DIFFERENTIAL GEOMETRY COURSE NOTES

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1. Review of topology and linear algebra

1.1. Review of topology.

Definition 1.1. *A* topological space *is a pair $(X, \mathcal{T})$ consisting of a set X and a collection $\mathcal{T} = \{U_\alpha\}$ of subsets of X, satisfying the following:*

(1) $\emptyset, X \in \mathcal{T}$,
(2) *if $U_\alpha, U_\beta \in \mathcal{T}$, then $U_\alpha \cap U_\beta \in \mathcal{T}$,*
(3) *if $U_\alpha \in \mathcal{T}$ for all $\alpha \in I$, then $\cup_{\alpha \in I} U_\alpha \in \mathcal{T}$. (Here I is an indexing set, and is not necessarily finite.)*

$\mathcal{T}$ is called a topology *for X and $U_\alpha \in \mathcal{T}$ is called an* open set *of X.*

Example 1: $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) $= \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}, i = 1, \dots, n\}$, called *real n-dimensional space*.

How to define a topology $\mathcal{T}$ on $\mathbb{R}^n$? We would at least like to include open balls of radius r about $y \in \mathbb{R}^n$:

$$B_r(y) = \{x \in \mathbb{R}^n \mid |x - y| < r\},$$

where

$$|x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Question: Is $\mathcal{T}_0 = \{B_r(y) \mid y \in \mathbb{R}^n, r \in (0, \infty)\}$ a valid topology for $\mathbb{R}^n$?

No, so you must add more open sets to $\mathcal{T}_0$ to get a valid topology for $\mathbb{R}^n$.

$$\mathcal{T} = \{U \mid \forall y \in U, \exists B_r(y) \subset U\}.$$

Example 2A: $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. A reasonable topology on S^1 is the topology *induced by the inclusion* $S^1 \subset \mathbb{R}^2$.

Definition 1.2. *Let $(X, \mathcal{T})$ be a topological space and let $f : Y \to X$. Then the* induced topology *$f^{-1}\mathcal{T} = \{f^{-1}(U) \mid U \in \mathcal{T}\}$ is a topology on Y.*

Example 2B: Another definition of S^1 is $[0, 1]/\sim$, where $[0, 1]$ is the closed interval (with the topology induced from the inclusion $[0, 1] \to \mathbb{R}$) and the equivalence relation identifies $0 \sim 1$. A reasonable topology on S^1 is the *quotient topology*.

Definition 1.3. *Let $(X, \mathcal{T})$ be a topological space, $\sim$ be an equivalence relation on X, $\overline{X} = X/\sim$ be the set of equivalence classes of X, and $\pi : X \to \overline{X}$ be the projection map which sends $x \in X$ to its equivalence class $[x]$. Then the* quotient topology *$\overline{\mathcal{T}}$ of $\overline{X}$ is the set of $V \subset \overline{X}$ for which $\pi^{-1}(V)$ is open.*

Definition 1.4. *A map $f : X \to Y$ between topological spaces is* continuous *if $f^{-1}(V) = \{x \in X | f(x) \in V\}$ is open whenever $V \subset Y$ is open.*

HW: Show that the inclusion $S^1 \subset \mathbb{R}^2$ is a continuous map. Show that the quotient map $[0, 1] \to S^1 = [0, 1]/\sim$ is a continuous map.

More generally,

(1) Given a topological space $(X, \mathcal{T})$ and a map $f : Y \to X$, the induced topology on Y is the "smallest"[1] topology which makes f continuous.
(2) Given a topological space $(X, \mathcal{T})$ and a surjective map $\pi : X \twoheadrightarrow Y$, the quotient topology on Y is the "largest" topology which makes π continuous.

When are two topological spaces equivalent? The following gives one notion:

Definition 1.5. *A map $f : X \to Y$ is a* homeomorphism *is there exists an inverse $f^{-1} : Y \to X$ for which f and f^{-1} are both continuous.*

HW: Show that the two incarnations of S^1 from Examples 2A and 2B are homeomorphic.

Zen of mathematics: Any world ("category") in mathematics consists of spaces ("objects") and maps between spaces ("morphisms").

Examples:

(1) (Topological category) Topological spaces and continuous maps.
(2) (Groups) Groups and homomorphisms.
(3) (Linear category) Vector spaces and linear transformations.

1.2. Review of linear algebra.

Definition 1.6. *A vector space V over a field $k = \mathbb{R}$ or $\mathbb{C}$ is a set V equipped with two operations $V \times V \to V$ (called addition) and $k \times V \to V$ (called scalar multiplication) s.t.*

(1) *V is an* abelian group *under addition.*
   (a) *(Identity) There is a zero element 0 s.t. $0 + v = v + 0 = v$.*
   (b) *(Inverse) Given $v \in V$ there exists an element $w \in V$ s.t. $v + w = w + v = 0$.*
   (c) *(Associativity) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$.*
   (d) *(Commutativity) $v + w = w + v$.*
(2) (a) $1v = v$.
   (b) $(ab)v = a(bv)$.
   (c) $a(v + w) = av + aw$.

[1]Figure out what "smallest" and "largest" mean.

(d) $(a+b)v = av + bv$.

Note: Keep in mind the Zen of mathematics — we have defined objects (vector spaces), and now we need to define maps between objects.

Definition 1.7. *A linear map $\phi : V \to W$ between vector spaces over k satisfies $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$ ($v_1, v_2 \in V$) and $\phi(cv) = c \cdot \phi(v)$ ($c \in k$ and $v \in V$).*

Now, when are two vector spaces equivalent in the linear category?

Definition 1.8. *A linear map $\phi : V \to W$ is an* isomorphism *if there exists a linear map $\psi : W \to V$ such that $\phi \circ \psi = id$ and $\psi \circ \phi = id$. (We often also say ϕ is* invertible.*)*

If V and W are finite-dimensional*,[2] then we may take bases* $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ and represent a linear map $\phi : V \to W$ as an $m \times n$ matrix A. ϕ is then invertible if and only if $m = n$ and $\det(A) \neq 0$.*

Examples of vector spaces: Let $\phi : V \to W$ be a linear map of vector spaces.

(1) The *kernel* $\ker \phi = \{v \in V \mid \phi(v) = 0\}$ is a vector subspace of V.
(2) The *image* $\operatorname{im} \phi = \{\phi(v) \mid v \in V\}$ is a vector subspace of W.
(3) Let $V \subset W$ be a subspace. Then the *quotient* $W/V = \{w + V \mid w \in W\}$ can be given the structure of a vector space. Here $w + V = \{w + v \mid v \in V\}$.
(4) The *cokernel* $\operatorname{coker} \phi = W / \operatorname{im} \phi$.

[2] $\star$ means you should look up its definition.

2. Review of differentiation

2.1. **Definitions.** Let $f:\mathbb{R}^m\to\mathbb{R}^n$ be a map. The discussion carries over to $f:U\to V$ for open sets $U\subset\mathbb{R}^m$ and $V\subset\mathbb{R}^n$.

Definition 2.1. *The map* $f:\mathbb{R}^m\to\mathbb{R}^n$ *is* differentiable *at a point* $x\in\mathbb{R}^m$ *if there exists a linear map* $L:\mathbb{R}^m\to\mathbb{R}^n$ *satisfying*

$$\lim_{h\to 0}\frac{|f(x+h)-f(x)-L(h)|}{|h|}=0, \tag{1}$$

where $h\in\mathbb{R}^m-\{0\}$. L *is called the* derivative of f at x *and is usually written as* $df(x)$.

HW: Show that if $f:\mathbb{R}^m\to\mathbb{R}^n$ is differentiable at $x\in\mathbb{R}^m$, then there is a unique L which satisfies Equation (1).

Fact 2.2. *If f is differentiable at x, then* $df(x):\mathbb{R}^m\to\mathbb{R}^n$ *is a linear map which satisfies*

$$df(x)(v)=\lim_{t\to 0}\frac{f(x+tv)-f(x)}{t}. \tag{2}$$

We say that the *directional derivative* of f at x in the direction of v exists if the right-hand side of Equation (2) exists. What Fact 2.2 says is that if f is differentiable at x, then the directional derivative of f at x in the direction of v exists and is given by $df(x)(v)$.

2.2. **Partial derivatives.** Let e_j be the usual basis element $(0,\dots,1,\dots,0)$, where 1 is in the jth position. Then $df(x)(e_j)$ is usually called the *partial derivative* and is written as $\frac{\partial f}{\partial x_j}(x)$ or $\partial_j f(x)$.

