A note on Morse theory of harmonic 1-forms

Ko Honda

Revised Jan. 20, 1998

In this paper we prove the general form of the theorem conjectured by Farber, Katz, and Levine in [2] regarding allowable Morse singularities of harmonic 1-forms. Let M be a closed, connected, oriented n-manifold. Calabi, in [1], gave an intrinsic characterization for a closed 1-form to be harmonic, which we will describe presently.

Call a closed 1-form ω *generic* if ω, as a section of T^*M, is transverse to the zero section. This is equivalent to the critical points of f being Morse, where f is any local function with $df = \omega$. Using a Sard argument it can be shown that the 'generic' closed 1-form is transverse to the zero section - it is true even for harmonic 1-forms (cf. [4]). We may talk about the *index* of each zero of ω - this is the Morse index of any local function f satisfying $df = \omega$.

Theorem 1 (Calabi) *Given a generic closed 1-form ω, there exists a metric g with respect to which ω is harmonic, if and only if (1) ω does not have any zeros of index 0 or n, and (2) given any two points p, q on M, which are not zeros of ω, there exists a path $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$ such that $\omega(\gamma(t))(\dot\gamma(t)) > 0$ for all $t \in [0,1]$.*

We shall call such a closed 1-form an *intrinsically harmonic* 1-form, and a positive path as in (2) an ω*-positive* path. Using Calabi's characterization of intrinsically harmonic 1-forms, we prove the following theorem:

Theorem 2 *Let ω_0 be a generic closed 1-form with no zeros of index 0 or n. Then there exists a family of generic closed 1-forms ω_t, $t \in [0,1]$, with $[\omega_t] \in H^1(M;\mathbf{R})$ fixed, such that ω_1 is intrinsically harmonic and each ω_t has the same number of zeros of each index.*

This theorem tells us that, in studying the Morse theory of closed 1-forms as in [6], the assumption of harmonicity does not give rise to additional constraints regarding the number of critical points.

We would like to emphasize the relationship to four-dimensional geometry. The generic self-dual (or anti-self-dual) harmonic 2-form μ on a closed, oriented 4-manifold X is symplectic away from a disjoint union of circles, on which μ vanishes. (For a proof of this fact, see [4].) We view the circles as an obstruction to the existence of a symplectic form on

X. Note that the primary obstruction is bundle-theoretic (i.e. the existence of an almost symplectic or almost complex structure), but even if this is satisfied, there is still an obstruction to the existence of a symplectic structure, which can presently be detected only by Seiberg-Witten invariants. Let $\#\mu$ be the number of circles on which μ vanishes, ignoring sign.

On $X = M \times S^1$, with M a 3-manifold, we can construct an S^1-invariant self-dual (SD) harmonic 2-form $\mu = *_3\omega + \omega \wedge d\theta$, where ω is a harmonic 1-form on M, $*_3$ is the $*$-operator on M, and the metric on X is split. In this situation, the zero set of μ is $\bigcup_p \{p\} \times S^1$, where p runs over all the zeros of the harmonic 1-form ω, and $\#\mu = \#\omega$, where we define $\#\omega$ to be the (unsigned) number of zeros of ω. We then conjecture the following:

Conjecture 1 *If $X = M \times S^1$, then, given $\mu = *_3\omega + \omega \wedge d\theta$, with ω a harmonic 1-form on M,*

$$\min_{\tilde{\mu} \in [\mu] \text{ SD}} (\#\tilde{\mu}) = \min_{\widetilde{\omega} \in [\omega]} (\#\widetilde{\omega}),$$

where we assume that $\tilde{\mu}$ and $\widetilde{\omega}$ have regular zeros. Here, $[\cdot]$ denotes cohomology classes.

Theorem 2 tells us that

$$\min_{\widetilde{\omega} \in [\omega]} (\#\widetilde{\omega}) = \min_{\widetilde{\omega} \in [\omega] \text{ harmonic}} (\#\widetilde{\omega}),$$

in the case $[\omega] \in H^1(M; \mathbf{Q})$, and where we assume that the $\widetilde{\omega}$ are generic. This follows from a standard cancellation argument for index 0 and index n, found in [5].

We now turn to the proof of Theorem 2. Observe that if ω is harmonic with respect to g, given any local function f with $df = \omega$, f is harmonic and, by the maximum principle, ω cannot have any zeros of index 0 or n.

Proof: First note that the closed form $\omega = \omega_0$ gives rise to an $(n-1)$-dimensional foliation consisting of integral submanifolds of ω, away from the zeros of ω. For a closed 1-form ω with zeros, define a *leaf* L_p through $p \in M$ by $L_p = \{x \in M |\, \exists$ a smooth path $\gamma : [0,1] \to M$ from p to x with $\omega(\gamma(t))(\dot{\gamma}(t)) = 0$ for all $t\}$. If a leaf does not pass through a zero of ω, it is called a *nonsingular leaf*; otherwise, it is a *singular leaf*. If L is a singular leaf, then let the *components* of L be the closures in L of the connected components of L restricted to $M - \{x | \omega(x) = 0\}$.

The proof then breaks up into the following components:

(1) The compact leaf case. Here we assume all the leaves of ω are compact, and reduce the problem to a problem in graph theory.

