 + dt \wedge \tilde{\omega}_2),$$

with $\tilde{\mu}_2$ not identically zero. Then,

$$*\omega(0,x) = (*_3\tilde{\mu}_2 + dt \wedge *_3\mu_1)(0,x),$$

and there exist $\xi_j(t,x) = c_j f_j \tilde{\mu}_2(0,x)$ on N_j such that

$$\int_{N_j} c_j f_j \tilde{\mu}_2 \wedge *_3\tilde{\mu}_2 = \int_{N_j} \alpha.$$

We then damp ξ_j outside of $I \times N_j$, and solve for $\xi \wedge *\omega = \alpha - \sum \xi_j \wedge *\omega$, where the sum runs over all non-semi-contact N_j. Here, we may need to modify α using Claim 1, so that $(\alpha - \sum \xi_j \wedge *\omega)|_{N_i} = 0$ for every component N_i of N. Finally, we can write $\alpha = (\xi + \sum \xi_j) \wedge *\omega$. $\square$

We will now complete the proof of Theorem 18. Refer back to Equation 4.8. Observe that $i^*_{N'}[\xi \wedge *\omega] = 0 \in H^3(N')$ if N' is the union of the semi-contact components. Hence, given β with $[\beta] = 0 \in H^4(M;\mathbf{R})$, for $\beta = d\alpha$ with $\alpha = \xi \wedge *\omega$ to be satisfied, we need $i^*_{N'}[\alpha] = 0 \in H^3(N')/i(H^3(M))$. This condition is also sufficient, since $i^*_{N'}[\alpha] = 0 \in H^3(N')/i(H^3(M))$ implies that there exists a representative α with $i^*_{N_j}[\alpha] = 0 \in H^3(N_j)$ for all N_j semi-contact, and we can apply Claim 3. Finally, dim $H^3(N')/i(H^3(M)) =$ dim $H^4(M,N') -$ dim $H^4(M) = (\#$ of components of $M - N') - 1$. Thus, dim $H^4(M,\mathcal{I}) = (\#$ of connected components of $M - N')$. $\square$

Remark 1: I know of no explicit examples of a generic non-SD/ASD harmonic 2-form on M^4 and don't know whether the connectivity of $M - N'$ is vacuous.

Remark 2: We have two differential ideals, $\mathcal{I} = (*\omega)$ and $\mathcal{J} = (\omega)$, whose fates seem interconnected. It would be interesting to find out how they are related.

Remark 3: The computations of the singular differential ideals seem generalizable to higher dimensions, provided we have sufficient genericity.

Analysis near $\bigcup S^1$

In the previous section we saw that, if $M - N'$ is connected, then we have a solution to $(\eta' + *\eta + \mu) \wedge *\omega = 0$. Note that we can set $\mu = 0$ since $*\eta \wedge *\omega = \eta \wedge \omega$ is exact on M. Then, by Theorem 18, we find that there exist a 1-form ξ such that $d(\xi \wedge *\omega) = *\eta \wedge *\omega$, and hence we can set $\eta' = d\xi$.

We now need to perform a more careful analysis near $C = \bigcup S^1$ in order to finish the proof of Theorem 17. Consider a connected component S^1 of C and let $N(S^1) = S^1 \times D^3$ have coordinates θ, x_1, x_2, x_3, which are orthonormal at $S^1 \times \{0\}$. Without loss of generality, let ω be SD on S^1. Fix an exact η, and we will solve for η' satisfying $(\eta' + *\eta) \wedge *\omega = 0$ on $S^1 \times D^3$, with the additional constraint that $\eta' + *\eta$ be ASD on S^1.

Lemma 14 *There exists an exact η'_1 such that $\eta'_1 + *\eta$ is ASD on S^1.*

Proof: Let $\eta'_1 = -\eta$. Then η is exact and $-\eta + *\eta$ is ASD. □

Next, consider

$$\Omega^1(N(S^1)) \xrightarrow{\mathcal{A}} \Omega^3(N(S^1)) \xrightarrow{d} \Omega^4(N(S^1))$$
$$\xi \mapsto \xi \wedge *\omega \mapsto d(\xi \wedge *\omega) = d\xi \wedge *\omega.$$

Lemma 15 *Given $F\omega \wedge *\omega \in \Omega^4(N(S^1))$ with $F|_{S^1} = 0$, there exists a $\xi \in \Omega^1(N(S^1))$ with $\xi|_{S^1} = 0$ and $d\xi|_{S^1} = 0$ such that $d \circ \mathcal{A}(\xi) = F\omega \wedge *\omega$.*