More explicitly, if we write $f=(f_1,\dots,f_n)^T$ (here T means transpose), where $f_i:\mathbb{R}^m\to\mathbb{R}$, then

$$\frac{\partial f}{\partial x_j}(x)=\left(\frac{\partial f_1}{\partial x_j}(x),\dots,\frac{\partial f_n}{\partial x_j}(x)\right)^T,$$

and $df(x)$ can be written in matrix form as follows:

$$df(x)=\begin{pmatrix}\frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_m}(x)\\ \vdots & \vdots & \vdots\\ \frac{\partial f_n}{\partial x_1}(x) & \dots & \frac{\partial f_n}{\partial x_m}(x)\end{pmatrix}$$

The matrix is usually called the *Jacobian matrix*.

Facts:

1. If $\partial_i(\partial_j f)$ and $\partial_j(\partial_i f)$ are continuous on an open set $\ni x$, then $\partial_i(\partial_j f)(x)=\partial_j(\partial_i f)(x)$.
2. $df(x)$ exists if all $\frac{\partial f_i}{\partial x_j}(y)$, $i=1,\dots,n$, $j=1,\dots,m$, exist on an open set $\ni x$ and each $\frac{\partial f_i}{\partial x_j}$ is continuous at x.

Shorthand: Assuming f is smooth, we write $\partial^\alpha f=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_k^{\alpha_k}f$ where $\alpha=(\alpha_1,\dots,\alpha_k)$.

Definition 2.3.

(1) *f is* smooth *or* of class C^∞ *at $x \in \mathbb{R}^m$ if all partial derivatives of all orders exist at x.*
(2) *f is* of class C^k *at $x \in \mathbb{R}^m$ if all partial derivatives up to order k exist on an open set $\ni x$ and are continuous at x.*

2.3. **The Chain Rule.**

Theorem 2.4 (Chain Rule)**.** *Let $f : \mathbb{R}^\ell \to \mathbb{R}^m$ be differentiable at x and $g : \mathbb{R}^m \to \mathbb{R}^n$ be differentiable at $f(x)$. Then $g \circ f : \mathbb{R}^\ell \to \mathbb{R}^n$ is differentiable at x and*

$$d(g \circ f)(x) = dg(f(x)) \circ df(x).$$

Draw a picture of the maps and derivatives.

Definition 2.5. *A map $f : U \to V$ is a C^∞-diffeomorphism if f is a smooth map with a smooth inverse $f^{-1} : V \to U$. (C^1-diffeomorphisms can be defined similarly.)*

One consequence of the Chain Rule is:

Proposition 2.6. *If $f : U \to V$ is a diffeomorphism, then $df(x)$ is an isomorphism for all $x \in U$.*

Proof. Let $g : V \to U$ be the inverse function. Then $g \circ f = \mathrm{id}$. Taking derivatives, $dg(f(x)) \circ df(x) = \mathrm{id}$ as linear maps; this give a left inverse for $df(x)$. Similarly, a right inverse exists and hence $df(x)$ is an isomorphism for all x. □

3. MANIFOLDS

3.1. Topological manifolds.

Definition 3.1. *A* topological manifold *of* dimension n *is a pair consisting of a topological space* X *and a collection* $\mathcal{A} = \{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha\in I}$ *of maps (called an* atlas *of* X*) such that:*

1. U_α *is an open set of* X *and* $\cup_{\alpha\in I} U_\alpha = X$,
2. ϕ_α *is a homeomorphism onto an open subset* $\phi_\alpha(U_\alpha)$ *of* $\mathbb{R}^n$.
3. *(Technical condition 1)* X *is Hausdorff.*
4. *(Technical condition 2)* X *is second countable.*

Each $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ *is called a* coordinate chart.

Definition 3.2. *A topological space* X *is* Hausdorff *if for any* $x \neq y \in X$ *there exist open sets* U_x *and* U_y *containing* x, y *respectively such that* $U_x \cap U_y = \emptyset$.

Definition 3.3. *A topological space* $(X, \mathcal{T})$ *is* second countable *if there exists a countable subcollection* $\mathcal{T}_0$ *of* $\mathcal{T}$ *and any open set* $U \in \mathcal{T}$ *is a union (not necessarily finite) of open sets in* $\mathcal{T}_0$.

HW: Show that S^1 from Example 2A or 2B from Day 1 (already shown to be homeomorphic from an earlier exercise) is a topological manifold.

Non-example. A topological manifold which satisfies all the axioms except for the Hausdorff condition: Take $\mathbb{R} \times \{0,1\}/\sim$, where $(x,0) \sim (x,1)$ for all $x \neq 0$, with the quotient topology. For any open sets U_i, $i = 0,1$, containing $(0,i)$, $U_0 \cap U_1 \neq \varnothing$.

Non-example. The *long line* is a topological manifold which satisfies all the axioms except for the second countable condition. It is $\omega_1 \times [0,1)$ with the smallest element deleted, where ω_1 is the "first uncountable ordinal" (in particular it is uncountable and has an ordering), and the topology is the order topology coming from the lexicographic ordering on $\omega_1 \times [0,1)$. For more details see Wikipedia or Munkres, "Topology".

HW: Give an example of a Hausdorff, second countable topological space X which is not a topological manifold. (You may have trouble proving that it is not a topological manifold, though. You may also want to find several different types of examples.)

Observe that in the land of topological manifolds, a square and a circle are the same, i.e., they are homeomorphic! That is not the world we will explore — in other words, we seek a category where squares are not the same as circles. In other words, we need derivatives!

3.2. Differentiable manifolds.

Definition 3.4. *A* smooth manifold *is a topological manifold* $(X, \mathcal{A} = \{\phi_\alpha : U_\alpha \to \mathbb{R}^n\})$ *satisfying the following: For every* $U_\alpha \cap U_\beta \neq \emptyset$,

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a smooth map. The maps $\phi_\beta \circ \phi_\alpha^{-1}$ *are called* transition maps.

Note: In the rest of the course when we refer to a “manifold”, we mean a “smooth manifold”, unless stated otherwise.

4. Examples of smooth manifolds

Today we give some examples of smooth manifolds. *For each of the examples, you should also verify the Hausdorff and second countable conditions!*

(1) $\mathbb{R}^n$ is a smooth manifold.[3] Atlas: $\{\mathrm{id} : U = \mathbb{R}^n \to \mathbb{R}^n\}$ consisting of only one chart.

(2) Any open subset U of a smooth manifold M is a smooth manifold. Given an atlas $\{\phi_\alpha : U_\alpha \to \mathbb{R}^n\})$ for M, an atlas for U is $\{\phi_\alpha|_{U\cap U_\alpha} : U_\alpha \cap U \to \mathbb{R}^n\}$.

(3) Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real entries, and let

$$GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}.$$

$GL(n, \mathbb{R})$ is an open subset of $M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$, hence is a smooth n^2-dimensional manifold. $GL(n, \mathbb{R})$ is called the *general linear group* of $n \times n$ real matrices.

(4) If M and N are smooth m- and n-dimensional manifolds, then their *product* $M \times N$ can naturally be given the structure of a smooth $(m + n)$-dimensional manifold. Atlas: $\{\phi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n\}$, where $\{\phi_\alpha : U_\alpha \to \mathbb{R}^m\}$ is an atlas for M and $\{\psi_\beta : V_\beta \to \mathbb{R}^n\}$ is an atlas for N.

(5) $S^1 = \{x^2 + y^2 = 1\}$ is a smooth 1-dimensional manifold.

- (i) One possible atlas: Open sets $U_1 = \{y > 0\}$, $U_2 = \{y < 0\}$, $U_3 = \{x > 0\}$, $U_4 = \{x < 0\}$, together with projections to the x-axis or the y-axis, as appropriate. *Check the transition maps!*
- (ii) Another atlas: Open sets $U_1 = \{y \neq 1\}$ and $U_2 = \{y \neq -1\}$, together with stereographic projections from U_1 to $y = -1$ and U_2 to $y = 1$. The map $\phi_1 : U_1 \to \mathbb{R}$ is defined as follows: Take the line $L_{(x,y)}$ which passes through $(0, 1)$ and $(x, y) \in U_1$. Then let ϕ_1 be the x-coordinate of the intersection point between $L_{(x,y)}$ and $y = -1$. The map $\phi_2 : U_2 \to \mathbb{R}$ is defined similarly by projecting from $(0, -1)$ to $y = 1$. *Check the transition maps!*

(6) $S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$. *Generalize the discussion from (5).*

(7) In dimension 2, S^2, T^2, genus g surface.