(2) Decompose M into two components M_c and M_∞, consisting of (roughly speaking) the compact leaves and the noncompact leaves, respectively.

(3) The general case. The noncompact leaf case is treated in [2]. Using this, and the same methods from (1), we obtain the general result.

We note that this method of proof was suggested in [2], and some of the methods and results from [2] will be repeated here for the sake of completeness. In particular, the idea of using graphs and self-indexing in (1), all of (2), and the noncompact leaf case in (3) are all due to [2]

(1) **Compact leaf case**

Let us first assume that all the leaves of ω are compact. If we introduce the equivalence relation $\sim$, where $p \sim q$ if and only if $L_p = L_q$, then $\Gamma = M/\sim$ is a graph. Critical points of index i, $1 < i < n-1$ do not give rise to true vertices of the graph Γ, since the surgeries corresponding to passing such critical points do not change the connectivity of the leaves. Near a critical point of index 1 we have the situation as in Figure 1, whereas near a critical point of index $n-1$, we have the situation as in Figure 2, provided (i) $n \geq 3$, (ii) ω does not have more than 1 critical point on each leaf, and (iii) the leaves which locally come together or split off were not parts of the same global leaf. When $n = 2$, at a point of index 1, both Figure 1 and Figure 2 are possible. The arrows represent the direction of increase for the local Morse function (f such that $df = \omega$).

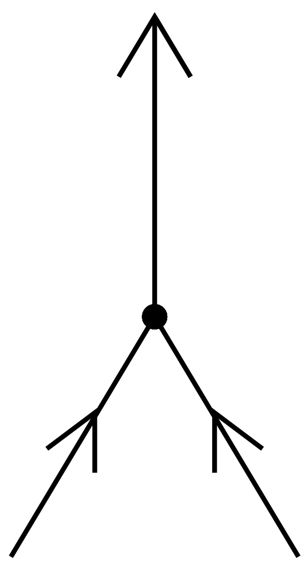

Figure 1

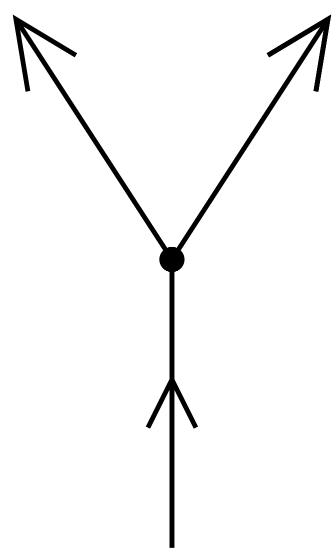

Figure 2

Perturb ω slightly in its cohomology class, so that each leaf contains at most one zero of ω, while keeping all the leaves of ω still compact. Then, $\Gamma = M/\sim$ can be viewed as a trivalent (= each vertex has exactly 3 edges), directed graph, and we can assign weights to each edge: If γ is an edge from p to q, then its weight is the ω-length of any path from $\pi^{-1}(p)$ to $\pi^{-1}(q)$ sitting inside $\pi^{-1}(\gamma)$, where $\pi : M \to \Gamma = M/\sim$. These weights represent $\pi_*\omega$ on $\Gamma = M/\sim$, i.e. they give a cohomology class $[\pi_*\omega]$ on Γ. Thus Γ is a collection $(V(\Gamma), E(\Gamma), W)$ of vertices, directed edges, and weights $W : E(\Gamma) \to \mathbf{R}^+$.

Note also that the vertices have either (i) two incoming edges and one outgoing, or (ii) one incoming and two outgoing, from our previous discussion, i.e. no vertices as in Figure 3.

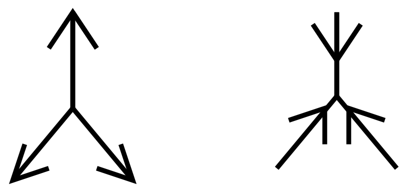

Figure 3

We introduce the two kinds of operations we shall be using:

(A_1). Graph modification to increase connectivity. Whenever we have a subgraph of the type pictured on the left-hand side of Figure 4, we may replace it with one on the right-hand side, provided that $\epsilon < d$.

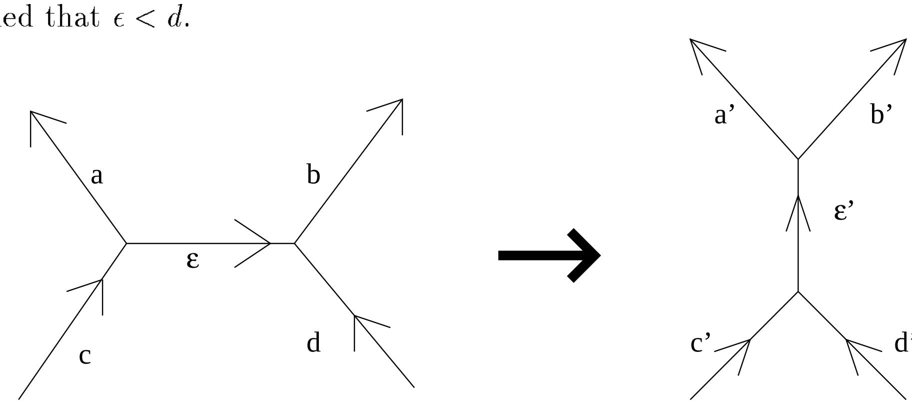

Figure 4

This modification corresponds to the self-indexing procedure for Morse functions, and clearly does not alter the homotopy type of the graph. In order to preserve the cohomology class, we need:

(i) $a + c = a' + \epsilon' + c'$
(ii) $b + d = b' + \epsilon' + d'$
(iii) $a - \epsilon + d = a' + \epsilon' + d'$
(iv) $b + \epsilon + c = b' + \epsilon' + c'$
(v) $d - \epsilon - c = d' - c'$
(vi) $b + \epsilon - a = b' - a'$

Upon a moment's consideration, these six equations are linear combinations of three of them, say (i), (iv), (v).

