Homework sheet, version 10/30/11

Each problem is worth 10 points unless stated otherwise.

(1) Show that the induced topology and the quotient topology satisfy the axioms of a topological space.

(2) Prove that $S^1 = \{x^2 + y^2 = 1\}$ with the induced topology (from $\mathbb{R}^2$) is homeomorphic to $S^1 = [0,1]/\sim$ with the quotient topology.

(3) Let $\phi : V \to W$ be a linear map between vector spaces (over a fixed ground field k). Prove the following:
  (a) $\mathrm{Ker}(\phi)$ is a vector subspace of V.
  (b) $\mathrm{Im}(\phi)$ is a vector subspace of W.

(4) Let $V \subset W$ be a vector subspace. Prove that the quotient W/V is a vector space.

(5) Let V^* (called the *dual of* V) be the set of linear maps from V to the base field k. Prove that V^* is a vector space over k. Assuming V is finite-dimensional, exhibit a basis for V^* in terms of a basis for V.

(6) Read pp. 15–34 of Spivak, *Calculus on Manifolds*.

(7) Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$, then there is a unique L which satisfies

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - L(h)|}{|h|} = 0.$$

(8) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by:

$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}}, & (x,y) \neq 0, \\ 0, & (x,y) = 0. \end{cases}$$

Show that f is not differentiable at $(0,0)$.

(9) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by:

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq 0, \\ 0, & (x,y) = 0. \end{cases}$$

Show that $\frac{\partial f}{\partial y}(x,0) = x$ for all x and $\frac{\partial f}{\partial x}(0,y) = -y$ for all y. Then show that $\partial_x \partial_y f(0,0) \neq \partial_y \partial_x f(0,0)$.

(10) Prove that S^1 (with either topology in Problem (2)) is a topological manifold.

(11) Give an example of a Hausdorff, second countable topological space which is not a topological manifold.

(12) Prove that $S^n = \{x_0^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$ is a smooth n-dimensional manifold, by taking stereographic projections.

(13) Complete the proof that $\mathbb{RP}^n$ is a smooth n-dimensional manifold.

(14) Define $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{(0,\dots,0)\})/\sim$, where $(z_0,\dots,z_n) \sim (tz_0,\dots,tz_n)$, $t \in \mathbb{C} - \{0\}$. Prove that $\mathbb{CP}^n$ is a smooth $2n$-dimensional manifold. (Recall that $\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2$, where $z = x + iy \mapsto (x,y)$.)

(15) Complete the proof that $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a smooth manifold of dimension 2.

(16) Prove that $S^n = \{x_0^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$ can be given the structure of an n-dimensional manifold by showing it is a regular value of some map.

(17) Prove that if M, N are manifolds, $f : M \to N$ is a submersion, and $U \subset M$ is open, then $f(U)$ is open in N.

(18) Show that if M is compact and N is connected, then every submersion $f : M \to N$ is surjective. Also show that there is no submersion from a compact manifold to $\mathbb{R}^n$.

(19) Define $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$, where $M_n(\mathbb{R})$ is the set of $n \times n$ matrices with real entries. Prove that $SL(n, \mathbb{R})$ is a submanifold of $M_n(\mathbb{R})$. What is the dimension of $SL(n, \mathbb{R})$?

(20) Let N be an n-dimensional submanifold of a manifold M of dimension m and let $x \in N$. Prove that there exists an open set U of M containing x and a local coordinate system $\{x_1, \ldots, x_m\}$ on U such that $U \cap N = \{x_{n+1} = 0, \ldots, x_m = 0\}$.

(21) Let $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = I\}$, be the *orthogonal group*, where A^T is the transpose of A. Consider the map

$$\phi : M_n(\mathbb{R}) \to \text{Sym}(n), \quad A \mapsto AA^T,$$

where Sym(n) is the set of symmetric $n \times n$ matrices (i.e., $B^T = B$).

(a) Show that Sym(n) is a manifold. Compute its dimension.

(b) Compute the derivative of ϕ and show that ϕ is a submersion.

(c) Prove that $O(n)$ is a submanifold of $M_n(\mathbb{R})$. What is the dimension of $O(n)$?

(d) Prove that $O(n)$ is compact.