(8) (Real projective space) $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{(0, \dots, 0)\})/\sim$, where

$$(x_0, x_1, \dots, x_n) \sim (tx_0, tx_1, \dots, tx_n), \quad t \in \mathbb{R} - \{0\}.$$

$\mathbb{RP}^n$ is called the *real projective space* of dimension n. The equivalence class of $(x_0, \dots, x_n)$ is denoted by $[x_0, \dots, x_n]$.

[3] Strictly speaking, this should say "can be given the structure of a smooth manifold". There may be more than one choice and we have not yet discussed when two manifolds are the same.

Consider $U_0 = \{x_0 \neq 0\}$ with the coordinate chart $\phi_0 : U_0 \to \mathbb{R}^n$ given by

$$[x_0, x_1, \ldots, x_n] = \left[1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right] \mapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

Similarly, take $U_i = \{x_i \neq 0\}$ and define $\phi_i : U_i \to \mathbb{R}^n$. What about transition maps $\phi_j \circ \phi_i^{-1}$? (Explain this in detail.)

(9) (Group actions) The 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The discrete group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by translation:

$$\mathbb{Z}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$$

$$((m, n), (x, y)) \mapsto (m + x, n + y).$$

Note that for each fixed (m, n), we have a diffeomorphism

$$\mathbb{R}^2 \to \mathbb{R}^2,$$

$$(x, y) \mapsto (m + x, n + y).$$

$\mathbb{R}^2/\mathbb{Z}^2$ is the set of orbits of $\mathbb{R}^2$ under the action of $\mathbb{Z}^2$. (One orbit is $(x, y) + \mathbb{Z}^2$.)

Equivalently, the 2-torus is obtained from the "fundamental domain" $[0, 1] \times [0, 1]$ by identifying $(0, y) \sim (1, y)$ and $(x, 0) \sim (x, 1)$, i.e., the sides and the top and the bottom. The assignment $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, $x \mapsto [x]$, is injective when restricted to the interior of the fundamental domain.

The n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is defined similarly.

Next time: Try to answer the question of what it means for two atlases of the same M to be "the same".

5. Smooth functions and smooth maps

Today we discuss smooth functions on a manifold and smooth maps between manifolds.

5.1. **Choice of atlas.** Let $(M, \mathcal{T})$ be the underlying topological space of a manifold, and $\mathcal{A}_1 = \{(U_\alpha, \phi_\alpha)\}$, $\mathcal{A}_2 = \{(V_\beta, \psi_\beta)\}$ be two atlases.

Question: When do they represent the *same* smooth manifold?

Definition 5.1. *Two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on M are* compatible *if*

$$\phi_\alpha(U_\alpha \cap V_\beta) \stackrel{\psi_\beta \circ \phi_\alpha^{-1}}{\longrightarrow} \psi_\beta(U_\alpha \cap V_\beta)$$

is a smooth map for all pairs $U_\alpha \cap V_\beta \neq \emptyset$.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible, then we can take $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ which is compatible with both $\mathcal{A}_1$ and $\mathcal{A}_2$.

Definition 5.2. *Given a smooth manifold $(M, \mathcal{A})$, its* maximal atlas $\mathcal{A}_{max} = \{(U_\alpha, \phi_\alpha)\}$ *is an atlas which is compatible with $\mathcal{A}$ and contains every atlas $\mathcal{A}' \supset \mathcal{A}$ which is compatible with $\mathcal{A}$.*

5.2. **Smooth functions.**

Some more zen: You can study an object (such as a manifold) either by looking at the object itself or by looking at the space of functions on the object. In the topological category, the space of functions would be $C^0(M)$, the space of continuous functions $f : M \to \mathbb{R}$. The function space perspective has been especially fruitful in algebraic geometry.

Question: What is the appropriate space of functions for a smooth manifold $(M, \mathcal{A})$?

Definition 5.3. *Given a smooth manifold $(M, \mathcal{A})$, a function $f : M \to \mathbb{R}$ is* smooth *if*

$$f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbb{R}$$

is smooth for each coordinate chart (U_α, ϕ_α) of $\mathcal{A}$.

Note that the definition of a smooth function on M depends on the atlas $\mathcal{A}$.

The space of smooth functions $f : M \to \mathbb{R}$ with respect to $\mathcal{A}$ is written as $C^\infty_{\mathcal{A}}(M)$. When $\mathcal{A}$ is understood, we write $C^\infty(M)$.

Lemma 5.4. *Two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible if and only if $C^\infty_{\mathcal{A}_1}(M) = C^\infty_{\mathcal{A}_2}(M)$.*

Proof. Suppose $\mathcal{A}_1 = \{(U_\alpha, \phi_\alpha)\}$ and $\mathcal{A}_2 = \{(V_\beta, \psi_\beta)\}$ are compatible. It suffices to show that $C^\infty_{\mathcal{A}_1}(M) \supset C^\infty_{\mathcal{A}_2}(M)$. If $f \in C^\infty_{\mathcal{A}_2}(M)$, then $f \circ \psi_\beta^{-1} : \psi_\beta(V_\beta) \to \mathbb{R}$ is smooth for all β. Now

$$f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap V_\beta) \to \mathbb{R} \tag{3}$$

can be written as $(f \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ \phi_\alpha^{-1})$, and each of $f \circ \psi_\beta^{-1}$ and $\psi_\beta \circ \phi_\alpha^{-1}$ is smooth (the latter is smooth because $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible); hence (3) is smooth for all α and β. This implies that $f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbb{R}$ is smooth for all α.

Suppose $C^\infty_{\mathcal{A}_1}(M) = C^\infty_{\mathcal{A}_2}(M)$. We use the existence of *bump functions*, i.e., smooth functions $h : \mathbb{R} \to [0, 1]$ such that $h(x) = 1$ on $[a, b]$ and $h(x) = 0$ on $\mathbb{R} - [c, d]$, where $c < a < b < d$. (The construction of bump functions is an exercise.)

In order to show that the transition maps

$$\psi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap V_\beta) \to \psi_\beta(U_\alpha \cap V_\beta) \subset \mathbb{R}^n$$

are smooth, we postcompose with the projection $\pi_j : \mathbb{R}^n \to \mathbb{R}$ to the jth $\mathbb{R}$ factor and show that $\pi_j \circ \psi_\beta \circ \phi_\alpha^{-1}$ is smooth. Given $x \in \psi_\beta(U_\alpha \cap V_\beta)$, let $B_\varepsilon(x) \subset B_{2\varepsilon}(x) \subset \psi_\beta(U_\alpha \cap V_\beta)$ be small open balls around x. Using the bump functions we can construct a function f on $U_\alpha \cap V_\beta$ such that $f \circ \psi_\beta^{-1}$ equals π_j on $B_\varepsilon(x)$ and 0 outside $B_{2\varepsilon}(x)$; f can be extended to the rest of M by setting $f = 0$. f is clearly in $C^\infty_{\mathcal{A}_2}(M)$. Since $C^\infty_{\mathcal{A}_1}(M) = C^\infty_{\mathcal{A}_2}(M)$, $f \circ \phi_\alpha^{-1} = (f \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ \phi_\alpha^{-1})$ is smooth. This is sufficient to show the smoothness of $\pi_j \circ \psi_\beta \circ \phi_\alpha^{-1}$ and hence of $\psi_\beta \circ \phi_\alpha^{-1}$. $\square$

Pullback: Let $\phi : X \to Y$ be a continuous map between topological spaces. Then there is a naturally defined *pullback map*

$$\phi^* : C^0(Y) \to C^0(X)$$

given by $f \mapsto f \circ \phi$. Note that pullback is *contravariant*, i.e., the direction is from Y to X, which is the opposite from the original map ϕ.

Consider the smooth manifold $(M, \mathcal{A})$. If $\psi : M \to M$ is a homeomorphism, then $\psi^* : C^0(M) \xrightarrow{\sim} C^0(M)$. Although $C^\infty_{\mathcal{A}}(M) \xrightarrow{\sim} \psi^*(C^\infty_{\mathcal{A}}(M))$, in general $C^\infty_{\mathcal{A}}(M) \neq \psi^*(C^\infty_{\mathcal{A}}(M))$.

Definition 5.5. *Two C^∞-structures $C^\infty_{\mathcal{A}_1}(M)$ and $C^\infty_{\mathcal{A}_2}(M)$ are* equivalent *if there exists a homeomorphism of M which takes $C^\infty_{\mathcal{A}_1}(M) \simeq C^\infty_{\mathcal{A}_2}(M)$.*

Amazing fact: (Milnor) S^7 has several inequivalent smooth structures! (Not amazingly, S^1 has only one smooth structure.)