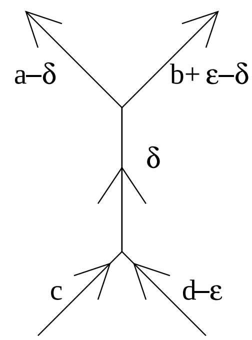

Figure 5

A possible solution is given in Figure 5, provided ϵ is small, i.e. $< d$. Here δ can be any positive number $\leq \min(a, b + \epsilon)$. Of course, the key point in solving for a', b', c', d', ϵ' was that they were *positive* numbers.

In terms of Morse theory, if f is a local function with $\omega = df$ on this subgraph, and c_1 and c_2 are the two critical points of index $n - 1$ and 1 respectively, then we may reduce $f(c_2)$ so that $f(c_2) - f(c_1) < a$, so long as $d > \epsilon$. We can then raise $f(c_1)$ using the same self-indexing procedure and obtain $f(c_2) < f(c_1)$. See Figure 6.

(A_2). Graph modification as in Figure 7. Morse-theoretically, this corresponds to reversing the heights of the two critical points of index 1. $d > \epsilon$ is sufficient for the modification to be valid. See Figure 8 for a Morse-theoretic interpretation.

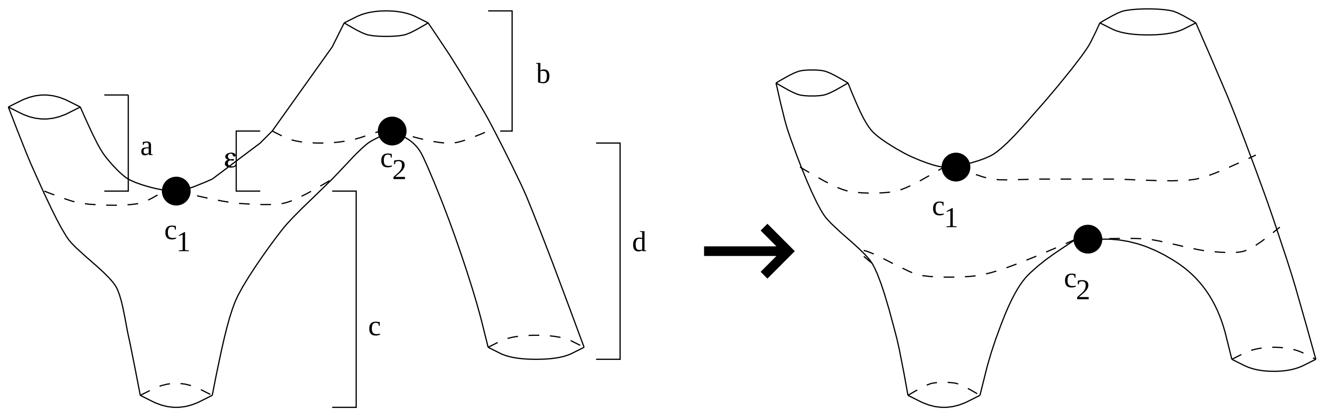

Figure 6

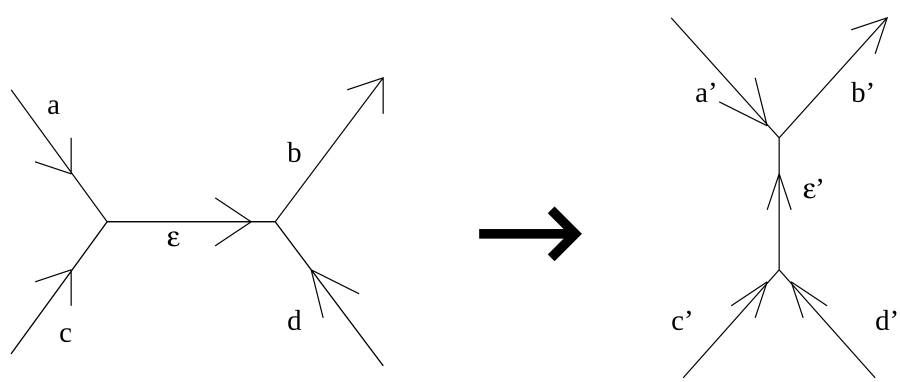

Figure 7

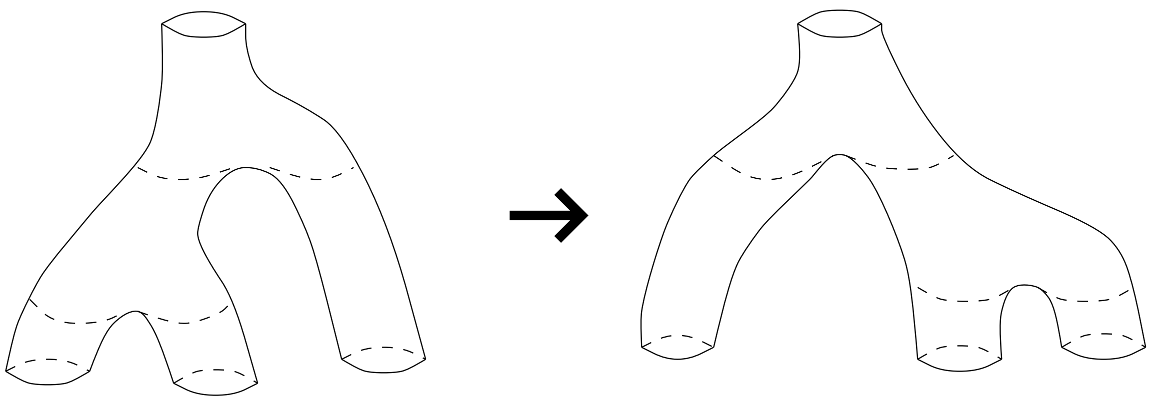

Figure 8

(B). We may alter the weights of the edges, provided the weights remain positive, and the total weight around each closed path remains the same (hence preserving the cohomology class). This corresponds to shortening a Morse function $f : f^{-1}([a,b]) \to [a,b]$ by excising $f^{-1}([r,r']) \subset f^{-1}([a,b])$ and regluing, as long as there are no critical values in $[r,r']$ (or its reverse process).

Observe that all the above operations can be performed through a 1-parameter family of closed 1-forms in the same cohomology class.