Proof: The key is to find $\alpha = \xi \wedge *\omega$ of the form $\alpha = \tilde{\alpha} \wedge d\theta$ with $\tilde{\alpha} = \sum_i \alpha_i dx_{(i)}$, such that

$$\alpha_j(\theta, 0) = 0, \ \frac{\partial}{\partial x_i}\alpha_j(\theta, 0) = 0, \text{ and } \frac{\partial}{\partial \theta}\alpha_j(\theta, 0) = 0,$$

where $1 \leq i, j \leq 3$, and $\theta \in S^1$. $d\alpha = d\tilde{\alpha} \wedge d\theta = d_3\tilde{\alpha} \wedge d\theta$, where d_3 is the differential with respect to $\{x_i\}$; on the other hand, $F\omega \wedge *\omega = f dx_1 dx_2 dx_3 d\theta$ for some f with $f|_{S^1} = 0$. Thus solving for $d\alpha = F\omega \wedge *\omega$ is equivalent to solving for $\sum_i \frac{\partial \alpha_i}{\partial x_i} = f$. It is clearly advantageous to us that this partial differential equation is very underdetermined. Let $\alpha_2 = \alpha_3 = 0$ on

$N(S^1)$. Then $\frac{\partial \alpha_1}{\partial x_1} = f$ can be solved with initial condition $\alpha_1(\theta, 0, x_2, x_3) = 0$. Since $f|_{S^1} = 0$, we can choose α with $\frac{\partial \alpha_i}{\partial x_j}(\theta, 0) = \frac{\partial \alpha_i}{\partial \theta}(\theta, 0) = 0$.

Thus, $\alpha = \xi \wedge *\omega$ has $\alpha(\theta, 0)$ and all of its first partials vanish on S^1. Under the linear map $\mathcal{A}^{-1}$, α will get sent to ξ, with $\xi(\theta, 0)$ and all of the first partials of ξ equal to zero on S^1. Thus, $\xi|_{S^1} = 0$ and $d\xi|_{S^1} = 0$. □

We find an η_1' as in Lemma 14 and an η_2' as in Lemma 15 such that $(\eta_2' + \eta_1' + *\eta) \wedge *\omega = 0$ on $N(S^1)$. Let $\eta'_{N(S^1)} = \eta_1' + \eta_2'$. This proves the following proposition:

Proposition 25 *Given any exact 2-form η, there exists an exact $\eta'_{N(S^1)}$ on $N(S^1)$ such that $\eta'_{N(S^1)} + *\eta$ is ASD on S^1 and $(\eta'_{N(S^1)} + *\eta) \wedge *\omega = 0$ on $N(S^1)$.*

Let η be an exact 2-form on M as before. On M we have $\eta' = d\xi$ such that $(\eta' + *\eta) \wedge *\omega = 0$, and on $N(C)$ there exists an $\eta'_{N(C)}$ such that $\eta'_{N(C)} + *\eta$ is SD/ASD on the various S^1 as appropriate, and satisfies $(\eta'_{N(C)} + *\eta) \wedge *\omega = 0$ on $N(C)$.

Now write $\eta' = d\xi$ and $\eta'_{N(C)} = d\xi_{N(C)}$. Then, $d((\xi - \xi_{N(C)}) \wedge *\omega) = 0$, and $(\xi - \xi_{N(C)}) \wedge *\omega$ must be *exact* on $N(C)$. Write $(\xi - \xi_{N(C)}) \wedge *\omega = d\gamma$ on $N(C)$, with γ defined on $N(C)$. Extend γ to all of M by damping out outside of $N(C)$. Since ω is symplectic on $\mathrm{Supp}(\gamma)$, we can write $d\gamma = \xi' \wedge *\omega$, and modify $\eta' \mapsto \eta' - d\xi' = d(\xi - \xi')$. Summarizing,

Proposition 26 *Assume $M - N'$ is connected. Then given an exact 2-form η on M, there exists an $\eta' = d\xi$ on M such that $\eta' + *\eta$ is SD/ASD on C and $(\eta' + *\eta) \wedge *\omega = 0$ on M.*

It remains to obtain a section h with $*\dot{*}_{g+th}\omega = \eta' + *\eta$. We use the following proposition with $\beta = \eta' + *\eta$ to complete our argument for Theorem 17.