Major open question: (Smooth Poincaré Conjecture) How many smooth structures does S^4 have?

5.3. **Smooth maps.** In the category of smooth manifolds, we need to define the appropriate maps, called *smooth maps*.

Definition 5.6. *A map $\phi : M \to N$ between manifolds is smooth if for any $p \in M$ there exist coordinate charts (U_α, ϕ_α), (V_β, ψ_β) such that $U_\alpha \ni p$, $V_\beta \ni f(p)$, and the composition*

$$\phi_\alpha(U_\alpha) \xrightarrow{\phi_\alpha^{-1}} U_\alpha \xrightarrow{\phi} V_\beta \xrightarrow{\psi_\beta} \psi_\beta(V_\beta)$$

is smooth.

Remark 5.7. *For the above definition, we need to take $U_\alpha \ni p$ which is "sufficiently small" so that $\phi(U_\alpha) \subset V_\beta$. So this means that we should be using a maximal atlas (or at least a "large enough" atlas).*

Lemma 5.8. *$\phi : M \to N$ is* smooth *if and only if $\phi^*(C^\infty(N)) \subset C^\infty(M)$.*

The proof is similar to that of Lemma 5.4.

Definition 5.9. *A smooth map* $\phi : M \to N$ *is a* diffeomorphism *if there exists a smooth inverse* $\phi^{-1} : M \to N$.

Upshot: Smooth maps between smooth manifolds can be "reduced" to smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

6. The Inverse Function Theorem

6.1. **Inverse function theorem.**

Definition 6.1. *A smooth map $f : M \to N$ between two manifolds is a* diffeomorphism *if there is a smooth inverse $f^{-1} : N \to M$.*

The inverse function theorem, given below, is the most important basic theorem in differential geometry. It says that an isomorphism in the linear category implies a local diffeomorphism in the differentiable category. Hence we can move from "infinitesimal" to "local".

Theorem 6.2 (Inverse function theorem)**.** *Let $f : U \to V$ be a C^1 map, where U and V are open sets of $\mathbb{R}^n$. If $df(x) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then f is a* local diffeomorphism *near x, i.e., there exist open sets $U_x \ni x$ and $V_{f(x)} \ni f(x)$ such that $f|_{U_x} : U_x \to V_{f(x)}$ is a diffeomorphism.*

Partial proof. Refer to Spivak, *Calculus on Manifolds* for a complete proof.

Assume without loss of generality that $x = 0$ and $f(0) = 0$. We will only show that for all $y \in V$ near 0 there exists $x' \in U$ near 0 such that $f(x') = y$. First pick x_1 such that $df(0)(x_1) = y$; this is possible since $df(0)$ is an isomorphism. We then compare $f(x_1)$ and $df(0)(x_1) = y$: By the differentiability of f, for any sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $|x_1| < \delta$ we have:

$$|f(x_1) - f(0) - df(0)(x_1)| = |f(x_1) - y| \leq \varepsilon |x_1|.$$

In other words, the error $|f(x_1) - y|$ is much smaller than $|x_1|$. Next we take x_2 such that $df(x_1)(x_2) = y - f(x_1)$. Then we have:

$$|f(x_1 + x_2) - f(x_1) - df(x_1)(x_2)| = |f(x_1 + x_2) - y| \leq \varepsilon |x_2|.$$

Now, since f is in the class C^1, $df(\tilde{x})$ is invertible for all $\tilde{x}$ near 0 and there exists a constant $C > 0$ such that the norm of $(df(\tilde{x}))^{-1}$ is $< C$. Hence

$$|x_2| < C|df(x_1)(x_2)| = C|y - f(x_1)| \leq C\varepsilon |x_1|.$$

We then repeat the process to obtain $x_1, x_2, \dots$, and $f(x_1 + x_2 + \dots) = y$. (This process is usually called *Newton iteration*.) □

6.2. **Illustrative example.**

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 + y^2$. We would like to analyze the *level sets* $f^{-1}(a)$, where $a > 0$. To that end, we consider

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (f(x, y), y).$$

Let us use coordinates (x, y) for the domain $\mathbb{R}^2$ and coordinates (u, v) for the range $\mathbb{R}^2$. We compute:

$$dF(x, y) = \begin{pmatrix} 2x & 2y \\ 0 & 1 \end{pmatrix}.$$

Let us restrict our attention to the portion $x > 0$. Since $\det(dF(x, y)) = 2x > 0$, the inverse function theorem applies and there is a local diffeomorphism between a neighborhood $U_{(x,y)} \subset \mathbb{R}^2$

of a point (x, y) on the level set $f(x, y) = a$ and a neighborhood $V_{F(x,y)}$ of $F(x, y)$ on the line $u = a$.

In particular, $f^{-1}(a) \cap U_{(x,y)}$ is mapped to $\{u = a\} \cap V_{F(x,y)}$; in other words, F is a local diffeomorphism which "straightens out" $f^{-1}(a)$. Hence $f^{-1}(a)$, restricted to $x > 0$, is a smooth manifold. *Check the transition functions!*

Interpreted slightly differently, the pair f, y can locally be used as coordinate functions on $\mathbb{R}^2$, provided $x > 0$.

6.3. **Rank.** Recall that the *dimension* of a vector space V is the cardinality of a basis for V. If V is finite-dimensional, then $V \simeq \mathbb{R}^m$ for some m, and $\dim V = m$.

Definition 6.3. *The* rank *of a linear map* $L : V \to W$ *is the dimension of* $\operatorname{im}(L)$.

Definition 6.4. *The* rank *of a smooth map* $f : \mathbb{R}^m \to \mathbb{R}^n$ *at* $x \in \mathbb{R}^m$ *is the rank of* $df(x) : \mathbb{R}^m \to \mathbb{R}^n$. *The map* f *has* constant rank *if the rank of* $df(x)$ *is constant.*

We can similarly define the rank of a smooth map $f : M \to N$ at a point $x \in M$ by using local coordinates.

Claim 6.5. *The rank at* $x \in M$ *is constant under change of coordinates.*

Proof. We compare the ranks of $d(\psi_\alpha \circ f \circ \phi_\alpha^{-1})$ and $d(\psi_\beta \circ f \circ \phi_\beta^{-1})$, where $\phi_\alpha : U_\alpha \to \mathbb{R}^m$, $\psi_\alpha : V_\alpha \to \mathbb{R}^n$, $U_\alpha \subset M$, $V_\beta \subset N$, and ϕ_β, ψ_β are defined similarly. The invariance of rank is due to the chain rule:

$$\begin{aligned} d(\psi_\beta \circ f \circ \phi_\beta^{-1}) &= d((\psi_\beta \circ \psi_\alpha^{-1}) \circ (\psi_\alpha \circ f \circ \phi_\alpha^{-1}) \circ (\phi_\alpha^{-1} \circ \phi_\beta)) \\ &= d(\psi_\beta \circ \psi_\alpha^{-1}) \circ d(\psi_\alpha \circ f \circ \phi_\alpha^{-1}) \circ d(\phi_\alpha^{-1} \circ \phi_\beta), \end{aligned}$$

and by observing that $d(\psi_\beta \circ \psi_\alpha^{-1})$ and $d(\phi_\alpha^{-1} \circ \phi_\beta)$ are linear isomorphisms. $\square$

7. Submersions and regular values

7.1. **Submersions.**

Definition 7.1. *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets. A smooth map $f : U \to V$ is a* submersion *if $df(x)$ is surjective for all $x \in U$. (Note that this means that $m \geq n$ and that f has full rank.)*

Definition 7.2. *Let M be a manifold with maximal atlas $\mathcal{A} = \{\phi_\alpha : U_\alpha \to \mathbb{R}^m\}$ and let N be a manifold with maximal atlas $\mathcal{B} = \{\psi_\beta : V_\beta \to \mathbb{R}^n\}$. Then a smooth map $f : M \to N$ is a* submersion *if all $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ are submersions, where defined.*

Prototype: $f : \mathbb{R}^m \to \mathbb{R}^n$, $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$, where $m \geq n$.