We shall modify the graph Γ using the above operations and inductively increasing the 'connectivity'. Pick a vertex p on $\Gamma = M/\sim$. Let U_p be the 'upland' of p, that is, the subgraph consisting of all the vertices and edges that can be reached from p by traveling in the positive direction. Define the 'lowland' L_p in a similar fashion. We stipulate that $p \notin U_p$ unless there is a positive path from p back to itself. (Same for p in L_p.)

It is sufficient to show that there exists a q such that $U_q = L_q = \Gamma$. We find it most convenient to use a q such that $q \in U_q$ (and hence $q \in L_q$). Indeed, start with p and keep traveling in some positive direction until we return to a vertex on the path. Simply take q to be this point.

Operation (B) may not appear practical, until we couple it with the following observation:

Key Observation: If $U_q \neq \Gamma$, then there exist disjoint subgraphs U_q and U_q', with $V(U_q) \bigcup V(U_q') = V(\Gamma)$ and $E(U_q) \bigcup E(U_q') \bigcup \{\text{edges between } U_q \text{ and } U_q'\} = E(\Gamma)$, such that all the edges between U_q and U_q' are directed from U_q' to U_q.

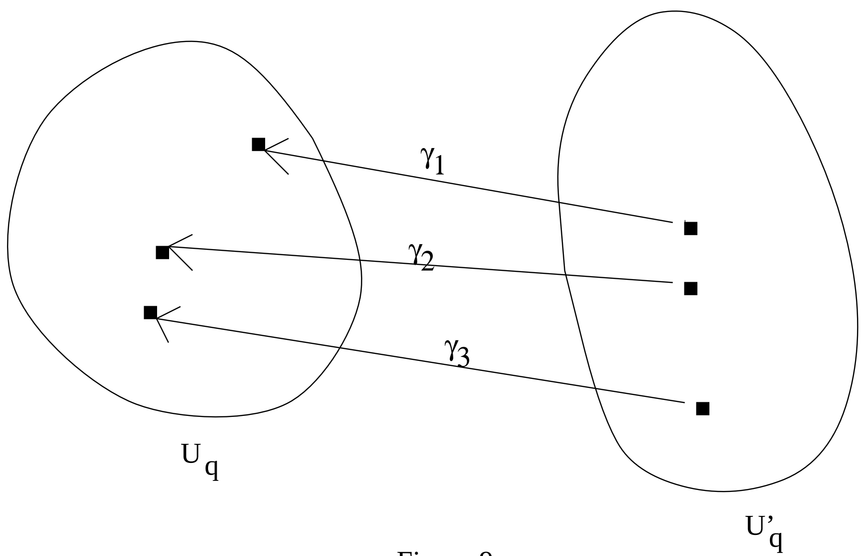

Figure 9

Let γ_1,..., γ_s be the edges from U'_q to U_q. Then, modifying each of the $W(\gamma_i)$ by the same constant (that is, modifying $W(\gamma_i) \mapsto W(\gamma_i) + C$ for some C) will not change the cohomology class represented by ω; this is due to the fact that each closed loop in Γ must traverse the same number of times from U'_q to U_q as from U_q to U'_q.

Hence, we may now apply (B) and assume without loss of generality that $W(\gamma_1) = \epsilon$ where ϵ can be made as small as we want.

