# An Openness Theorem for Harmonic 2-forms on 4-manifolds

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Revised June 29, 1998

**Abstract**

Let $M$ be a closed, oriented 4-manifold with $b_2^{\pm} > 0$. In this paper we show that the space of transverse intrinsically harmonic 2-forms in a fixed cohomology class is open in the space of closed 2-forms, subject to a condition which arises from cohomological considerations of a singular differential ideal.

## 1 Introduction

In this paper we address the question: When is a closed $i$-form $\omega$ on a closed manifold $M$ of dimension $n$ *intrinsically harmonic*, that is, there exists a Riemannian metric $g$ with respect to which $\omega$ is harmonic? In the case of 1-forms, an answer was given by Calabi in [2]:

**Theorem 1 (Calabi)** *Let $\omega$ be a closed 1-form on $M$. Assume that it is transverse to the zero section of $T^*M$. Then $\omega$ is intrinsically harmonic if and only if (i) $\omega$ does not have any zeros of index 0 or $n$, and (ii) given any two points $p$, $q \in M$ which are not zeros of $\omega$, there exists a path $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$, such that $\omega(\gamma(t))(\dot{\gamma}(t)) > 0$ for all $t \in [0,1]$.*

Let $\Omega^i(M)$ be the space of $i$-forms, and $\Omega^i_\alpha(M)$ be the subspace consisting of $i$-forms in the cohomology class $\alpha \in H^i(M;\mathbf{R})$. Denote by $\mathcal{H}^i_\alpha \subset \Omega^i(M)$ the space of intrinsically harmonic $i$-forms in the class $\alpha$, and let $\widetilde{\mathcal{H}}^i_\alpha \subset \mathcal{H}^i_\alpha$ be the harmonic $i$-forms in $\alpha$ which are transverse to all the strata of $\bigwedge^i(\mathbf{R}^n)^*$ under the action of $SO(n)$. Call elements in $\widetilde{\mathcal{H}}^i_\alpha$ *transverse*. For transversality results for harmonic forms we refer the reader to [8]. Calabi's theorem implies the following:

**Proposition 1** *$\widetilde{\mathcal{H}}^1_\alpha \subset \Omega^1_\alpha(M)$ is an open subset.*

1991 *Mathematics Subject Classification.* Primary: 58A10, 58A14, 58A15, 53C15.

In the case of 2-forms, the situation is quite subtle. There is no known analog of Calabi's theorem for 2-forms, and an intrinsic characterization of harmonic 2-forms is rather elusive. In this paper we prove an openness theorem for transverse harmonic 2-forms on a 4-manifold, which will illustrate some of the obstructions which may arise in the general case.

Let $M$ be a closed, oriented 4-manifold with $b_2^{\pm} > 0$. Then the generic harmonic 2-form $\omega$ in the class $\alpha \in H^2(M;\mathbf{R})$ (generic in the space of metrics) is neither self-dual (SD) nor anti-self-dual (ASD) (cf. Section 4.3 of [3]), and is transverse. In particular, recalling the stratification of $\bigwedge^2(\mathbf{R}^4)^*$ under the action of $SO(4)$,

(i) $\omega$ has no zeros.

(ii) The locus where $\omega$ is SD/ASD consists of a union of circles $C = \bigcup S^1$.

(iii) The locus where $\omega$ has rank 2 is a 3-manifold $N$ (possibly disconnected).

Note that $C$ and $N$ are disjoint. For a proof, we refer to [8].

In order to state the theorem, it would be convenient to introduce the following:

**Connectivity Condition:** Let $\{N_j\}_{j=1}^r$ be the set of connected components of $N$. $\omega$ is said to be *semi-contact* on $N_j$ if the pullback to $N_j$ of $*\omega$ is zero. Let $N'$ be the union of all the semi-contact $N_j$. Then $\omega$ satisfies the *connectivity condition* if $M - N'$ is connected.

We then have the following

**Theorem 2** *$\widetilde{\mathcal{H}}^2_\alpha \subset \Omega^2_\alpha$ is open on the set of transverse intrinsically harmonic 2-forms $\omega$ satisfying the connectivity condition.*

On the way to proving this theorem we encounter the cohomology of the singular differential ideal $\mathcal{I} = (*\omega)$, which naturally gives rise to our connectivity condition. We will compute the *infinitesimal harmonic perturbations* of a harmonic form $\omega$ (see Section 2), via the singular differential ideal, and pass from infinitesimal to local using the Nash-Moser iteration technique.

We remark that the SD harmonic 2-forms are quite interesting in their own right - for a discussion see [9].

# 2 Infinitesimal harmonic perturbations

Let $\mathrm{Met}(M)$ be the space of $C^\infty$-metrics on an $n$-dimensional manifold $M$, $T_g\mathrm{Met}(M) = \Gamma(\mathrm{Sym}^2(TM))$ be its tangent space at $g \in \mathrm{Met}(M)$, and $\Omega^i(M)$ consist of $C^\infty$ $i$-forms. Then define

$$\Phi_\alpha : \mathrm{Met}(M) \to \Omega^i_\alpha(M),$$

which sends the metric $g$ to the $i$-form $\omega$ with $\Delta_g\omega = 0$ and $[\omega] = \alpha$.

The derivative of $\Phi_\alpha$ is the *infinitesimal harmonic perturbation map*

$$d\Phi_\alpha(g) : \Gamma(\mathrm{Sym}^2(TM)) \to d\Omega^{i-1}(M),$$

which we shall now compute.

Consider a 1-parameter family $(\omega_t, g_t)$ of harmonic $i$-forms on $M$, with $g_0 = g$, $[\omega_t] \in \alpha$, $h = \frac{d}{dt}g_t|_{t=0}$, and $\eta = \frac{d}{dt}\omega_t|_{t=0}$ exact. We differentiate

$$d\omega_t = 0, d^*\omega_t = 0$$

to obtain

$$\begin{array}{ll} (i) & d\eta = 0, \\ (ii) & d^*\eta = \pm d^*(*\dot{*}_{g_t}\omega). \end{array}$$

The Hodge decomposition gives

$$\Omega^i = d\Omega^{i-1} \oplus d^*\Omega^{i+1} \oplus \mathcal{H}^i,$$

so we find that $\eta = \pm\pi_1(*\dot{*}_{g_t}\omega)$, where $\pi_1$ is the projection onto the $d\Omega^{i-1}$ factor. Hence, $d\Phi_\alpha(g)$ is the composite map

$$\Gamma(\mathrm{Sym}^2(TM)) \stackrel{*\dot{*}\omega}{\to} \Omega^i(M) \stackrel{\pi_1}{\to} d\Omega^{i-1}(M)$$

$$h \mapsto *\dot{*}_{g+th}\omega \mapsto \pi_1(*\dot{*}_{g+th}\omega).$$

Hence, in order to compute the image of $d\Phi_\alpha(g)$, we solve the equation

$$\eta + *\eta' + \mu = *\dot{*}_{g+th}\omega, \tag{1}$$

where the exact form $\eta$ is the given candidate for an infinitesimal harmonic perturbation, and we determine $\eta'$ exact, $\mu$ harmonic, and $h$, the metric perturbation.