Theorem 7.3 (Implicit function theorem, submersion version)**.** *Let $f : U \to V$ be a submersion, where $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets with $m \geq n$. Then for each $p \in U$ there exist $U \supset U_p \ni p$ and a diffeomorphism $F : U_p \xrightarrow{\sim} W \subset \mathbb{R}^m$ such that*

$$f \circ F^{-1} : W \to \mathbb{R}^n$$

is given by

$$(x_1, \dots, x_m) \mapsto (x_1, \dots, x_n).$$

Proof. Write $f = (f_1, \dots, f_n)$ where $f_i : U \to \mathbb{R}$, and define the map

$$F : U \to V \times \mathbb{R}^{m-n},$$

$$(x_1, \dots, x_m) \mapsto (f_1, \dots, f_n, x_{n+1}, \dots, x_m).$$

Here we choose the appropriate $x_{n+1}, \dots, x_m$ (after possibly permuting some variables) so that $dF(x)$ is invertible. Then F is a local diffeomorphism by the inverse function theorem,

$$f \circ F^{-1}(f_1, \dots, f_n, x_{n+1}, \dots, x_m) = (f_1, \dots, f_n),$$

and F satisfies the conditions of the theorem. □

Carving manifolds out of other manifolds: The implicit function theorem, submersion version, has the following corollary:

Corollary 7.4. *If $f : M \to N$ is a submersion, then $f^{-1}(y)$, $y \in N$, can be given the structure of a manifold of dimension* $\dim M - \dim N$.

Proof. The implicit function theorem above gives a coordinate chart with coordinate functions of the form $(x_{i_1}, \dots, x_{i_{n-m}})$ about each point in $f^{-1}(y)$. HW: Check the transition functions and verify the Hausdorff and second countable conditions!! □

Example: The easy way to prove that the circle $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ can be given the structure of a manifold is to consider the map

$$f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}, \quad f(x, y) = x^2 + y^2.$$

The Jacobian is $df(x,y) = (2x, 2y)$. Since x and y are never simultaneously zero, the rank of df is 1 at all points of $\mathbb{R}^2 - \{(0,0)\}$ and in particular on S^1. Using the implicit function theorem, it follows that S^1 is a manifold.

7.2. **Regular values and Sard's theorem.**

Definition 7.5. *Let $f : M \to N$ be a smooth map.*

(1) *A point $y \in N$ is a* regular value *of f if $df(x)$ is surjective for all $x \in f^{-1}(y)$.*
(2) *A point $y \in N$ is a* critical value *of f if $df(x)$ is not surjective for some $x \in f^{-1}(y)$.*
(3) *A point $x \in M$ is a* critical point *of f if $df(x)$ is not surjective.*

The implicit function theorem implies that $f^{-1}(y)$ can be given the structure of a manifold if y is a regular value of f.

Example: Let $M = \{x^3 + y^3 + z^3 = 1\} \subset \mathbb{R}^3$. Consider the map

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad (x,y,z) \mapsto x^3 + y^3 + z^3.$$

Then $M = f^{-1}(1)$. The Jacobian is given by $df(x,y,z) = (3x^2, 3y^2, 3z^2)$ and the rank of $df(x,y,z)$ is one if and only if $(x,y,z) \neq (0,0,0)$. Since $(0,0,0) \notin M$, it follows that 1 is a regular value of f. Hence M can be given the structure of a manifold.

HW: Prove that $S^n \subset \mathbb{R}^{n+1}$ is a manifold.

Example: Zero sets of homogeneous polynomials in $\mathbb{RP}^n$. A polynomial $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is *homogeneous of degree* d if $f(tx) = t^d f(x)$ for all $t \in \mathbb{R} - \{0\}$ and $x \in \mathbb{R}^{n+1}$. The zero set $Z(f)$ of f is given by $\{[x_0, \dots, x_n] \mid f(x_0, \dots, x_n) = 0\}$. By the homogeneous condition, $Z(f)$ is well-defined. We can check whether $Z(f)$ is a manifold by passing to local coordinates.

For example, consider the homogeneous polynomial $f(x_0, x_1, x_2) = x_0^3 + x_1^3 + x_2^3$ of degree 3 on $\mathbb{RP}^2$. Consider the open set $U = \{x_0 \neq 0\} \subset \mathbb{RP}^2$. If we let $x_0 = 1$, then on $U \simeq \mathbb{R}^2$ we have $f(x_1, x_2) = 1 + x_1^3 + x_2^3$. Check that 0 is a regular value of $f(x_1, x_2)$! The open sets $\{x_1 \neq 0\}$ and $\{x_2 \neq 0\}$ can be treated similarly.

More involved example: Let $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$. $SL(n, \mathbb{R})$ is called the *special linear group* of $n \times n$ real matrices. Consider the determinant map

$$f : \mathbb{R}^{n^2} \to \mathbb{R}, \quad A \mapsto \det(A).$$

We can rewrite f as follows:

$$f : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}, \quad (a_1, \dots, a_n) \mapsto \det(a_1, \dots, a_n),$$

where a_i are column vectors and $A = (a_1, \dots, a_n) = (a_{ij})$.

First we need some properties of the determinant:

(1) $f(e_1, \dots, e_n) = 1$.
(2) $f(a_1, \dots, c_i a_i + c_i' a_i', \dots, a_n) = c_i \cdot f(a_1, \dots, a_i, \dots, a_n) + c_i' \cdot f(a_1, \dots, a_i', \dots, a_n)$.
(3) $f(\dots, a_i, a_{i+1}, \dots) = -f(\dots, a_{i+1}, a_i, \dots)$.

(1) is a normalization, (2) is called *multilinearity*, and (3) is called the *alternating property*. It turns out that (1), (2), and (3) uniquely determine the determinant function.

We now compute $df(A)(B)$:

$$\begin{aligned} df(A)(B) &= \lim_{t\to 0} \frac{f(A+tB)-f(A)}{t} \\ &= \lim_{t\to 0} \frac{\det(a_1+tb_1,\dots,a_n+tb_n)-\det(a_1,\dots,a_n)}{t} \\ &= \lim_{t\to 0} \frac{\det(a_1,\dots,a_n)+t[\det(b_1,a_2,\dots,a_n)+\det(a_1,b_2,\dots,a_n)}{t} \\ &\quad \frac{+\cdots+\det(a_1,\dots,a_{n-1},b_n)]+t^2(\dots)-\det(a_1,\dots,a_n)}{t} \\ &= \det(b_1,a_2,\dots,a_n)+\cdots+\det(a_1,\dots,a_{n-1},b_n) \end{aligned}$$

It is easy to show that 1 is a regular value of df (it suffices to show that $df(A)$ is nonzero for any $A \in SL(n,\mathbb{R})$). For example, take $b_1 = ca_1$ where $c \in \mathbb{R}$ and $b_i = 0$ for all $i \neq 1$.

Theorem 7.6 (Sard's theorem)**.** *Let $f : U \to V$ be a smooth map. Then* almost every point $y \in \mathbb{R}^n$ *is a regular value.*

The notion of *almost every point* will be made precise later. But in the meantime:

Reality Check: In Sard's theorem what happens when $m < n$?

8. IMMERSIONS AND EMBEDDINGS

8.1. **Some more point-set topology.** We first review some more point-set topology. Let X be a topological space.

(1) A subset $V \subset X$ is *closed* if its complement $X - V = \{x \in X \mid x \notin V\}$ is open.
(2) The *closure* $\overline{V}$ of a subset $V \subset X$ is the smallest closed set containing V.
(3) A subset $V \subset X$ is *dense* if $U \cap V \neq \emptyset$ for every nonempty open set U. In other words, $\overline{V} = X$.
(4) A subset $V \subset X$ is *compact* if it satisfies the following *finite covering property*: any open cover $\{U_\alpha\}_{\alpha \in I}$ of V (i.e., the U_α are open and $\cup_{\alpha \in I} U_\alpha = V$) admits a finite subcover.
(5) A metric space is compact if and only if every sequence has a convergent subsequence.
(6) A subset of Euclidean space is compact if and only if it is closed and *bounded*, i.e., is a subset of some $B_r(y)$.
(7) A map $f : X \to Y$ is *proper* if the preimage $f^{-1}(V)$ of every compact set $V \subset Y$ is compact. (Remark: If $f : X \to Y$ is continuous, then the image of every compact set is compact.)

8.2. **Immersions.**

Definition 8.1. *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets. A smooth map $f : U \to V$ is an* immersion *if $df(x)$ is injective for all $x \in U$. (Note this means $n \geq m$.) A smooth map $f : M \to N$ between manifolds is an* immersion *if f is an immersion with respect to all local coordinates.*

Prototype: $f : \mathbb{R}^m \to \mathbb{R}^n$, $n \geq m$, $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$.