(22) Let $F(x_1, \ldots, x_n)$ be a *homogeneous function* of degree d in n real variables, i.e.,

$$F(tx_1, \ldots, tx_n) = t^d \cdot F(x_1, \ldots, x_n).$$

(a) Prove Euler's identity:

$$\sum_{i=1}^{n} x_i \cdot \frac{\partial F}{\partial x_i} = d \cdot F.$$

(b) Prove that the set $F^{-1}(a)$, $a \neq 0$, is a submanifold of $\mathbb{R}^n$.

(23) (30 points) Give a detailed proof of the equivalence of the three definitions of T_pM given in class. Pay special attention to good exposition.

(24) (20 points) Recall that $\mathcal{F}_p$ is the set of germs of functions on a manifold M which vanish at $p \in M$. Let $\mathcal{F}_p^k$ be the ideal of $C^\infty(p)$ generated by $f_1 \cdots f_k$, where $f_i \in \mathcal{F}_p$. (This means that every element of $\mathcal{F}_p^k$ is a sum $\sum_i g_i f_{i1} \cdots f_{ik}$, $g_i \in C^\infty(p)$, $f_{ij} \in \mathcal{F}_p$.)

(a) Prove that, in every coordinate system $(x_1, \ldots, x_n)$, an element $f \in \mathcal{F}_p^k$ has a Taylor expansion which vanishes up to order k.

(b) Compute the dimension of $\mathcal{F}_p^k / \mathcal{F}_p^{k+1}$.

(c) Construct a smooth manifold $E \xrightarrow{\pi} M$ whose fiber at $p \in M$ is $\mathcal{F}_p^1/\mathcal{F}_p^3$. (This involves writing down coordinate charts and computing transition functions.)

(25) Consider the cotangent bundle $\pi : T^*M \to M$. In class we gave an atlas for T^*M in terms of $\pi^{-1}(U_\alpha)$, where $\{U_\alpha\}$ was an atlas for M. Compute the Jacobian for the transition functions on the overlaps $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$.

(26) Prove that $d(fg) = fdg + gdf$.

(27) Consider the map $i : S^1 = [0, 2\pi]/(0 \sim 2\pi) \to \mathbb{R}^2$, $\theta \mapsto (\cos\theta, \sin\theta)$. Compute $i^*((x^2 + y)dx + (3 + xy^2)dy)$.

(28) In class we defined the *derivative map* as follows: Let $\phi : M \to N$ be a smooth map between manifolds. Then the derivative $\phi_* : T_pM \to T_{\phi(p)}N$ is given by $X \mapsto X \circ \phi^*$, where $X : C^\infty(p) \to \mathbb{R}$ is a derivation and ϕ^* is the pullback $C^\infty(\phi(p)) \to C^\infty(p)$. Give an equivalent definition for ϕ_* in terms of Definition 1 of a tangent space.

(29) Let $\phi : M \to N$ be a smooth map between manifolds. Prove that the following diagram commutes:

$$\begin{array}{ccc} \Omega^0(N) & \stackrel{\phi^*}{\to} & \Omega^0(M) \\ d\downarrow & \circlearrowleft & \downarrow d \\ \Omega^1(N) & \stackrel{\phi^*}{\to} & \Omega^1(M) \end{array}$$

(30) Let $\phi : [a, b] \to [c, d]$ be a diffeomorphism with coordinates s for $[a, b]$ and t for $[c, d]$. A global 1-form ω on $[c, d]$ can be written as $f(t)dt$, for some smooth function $f(t)$.
   (a) Write $\phi^*\omega$ in terms of coordinates s on $[a, b]$.
   (b) Now define the *integral* of ω on $[c, d]$ to be $\int_{[c,d]} \omega = \int_c^d f(t)dt$. Similarly define $\int_{[a,b]} \phi^*\omega$. Prove that

$$\int_{[c,d]} \omega = \int_{[a,b]} \phi^*\omega.$$

(31) Prove that $SO(2)$ consists of the 2×2 matrices $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, where $\theta \in \mathbb{R}$. Show that $SO(2)$ is diffeomorphic to S^1.

(32) Explain why $SO(n)$ is a Lie group. Explain why $GL(n, \mathbb{C})$ is a Lie group.

(33) Show that $GL(n, \mathbb{C}) \subset GL^+(2n, \mathbb{R})$.

(34) Let $f : M \to N$ and $g : L \to M$ be smooth maps between smooth manifolds, and let ω be a 1-form on N. Then prove that $(f \circ g)^*\omega = g^* \circ (f^*\omega)$.