# 3 Infinitesimal computation for non-self-dual (or anti-self-dual) harmonic 2-forms on a 4-manifold

Let us now specialize to the 4-manifold $M$ with $b_2^\pm > 0$. We then prove the following microlocal result:

**Theorem 3** *Let $(\omega, g)$ be a transverse harmonic 2-form on $M^4$ in the class $\alpha \in H^2(M;\mathbf{R})$. If $\omega$ satisfies the connectivity condition, then $d\Phi_\alpha(g)$ is surjective, i.e. all the exact 2-forms on $M$ are infinitesimal harmonic perturbations.*

Observe that a transverse harmonic 2-form must necessarily be non-SD/ASD, when both $b_2^+ > 0$ and $b_2^- > 0$.

In order to make use of Equation 1, we must first compute the image of the map

$$i_{\omega(x)} : \mathcal{S} \to \bigwedge^2(\mathbf{R}^4)^*,$$

$$h \mapsto *\dot{*}_{g+th}\omega(x),$$

where $\mathcal{S}$ is the set of symmetric $n \times n$ matrices, and we assume that the bundle $T^*M$ has been trivialized near $x$. In [8] we computed

1. Im $i_{\omega(x)} = 0$, if $\omega(x) = 0$.
2. Im $i_{\omega(x)} = \{$ASD (SD) 2-forms$\}$, if $\omega(x)$ is SD (ASD).
3. Im $i_{\omega(x)} = (*\omega(x))^\perp$, otherwise.

For a transverse 2-form there are no points $x$ where $\omega(x) = 0$. The primary difficulty with the transverse non-SD/ASD harmonic 2-form on a 4-manifold is that Im $i_{\omega(x)}$ is never surjective.

It is most convenient to rewrite Equation 1 as follows: Noting that Im $i_{\omega(x)} \subset (*\omega(x))^\perp$ whenever $\omega(x) \neq 0$, and Im $i_{\omega(x)} = (*\omega(x))^\perp$ in particular when $\omega(x)$ is not SD/ASD, we obtain, after taking $*$,

$$\eta' + *\eta + \mu \perp \omega, \tag{2}$$

where $\perp$ is the pointwise inner product with respect to $g$, and $\mu$ is some harmonic 2-form which may not be the same $\mu$ as in Equation 1. This can be rephrased as

$$(\eta' + *\eta + \mu) \wedge *\omega = 0. \tag{3}$$

We will thus compute the image of $d\Phi_\alpha(g)$ in the following fashion: Fix $\eta \in d\Omega^1(M)$, and solve for $\eta' = d\xi$ and $\mu$ harmonic in Equation 3, where we additionally require on each component $S^1$ of $C$ that $(\eta' + *\eta + \mu)|_{S^1}$ be ASD whenever $\omega|_{S^1}$ is SD (and vice versa). If there exist such $\eta'$ and $\mu$, then, by linear algebra, we can find an $h$ solving Equation 1. Neighborhoods of $C$ require a little care when solving for $h$.

## 3.1 Singular differential ideal

We want to compute the image of the following composite map:

$$\Omega^1(M) \xrightarrow{\mathcal{A}} \Omega^3(M) \xrightarrow{d} \Omega^4(M)$$

$$\xi \mapsto \xi \wedge *\omega \mapsto d(\xi \wedge *\omega) = d\xi \wedge *\omega.$$

We shall relate the image of this map to the cohomology $H^4(M,\mathcal{I})$ of a singular differential ideal, and compute it in this section. Let $\mathcal{I} = (*\omega)$ be the differential ideal generated by $*\omega$. The ideal has the following chain complex:

$$0 \to \mathcal{I}^0 = 0 \to \mathcal{I}^1 = 0 \to \mathcal{I}^2 \to \mathcal{I}^3 \to \mathcal{I}^4 \to 0.$$

Observe that $\mathcal{I}^4 = \Omega^4$: As long as $*\omega$ has no zeros, there exists a 2-form $\xi$ such that $\xi \wedge *\omega = F\omega \wedge *\omega$ for given $F$. Also noting that $\mathcal{I}^3 = \{\xi \wedge *\omega | \xi \in \Omega^1\}$, we have

**Lemma 1** $H^4(M,\mathcal{I}) = \Omega^4(M)/Im\ d \circ \mathcal{A}$.

Hence, our problem is equivalent to computing $H^4(M,\mathcal{I})$ of a *singular differential ideal.*

**Proposition 2** $H^4(U,\mathcal{I}) = H^4(U,\mathbf{R})$*, if* $U \subset \{x \in M | \omega^2(x) \neq 0\}$.

**Proof:** This follows from observing that if $\omega$ is symplectic at $x$, then $\xi \mapsto \xi \wedge *\omega$ gives an isomorphism $\bigwedge^1(\mathbf{R}^4)^* \simeq \bigwedge^3(\mathbf{R}^4)^*$. □

**Corollary 1** *If* $\omega$ *is symplectic, then* $H^4(M,\mathcal{I}) = \mathbf{R}$.

**Corollary 2** *If* $\omega(x)$ *is of generic type for all* $x \in M$*, then* $d\Phi_\alpha(g)$ *is surjective.*

**Proof:** Note that $*\eta \wedge *\omega = \eta \wedge \omega$, with $\eta$ exact. Hence $[*\eta \wedge *\omega] = 0 \in H^4(M;\mathbf{R})$. That is, we can let $\mu = 0$ and solve for $d\xi \wedge *\omega = *\eta \wedge *\omega$, which has a solution $d\xi$ by the proposition. □

Let us now examine $\mathcal{I} = (*\omega)$ near the rank 2 submanifold $N$. Let $I \times N$ be a neighborhood of $N$, with coordinates $(t,x)$. We can write

$$\omega = (\mu_1 + dt \wedge \tilde{\mu}_2) + t(\omega_1 + dt \wedge \tilde{\omega}_2),$$

with $\mu_1$, $\omega_1$ 2-forms, and $\tilde{\mu}_2$, $\tilde{\omega}_2$ 1-forms, all without a $dt$-term. Here $\mu_1$, $\tilde{\mu}_2$ do not depend on $t$.