Theorem 8.2 (Implicit function theorem, immersion version)**.** *Let $f : U \to V$ be an immersion, where $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets. Then for any $p \in U$, there exist open sets $U \supset U_p \ni p$, $V \supset V_{f(p)} \ni f(p)$ and a diffeomorphism $G : V_{f(p)} \xrightarrow{\sim} W \subset \mathbb{R}^n$ such that*

$$G \circ f : U_p \to \mathbb{R}^n$$

is given by

$$(x_1, \dots, x_m) \to (x_1, \dots, x_m, 0, \dots, 0).$$

Proof. The proof is similar to that of the submersion version. Define the map

$$F : U \times \mathbb{R}^{n-m} \to \mathbb{R}^n,$$

$$(x_1, \dots, x_m, y_{m+1}, \dots, y_n) \mapsto (f_1(x), \dots, f_m(x), f_{m+1}(x) + y_{m+1}, \dots, f_n(x) + y_n),$$

where $x = (x_1, \dots, x_m)$ and $f(x) = (f_1(x), \dots, f_n(x))$. We can check that dF is nonsingular after possibly reordering the $f_1, \dots, f_n$ and that $F^{-1} \circ f : U_p \to \mathbb{R}^n$ is given by

$$(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0).$$

We then set $G = F^{-1}$. □

Zen: The implicit function theorem tells us that under a constant rank condition we may assume that locally we can straighten our manifolds and maps and pretend we are doing linear algebra.

Examples of immersions:

(1) Circle mapped to figure 8 in $\mathbb{R}^2$.
(2) The map $f : \mathbb{R} \to \mathbb{C}$, $t \mapsto e^{it}$, which wraps around the unit circle $S^1 \subset \mathbb{C}$ infinitely many times.
(3) The map $f : \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$, $t \mapsto (at, bt)$, where b/a is irrational. The image of f is *dense* in $\mathbb{R}^2/\mathbb{Z}^2$.

8.3. **Embeddings and submanifolds.** We upgrade immersions $f : M \to N$ as follows:

Definition 8.3. *An* embedding $f : M \to N$ *is an immersion which is one-to-one and proper. The image of an embedding is called a* submanifold *of* N.

The "pathological" examples above are immersions but not embeddings. Why? (1) and (2) are not one-to-one and (3) is not proper.

Proposition 8.4. *Let* M *and* N *be manifolds of dimension* m *and* n *with topologies* $\mathcal{T}$ *and* $\mathcal{T}'$. *If* $f : M \to N$ *is an embedding, then* $f^{-1}(\mathcal{T}') = \mathcal{T}$.

Proof. It suffices to show that $f^{-1}(\mathcal{T}') \supset \mathcal{T}$, since a continuous map f satisfies $f^{-1}(\mathcal{T}') \subset \mathcal{T}$. Let $x \in M$ and U be a small open set containing x. Then by the implicit function theorem f can be written locally as $U \to \mathbb{R}^n$, $x' \mapsto (x', 0)$ (where we are using x' to avoid confusion with x). We claim that there is an open set $V \subset \mathbb{R}^n$ such that $V \cap f(M) = f(U)$: Arguing by contradiction, suppose there exist $y \in f(U)$ and a sequence $\{x_i\}_{i=1}^{\infty} \subset M$ such that $f(x_i) \to y$ but $f(x_i) \notin f(U)$. The set $\{y\} \cup \{f(x_i)\}_{i=1}^{\infty}$ is compact, so $\{f^{-1}(y)\} \cup \{x_i\}_{i=1}^{\infty}$ is compact by properness, where we are recalling that f is one-to-one. By compactness, there is a subsequence of $\{x_i\}$ which converges to $f^{-1}(y)$. This implies that $x_i \in U$ and $f(x_i) \in f(U)$ for sufficiently large i, a contradiction. □

9. Tangent spaces, Day I

9.1. **Concrete example.** Consider $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. We recall the definition/computation of the *tangent plane* $T_{(a,b,c)}S^2$ from multivariable calculus. We use the fact that S^2 is the preimage of the regular value 1 of f, where

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x, y, z) = x^2 + y^2 + z^2.$$

The derivative of f at the point (a, b, c) is:

$$df(a, b, c)(x, y, z)^T = (2a, 2b, 2c)(x, y, z)^T.$$

The tangent directions are directions (x, y, z) where $df(a, b, c)(x, y, z)^T = 0$.

If you want to think of the tangent plane as a vector space, then $T_{(a,b,c)}S^2 = \{ax + by + cz = 0\}$. If you want to think of it as an affine space, then the tangent plane is the plane through (a, b, c) which is parallel to $ax + by + cz = 0$, i.e., $T_{(a,b,c)}S^2 = \{ax + by + cz = a^2 + b^2 + c^2 = 1\}$.

Definition 9.1. *Let M be a (codimension m) submanifold of $\mathbb{R}^n$. Then we can define T_pM as follows: Pick a small neighborhood $U_p \subset \mathbb{R}^n$ of p and a submersion $f : U_p \to \mathbb{R}^m$ such that $M \cap U_p$ is a level set of f. Then T_pM is the set of vectors $v \in \mathbb{R}^n$ such that $df(p)(v) = 0$.*

Some issues with this definition:

1. Need to verify that T_pM does not depend on the choice of $f : U_p \to \mathbb{R}^m$. (Not so serious.)
2. The definition seems to depend on how M is embedded in $\mathbb{R}^n$. In other words, the definition is *not intrinsic*.

We will give several definitions of T_pM which are intrinsic, in increasing order of abstraction!!

9.2. **First definition.** Let M be a smooth n-dimensional manifold. If $U \subset M$ is an open set, then let $C^\infty(U)$ be the set of smooth functions $f : U \to \mathbb{R}$.

Notation: Let $f, g : U \to \mathbb{R}$ where $0 \in U \subset \mathbb{R}$. Then $f = O(g)$ if there exists a constant C such that $|f(t)| \leq C|g(t)|$ for all t sufficiently close to 0. For example, $t = \sin t + O(t^3)$ near $t = 0$.

Definition 9.2 (First definition). *The* tangent space $T_p^{(1)}M$ *(here (1) is to indicate that it's the first definition) to M at p is the set of equivalence classes*

$$T_p^{(1)}(M) = \{\text{smooth curves } \gamma : (-\varepsilon_\gamma, \varepsilon_\gamma) \to M, \gamma(0) = p\}/\sim,$$

where $\gamma_1 \sim \gamma_2$ if $f \circ \gamma_1(t) = f \circ \gamma_2(t) + O(t^2)$ for all pairs (f, U) where U is an open set containing p and $f \in C^\infty(U)$. Here $\varepsilon_\gamma > 0$ is a constant which depends on γ.

Let $x_1, \dots, x_n$ be coordinate functions for an open set $U \subset M$.

Theorem 9.3 (Taylor's Theorem). *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth function and $0 \in U$. Then we can write*

$$f(x) = a + \sum_i a_i x_i + \sum_{i,j} a_{ij}(x) x_i x_j,$$

on an open rectangle $(-a_1, b_1) \times \cdots \times (-a_n, b_n) \subset U$ which contains 0, where a, a_i are constants and $a_{ij}(x)$ are smooth functions.

Proof. We prove the theorem for one variable x. By the Fundamental Theorem of Calculus,

$$g(1) - g(0) = \int_0^1 g'(t)dt.$$

Substituting $g(t) = f(tx)$ and integrating by parts (i.e., $\int udv = uv - \int vdu$) with $u = \frac{d}{dt}f(tx)$ and $v = t - 1$ we obtain:

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt} f(tx)dt \\ &= x \cdot f'(tx) \cdot (t-1)\big|_{t=0}^{t=1} - \int_0^1 (t-1)x^2 \cdot f''(tx)dt \\ &= -xf'(0)(-1) - x^2 \int_0^1 (t-1)f''(tx)dt \\ &= f'(0) \cdot x + h(x) \cdot x^2. \end{aligned}$$

Here we write $h(x) = -\int_0^1 (t-1)f''(tx)dt$. This gives us the desired result

$$f(x) = f(0) + f'(0) \cdot x + h(x) \cdot x^2$$

for one variable. □

Corollary 9.4. $\gamma_1 \sim \gamma_2$ *if and only if* $x_i(\gamma_1(t)) = x_i(\gamma_2(t)) + O(t^2)$ *for* $i = 1, \ldots, n$.