Proposition 27 *There exists a solution h to the equation $i_\omega(h) = \beta$, provided $\beta|_{S^1}$ is ASD and $\beta \wedge *\omega = 0$ on $N(S^1) = S^1 \times D^3$.*

Proof: Decompose $\omega = \omega_+ + \omega_-$ and $\beta = \beta_+ + \beta_-$ into the SD and ASD parts. If $i_\omega(h) = \beta$, then

$$\begin{aligned} i_{\omega_+}(h) &= \beta_- \\ i_{\omega_-}(h) &= \beta_+. \end{aligned}$$

We expand $\omega_1^+ = \omega_+$ to a basis $\{\omega_1^+, \omega_2^+, \omega_3^+\}$ for the SD forms near S^1. Since $T_g\mathrm{Met}(M) \simeq \mathrm{Hom}(\wedge^+, \wedge^-)$, in order to specify h it suffices to specify

$$\begin{aligned} \omega_1^+ &\mapsto \beta_1^- = \beta_- \\ \omega_2^+ &\mapsto \beta_2^- \\ \omega_3^+ &\mapsto \beta_3^- \end{aligned}$$

in a manner consistent with $\omega_- \mapsto \beta_+$.

Claim: $h : \bigwedge^+ \oplus \bigwedge^- \to \bigwedge^- \oplus \bigwedge^+$ satisfies $< h(\alpha_+), \alpha_- >= - < \alpha_+, h(\alpha_-) >$, where $\alpha_\pm \in \bigwedge^\pm$.

The claim is an easy exercise. We then see that the consistency condition is $< \beta_i^-, \omega_- >= - < \omega_i^+, \beta_+ >$, or, equivalently, $\beta_i^- \wedge \omega_- = \omega_i^+ \wedge \beta_+$. We check that $\beta \wedge *\omega = 0$ implies $(\beta_+ + \beta_-) \wedge (\omega_+ - \omega_-) = \beta_+ \wedge \omega_+ - \beta_- \wedge \omega_- = 0$, giving us $\beta_- \wedge \omega_- = \omega_+ \wedge \beta_+$.

Let us now show that there exist β_2^-, β_3^- satisfying the consistency conditions. Write $\omega_- = \sum_k x_k \omega_k^-$ and $\beta_i^- = \sum_j b_{ij}\omega_j^-$, $i = 2, 3$, where $\{\omega_1^-, \omega_2^-, \omega_3^-\}$ is a basis for $\bigwedge^-$ on $N(S^1)$, $\omega_i^- \wedge \omega_j^- = a_{ij} dv_{N(S^1)}$, and $dv_{N(S^1)}$ is the volume form on $N(S^1)$. Then

$$\begin{aligned}
\beta_i^- \wedge \omega_- &= \sum_{jk} b_{ij}\omega_j^- \wedge x_k \omega_k^- = \sum_{jk} b_{ij} a_{jk} x_k dv_{N(S^1)} \\
\omega_i^+ \wedge \beta_+ &= \sum_i r_i x_i dv_{N(S^1)} \text{ for some } r_i,
\end{aligned}$$

and solving for β_i^- in $\beta_i^- \wedge \omega_- = \omega_i \wedge \beta_+$ would be tantamount to solving for b_{ij} in $\sum_k b_{ij} a_{jk} = r_k$. But here a_{ij} is invertible since $\{\omega_1^-, \omega_2^-, \omega_3^-\}$ is a basis for $\bigwedge^-$. □

Thus we have proved Theorem 17. In fact, we have a slightly stronger version of the theorem:

Theorem 19 *If (ω, g) is a generic non-SD/ASD harmonic 2-form on M^4 in the class α, and $M - N'$ is connected, then $d\Phi_\alpha(g)$ is surjective.*

Analysis near N

Although it is not necessary for our theorem, it is instructive to study the neighborhood $I \times N$ of N. Assume N is connected and the metric g on $I \times N$ is the product metric for simplicity. Take coordinates (t, x) on $I \times N$. Write

$$\omega = (\mu_1 + dt \wedge *_3\mu_2) + t(\omega_1 + dt \wedge *_3\omega_2),$$

where μ_1, μ_2 do not depend on t, ω_1, ω_2 depend on t, and μ_i, ω_i are all 2-forms without a dt-term. Write d, $*$ on N as d_3, $*_3$.

It turns out that ω_1, ω_2 are completely determined by μ_1, μ_2 because of the harmonicity ($d\omega = 0,\ d * \omega = 0$).