On $I \times N$, we can solve for $\alpha$ in $d\alpha = *\eta \wedge *\omega$. Since $\alpha$ must satisfy $\alpha = \xi \wedge *\omega$ for some 1-form $\xi$, we require $\alpha|_N = 0$. Let us then modify $\alpha \mapsto \alpha - \delta\alpha$ so that $\alpha - \delta\alpha|_N = 0$. We write

$$\begin{aligned} \alpha &= \alpha_1(t,x) + dt \wedge \tilde{\alpha}_2(t,x) && (4) \\ &= \alpha_1(0,x) + dt \wedge \tilde{\alpha}_2(0,x) + \text{ h.o. in } t. && (5) \end{aligned}$$

Here, $\alpha_1$ is a 3-form and $\tilde{\alpha}_2$ is a 2-form, both without $dt$-terms.

If we let $\delta\alpha(t,x) = \alpha_1(0,x) + d(t\tilde{\alpha}_2(0,x))$, then $(\alpha - \delta\alpha)|_N = 0$; since $\delta\alpha$ is closed, we still have $d(\alpha - \delta\alpha) = *\eta \wedge *\omega$. It is not difficult to see that $(\alpha - \delta\alpha)|_N = 0$ is sufficient to ensure the existence of a $\xi$ such that $\xi \wedge *\omega = \alpha - \delta\alpha$. This follows from the transversality of $\omega$ near $N$. Summarizing,

**Proposition 3** $H^4(I \times N, \mathcal{I}) = 0$.

Having taken care of the local aspects, we can pass from local to global. As before, let $\{N_j\}_{j=1}^r$ be the set of connected components of $N$. $\omega$ is said to be *semi-contact* on $N_j$ if $\omega = \mu_1 + t(\omega_1 + dt \wedge \tilde{\omega}_2)$, with $\mu_1$ nonsingular and closed on $N_j$, i.e. $i^*_{N_j}(*\omega) = 0$, where $i_{N_j} : N_j \to M$ is the inclusion. Let $N'$ be the union of all the semi-contact $N_j$. Then we have the following theorem:

**Theorem 4** $\dim H^4(M, \mathcal{I}) = (\#$ *of connected components of* $M - N')$.

**Proof:** If $[\beta] = 0 \in H^4(M; \mathbf{R})$, then there exists a global $\alpha$ such that $d\alpha = \beta$.

**Claim 1:** If $i^*_{N_j}[\alpha] = 0 \in H^3(N_j; \mathbf{R})$, then we can modify $\alpha$ so that $\alpha|_{N_j} = 0$.

**Proof of Claim 1:** Recall Equation 4 in the proof of Proposition 3. If $i^*_{N_j}[\alpha] = 0$, then we can write $\alpha_1(0, x) = d_3\gamma_j$ on $N_j$. Extend $\gamma_j$ to $I \times N_j$ so that $\gamma_j(t, x) = \gamma_j(0, x)$, and damp $\gamma_j + t\tilde{\alpha}_2(0, x)$ out outside of $I \times N_j$. Finally, modify $\alpha \mapsto \alpha - \delta\alpha$, where $\delta\alpha = d(\gamma_j + t\tilde{\alpha}_2(0, x))$.
□

**Claim 2:** If $M - N'$ is connected, then we can modify $\alpha \mapsto \alpha + \delta\alpha$ with $\delta\alpha \in H^3(M; \mathbf{R})$ so that $i^*_{N_j}[\alpha + \delta\alpha] = 0 \in H^3(N_j; \mathbf{R})$ for all $N_j$ semi-contact.

**Proof of Claim 2:** Consider the exact sequence

$$H^3(M) \xrightarrow{i} H^3(N') \to H^4(M, N') \to H^4(M) \to 0. \tag{6}$$

Since $M - N'$ is connected, $H^4(M, N') \simeq H_0(M - N') \simeq \mathbf{R}$. This implies that $i$ is surjective, and that there exists a $\delta\alpha \in H^3(M; \mathbf{R})$ such that $i^*_{N_j}[\alpha + \delta\alpha] = 0 \in H^3(N_j; \mathbf{R})$ for all $N_j$ semi-contact. □

**Claim 3:** If $i^*_{N_i}[\alpha] = 0$ for all $N_i$ semi-contact, then there exists an $\alpha = \xi \wedge *\omega$ such that $d\alpha = \beta$.

**Proof of Claim 3:** Let $\alpha$ satisfy $d\alpha = \beta$, with the additional condition that $i^*_{N_i}[\alpha] = 0$ for all $N_i$ which is semi-contact. By Claim 1, we may also assume that $\alpha|_{N_i} = 0$ for all $N_i$ semi-contact. Now assume $N_j$ is not semi-contact. Then we can write

$$\omega = (\mu_1 + dt \wedge \tilde{\mu}_2) + t(\omega_1 + dt \wedge \tilde{\omega}_2),$$

with $\tilde{\mu}_2$ not identically zero. Then,

$$*\omega(0, x) = (*_3\tilde{\mu}_2 + dt \wedge *_3\mu_1)(0, x),$$

and there exist $\xi_j(t,x) = c_j f_j \tilde{\mu}_2(0,x)$ on $N_j$ such that

$$\int_{N_j} c_j f_j \tilde{\mu}_2 \wedge *_3 \tilde{\mu}_2 = \int_{N_j} \alpha.$$

We then damp $\xi_j$ outside of $I \times N_j$, and solve for $\xi \wedge *\omega = \alpha - \sum \xi_j \wedge *\omega$, where the sum runs over all non-semi-contact $N_j$. Here, we may need to modify $\alpha$ using Claim 1, so that $(\alpha - \sum \xi_j \wedge *\omega)|_{N_i} = 0$ for every component $N_i$ of $N$. Finally, we can write $\alpha = (\xi + \sum \xi_j) \wedge *\omega$.
□

We will now complete the proof of Theorem 4. Refer back to Equation 6. Observe that $i^*_{N'}[\xi \wedge *\omega] = 0 \in H^3(N')$ if $N'$ is the union of the semi-contact components. Hence, given $\beta$ with $[\beta] = 0 \in H^4(M; \mathbf{R})$, for $\beta = d\alpha$ with $\alpha = \xi \wedge *\omega$ to be satisfied, we need $i^*_{N'}[\alpha] = 0 \in H^3(N')/i(H^3(M))$. This condition is also sufficient, since $i^*_{N'}[\alpha] = 0 \in H^3(N')/i(H^3(M))$ implies that there exists a representative $\alpha$ with $i^*_{N_j}[\alpha] = 0 \in H^3(N_j)$ for all $N_j$ semi-contact, and we can apply Claim 3. Finally, $\dim H^3(N')/i(H^3(M)) = \dim H^4(M, N') - \dim H^4(M) =$ (# of components of $M - N'$) $- 1$. Thus, $\dim H^4(M, \mathcal{I}) =$ (# of connected components of $M - N'$). □

**Remark 1:** We have two differential ideals, $\mathcal{I} = (*\omega)$ and $\mathcal{J} = (\omega)$, whose fates seem interconnected. It would be interesting to find out how they are related.