Corollary 9.5. *If* M *is a submanifold of* $\mathbb{R}^m$*, then* $\gamma_1 \sim \gamma_2$ *if and only if* $\gamma_1'(0) = \gamma_2'(0)$.

Remark: At this point it's not clear whether $T_p^{(1)}M$ has a canonical vector space structure. Here's one possible definition: Choose coordinates $x = (x_1, \ldots, x_n)$ about $p = 0$. If we write $\gamma_1, \gamma_2 \in T_p^{(1)}M$ with respect to x, then we can do addition $x \circ \gamma_1 + x \circ \gamma_2$ and scalar multiplication $cx \circ \gamma_1$. However, the above vector space structure depends on the choice of coordinates. Of course we can show that the definition does not depend on the choice of coordinates, but the vector space structure is clearly canonical in the definition of $T_p^{(2)}M$ from next time.

10. Tangent spaces, Day II

10.1. **Sheaf-theoretic notions.** We first discuss some sheaf-theoretic notions.

The set $C^\infty(U)$ of smooth functions $f : U \to \mathbb{R}$ is an *algebra over* $\mathbb{R}$, i.e., it has operations $c \cdot f$, $f \cdot g$, $f + g$, where $c \in \mathbb{R}$ and $f, g \in C^\infty(U)$.

Let $V \subset M$ be any set, not necessarily open. Then we define

$$C^\infty(V) = \{(f, U) \mid U \supset V, f : U \to \mathbb{R} \text{ smooth}\}/\sim .$$

Here $(f_1, U_1) \sim (f_2, U_2)$ if there exists $V \subset U \subset U_1 \cap U_2$ for which $f_1|_U = f_2|_U$. When we refer to a *smooth function on* V, what we really mean is an element of $C^\infty(V)$, since we need open sets to define derivatives.

Given open sets $U_1 \subset U_2$, there exists a natural *restriction map* $\rho^{U_2}_{U_1} : C^\infty(U_2) \to C^\infty(U_1)$, $f \mapsto f|_{U_1}$. Then $C^\infty(V)$ is the *direct limit** of $C^\infty(U)$ for all U containing V.

Example: When $V = \{p\}$, $C^\infty(\{p\})$ (written simply as $C^\infty(p)$), is called the *stalk* at the point p or the set of *germs of smooth functions* at p.

10.2. **Second definition.**

Definition 10.1 (Second definition)**.** *A* derivation *is an* $\mathbb{R}$*-linear map* $X : C^\infty(p) \to \mathbb{R}$ *which satisfies the Leibniz rule:*

$$X(fg) = X(f) \cdot g(p) + f(p) \cdot X(g).$$

The tangent space $T^{(2)}_p M$ *is the set of derivations at* p*.*

By definition, $T^{(2)}_p M$ is clearly a vector space over $\mathbb{R}$.

Remark: It does not matter whether M is a manifold — it could have been Euclidean space instead, since $C^\infty(p)$ only depends on a small neighborhood of p.

HW: If X is a derivation, then $X(c) = 0$, $c \in \mathbb{R}$.

Examples.

(1) $X_i = \frac{\partial}{\partial x_i}$. Take local coordinate functions $x_1, \dots, x_n$ near $p = 0$. Then let $X(f) = \frac{\partial f}{\partial x_i}(0)$. Check: this is indeed a derivation and $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$.

(2) Given $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, define $X(f) = (f \circ \gamma)'(0)$. This is usually called the *directional derivative of* f *in the direction* γ. Easy check: two $\gamma \sim \gamma'$ give rise to the same directional derivative.

Lemma 10.2. *If* M *is an* n*-dimensional manifold, then* $\dim T^{(2)}_p M = n$*.*

Proof. Take local coordinates $x_1, \dots, x_n$ so that $p = 0$. Let $X_i = \frac{\partial}{\partial x_i}$. Then $X_i(x_j) = \delta_{ij}$ and clearly the X_i are linearly independent. Thus $\dim T^{(2)}_p M \geq n$. Now, given some derivation X, suppose $X(x_i) = b_i$. Then the derivation $Y = X - \sum_j b_j X_j$ satisfies $Y(x_i) = 0$ for all

x_i. By Taylor's Theorem, any $f \in C^\infty(p)$ can be written as $a + \sum a_i x_i + \sum a_{ij} x_i x_j$. By the derivation property, all the quadratic and higher terms vanish, and hence $Y(f) = 0$. Therefore $\dim T_p^{(2)} M = n$. □

Lemma 10.3. *The first two definitions of T_pM are equivalent.*

Proof. Given $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, we define $X(f) = (f \circ \gamma)'(0)$ as in the example. Since we already calculated $\dim T_pM = n$ for the first and second definitions, we see this map is surjective and hence an isomorphism. □

10.3. **Third definition.** Define $\mathcal{F}_p \subset C^\infty(p)$ to be germs of functions which are 0 at p. $\mathcal{F}_p$ is an *ideal* of $C^\infty(p)$. This means that if $f \in \mathcal{F}_p$ and $g \in C^\infty(p)$, then $fg \in \mathcal{F}_p$. Let $\mathcal{F}_p^2 \subset \mathcal{F}_p$ be the ideal of $C^\infty(p)$ generated by products of elements of $\mathcal{F}_p$, i.e., consisting of elements $\sum f_i \phi_j \phi_k$, where $\phi_j, \phi_k \in \mathcal{F}_p$.

Definition 10.4 (Third definition). $T_p^{(3)} M = (\mathcal{F}_p / \mathcal{F}_p^2)^*$*, i.e.,* $T_p^{(3)} M$ *is the dual vector space (the space of* $\mathbb{R}$*-linear functionals) of* $\mathcal{F}_p / \mathcal{F}_p^2$*.*

Equivalence of $T_p^{(2)} M$ and $T_p^{(3)} M$: Recall that a derivation $X : C^\infty(p) \to \mathbb{R}$ satisfies $X(const) = 0$. Show that $X : \mathcal{F}_p \to \mathbb{R}$ factors through $\mathcal{F}_p^2$. (Pretty easy, since it's a derivation.) Hence we have a linear map $T_p^{(2)} M \to T_p^{(3)} M$. Now note that $\dim(\mathcal{F}_p / \mathcal{F}_p^2) = n$ by Taylor's Theorem.

$\mathcal{F}_p / \mathcal{F}_p^2$ is called the *cotangent space* at p, and is denoted $T_p^* M$. If $f \in C^\infty(p)$, then $f - f(p) \in \mathcal{F}_p$, and is denoted $df(p)$, when viewed as an element $[f - f(p)] \in \mathcal{F}_p / \mathcal{F}_p^2$.

11. THE TANGENT BUNDLE

11.1. **The tangent bundle.** Let $TM = \sqcup_{p\in M} T_pM$. This is called the *tangent bundle*. We explain how to topologize (i.e., give a topology) and give a smooth structure on the tangent bundle.

Consider the projection $\pi : TM \to M$ which sends any $q \in T_pM$ to p. Let $U \subset M$ be an open set with coordinates $x_1, \dots, x_n$. Since an element $q \in T_pM$ is written as $\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, we identify $\pi^{-1}(U) \simeq U \times \mathbb{R}^n$ by sending the point $q \in TM$ to $(x_1, \dots, x_n, a_1, \dots, a_n)$. By this identification we can induce a topology on $\pi^{-1}(U)$ from $U \times \mathbb{R}^n$; at the same time, we also obtain a local chart for $\pi^{-1}(U)$.

Check the transition functions. Let $x = (x_1, \dots, x_n)$ be coordinates on U and $y = (y_1, \dots, y_n)$ be coordinates on V. We need to show that the induced topologies on $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are consistent and that the transition functions on $\pi^{-1}(U \cap V)$ are smooth.

Let (x, a) be coordinates on $\pi^{-1}(U)$ and (y, b) be coordinates on $\pi^{-1}(V)$. Think of y as a function of x on $U \cap V$. Write $\frac{\partial y}{\partial x} = (\frac{\partial y_i}{\partial x_j})$. In terms of y coordinates,

$$\sum a_i \frac{\partial}{\partial x_i} = \sum a_i \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}.$$

This is easily verified by thinking of evaluation on functions. Thus, $b_j = \sum_i a_i \frac{\partial y_j}{\partial x_i}$.

Then $q \in TM$ corresponds to (x, a) or $(y, \frac{\partial y}{\partial x} a)$, where a is viewed as a column matrix.