(35) Let V, W be finite-dimensional $\mathbb{R}$-vector spaces.
   (a) Prove that $Hom(V, W)$ is an $\mathbb{R}$-vector space.
   (b) Show that $\dim(Hom(V, W)) = \dim V \cdot \dim W$.
   (c) Find a natural linear map $V^* \otimes W \to Hom(V, W)$ and prove that it is an isomorphism. It follows that $V \otimes W$ has dimension $\dim V \cdot \dim W$. (Here, a *natural isomorphism* is an isomorphism which does not depend on a choice of basis.)

(36) Let V, W, and U be $\mathbb{R}$-vector spaces. Prove that $V \otimes W$ is naturally isomorphic to $W \otimes V$ and that $(V \otimes W) \otimes U$ is naturally isomorphic to $V \otimes (W \otimes U)$.

(37) Let V be a 2-dimensional vector space with basis $\{v_1, v_2\}$ and $A : V \to V$ be a linear map given by $v_1 \mapsto 5v_1 + 6v_2$, $v_2 \mapsto 3v_1 + 2v_2$. Then write a matrix for $A \otimes A : V \otimes V \to V \otimes V$ in terms of the basis $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$.

(38) Let V be a vector space of dimension n. Show that $\bigwedge^n V \simeq \mathbb{R}$ as follows:

(a) Find an alternating multilinear form $\phi : V \times \cdots \times V \to \mathbb{R}$, ($n$ copies of V). (Prove it's alternating.)

(b) Explain how to use the universal property to show $\bigwedge^n V \simeq \mathbb{R}$.

Note that ϕ is the determinant map, up to a normalization.

(39) Let V be a vector space. Show there exists a (well-defined) linear map $\bigwedge^k V \otimes \bigwedge^l V \to \bigwedge^{k+l} V$ which sends $(v_1 \wedge \cdots \wedge v_k) \otimes (w_1 \wedge \cdots \wedge w_l) \mapsto v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l$.

(40) If V is a finite-dimensional vector space of dimension n, then show there exists a natural isomorphism $\bigwedge^{n-k} V \simeq (\bigwedge^k V)^*$.

(41) Let M be a manifold. Prove that d satisfies the formula $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$, where $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$.

(42) Let $\phi : M \to N$ be a smooth map between manifolds and $\omega \in \Omega^k(N)$. With respect to local coordinates $x_1, \dots, x_m$ for M and $y_1, \dots, y_n$ for N, we defined

$$\phi^*\omega = \sum_{i_1,\dots,i_k} f_{i_1,\dots,i_k}(y(x)) dy_{i_1}(x) \dots dy_{i_k}(x)$$

given

$$\omega = \sum_{i_1,\dots,i_k} f_{i_1,\dots,i_k}(y) dy_{i_1} \dots dy_{i_k}.$$

Show that $\phi^*\omega$ is well-defined.

(43) Compute all the de Rham cohomology groups of S^2. Then inductively compute all the de Rham cohomology groups of S^n.

(44) Suppose the manifold M is the disjoint union of manifolds M_1 and M_2. Then prove that $H^k_{dR}(M) = H^k_{dR}(M_1) \oplus H^k_{dR}(M_2)$.

(45) Let $\phi : M \to N$ be a smooth map between manifolds. If $\omega \in \Omega^k(N)$, then prove that $d(\phi^*\omega) = \phi^*(d\omega)$.

(46) (30 points) Complete the proof that the short exact sequence of cochain maps

$$0 \to \mathcal{A} \stackrel{\phi}{\to} \mathcal{B} \stackrel{\psi}{\to} \mathcal{C} \to 0$$

gives rise to a long exact sequence on cohomology. (This problem should be done carefully and completely — it's worth 30 points.)

(47) Let M be an oriented manifold of dimension n and $\omega \in \Omega^n(M)$. Let $\{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ be an oriented atlas and $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. We defined

$$\int_M \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(f_\alpha \omega).$$

Prove that this definition does not depend on the choice of oriented atlas as well as on the choice partition of unity.