**Remark 2:** The computations of the singular differential ideals seem generalizable to higher dimensions, provided we have sufficient genericity.

Let us finally close this section with the following:

**Conjecture:** The connectivity condition is non-vacuous, i.e. there exists a transverse intrinsically harmonic form $\omega$ on a manifold $M$ which does not satisfy the connectivity condition. Although we know of no explicit examples where the connectivity condition is necessary, the condition arises in such a natural fashion as a necessary condition for the surjectivity of the derivative map that we suspect that there indeed exist examples.

## 3.2 Analysis near $\bigcup S^1$

In the previous section we saw that, if the connectivity condition is met, then we have a solution to $(\eta' + *\eta + \mu) \wedge *\omega = 0$. Note that we can set $\mu = 0$ since $*\eta \wedge *\omega = \eta \wedge \omega$ is exact on $M$. Under the conditions for Theorem 3, we find that there exist a 1-form $\xi$ such that $d(\xi \wedge *\omega) = *\eta \wedge *\omega$ by Theorem 4, and hence we can set $\eta' = d\xi$.

We now need to perform a more careful analysis near $C = \bigcup S^1$ in order to finish the proof of Theorem 3. Consider a connected component $S^1$ of $C$ and let $N(S^1) = S^1 \times D^3$

have coordinates $\theta, x_1, x_2, x_3$, which are orthonormal at $S^1 \times \{0\}$. Without loss of generality, let $\omega$ be SD on $S^1$. Fix an exact $\eta$, and we will solve for $\eta'$ satisfying $(\eta' + *\eta) \wedge *\omega = 0$ on $S^1 \times D^3$, with the additional constraint that $\eta' + *\eta$ be ASD on $S^1$.

**Lemma 2** *There exists an exact $\eta'_1$ such that $\eta'_1 + *\eta$ is ASD on $S^1$.*

**Proof:** Let $\eta'_1 = -\eta$. Then $\eta$ is exact and $-\eta + *\eta$ is ASD. □

Next, consider

$$\Omega^1(N(S^1)) \xrightarrow{\mathcal{A}} \Omega^3(N(S^1)) \xrightarrow{d} \Omega^4(N(S^1))$$
$$\xi \mapsto \xi \wedge *\omega \mapsto d(\xi \wedge *\omega) = d\xi \wedge *\omega.$$

**Lemma 3** *Given $F\omega \wedge *\omega \in \Omega^4(N(S^1))$ with $F|_{S^1} = 0$, there exists a $\xi \in \Omega^1(N(S^1))$ with $\xi|_{S^1} = 0$ and $d\xi|_{S^1} = 0$ such that $d \circ \mathcal{A}(\xi) = F\omega \wedge *\omega$.*

**Proof:** The key is to find $\alpha = \xi \wedge *\omega$ of the form $\alpha = \tilde{\alpha} \wedge d\theta$ with $\tilde{\alpha} = \sum_i \alpha_i dx_{(i)}$, such that

$$\alpha_j(\theta, 0) = 0, \ \frac{\partial}{\partial x_i}\alpha_j(\theta, 0) = 0, \text{ and } \frac{\partial}{\partial \theta}\alpha_j(\theta, 0) = 0,$$

where $1 \le i, j \le 3$, and $\theta \in S^1$. $d\alpha = d\tilde{\alpha} \wedge d\theta = d_3\tilde{\alpha} \wedge d\theta$, where $d_3$ is the differential with respect to $\{x_i\}$; on the other hand, $F\omega \wedge *\omega = f dx_1 dx_2 dx_3 d\theta$ for some $f$ with $f|_{S^1} = 0$. Thus solving for $d\alpha = F\omega \wedge *\omega$ is equivalent to solving for $\sum_i \frac{\partial \alpha_i}{\partial x_i} = f$. It is clearly advantageous to us that this partial differential equation is very underdetermined. Let $\alpha_2 = \alpha_3 = 0$ on $N(S^1)$. Then $\frac{\partial \alpha_1}{\partial x_1} = f$ can be solved with initial condition $\alpha_1(\theta, 0, x_2, x_3) = 0$. Since $f|_{S^1} = 0$, we can choose $\alpha$ with $\frac{\partial \alpha_i}{\partial x_j}(\theta, 0) = \frac{\partial \alpha_i}{\partial \theta}(\theta, 0) = 0$.

Thus, $\alpha = \xi \wedge *\omega$ has $\alpha(\theta, 0)$ and all of its first partials vanish on $S^1$. Under the linear map $\mathcal{A}^{-1}$, $\alpha$ will get sent to $\xi$, with $\xi(\theta, 0)$ and all of the first partials of $\xi$ equal to zero on $S^1$. Thus, $\xi|_{S^1} = 0$ and $d\xi|_{S^1} = 0$. □

We find an $\eta'_1$ as in Lemma 2 and an $\eta'_2$ as in Lemma 3 such that $(\eta'_2 + \eta'_1 + *\eta) \wedge *\omega = 0$ on $N(S^1)$. Let $\eta'_{N(S^1)} = \eta'_1 + \eta'_2$. This proves the following proposition:

**Proposition 4** *Given any exact 2-form $\eta$, there exists an exact $\eta'_{N(S^1)}$ on $N(S^1)$ such that $\eta'_{N(S^1)} + *\eta$ is ASD on $S^1$ and $(\eta'_{N(S^1)} + *\eta) \wedge *\omega = 0$ on $N(S^1)$.*

Let $\eta$ be an exact 2-form on $M$ as before. On $M$ we have $\eta' = d\xi$ such that $(\eta' + *\eta) \wedge *\omega = 0$, and on $N(C)$ there exists an $\eta'_{N(C)}$ such that $\eta'_{N(C)} + *\eta$ is SD/ASD on the various $S^1$ as appropriate, and satisfies $(\eta'_{N(C)} + *\eta) \wedge *\omega = 0$ on $N(C)$.