Proposition 28 *ω_1 and ω_2 are given by*

$$\omega_1(t, x) = \frac{1}{t}\left(\frac{e^{(d_3*_3)t} + e^{-(d_3*_3)t}}{2} - 1\right)\mu_1 + \frac{1}{t}\left(\frac{e^{(d_3*_3)t} - e^{-(d_3*_3)t}}{2}\right)\mu_2,$$

$$\omega_2(t,x) = \frac{1}{t}\left(\frac{e^{(d_3*_3)t} - e^{-(d_3*_3)t}}{2}\right)\mu_1 + \frac{1}{t}\left(\frac{e^{(d_3*_3)t} + e^{-(d_3*_3)t}}{2} - 1\right)\mu_2,$$

provided $e^{\pm(d_3*_3)t}(\mu_1)$ *and* $e^{\pm(d_3*_3)t}(\mu_2)$ *make sense.*

Proof: (A) $d\omega = 0$ implies

(1) $d_3\mu_1 = -td_3\omega_1$.

(2) $t\dot{\omega}_1 + \omega_1 = d_3 *_3 \mu_2 + td_3 *_3 \omega_2$.

(B) $d * \omega = 0$ implies

(3) $d_3\mu_2 = -td_3\omega_2$.

(4) $t\dot{\omega}_2 + \omega_2 = d_3 *_3 \mu_1 + td_3 *_3 \omega_1$.

Observe that (1), (3) imply that $d_3\mu_1 = d_3\mu_2 = d_3\omega_1 = d_3\omega_2 = 0$ because the μ_i are t-independent.

Let us first integrate (2) and (4) using $(tf)' = tf'(t) + f(t) = h(t)$ as the model, with $f(t) = \frac{1}{t}\left(c + \int_0^t h(s)ds\right)$ as its general solution. If we require $f(0)$ to be finite, $c = 0$, and we have $f(t) = \frac{1}{t}\int_0^t h(s)ds$. Thus,

$$\begin{aligned}
\omega_1(t,x) &= \frac{1}{t}\int_0^t \left[d_3 *_3 \mu_2(s,x) + sd_3 *_3 \omega_2(s,x)\right] ds \\
&= d_3 *_3 \mu_2(0,x) + \frac{1}{t}\int_0^t sd_3 *_3 \omega_2(s,x)ds, \\
\omega_2(t,x) &= d_3 *_3 \mu_1(0,x) + \frac{1}{t}\int_0^t sd_3 *_3 \omega_1(s,x)ds.
\end{aligned}$$

Plugging ω_1 into the right-hand side of ω_2 (and vice versa), and iterating, we obtain

$$\begin{aligned}
\omega_1(t,x) &= (d_3*_3)\mu_2 + \frac{t}{2}(d_3*_3)^2\mu_1 + \frac{t^2}{6}(d_3*_3)^3\mu_2 + \ldots \\
&= \frac{1}{t}\left(\frac{e^{(d_3*_3)t} + e^{-(d_3*_3)t}}{2} - 1\right)\mu_1 + \frac{1}{t}\left(\frac{e^{(d_3*_3)t} - e^{-(d_3*_3)t}}{2}\right)\mu_2 \\
\omega_2(t,x) &= \frac{1}{t}\left(\frac{e^{(d_3*_3)t} - e^{-(d_3*_3)t}}{2}\right)\mu_1 + \frac{1}{t}\left(\frac{e^{(d_3*_3)t} + e^{-(d_3*_3)t}}{2} - 1\right)\mu_2
\end{aligned}$$

□

Example: (Contact case) This is the situation where $\omega = \mu_1 + t(\omega_1 + dt \wedge *_3\omega_2)$, with $*_3\mu_1 = \xi$, a contact 1-form, and $d_3 *_3 \mu_1 = d\xi = \mu_1$. Then we obtain

$$\begin{aligned}
\omega &= (e^t + e^{-t})\mu_1 + (e^t - e^{-t})dt \wedge *_3\mu_1 \\
&= d((e^t + e^{-t})\xi).
\end{aligned}$$

4.3.2 Local considerations

In this section, $\Phi_\alpha : \mathrm{Met}^\infty(M) \to C^\infty(\Omega^2_\alpha(M))$. With the help of the Nash-Moser iteration technique, we are able to pass from the infinitesimal computation to a local statement:

Theorem 20 *Φ_α is surjective near (ω, g), whenever (ω, g) satisfies the conditions of Theorem 19.*

The importance of Theorem 20 is that it says the space of generic harmonic 2-forms $\widetilde{\mathcal{H}}^2_\alpha(M) \subset \mathcal{H}^2_\alpha(M)$ is open in $\Omega^2_\alpha(M)$, with the exception of some harmonic forms with 'non-generic' behavior, namely that the harmonic form (ω, g) has semi-contact rank 2 components and the union N' of all the semi-contact components makes $M - N'$ disconnected. Here we are assuming that b_2^+ and b_2^- are both positive, so that there exist non-SD/ASD harmonic 2-forms.