(48) Using the version of Stokes' Theorem given in class, prove the classical Stokes' Theorem: Let S be a compact, oriented 2-manifold (i.e., a surface) with boundary in $\mathbb{R}^3$ and let $F = (F_1, F_2, F_3)$ be a smooth vector field defined on a neighborhood of S. Then:

$$\int_S \langle \operatorname{curl} F, n \rangle dA = \int_{\partial S} F_1 dx + F_2 dy + F_3 dz,$$

where n is a unit normal to S, $\langle \cdot \rangle$ is the standard inner product, $dA = n_1 dydz + n_2 dzdx + n_3 dxdy$, $\operatorname{curl} F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$.

The Cauchy Integral Formula. We will now define complex-valued forms on a complex manifold M. Complex-valued differential forms are simply sums $\omega = \omega_1 + i\omega_2$, where ω_1 and ω_1 are real. The wedge product extends in the obvious way, and

$$d\omega \stackrel{def}{=} d\omega_1 + id\omega_2, \quad \int_M \omega \stackrel{def}{=} \int_M \omega_1 + i \int_M \omega_2.$$

Note that Stokes' Theorem is valid for complex-valued forms since it is valid separately for the real part and the complex part.

On $\mathbb{C} = \mathbb{R}^2$, let $z = x + iy$ be the complex coordinate and $\overline{z} = x - iy$ be its complex conjugate. Then $dz = dx + idy$ and $d\overline{z} = dx - idy$.

(49) Consider the function $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto f(z)$. Show that $\omega = f(z)dz$ is closed ($d\omega = 0$) if and only if $f(z) = f(x, y)$ (f is viewed as $\mathbb{R}^2 \to \mathbb{R}^2$) satisfies the Cauchy-Riemann equation

$$\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}.$$

Recall that a smooth function f satisfies the Cauchy-Riemann equation if and only if f is holomorphic. Also recall that rational functions in z are holomorphic where defined. For example, $f(z) = \frac{1}{z-a}$ is holomorphic on $\mathbb{C} - \{a\}$.

(50) Suppose f is a holomorphic function on a domain $\Omega \subset \mathbb{C}$. Prove that if γ_0 and γ_1 are homotopic curves in Ω, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Two curves $\gamma_0, \gamma_1 : S^1 \to \Omega$ are said to be *homotopic* if there exists a smooth $\Gamma : S^1 \times [0, 1] \to \Omega$ with $\Gamma(\theta, t) = \gamma_t(\theta)$.

(51) Let C be a circle of radius R around $a \in \mathbb{C}$. Then prove that

$$\int_C \frac{1}{z-a} dz = 2\pi i.$$

(52) Let $f(z)$ be a holomorphic function on Ω, let $C \subset \Omega$ be a circle of radius R around $a \in \Omega$, and let γ be a smooth closed curve which is homotopic to C inside $\Omega - \{a\}$. Then prove that

$$\int_\gamma \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

[Hint: By Exercise (46), the integral only depends on the curve up to homotopy. Therefore, we may assume $\gamma = C$. Shrink the radius R and take the limit.]

(53) The vector field $(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ (defined almost everywhere) on $\mathbb{R}^2$ has curl zero, but it cannot be written as the gradient of any function. Explain what this means in terms of de Rham cohomology.

(54) Let V be an $\mathbb{R}$-vector space. Prove that the interior product $i_v : \bigwedge^k V^* \to \bigwedge^{k-1} V^*$, $v \in V$, where $f_1 \wedge \cdots \wedge f_k \mapsto \sum_l (-1)^{l+1} f_1 \wedge \ldots f_l(v) \cdots \wedge f_k$, is well-defined.

(55) Let M be a manifold and X a global vector field on M. Show that $L = d \circ i_X + i_X \circ d : \Omega^k(M) \to \Omega^k(M)$ is a derivation, i.e., it satisfies $L(\alpha \wedge \beta) = L(\alpha) \wedge \beta + \alpha \wedge L(\beta)$. Here i_X denotes the interior product with X.

(56) Compute the de Rham cohomology of a compact 2-dimensional manifold (surface) of genus g without boundary. [For this problem, you may feel free to draw pictures of the surface of genus g and its decomposition into pieces. You will probably need to use the Mayer-Vietoris sequence a couple of times, as well as the homotopy properties.] Hint: remove a disk from the surface and see what you are left with.

(57) For each integer n, exhibit a smooth map $S^1 \to S^1$ of degree n. (You must prove that your map has degree n.)