Now write $\eta' = d\xi$ and $\eta'_{N(C)} = d\xi_{N(C)}$. Then, $d((\xi - \xi_{N(C)}) \wedge *\omega) = 0$, and $(\xi - \xi_{N(C)}) \wedge *\omega$ must be *exact* on $N(C)$. Write $(\xi - \xi_{N(C)}) \wedge *\omega = d\gamma$ on $N(C)$, with $\gamma$ defined on $N(C)$. Extend $\gamma$ to all of $M$ by damping out outside of $N(C)$. Since $\omega$ is symplectic on Supp$(\gamma)$, we can write $d\gamma = \xi' \wedge *\omega$, and modify $\eta' \mapsto \eta' - d\xi' = d(\xi - \xi')$. Summarizing,

**Proposition 5** *Assume $\omega$ satisfies the connectivity condition. Then given an exact 2-form $\eta$ on $M$, there exists an $\eta' = d\xi$ on $M$ such that $\eta' + *\eta$ is SD/ASD on $C$ and $(\eta' + *\eta) \wedge *\omega = 0$ on $M$.*

It remains to obtain a section $h$ with $**_{g+th}\omega = \eta' + *\eta$. We use the following proposition with $\beta = \eta' + *\eta$ to complete our argument for Theorem 3.

**Proposition 6** *There exists a smooth solution $h$ to the equation $i_\omega(h) = \beta$, provided $\beta|_{S^1}$ is ASD and $\beta \wedge *\omega = 0$ on $N(S^1) = S^1 \times D^3$.*

**Proof:** Decompose $\omega = \omega_+ + \omega_-$ and $\beta = \beta_+ + \beta_-$ into the SD and ASD parts. If $i_\omega(h) = \beta$, then

$$\begin{aligned} i_{\omega_+}(h) &= \beta_- \\ i_{\omega_-}(h) &= \beta_+. \end{aligned}$$

We expand $\omega_1^+ = \omega_+$ to a basis $\{\omega_1^+, \omega_2^+, \omega_3^+\}$ for the SD forms near $S^1$. Since $T_g\mathrm{Met}(M) \simeq \mathrm{Hom}(\Lambda^+, \Lambda^-)$, in order to specify $h$ it suffices to specify

$$\begin{aligned} \omega_1^+ &\mapsto \beta_1^- = \beta_- \\ \omega_2^+ &\mapsto \beta_2^- \\ \omega_3^+ &\mapsto \beta_3^- \end{aligned}$$

in a manner consistent with $\omega_- \mapsto \beta_+$.

**Claim:** $h : \Lambda^+ \oplus \Lambda^- \to \Lambda^- \oplus \Lambda^+$ satisfies $\langle h(\alpha_+), \alpha_- \rangle = -\langle \alpha_+, h(\alpha_-) \rangle$, where $\alpha_\pm \in \Lambda^\pm$.

The claim is an easy exercise. We then see that the consistency condition is $\langle \beta_i^-, \omega_- \rangle = -\langle \omega_i^+, \beta_+ \rangle$, or, equivalently, $\beta_i^- \wedge \omega_- = \omega_i^+ \wedge \beta_+$. We check that $\beta \wedge *\omega = 0$ implies $(\beta_+ + \beta_-) \wedge (\omega_+ - \omega_-) = \beta_+ \wedge \omega_+ - \beta_- \wedge \omega_- = 0$, giving us $\beta_- \wedge \omega_- = \omega_+ \wedge \beta_+$.

Let us now show that there exist $\beta_2^-$, $\beta_3^-$ satisfying the consistency conditions. Write $\omega_- = \sum_l x_l \omega_l^-$ and $\beta_i^- = \sum_j b_{ij} \omega_j^-$, $i = 2, 3$, where $\{\omega_1^-, \omega_2^-, \omega_3^-\}$ is a basis for $\Lambda^-$ on $N(S^1)$, $\omega_i^- \wedge \omega_j^- = a_{ij} dv_{N(S^1)}$, and $dv_{N(S^1)}$ is the volume form on $N(S^1)$. Then

$$\begin{aligned} \beta_i^- \wedge \omega_- &= \sum_{jl} b_{ij} \omega_j^- \wedge x_l \omega_l^- = \sum_{jl} b_{ij} a_{jl} x_l dv_{N(S^1)} \\ \omega_i^+ \wedge \beta_+ &= \sum_l r_{il} x_l dv_{N(S^1)} \text{ for some } r_{il}, \end{aligned}$$

and solving for $\beta_i^-$ in $\beta_i^- \wedge \omega_- = \omega_i \wedge \beta_+$ would be tantamount to solving for $b_{ij}$ in $\sum_l b_{ij} a_{jl} = r_{il}$. But here $a_{ij}$ is invertible since $\{\omega_1^-, \omega_2^-, \omega_3^-\}$ is a basis for $\Lambda^-$. □

This completes the proof of Theorem 3.

### 3.3 Analysis near $N$

Although it is not necessary for our theorem, it is instructive to study the neighborhood $I \times N$ of $N$. Assume $N$ is connected and the metric $g$ on $I \times N$ is the product metric for simplicity. Take coordinates $(t, x)$ on $I \times N$. Write

$$\omega = (\mu_1 + dt \wedge *_3\mu_2) + t(\omega_1 + dt \wedge *_3\omega_2),$$

where $\mu_1$, $\mu_2$ do not depend on $t$, $\omega_1$, $\omega_2$ depend on $t$, and $\mu_i$, $\omega_i$ are all 2-forms without a $dt$-term. Write $d$, $*$ on $N$ as $d_3$, $*_3$.

It turns out that $\omega_1$, $\omega_2$ are completely determined by $\mu_1$, $\mu_2$ because of the harmonicity $(d\omega = 0,\ d * \omega = 0)$.

**Proposition 7** *$\omega_1$ and $\omega_2$ are given by*

$$\omega_1(t,x) = \frac{1}{t}\left(\frac{e^{(d_3*_3)t} + e^{-(d_3*_3)t}}{2} - 1\right)\mu_1 + \frac{1}{t}\left(\frac{e^{(d_3*_3)t} - e^{-(d_3*_3)t}}{2}\right)\mu_2,$$

$$\omega_2(t,x) = \frac{1}{t}\left(\frac{e^{(d_3*_3)t} - e^{-(d_3*_3)t}}{2}\right)\mu_1 + \frac{1}{t}\left(\frac{e^{(d_3*_3)t} + e^{-(d_3*_3)t}}{2} - 1\right)\mu_2,$$

*provided $e^{\pm(d_3*_3)t}(\mu_1)$ and $e^{\pm(d_3*_3)t}(\mu_2)$ make sense.*

**Proof:** (A) $d\omega = 0$ implies

(1) $d_3\mu_1 = -td_3\omega_1$.

(2) $t\dot{\omega}_1 + \omega_1 = d_3 *_3 \mu_2 + td_3 *_3 \omega_2$.

(B) $d * \omega = 0$ implies

(3) $d_3\mu_2 = -td_3\omega_2$.

(4) $t\dot{\omega}_2 + \omega_2 = d_3 *_3 \mu_1 + td_3 *_3 \omega_1$.

Observe that (1), (3) imply that $d_3\mu_1 = d_3\mu_2 = d_3\omega_1 = d_3\omega_2 = 0$ because the $\mu_i$ are $t$-independent.