We will prove the following theorem:

Theorem 21 *Let $g_0 \in Met^\infty(M)$ be a metric for which $\omega_0 = \Phi_\alpha(g_0)$ is generic. Then there exist constants $C_k > 0$ and $\delta > 0$ with the following property: Given $\eta \in C^\infty(d\Omega^1)$ and $|g - g_0|_1 \le \delta$, there exists an $h \in \Gamma^\infty(Sym^2(TM))$ such that $d\Phi_\alpha(g)(h) = \eta$ and $|h|_{k-1} \le C_k(|\eta|_k + |\eta|_0|g|_k)$.*

Theorem 21 implies Theorem 20 by the Nash-Moser iteration process. We will now quickly review the setup for Nash-Moser à la Hamilton [Ham].

Tame maps

Definition: A *graded* Frèchet space is a Frèchet space, together with seminorms $\{|\cdot|_k\}$ satisfying $|f|_0 \le |f|_1 \le |f|_2 \le \cdots$.

Definition: Let F, G be graded Frèchet spaces, $U \subset F$ an open subset, and $L : U \to G$ a (not necessarily linear) map. L is *tame of degree* r, if there exist r, N such that $k \ge N$ implies the estimate

$$|Lf|_k \le C_k(1 + |f|_{k+r})$$

for all $f \in U$.

Model Frèchet space $\Sigma(B)$: Let B be a Banach space with norm $|\cdot|$. Then $\Sigma(B)$ is the set of rapidly decreasing sequences $\{f_i\}_{i=1}^\infty$, with $f_i \in B$, such that

$$|\{f_k\}|_n = \sum_{k=1}^\infty e^{nk}|f_k| < \infty$$

for all n. The $|\cdot|_n$ give the grading for $\Sigma(B)$.

Definition: A graded Frèchet space F is *tame* if it is a tame direct summand of a model graded Frèchet space $\Sigma(B)$, i.e. there exists a graded Frèchet space $\widetilde{F}$ so that $F \oplus \widetilde{F} \simeq \Sigma(B)$ as Frèchet spaces.

Definition: A *tame Frèchet* manifold is a Frèchet manifold whose coordinate functions map to tame Frèchet spaces and whose transition functions are tame.

Example: If M is a compact manifold, then $C^\infty(M)$ is tame with respect to (a) the sup norms $|f|_{C^n}$, (b) Hölder norms $|f|_{C^{n+\alpha}}$, or (c) Sobolev norms $|f|_{L^n_p}$. All of the norms are equivalent, i.e.

$$(C^\infty(M), |\cdot|_{C^n}) \xrightarrow{id} (C^\infty(M), |\cdot|_{C^{n+\alpha}})$$

is a tame map, and so on.

Definition: A *smooth tame* map $L : F \to G$ of tame Frèchet manifolds is a tame map whose derivatives are all tame.

Let V, W be vector bundles over M, and $C^\infty(V)$, $C^\infty(W)$ be tame Frèchet spaces of C^∞-sections over M. Consider $D^r(V, W)$, whose sections are differential operators of degree r from V to W. Locally we can write a differential operator of degree r as

$$L(\phi)(f) = \sum_{|\alpha| \leq r} \phi_\alpha(D_\alpha f).$$

Here α is a multiindex $(\alpha_1, ..., \alpha_n)$ and $D_\alpha = \partial^{\alpha_1} \cdots \partial^{\alpha_n}$. We can think of $\phi = \{\phi_\alpha\}$ as a section of $D^r(V, W)$. Then we have a map

$$L : C^\infty(D^r(V, W)) \times C^\infty(V) \to C^\infty(W),$$

$$(\phi, f) \mapsto L(\phi)(f).$$

Proposition 29 *L is a smooth tame map.*

Now consider an open set $U \subset C^\infty(D^r(V, W))$ consisting of $\phi = \{\phi_\alpha\}$ such that $L(\phi)$ is elliptic and invertible. Then we have

$$L^{-1} : U \times C^\infty(W) \to C^\infty(V),$$

$$(\phi, g) \mapsto [L(\phi)]^{-1}(g).$$

Proposition 30 *L^{-1} is a smooth tame map of degree $-r$.*

Proposition 31 *$\Phi_\alpha : Met^\infty(M) \to C^\infty(\Omega^i_\alpha(M))$ is a smooth tame map of degree* 0.