(58) Let M be a manifold. Consider the map $\wedge : H^k(M) \times H^l(M) \to H^{k+l}(M)$, $([\omega], [\eta]) \mapsto [\omega \wedge \eta]$. Prove that $\wedge$ is well-defined (on the level of cohomology). [Hence $H^*(M) \stackrel{def}{=} \oplus_{i=0}^{\dim M} H^i(M)$ has the structure of an *algebra*.]

(59) Let T^2 be the 2-dimensional torus. Compute the map $\wedge : H^1(T^2) \times H^1(T^2) \to H^2(T^2)$. (Give a basis $\{\omega_1, \ldots, \omega_k\}$ for $H^1(T^2)$ and compute $[\omega_i \wedge \omega_j]$.) In particular, prove that $\wedge$ is surjective.

(60) Prove that every smooth map $f : S^2 \to T^2$ has degree zero. (Hint: Use the previous exercise and the fact that $H^1(S^2) = 0$.)

(61) Let M be a compact orientable manifold of dimension $2n$ without boundary, and let ω be a *symplectic form* on M, i.e., a closed 2-form such that $\omega^n = \omega \wedge \cdots \wedge \omega$ is nowhere zero. Prove that $H^2(M) \neq 0$.

(62) Prove that if $0 \to C_1 \to C_2 \to \cdots \to C_k \to 0$ is an exact sequence, then

$$\sum_{i=1}^k (-1)^i \dim C_i = 0.$$

(63) Verify that Lie brackets satisfy anticommutativity and the Jacobi identity.

(64) Prove that if $X, X_1, \ldots, X_k$ are vector fields on M and $\omega \in \Omega^k(M)$, then

$$\mathcal{L}_X(\omega(X_1, \ldots, X_k)) = (\mathcal{L}_X\omega)(X_1, \ldots, X_k) + \sum_i \omega(X_1, \ldots, [X, X_i], \ldots, X_k).$$

(65) Prove that if $\omega \in \Omega^k(M)$, and $X_1, \ldots, X_{k+1}$ are vector fields on M, then

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) &= \sum_i (-1)^{i+1} X_i(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}). \end{aligned}$$

Here $\widehat{X_i}$ means omit the term with X_i.

(66) Consider the vector field $X = x^2 \frac{d}{dx}$ on $\mathbb{R}$. Compute its integral curves. Explain why X does not admit a global flow $\Phi : \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{R}$.

(67) Find a piecewise smooth curve connecting $(0,0,0)$ to any point (x, y, z), where each smooth piece an integral curve of the 2-plane field distribution $\xi = \ker(xdy - ydx + dz)$. (Such curves are far from unique.)

(68) (20 points) Write out the proof of Frobenius' Theorem in the general case.

(69) Let E be a rank k vector bundle over M which is parallelizable. If $s_1, \ldots, s_k$ are global sections which span E_p at every point $p \in M$, then we can write any $s \in \Gamma(E, M)$ as $s = \sum_i f_i s_i$, where $f_i \in C^\infty(M)$. Show $\nabla_X s = \sum_i (X f_i) s_i$ is a connection.

(70) Let $U \subset M$ be an open subset and let E be a rank k vector bundle over M. Consider two connections ∇' and ∇'' on $E|_U$. If $\lambda_1, \lambda_2 \in C^\infty(U)$ satisfy $\lambda_1 + \lambda_2 = 1$, then prove that $\lambda_1 \nabla' + \lambda_2 \nabla''$ is a connection on $E|_U$.

(71) Prove that $R(X, Y)s$ is tensorial in X and Y. Here R is the curvature of a connection ∇ and X, Y are vector fields.

(72) Complete the proof (started in class) that $R = dA + A \wedge A$.

(73) If ∇ is the Levi-Civita connection on a Riemannian manifold (M, g) and $R = R_\nabla$ is its curvature, then show that R satisfies the following properties:
  (a) $\langle R(X, Y)Z, W\rangle = -\langle R(X, Y)W, Z\rangle$.
  (b) $\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$.
  (c) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.

(74) Let $\Sigma = \{x^2 + y^2 + z^2 = R^2\} \subset \mathbb{R}^3$, and let $\overline{g}$ be the metric induced from the standard Euclidean metric on $\mathbb{R}^3$ to Σ. Compute:
  (a) the induced metric $\overline{g}$ and
  (b) the Levi-Civita connection of $(\Sigma, \overline{g})$,

locally near $(0, 0, R)$ using the coordinates (x, y), given by projecting onto the first two coordinates of $\mathbb{R}^3$.