Let us first integrate (2) and (4) using $(tf)' = tf'(t) + f(t) = h(t)$ as the model, with $f(t) = \frac{1}{t}\left(c + \int_0^t h(s)ds\right)$ as its general solution. If we require $f(0)$ to be finite, $c = 0$, and we have $f(t) = \frac{1}{t}\int_0^t h(s)ds$. Thus,

$$\begin{aligned}
\omega_1(t,x) &= \frac{1}{t}\int_0^t \left[d_3 *_3 \mu_2(s,x) + sd_3 *_3 \omega_2(s,x)\right] ds \\
&= d_3 *_3 \mu_2(0,x) + \frac{1}{t}\int_0^t sd_3 *_3 \omega_2(s,x)ds, \\
\omega_2(t,x) &= d_3 *_3 \mu_1(0,x) + \frac{1}{t}\int_0^t sd_3 *_3 \omega_1(s,x)ds.
\end{aligned}$$

Plugging $\omega_1$ into the right-hand side of $\omega_2$ (and vice versa), and iterating, we obtain

$$\begin{aligned}
\omega_1(t,x) &= (d_3{*_3})\mu_2 + \frac{t}{2}(d_3{*_3})^2\mu_1 + \frac{t^2}{6}(d_3{*_3})^3\mu_2 + \ldots \\
&= \frac{1}{t}\left(\frac{e^{(d_3*_3)t}+e^{-(d_3*_3)t}}{2} - 1\right)\mu_1 + \frac{1}{t}\left(\frac{e^{(d_3*_3)t}-e^{-(d_3*_3)t}}{2}\right)\mu_2 \\
\omega_2(t,x) &= \frac{1}{t}\left(\frac{e^{(d_3*_3)t}-e^{-(d_3*_3)t}}{2}\right)\mu_1 + \frac{1}{t}\left(\frac{e^{(d_3*_3)t}+e^{-(d_3*_3)t}}{2} - 1\right)\mu_2
\end{aligned}$$ □

**Example:** (Contact case) This is the situation where $\omega = \mu_1 + t(\omega_1 + dt \wedge *_3\omega_2)$, with $*_3\mu_1 = \xi$, a contact 1-form, and $d_3 *_3 \mu_1 = d\xi = \mu_1$. Then we obtain

$$\begin{aligned}
\omega &= (e^t + e^{-t})\mu_1 + (e^t - e^{-t})dt \wedge *_3\mu_1 \\
&= d((e^t + e^{-t})\xi).
\end{aligned}$$

# 4 Local considerations

In this section, $\mathrm{Met}(M)$ and $\Omega^2_\alpha(M)$ are Frèchet spaces of smooth sections, with a *grading* given by Hölder norms $|\cdot|_{C^k}$. With the help of the Nash-Moser iteration technique, we now pass from the microlocal computation to a local statement:

**Theorem 5** *$\Phi_\alpha$ is surjective near an $(\omega, g)$ which satisfies the connectivity condition.*

It is evident that Theorem 5 implies Theorem 2. Theorem 5, in turn, follows from the following:

**Theorem 6** *Let $g_0 \in Met(M)$ be a metric for which $(\omega_0 = \Phi_\alpha(g_0), g_0)$ satisfies the connectivity condition. Then there exist constants $C_k > 0$ and $\delta > 0$ with the following property: Given $\eta \in d\Omega^1$ and $|g - g_0|_1 \leq \delta$, there exists an $h \in \Gamma(Sym^2(TM))$ such that $d\Phi_\alpha(g)(h) = \eta$ and $|h|_{k-2} \leq C_k(|\eta|_k + |\eta|_0|g|_k)$.*

Theorem 6 implies Theorem 5 by the Nash-Moser iteration process, which we describe in the next two sections.

## 4.1 Tame maps

We will use the notion of tame maps between tame Frèchet manifolds, following R. Hamilton [7]. We refer the reader to [7] for definitions and a thorough discussion. Note that a *smooth tame map* $L : F \to G$ of tame Frèchet manifolds is a tame map all of whose derivatives are tame.

Let $V$, $W$ be vector bundles over $M$, and $\Gamma(V)$, $\Gamma(W)$ be tame Frèchet spaces of $C^\infty$-sections over $M$. Consider $D^r(V,W)$, whose sections are differential operators of degree $r$ from $V$ to $W$. Locally we can write a differential operator of degree $r$ as

$$L(\phi)(f) = \sum_{|\alpha|\leq r} \phi_\alpha(D_\alpha f).$$

Here $\alpha$ is a multiindex $(\alpha_1, ..., \alpha_n)$ and $D_\alpha = \partial^{\alpha_1}\cdots\partial^{\alpha_n}$. We can think of $\phi = \{\phi_\alpha\}$ as a section of $D^r(V,W)$. Then we have a map

$$L : \Gamma(D^r(V,W)) \times \Gamma(V) \to \Gamma(W),$$

$$(\phi, f) \mapsto L(\phi)(f).$$

**Proposition 8** *$L$ is a smooth tame map.*

Now consider an open set $U \subset \Gamma(D^r(V,W))$ consisting of $\phi = \{\phi_\alpha\}$ such that $L(\phi)$ is elliptic and invertible. Then we have

$$L^{-1} : U \times \Gamma(W) \to \Gamma(V),$$

$$(\phi, g) \mapsto [L(\phi)]^{-1}(g).$$

**Proposition 9** *$L^{-1}$ is a smooth tame map of degree $-r$.*

**Proposition 10** *$\Phi_\alpha : Met(M) \to \Omega^i_\alpha(M)$ is a smooth tame map of degree 0.*

**Proof:** By the previous proposition,

$$L^{-1} : U \times \Omega^i(M) \to \Omega^i(M)$$

is a smooth tame map of degree $-2$, where $U \subset \Gamma(D^2(\Lambda^i, \Lambda^i))$ consists of elliptic and invertible degree 2 operators.