Proof: By the previous proposition,

$$L^{-1} : U \times C^\infty(\Omega^i) \to C^\infty(\Omega^i)$$

is a smooth tame map of degree -2, where $U \subset C^\infty(D^2(\Omega^i, \Omega^i))$ consists of elliptic and invertible degree 2 operators.

Now, consider the inclusion

$$\mathrm{Met}^\infty(M) \times \mathbf{C} \to C^\infty(D^2(\Omega^i, \Omega^i)),$$

$$(g, \lambda) \mapsto \Delta_g + \lambda,$$

which is a smooth tame map of degree 2. Since the composition of tame maps is tame, we have

$$G : ((\mathrm{Met}^\infty(M) \times \mathbf{C}) \bigcap U) \times C^\infty(\Omega^i) \to C^\infty(\Omega^i)$$

$$[(g, \lambda), \omega] \mapsto (\Delta_g + \lambda)^{-1}\omega = G_g(\lambda)\omega$$

is a smooth tame map of degree 0. Next,

$$\Pi : \mathrm{Met}^\infty(M) \times C^\infty(\Omega^i) \to C^\infty(\Omega^i)$$

$$(g, \omega) \mapsto \pi_g(\omega)$$

is a smooth tame map because

$$\pi_g(\omega) = -\frac{1}{2\pi i}\int_C G_g(\lambda)\omega d\lambda$$

and $C \subset \mathbf{C}$ can be fixed on a small neighborhood of g. Finally, composing with

$$i : \mathrm{Met}^\infty(M) \to \mathrm{Met}^\infty(M) \times C^\infty(\Omega^i)$$

$$g \to (g, \omega_0)$$

as before, we find that Φ_α is a smooth tame map of degree 0. □

Nash-Moser iteration scheme

The following is the version of Nash-Moser that we will use:

Theorem 22 (Nash-Moser) *Let F, G be tame Frèchet spaces and $U \subset F$ an open set. Suppose $L : U \to G$ is a smooth tame map, $dL(f)$ is surjective for all $f \in U$, and there exists a family of right inverses $(dL)^{-1} : U \times G \to F$ which is a tame map. Then L is locally surjective.*

We already know that $\Phi_\alpha : \mathrm{Met}^\infty(M) \to C^\infty(\Omega^2_\alpha(M))$ is a smooth tame map and that $d\Phi_\alpha$ is surjective near (ω_0, g_0). The conditions

$$|h|_{k-1} \leq C(|\eta|_k + |\eta|_0|g|_k) \tag{4.9}$$

would assure us that $(dL)^{-1}$ is tame. Applying the Nash-Moser iteration process, we see that Theorem 21 would imply Theorem 20.

Estimates

We will prove Estimates 4.9 above by carefully retracing the argument in Theorem 19. Keep in mind that $|g-g_0|_1 \leq \delta$ throughout.

The following interpolation lemma is useful in our estimates:

Lemma 16 (Interpolation) *If f_1, f_2 are functions on a compact X, then*

$$|f_1 f_2|_k \leq C(|f_1|_0|f_2|_k + |f_1|_k|f_2|_0).$$

In the proof of Theorem 18, we first solve for $d\alpha = *\eta \wedge *\omega$. Noting that $|\omega|_k \leq C(1+|g|_k)$ since Φ_α is smooth tame of degree 0, we obtain bounds

$$|d\alpha|_k \leq C(|\eta|_k|g|_0 + |g|_k|\eta|_0) \leq C(|\eta|_k + |\eta|_0|g|_k)$$

by interpolation.

Lemma 17 *Given an exact i-form β on M, there exists an $\alpha \in \Omega^{i-1}(M)$ such that $d\alpha = \beta$ and $|\alpha|_{k+1} \leq C|\beta|_k$.*

Proof: We make use of the Green's function, say at g_0, and write $\alpha = d^{*_0}G_{g_0}\beta$. $d\alpha = \beta$, and □

$$|\alpha|_{k+1} = |d^{*_0}G_{g_0}\beta|_{k+1} \leq C|G_{g_0}\beta|_{k+2} \leq C|\beta|_k.$$

Thus, there exists an α such that $d\alpha = *\eta \wedge *\omega$ and $|\alpha|_{k+1} \leq C(|\eta|_k + |\eta|_0|g|_k)$.