There are two possibilities for Γ near γ_1:

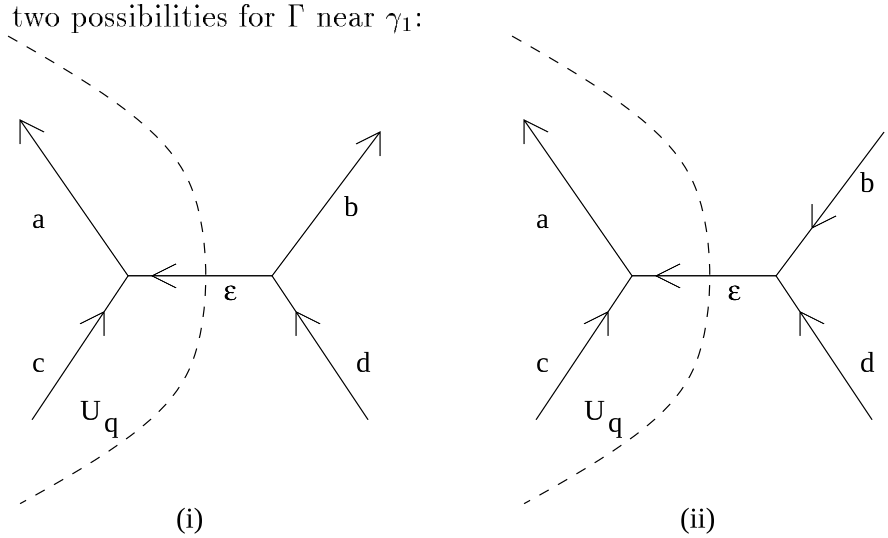

Figure 10

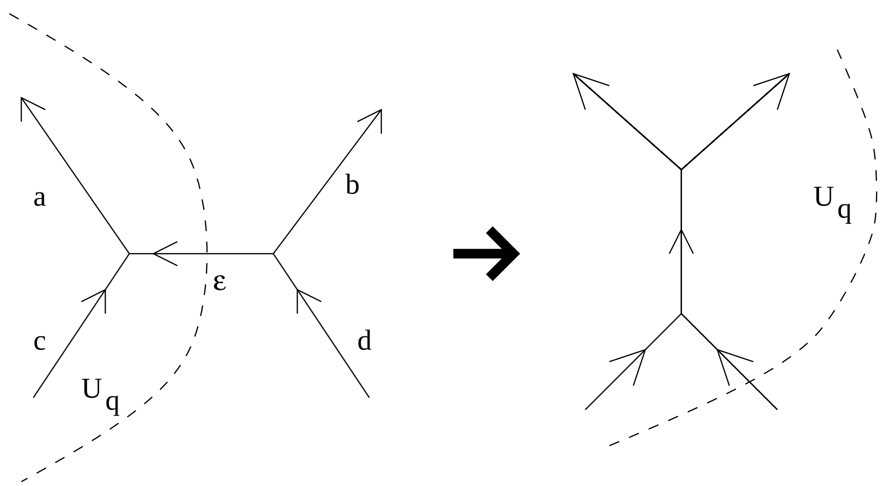

Figure 11

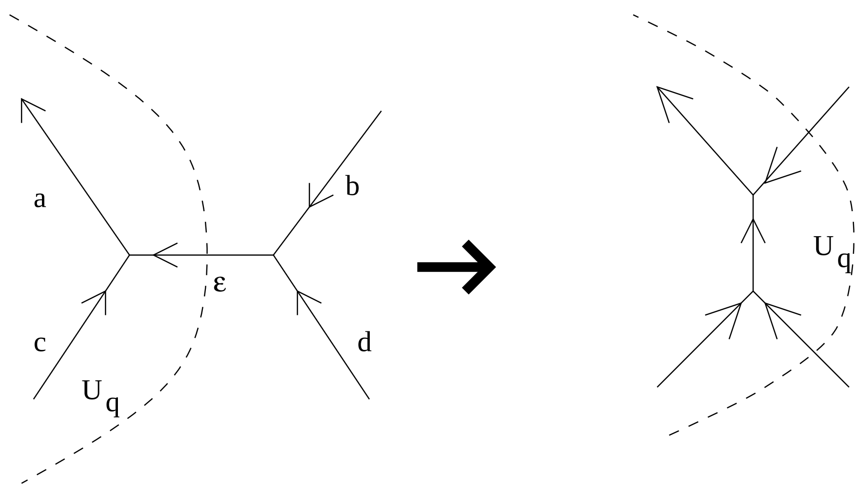

Figure 12

(i) Since ϵ can be made arbitrarily small, we simply make $\epsilon < c$, apply (A_1), and increase the number of vertices in U_q as in Figure 11.

(ii) Apply (A_2) with $\epsilon < c$. See Figure 12.

Since the number of vertices in U_q increases after each modification (whereas the total number of vertices of Γ remains the same), by induction, we can find a modified Γ with $U_q = \Gamma$. Next, apply the same procedure for L_q - here we simply note that neither (A_1) nor (A_2) decreases the circulation, and modifying Γ to make L_q larger will not affect $U_q = \Gamma$.

(2) **Compact and noncompact leaves**

Here we repeat much of the discussion in [2], included for the convenience of the reader.

Let ω now be a generic closed 1-form. We describe how M is split into M_c, which is roughly the union of compact leaves, and M_∞, roughly the union of noncompact leaves.