Now, consider the inclusion

$$\mathrm{Met}(M) \times \mathbf{C} \to \Gamma(D^2(\bigwedge^i, \bigwedge^i)),$$

$$(g, \lambda) \mapsto \Delta_g + \lambda,$$

which is a smooth tame map of degree 2. Since the composition of tame maps is tame, we have

$$G : ((\mathrm{Met}(M) \times \mathbf{C}) \bigcap U) \times \Omega^i(M) \to \Omega^i(M)$$

$$[(g, \lambda), \omega] \mapsto G_g(\lambda)\omega \overset{def}{=} (\Delta_g + \lambda)^{-1}\omega$$

is a smooth tame map of degree 0. Next, consider

$$\Pi : \mathrm{Met}(M) \times \Omega^i(M) \to \Omega^i(M)$$

$$(g, \omega) \mapsto \pi_g(\omega),$$

where $\pi_g : \Omega^i(M) \to \mathcal{H}^i_g$ is the orthogonal projection onto the harmonic space $\mathcal{H}^i_g$. $\Pi$ is a smooth tame map because

$$\pi_g(\omega) = -\frac{1}{2\pi i}\int_C G_g(\lambda)\omega d\lambda,$$

and $C \subset \mathbf{C}$ can be fixed on a small neighborhood of $g$. Finally, composing $\Pi$ with

$$i : \mathrm{Met}(M) \to \mathrm{Met}(M) \times \Omega^i(M)$$

$$g \to (g, \omega_0),$$

we find that $\Phi_\alpha$ is a smooth tame map of degree 0. □

## 4.2 Nash-Moser iteration scheme

The following is the version of Nash-Moser that we will use:

**Theorem 7 (Nash-Moser)** *Let $F$, $G$ be tame Frèchet spaces and $U \subset F$ an open set. Suppose $L : U \to G$ is a smooth tame map, $dL(f)$ is surjective for all $f \in U$, and there exists a family of right inverses $(dL)^{-1} : U \times G \to F$ which is a tame map. Then $L$ is locally surjective.*

We already know that $\Phi_\alpha : \mathrm{Met}(M) \to \Omega^2_\alpha(M)$ is a smooth tame map and that $d\Phi_\alpha$ is surjective near $(\omega_0, g_0)$. The conditions

$$|h|_{k-2} \leq C(|\eta|_k + |\eta|_0|g|_k) \tag{7}$$

would assure us that $(dL)^{-1}$ is tame. Applying the Nash-Moser iteration process, we see that Theorem 6 would imply Theorem 5.

## 4.3 Estimates

We will prove Estimates 7 above by carefully retracing the argument in Theorem 3. Keep in mind that $|g - g_0|_1 \leq \delta$ throughout.

The following interpolation lemma is useful in our estimates:

**Lemma 4 (Interpolation)** *If $f_1$, $f_2$ are functions on a compact manifold $X$, then*

$$|f_1 f_2|_k \leq C(|f_1|_0|f_2|_k + |f_1|_k|f_2|_0).$$

In the proof of Theorem 4, we first solve for $d\alpha = *\eta \wedge *\omega$. Noting that $|\omega|_k \leq C(1+|g|_k)$ since $\Phi_\alpha$ is smooth tame of degree 0, we obtain bounds

$$|d\alpha|_k \leq C(|\eta|_k|g|_0 + |g|_k|\eta|_0) \leq C(|\eta|_k + |\eta|_0|g|_k)$$

by interpolation.

**Lemma 5** *Given an exact i-form $\beta$ on a compact manifold $X$, there exists an $\alpha \in \Omega^{i-1}(X)$ such that $d\alpha = \beta$ and $|\alpha|_{k+1} \leq C|\beta|_k$.*

**Proof:** We make use of the Green's function $G_{g_0}$ at $g_0$, and write $\alpha = d^{*_0}G_{g_0}\beta$. $d\alpha = \beta$, and

$$|\alpha|_{k+1} = |d^{*_0}G_{g_0}\beta|_{k+1} \leq C|G_{g_0}\beta|_{k+2} \leq C|\beta|_k. \qquad \square$$

Thus, there exists an $\alpha$ such that $d\alpha = *\eta \wedge *\omega$ and $|\alpha|_{k+1} \leq C(|\eta|_k + |\eta|_0|g|_k)$.

**Claim 2 bounds:** Next, we bound the $\alpha$ modified as in Claim 2 of Theorem 4. Observe that, as long as $|g - g_0|_1 \leq \delta$, for $\delta$ small, $|\omega - \omega_0|_1$ is small, and the harmonic form remains transverse. Hence, the rank 2 subsets $N$ remain submanifolds, and are close together, provided the $|g - g_0|_1$ are kept small.

Take a basis $\{[dv_{N_i}]\}$ for $H^3(N')$, with $dv_{N_i}$ a volume form of unit volume on $N_i$. Let $[\gamma_i] \in H^3(M)$ satisfy $i^*_{N_i}[\gamma_j] = [\delta_{ij}dv_{N_i}]$. Fix representatives $\gamma_i \in [\gamma_i]$. Then $\delta\alpha = -\sum_i a_i\gamma_i$, with $|\gamma_i|_{k+1}$ fixed constants, and $a_i = \int_{N_i}\alpha$. Hence,

$$|\delta\alpha|_{k+1} \leq C\sum_i |\alpha|_0|\gamma_i|_{k+1} \leq C|\alpha|_0 \leq C(|\eta|_k + |\eta|_0|g|_k).$$

**Claim 3 bounds:** We now have bounds for $\alpha$, where $d\alpha = *\eta \wedge *\omega$ and $i^*_{N_i}[\alpha] = 0$ for all $N_i$ semi-contact. Take $N_j$ not semi-contact, and we first estimate $\xi_j$ on $I \times N_j$. $\xi_j(t,x) = c_j f_j \tilde{\mu}_2(0,x)$, with $\int_{N_j} \xi_j \wedge *_3\tilde{\mu}_2 = \int_{N_j}\alpha$, where we are using the same $f_j\tilde{\mu}_2(0,x) = f_j\tilde{\mu}_2(g_0)(0,x)$ for all $|g - g_0| \leq \delta$, and we are simply varying the scaling factor $c_j$. Thus,

$$|\xi_j|_{k+1} \leq C|\alpha|_0|\omega_0|_{k+1} \leq C|\alpha|_0,$$

on $I \times N_j$.

We now give bounds for the damping out process. Let $\phi(t)$ be a smooth function on $\mathbf{R}$ such that

$$\phi(t) = \begin{cases} 1 & \text{on } [\frac{-1}{2}, \frac{1}{2}] \\ 0 & \text{outside } [-1,1] \end{cases},$$

and $0 \leq \phi(t) \leq 1$ on $[-1, \frac{-1}{2}] \bigcup [\frac{1}{2}, 1]$.

Then, modify $\xi_j \mapsto \xi_j\phi$. We find that

$$|\xi_j\phi|_{k+1} \leq C(|\xi_j|_{k+1}|\phi|_0 + |\phi|_{k+1}|\xi_j|_0) \leq C|\xi_j|_{k+1},$$

since $\phi$ is fixed throughout. With this new $\xi_j$,

$$|\alpha - \sum \xi_j \wedge *\omega|_{k+1} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

**Claim 1 bounds:** We may now assume that $i^*_{N_i}[\alpha] = 0$ for all $N_i$. We then modify $\alpha \mapsto \alpha - \delta\alpha$ so that $(\alpha - \delta\alpha)|_{N_i} = 0$. If we write $\alpha_1(0,x) = d_3\gamma_j$ on $N_j$, then

$$|\gamma_j|_{k+2,N} \leq C|\alpha|_{k+1}$$

by Lemma 5. However, we can only bound $|\gamma_j + t\tilde{\alpha}_2(0,x)|_{k+1} \leq C|\alpha|_{k+1}$ because of the term $t\tilde{\alpha}_2$ - we lose one derivative here unless we are careful.