Claim 2 bounds: Next, we bound the α modified as in Claim 2 of Theorem 18. Observe that, as long as $|g-g_0|_1 \leq \delta$, for δ small, $|\omega-\omega_0|_1$ is small, and the harmonic form remains generic (i.e. transverse to the relevant strata). Hence, the rank 2 subsets N remain submanifolds, and are close together, provided the $|g-g_0|_1$ are kept small.

Take a basis $\{[dv_{N_i}]\}$ for $H^3(N')$, with dv_{N_i} a volume form of unit volume on N_i. Let $[\gamma_i] \in H^3(M)$ satisfy $i^*_{N_i}[\gamma_j] = [\delta_{ij}dv_{N_i}]$. Fix representatives $\gamma_i \in [\gamma_i]$. Then $\delta\alpha = -\sum_i a_i\gamma_i$, with $|\gamma_i|_{k+1}$ fixed constants, and $a_i = \int_{N_i}\alpha$. Hence,

$$|\delta\alpha|_{k+1} \leq C\sum_i |\alpha|_0|\gamma_i|_{k+1} \leq C|\alpha|_0 \leq C(|\eta|_k + |\eta|_0|g|_k).$$

Claim 3 bounds: We now have bounds for α, where $d\alpha = *\eta \wedge *\omega$ and $i^*_{N_i}[\alpha] = 0$ for all N_i semi-contact. Take N_j not semi-contact, and we first estimate ξ_j on $I \times N_j$. $\xi_j(t,x) = c_j f_j \tilde{\mu}_2(0,x)$, with $\int_{N_j} \xi_j \wedge *_3\tilde{\mu}_2 = \int_{N_j}\alpha$, where we are using the same $f_j\tilde{\mu}_2(0,x) = f_j\tilde{\mu}_2(g_0)(0,x)$ for all $|g-g_0| \leq \delta$, and we are simply varying the scaling factor c_j. Thus,

$$|\xi_j|_{k+1} \leq C|\alpha|_0|\omega_0|_{k+1} \leq C|\alpha|_0|g_0|_{k+1} \leq C|\alpha|_0,$$

on $I \times N_j$.

We now give bounds for the damping out process. Let $\phi(t)$ be a smooth function on $\mathbf{R}$ such that

$$\phi(t) = \begin{cases} 1 & \text{on } [\frac{-1}{2}, \frac{1}{2}] \\ 0 & \text{outside } [-1, 1] \end{cases},$$

and $0 \leq \phi(t) \leq 1$ on $[-1, \frac{-1}{2}] \bigcup [\frac{1}{2}, 1]$.

Then, modify $\xi_j \mapsto \xi_j \phi$. We find that

$$|\xi_j \phi|_{k+1} \leq C(|\xi_j|_{k+1}|\phi|_0 + |\phi|_{k+1}|\xi_j|_0) \leq C|\xi_j|_{k+1},$$

since ϕ is fixed throughout. With this new ξ_j,

$$|\alpha - \sum \xi_j \wedge *\omega|_{k+1} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

Claim 1 bounds: We may now assume that $i^*_{N_i}[\alpha] = 0$ for all N_i. We then modify $\alpha \mapsto \alpha - \delta\alpha$ so that $(\alpha - \delta\alpha)|_{N_i} = 0$. If we write $\alpha_1(0, x) = d_3\gamma_j$ on N_j, then

$$|\gamma_j|_{k+2,N} \leq C|\alpha|_{k+1}$$

by Lemma 17. However, we can only bound $|\gamma_j + t\widetilde{\alpha}_2(0, x)|_{k+1} \leq C|\alpha|_{k+1}$ because of the term $t\widetilde{\alpha}_2$ - we lose one derivative here unless we are careful.

Instead, use $\psi_\varepsilon(t)\widetilde{\alpha}_2(0, x)$, where

$$\psi_\varepsilon(t) = \begin{cases} t & \text{on } [-\varepsilon, \varepsilon] \\ 0 & \text{outside } [-1, 1] \end{cases},$$

and ψ_ε damps out slowly to 0 on $[-1, -\varepsilon] \bigcup [\varepsilon, 1]$. It is not difficult to see that for ε small, there exist ψ_ε with $|\psi_\varepsilon|_0$ arbitrarily small, and $|\psi_\varepsilon|_i \leq |\psi|_i$, for $i > 1$, where

$$\psi(t) = \begin{cases} t & \text{on } [-\frac{1}{2}, \frac{1}{2}] \\ 0 & \text{outside } [-1, 1] \end{cases},$$

and ψ damps out slowly to 0 on $[-1, -\frac{1}{2}] \bigcup [\frac{1}{2}, 1]$. $\psi_\varepsilon(t)\widetilde{\alpha}_2(0, x)$ will clearly do the job of $t\widetilde{\alpha}_2(0, x)$, with the advantage that we can find an ε (dependent on g) with

$$|\psi_\varepsilon \widetilde{\alpha}_2(0, x)|_{k+2} \leq C|\alpha|_{k+2},$$

$$|\delta\alpha|_{k+1} = |d(\gamma_j + \psi_\varepsilon(t)\widetilde{\alpha}_2(0, x))|_{k+1} \leq C|\alpha|_{k+1}.$$