First, if L is a compact leaf, then there exists an open neighborhood of L consisting solely of compact leaves. This is because we can integrate ω to give f with $df = \omega$ near L. Hence, the union of compact leaves in M is open.

Next, the union of all compact leaves (nonsingular or singular), as well as compact components of noncompact singular leaves, is closed in M. This is because all the above leaves and leaf components are closed leaves when restricted to $M - \{x | \omega(x) = 0\}$. We use a result of Haefliger's (c.f. [3]): Given a codimension 1 foliation on a (not necessarily closed) manifold N with dim $H^1(N; \mathbf{R})$ finite, the union of closed leaves is closed.

These tell us that $M = M_c \bigcup M_\infty$, where

$$M_c = \overline{\bigcup_{L_\alpha \text{ compact}} L_\alpha}, \quad M_\infty = \overline{\bigcup_{\substack{L_\alpha \text{ nonsingular} \\ \text{noncompact}}} L_\alpha}$$

are both n-dimensional submanifolds with (non-smooth) boundary, and $\partial M_c = \partial M_\infty = M_c \bigcap M_\infty$ is a union of compact components of noncompact leaves.

Lemma 1 *There exists a small perturbation of ω in its cohomology class so that each leaf contains at most one zero of ω.*

Proof: It suffices to modify ω by $\sum_i df_i$, where f_i is a compactly supported function near a zero p_i. Consider $\int \omega : H_1(M; \mathbf{Z}) \to \mathbf{R}$, and let $S = \mathrm{Im}(\int \omega)$. Since S is a countable set in $\mathbf{R}$, there exist f_i, new zeros $\tilde{p}_i$, and paths γ_{ij} from $\tilde{p}_i$ to $\tilde{p}_j$ such that $\omega(\gamma_{ij}) \notin S$. This prevents $\tilde{p}_i$ and $\tilde{p}_j$ from lying on the same leaf of $\omega + \sum_i df_i$. □

Lemma 2 *If all the zeros of ω lie on distinct leaves, then $\partial M_c = \partial M_\infty$ is the union of all compact components of singular noncompact leaves.*

Proof: Consider $\omega = df$ near the singular point of a singular noncompact leaf with a compact component. Without loss of generality assume f has index 1 and $f(x_1, ..., x_n) = -x_1^2 + \sum_{i=2}^n x_i^2$ near the zero of ω. If we integrate (in the negative direction) a gradient flow emanating from the compact component, we see that all the neighboring leaves of ω 'below' the compact component are compact nonsingular leaves. On the other hand, all the leaves 'above' the singular noncompact leaf will be noncompact and all the leaves 'below' the noncompact component of the noncompact leaf are also noncompact. □

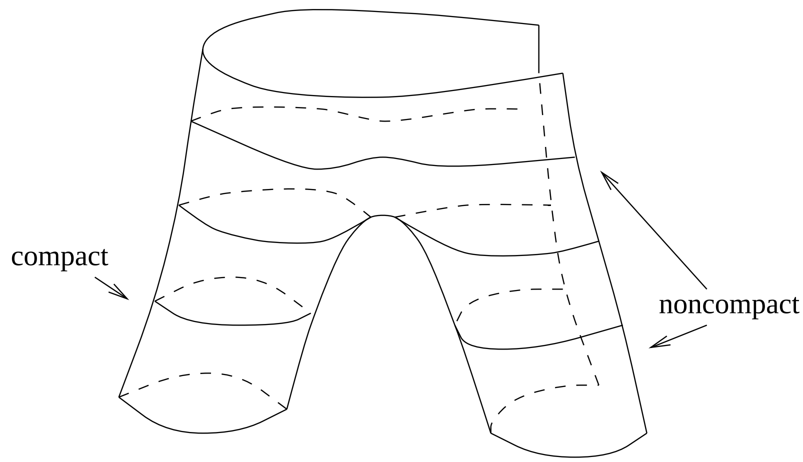

Figure 13

(3) **General result**

We start with a lemma:

Lemma 3 (Farber-Katz-Levine) *If L is a nonsingular noncompact leaf of ω, then there is an ω-positive path between any two points of L.*

Proof: We invoke the standard argument used to find a closed transversal to a non-closed leaf L. Let $p \in M$ be a limit point of L. Then there exists a (distinguished) neighborhood $U \subset M$ of p, with respect to which $L \bigcap U$ consists of hyperplanes converging to the hyperplane containing p. Which also means that there exist q_1 and q_2 in L and an ω-positive path from q_1 to q_2. Now take a path on L from any r_1 to q_1, the ω-positive path from q_1 to q_2, and some path on L from q_2 to r_2, and perturb so that the composite path is ω-positive and smooth. $\square$

Let ω be a generic closed 1-form with no zeros of index 0 or n. We may assume that all the zeros of ω lie on distinct leaves by Lemma 1.