Instead, use $\psi_\varepsilon(t)\tilde{\alpha}_2(0,x)$, where

$$\psi_\varepsilon(t) = \begin{cases} t & \text{on } [-\varepsilon, \varepsilon] \\ 0 & \text{outside } [-1,1] \end{cases},$$

and $\psi_\varepsilon$ damps out slowly to 0 on $[-1,-\varepsilon]\bigcup[\varepsilon,1]$. It is not difficult to see that for $\varepsilon$ small, there exist $\psi_\varepsilon$ with $|\psi_\varepsilon|_0$ arbitrarily small, and $|\psi_\varepsilon|_i \leq |\psi|_i$, for $i > 1$, where

$$\psi(t) = \begin{cases} t & \text{on } [-\frac{1}{2}, \frac{1}{2}] \\ 0 & \text{outside } [-1,1] \end{cases},$$

and $\psi$ damps out slowly to 0 on $[-1,-\frac{1}{2}]\bigcup[\frac{1}{2},1]$. $\psi_\varepsilon(t)\tilde{\alpha}_2(0,x)$ will clearly do the job of $t\tilde{\alpha}_2(0,x)$, with the advantage that we can find an $\varepsilon$ (dependent on $g$) with

$$|\psi_\varepsilon\tilde{\alpha}_2(0,x)|_{k+2} \leq C|\alpha|_{k+1},$$

$$|\delta\alpha|_{k+1} = |d(\gamma_j + \psi_\varepsilon(t)\tilde{\alpha}_2(0,x))|_{k+1} \leq C|\alpha|_{k+1}.$$

As before, we do not lose any derivatives by damping out $\gamma_j + \psi_\varepsilon(t)\tilde{\alpha}_2(0,x)$.

Thus,

$$|\alpha - \delta\alpha|_{k+1} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

**Bounds for $\eta'$:** Finally, $\alpha|_{N_i} = 0$ for all $N_i$, and we solve for $\xi \wedge *\omega = \alpha$. We do not lose any derivatives where $\mathcal{A}$ is an isomorphism. However, near the $N_i$'s we lose one derivative, i.e.

$$|\xi|_k \leq C(|\eta|_k + |\eta|_0|g|_k),$$

and

$$|\eta'|_{k-1} \le C|d\xi|_{k-1} \le C(|\eta|_k + |\eta|_0|g|_k).$$

**Estimates near $S^1$:** On $N(S^1)$, we have bounds

$$|\eta'|_k \le |\xi|_{k+1} \le C(|\eta|_k + |\eta|_0|g|_k).$$

Let us compute bounds on $\eta'_{N(S^1)}$ and $\xi_{N(S^1)}$. $\eta'_1 = -\eta$, so $|\eta'_1|_k \le C|\eta|_k$. For bounds on $\eta'_2$ satisfying $\eta'_2 \wedge *\omega = -(\eta'_1 + *\eta) \wedge *\omega$ on $N(S^1)$ and $-(\eta'_1 + *\eta) \wedge *\omega|_{S^1} = F\omega \wedge *\omega|_{S^1} = 0$, we look to the proof of Lemma 3. Clearly, $|F|_k \le C(|\eta|_k + |\eta|_0|g|_k)$. Solving for $\alpha = \tilde{\alpha} \wedge d\theta$ with $d\alpha = F\omega \wedge *\omega$, we have

$$|\alpha|_k \le C|F|_k \le C(|\eta|_k + |\eta|_0|g|_k),$$

and hence

$$|\xi_{N(S^1)}|_k \le C(|\eta|_k + |\eta|_0|g|_k).$$

Note that we have lost one derivative - had we worked a bit harder, that would not have been necessary, unlike the loss of derivative near $N_i$, which seems inherent to the problem.

Finally, we write $d\gamma = (\xi - \xi_{N(C)}) \wedge *\omega$ on $N(C)$. By compactifying $S^1 \times D^3$ to $S^1 \times S^3$, for example, we can use Lemma 3 and obtain a $\gamma$ satisfying

$$|\gamma|_{k+1} \le C|(\xi - \xi_{N(C)}) \wedge *\omega|_k \le C(|\eta|_k + |\eta|_0|g|_k).$$

Damping $\gamma$ out, we do not lose any derivatives, and hence

$$|\eta' - d\xi|_{k-1} \le C(|\eta|_k + |\eta|_0|g|_k).$$

In order the complete the proof of Theorem 6, we are left to prove:

**Lemma 6** *There exists an $h$ on $M$ such that $|h|_{k-1} \le C(|\eta|_k + |\eta|_0|g|_k)$.*

**Proof:** Consider $h$ away from $N(S^1)$. Since $\eta + *\eta' = \{h, \omega\}$, the anticommutator of $h$ and $\omega$ viewed as matrices, and $i_\omega$ has constant rank throughout, we are able to bound

$$|h|_{k-1} \le C(|\eta + *\eta'|_{k-1}|\omega|_0 + |\eta + *\eta'|_0|\omega|_{k-1}) \le C(|\eta|_k + |\eta|_0|g|_k),$$

by interpolation.

We next find $h$ on $N(S^1)$. Writing $\beta = \eta' + *\eta$ and $\beta = \beta_+ + \beta_-$,

$$|\beta_1^-|_{k-1} = |\beta_-|_{k-1} \le |\beta|_{k-1} \le C(|\eta|_k + |\eta|_0|g|_k).$$

$\beta_2^-$, $\beta_3^-$ come from solving $\beta_i^- \wedge \omega_- = \omega_i^+ \wedge \beta_+$. Hence,

$$|r_{il}|_{k-2} \leq C|\omega_i^+ \wedge \beta_+|_{k-1} \leq (|\eta|_k + |\eta|_0|g|_k),$$

and we lose a derivative. Since $b_{ij}a_{jl} = r_{il}$, we have

$$|\beta_i^-|_{k-2} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

Hence $|h|_{k-2} \leq C(|\eta|_k + |\eta|_0|g|_k)$ on $N(S^1)$. We finally interpolate the $h$ that we find on $N(S^1)$ to the $h$ on $M - N(S^1)$, while keeping $|h|_{k-2} \leq C(|\eta|_k + |\eta|_0|g|_k)$. □

*Acknowledgements:* I would like to thank Phillip Griffiths for his interest and support throughout this project.

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