As before, we do not lose any derivatives by damping out $\gamma_j + \psi_\varepsilon(t)\widetilde{\alpha}_2(0, x)$.

Thus,

$$|\alpha - \delta\alpha|_{k+1} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

Bounds for η': Finally, $\alpha|_{N_i} = 0$ for all N_i, and we solve for $\xi \wedge *\omega = \alpha$. We do not lose any derivatives where $\mathcal{A}$ is an isomorphism. However, near the N_i's we lose one derivative, i.e.

$$|\xi|_k \leq C(|\eta|_k + |\eta|_0|g|_k),$$

and

$$|\eta'|_{k-1} \leq C|d\xi|_{k-1} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

Estimates near S^1: On $N(S^1)$, we have bounds

$$|\eta'|_k \leq |\xi|_{k+1} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

Let us compute bounds on $\eta'_{N(S^1)}$ and $\xi_{N(S^1)}$. $\eta'_1 = -\eta$, so $|\eta'_1|_k \leq C|\eta|_k$. For bounds on η'_2 satisfying $\eta'_2 \wedge *\omega = -(\eta'_1 + *\eta) \wedge *\omega$ on $N(S^1)$ and $-(\eta'_1 + *\eta) \wedge *\omega|_{S^1} = F\omega \wedge *\omega|_{S^1} = 0$, we look to the proof of Lemma 15. Clearly, $|F|_k \leq C(|\eta|_k + |\eta|_0|g|_k)$. Solving for $\alpha = \widetilde{\alpha} \wedge d\theta$ with $d\alpha = F\omega \wedge *\omega$, we have

$$|\alpha|_k \leq C|F|_k \leq C(|\eta|_k + |\eta|_0|g|_k),$$

and hence

$$|\xi_{N(S^1)}|_k \leq C(|\eta|_k + |\eta|_0|g|_k).$$

Note that we have lost one derivative - had we worked a bit harder, that would not have been necessary, unlike the loss of derivative near N_i, which seems inherent to the problem.

Finally, we write $d\gamma = (\xi - \xi_{N(C)}) \wedge *\omega$ on $N(C)$. By compactifying $S^1 \times D^3$ to $S^1 \times S^3$, for example, we can use Lemma 15 and obtain a γ satisfying

$$|\gamma|_{k+1} \leq C|(\xi - \xi_{N(C)}) \wedge *\omega|_k \leq C(|\eta|_k + |\eta|_0|g|_k).$$

Damping γ out, we do not lose any derivatives, and hence

$$|\eta' - d\xi|_{k-1} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

In order the complete the proof of Theorem 21, we are left to prove:

Lemma 18 *There exists an h on M such that $|h|_{k-1} \leq C(|\eta|_k + |\eta|_0|g|_k)$.*

Proof: Consider h away from $N(S^1)$. Since $\eta + *\eta' = \{h, \omega\}$ and i_ω has constant rank throughout, we are able to bound

$$|h|_{k-1} \leq C(|\eta + *\eta'|_{k-1}|\omega|_0 + |\eta + *\eta'|_0|\omega|_{k-1}) \leq C(|\eta|_k + |\eta|_0|g|_k),$$

by interpolation.

We next find h on $N(S^1)$. Writing $\beta = \eta' + *\eta$ and $\beta = \beta_+ + \beta_-$,

$$|\beta_1^-|_{k-1} = |\beta_-|_{k-1} \leq |\beta|_{k-1} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

β_2^-, β_3^- come from solving $\beta_i^- \wedge \omega_- = \omega_i^+ \wedge \beta_+$, and

$$|\beta_i^-|_{k-1} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

Hence $|h|_{k-1} \leq C(|\eta|_k + |\eta|_0|g|_k)$ on $N(S^1)$. We finally interpolate the h that we find on $N(S^1)$ to the h on $M - N(S^1)$, while keeping $|h|_{k-1} \leq C(|\eta|_k + |\eta|_0|g|_k)$. $\square$

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