Lemma 4 *The interior of each connected component of M_∞ is ω-transitive (i.e. we can get from any point to any other via an ω-positive path, with the possible exception of zeros of ω.)*

Proof: Let M' be a component of M_∞, and pick $p \in \text{int}(M')$, the interior of M'. Then consider the *upland* $U_{p,M'} \subset M'$ of p, defined as follows: $U_{p,M'} = \{q \in M' | \exists$ an ω-positive path from p to $q\}$. Using the same methods as in Lemma 3, we find that $\partial U_{p,M'}$ must be compact, hence a union of compact components of singular noncompact leaves. Thus, if we assume that the zeros of ω lie on distinct leaves, then Lemma 2 implies that $\overline{U_{p,M'}} = M'$. Since this holds for all $p \in \mathrm{int}(M')$, we have proved the lemma. $\square$

Hence, we can graph-theoretically represent each connected component of M_∞ by a single vertex, and take $\Gamma = M/\sim$, where $\sim$ is the equivalence relation: (i) $p \sim q$ if $p, q \in L$, L compact, and (ii) $p \sim q$ if p, q belong to the same connected component of M_∞. Note that the resulting graph is no longer trivalent, and we must introduce new characters - the subgraphs pictured below, where a black dot denotes a component of M_∞:

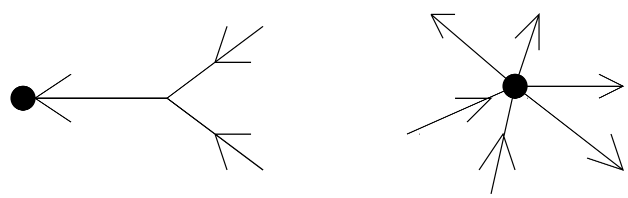

Figure 14

Let us start with $p \in M$ on a nonsingular noncompact leaf L. The same graph-theoretic maneuvers carry over to this case with a few differences. Whenever one of the critical points sits in a noncompact singular leaf (say the higher critical point), we have the following situations:

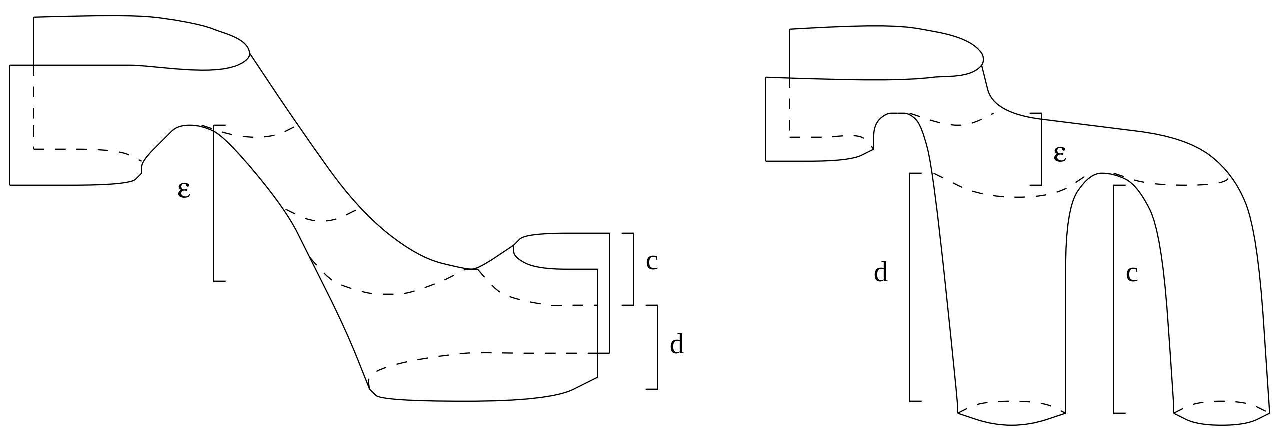

Figure 15

In these cases, as long as ϵ can be made small, moving a critical point of higher index above one of lower index is easily done. This is because one only needs to modify the Morse function in an arbitrarily small neighborhood of the trajectories (for a gradient-like flow into and out of a critical point) in order to change the height of the critical point.

Let us introduce additional graph-theoretic operations in the case one of the critical points sits in a noncompact singular leaf. The analogs of (A_1) are:

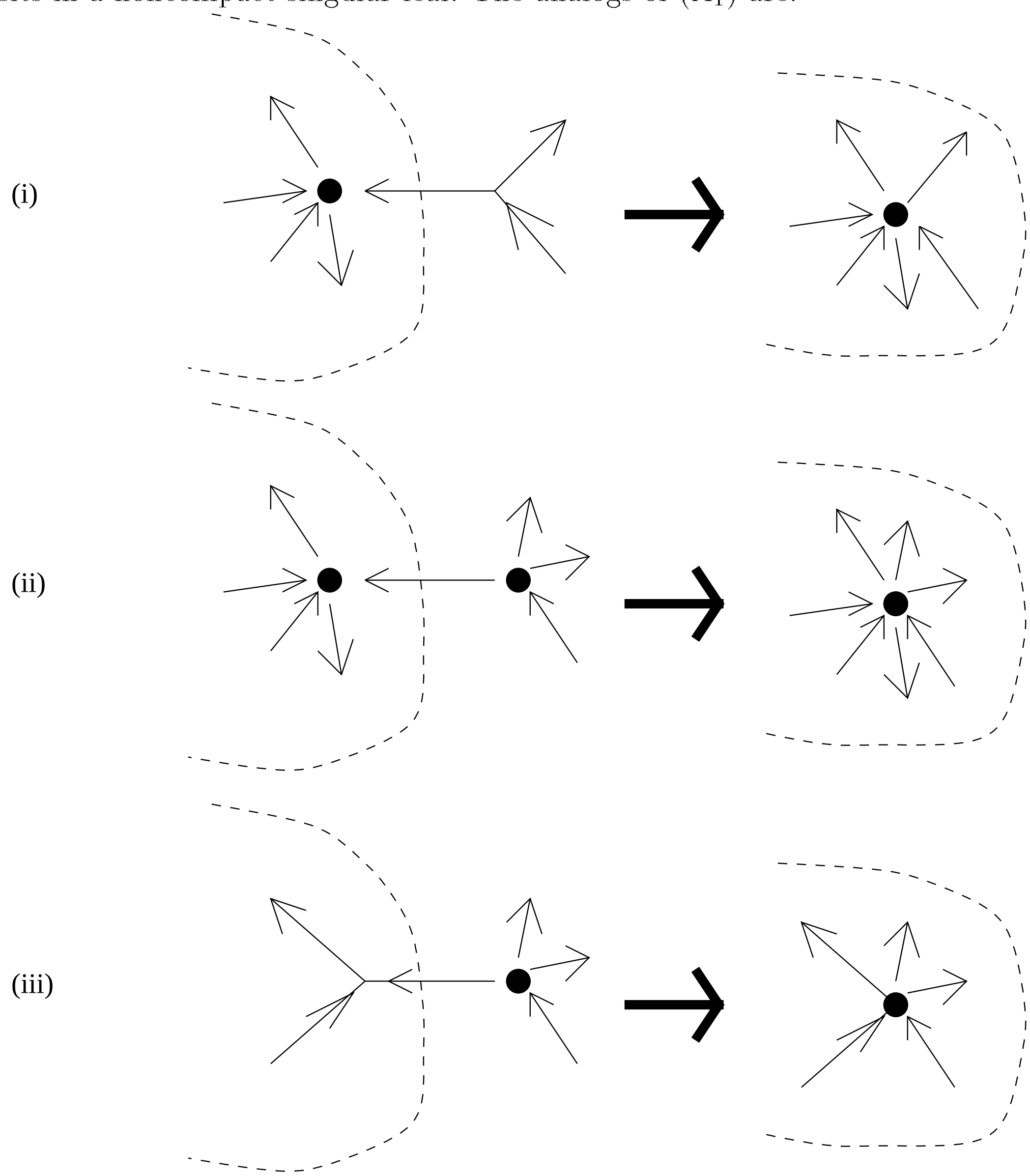

Figure 16

depending on whether the upper critical point sits on a compact singular leaf or a noncompact singular leaf (and the same for the lower critical point.)

The analogs of (A_2) are:

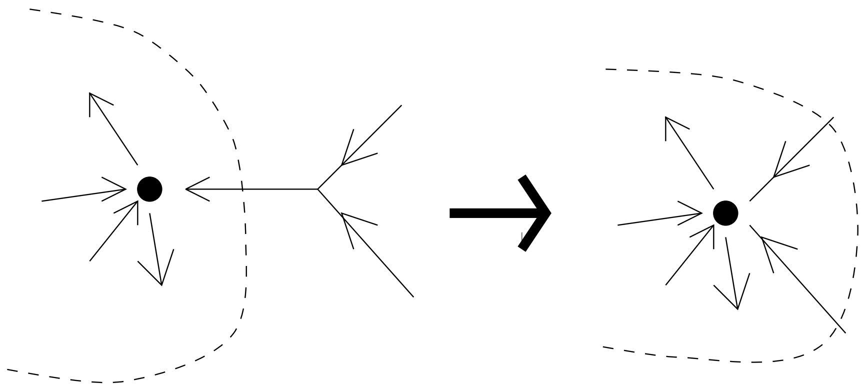

Figure 17

as well as (ii) and (iii) of Figure 16.

The above modifications decrease the total number of vertices in Γ by 1, while keeping the number of vertices in U_q at least the same. On the other hand, the modifications in the compact leaf case preserve the total number of vertices of Γ, while increasing the number of vertices in U_q. By induction on $\#V(\Gamma) - \#V(U_q)$, we complete the proof of Theorem 2. $\square$

Acknowledgements: I would like to thank my advisor Phillip Griffiths for his constant encouragement and support, and Derek Smith for his help on graph theory.

References

[1] E. Calabi: *An intrinsic characterization of harmonic one-forms,* Global Analysis (Papers in Honor of K. Kodaira), pp. 101-117, Princeton University Press, 1969.

[2] M. Farber, G. Katz, J. Levine: *Morse theory of harmonic forms,* to appear in Topology.

[3] A. Haefliger: *Variétés feuilletées*, Ann. Scuola Norm. Pisa **16** (1962), pp. 367-397.

[4] K. Honda: *Transversality theorems for harmonic forms,* preprint.

[5] J. Milnor: *Lectures on the h-cobordism theorem*, Princeton University Press, 1965.

[6] S. Novikov: *Multivalued functions and functionals. An analogue of the Morse theory,* Soviet Math. Doklady **24** (1981), pp. 222-226.

Mathematics Department
Duke University
Durham, NC 27708, USA
honda@math.duke.edu