Computation of the Jacobian of the transition function. The Jacobian matrix of the transition map $(x, a) \mapsto (y, \frac{\partial y}{\partial x} a)$ is:

$$\begin{pmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial a} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial a} \end{pmatrix} = \begin{pmatrix} \frac{\partial y}{\partial x} & 0 \\ \sum_k \frac{\partial^2 y_i}{\partial x_j \partial x_k} a_k & \frac{\partial y}{\partial x} \end{pmatrix}.$$

The two terms on the bottom are obtained by differentiating $b_i = \sum_k \frac{\partial y_i}{\partial x_k} a_k$.

Thus we obtain a smooth manifold TM and a C^∞-function $TM \xrightarrow{\pi} M$.

11.2. **Examples of tangent bundles.**

Example: The tangent bundle of $U \subset \mathbb{R}^n$. $TU \simeq U \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$. A tangent vector is described by a point $(x_1, \dots, x_n) \in U$, together with a tangent direction $(a_1, \dots, a_n) \in \mathbb{R}^n$, written as $\sum_i a_i \frac{\partial}{\partial x_i}$.

Example: $S^2 \subset \mathbb{R}^3$, $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$. A multivariable calculus definition of TS^2 is the following: Think of $TS^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ with coordinates (x, y) where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. At $x \in S^2$, T_xS^2 is the set of points y such that $x \cdot y = 0$. Therefore:

$$TS^2 = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |x| = 1, x \cdot y = 0\}.$$

HW: Show that the tangent bundle TS^2 defined in this way is diffeomorphic to our "official definition" of the tangent bundle.

11.3. **Complex manifolds.**

More abstract example: S^2 defined by gluing coordinate charts. Let $U = \mathbb{R}^2$ and $V = \mathbb{R}^2$, with coordinates (x_1, y_1), (x_2, y_2), respectively. Alternatively, think of $\mathbb{R}^2 = \mathbb{C}$. Take $U \cap V = \mathbb{C} - \{0\}$. The transition function is:

$$U - \{0\} \xrightarrow{\phi_{UV}} V - \{0\},$$
$$z \mapsto \frac{1}{z},$$

with respect to complex coordinates $z = x + iy$. With respect to real coordinates,

$$(x, y) \mapsto \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2} \right).$$

S^2 has the structure of a *complex manifold*.

Definition 11.1. *A function $\phi : \mathbb{C} \to \mathbb{C}$ is* holomorphic *(or* complex analytic*) if*

$$\frac{d\phi}{dz} = \lim_{h \to 0} \frac{\phi(z + h) - \phi(z)}{h}$$

exists for all $z \in \mathbb{C}$. Here $h \in \mathbb{C} - \{0\}$.

A function $\phi : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic if

$$\frac{\partial \phi}{\partial z_i} = \lim_{h \to 0} \frac{\phi(z_1, \ldots, z_i + h, \ldots, z_n) - \phi(z_1, \ldots, z_n)}{h}$$

exists for all $z = (z_1, \ldots, z_n)$ and $i = 1, \ldots n$.

Definition 11.2. *A complex manifold is a topological manifold with an atlas $\{U_\alpha, \phi_\alpha\}$, where $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ and $\phi_\beta \circ \phi_\alpha^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic map.*

Remark: A *holomorphic map* $f : \mathbb{C} \to \mathbb{C}$, when viewed as a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, is a smooth map. Hence a complex manifold is automatically a smooth manifold.

Compute the Jacobians. Rewriting as a map $\phi_{UV} : \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$, we compute:

$$J_{\phi_{UV}} = \begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) & \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) \\ \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) & \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \end{pmatrix} = \frac{1}{(x^2+y^2)^2} \begin{pmatrix} y^2 - x^2 & -2xy \\ 2xy & y^2 - x^2 \end{pmatrix}.$$

Remark: It is not a coincidence that $a_{11} = a_{22}$ and $a_{21} = -a_{12}$.

Explain that TS^2 is obtained by gluing two copies of $(\mathbb{R}^2 - \{0\}) \times \mathbb{R}^2$ together via a map which sends $(a_1, a_2)^T$ over (x_1, y_1) to $J(a_1, a_2)^T$ over (x_2, y_2).

12. Cotangent bundles and 1-forms

12.1. **The cotangent bundle.** An element of the cotangent space $T_p^*M = \mathcal{F}_p/\mathcal{F}_p^2$ is $df(p) = [f - f(p)]$, which we often write without brackets. It is not hard to see that if $x_1, \dots, x_n$ are local coordinates near p, then $dx_1(p), \dots, dx_n(p)$ are linearly independent and hence form a basis for T_p^*M. Therefore, an element of T_p^*M can be written as $\sum a_i dx_i$.

We now "topologize" the *cotangent bundle* $T^*M = \sqcup_p T_p^*M$. Again we have a projection $\pi : T^*M \to M$. Given a coordinate chart $U \subset M$, we identify $\pi^{-1}(U) \simeq U \times \mathbb{R}^n$. This induces the topology and smooth structure on $\pi^{-1}(U)$.

Transition functions. Take charts $U, V \subset M$, and coordinatize $\pi^{-1}(U)$ and $\pi^{-1}(V)$ by (x, a) and (y, b), where a corresponds to $\sum a_i dx_i$ and b corresponds to $\sum b_i dy_i$.

Lemma 12.1. *On the overlap $U \cap V$, $dy_i = \sum_j \frac{\partial y_i}{\partial x_j} dx_j$.*

Similarly, if f is a smooth function on U, then $df = \sum_j \frac{\partial f}{\partial x_j} dx_j$.

Proof. If $p \in U \cap V$, then, using Taylor's Theorem,

$$dy_i(p) = y_i - y_i(p) = \sum_j \frac{\partial y_i}{\partial x_j}(p)(x_j - x_j(p)) = \sum_j \frac{\partial y_i}{\partial x_j}(p) dx_j(p).$$

□

We denote this more simply as $dy = \frac{\partial y}{\partial x} dx$. Then $dx = \left(\frac{\partial y}{\partial x}\right)^{-1} dy$. Hence $\sum a_i dx_i = \sum a_i [(\frac{\partial y}{\partial x})^{-1}]_{ij} dy_j$ and $(x, a) \mapsto (y, ((\frac{\partial y}{\partial x})^{-1})^T a)$, if we write a as a column matrix.

HW: Compute the Jacobian of the transition function.

12.2. **Functoriality.** Let $f : M \to N$ be a smooth map between manifolds. Then we can define the following natural maps:

Contravariant functor. f induces the map $C^\infty(f(p)) \xrightarrow{f^*} C^\infty(p)$ which restricts to $\mathcal{F}_{f(p)} \xrightarrow{f^*} \mathcal{F}_p$. (Check this!) This descends to the quotient

$$f^* : T_{f(p)}^*N \to T_p^*M,$$
$$dg \mapsto dg \circ f.$$

(The functor takes the category of "pointed smooth manifolds", i.e., pairs (M, p) consisting of a smooth manifold and a point $p \in M$ and smooth maps $f : (M, p) \to (N, f(p))$, to the category of $\mathbb{R}$-vector spaces. The functor is contravariant, i.e., reverses directions.)

Covariant functor. f also induces the map

$$f_* : T_pM \to T_{f(p)}N,$$

given by $X \mapsto X \circ f^*$. (Here we are using $T_pM = T_p^{(3)}M$.) This makes sense:

$$T_{f(p)}^*N \xrightarrow{f^*} T_p^*M \xrightarrow{X} \mathbb{R}.$$

HW: Define the derivative map in terms of $T_p^{(1)}M$ and show the equivalence with the definition just given.

Derivative map. The maps $f_* : T_pM \to T_{f(p)}N$ can be combined into the *derivative map*

$$f_* : TM = \sqcup_{p\in M} T_pM \to TN = \sqcup_{r\in N} T_rN.$$

HW: Show that the derivative map is a smooth map between tangent bundles.

12.3. **Properties of 1-forms.**

Definition 12.2. *Let* $T^*M \xrightarrow{\pi} M$ *be the cotangent bundle. A* 1-form over $U \subset M$ *is a smooth map* $s : U \to T^*M$ *such that* $\pi \circ s = id$*.*

Note that a 1-form assigns an element of T_p^*M to a given $p \in M$ in a smooth manner. The space of 1-forms on U is denoted by $\Omega^1(U)$ and is an $\mathbb{R}$-vector space.

1. We often write $\Omega^0(M) = C^\infty(M)$. Then there exists a map $d : \Omega^0(M) \to \Omega^1(M)$, $g \mapsto dg$. HW: Verify that dg is a 1-form on M, i.e., check that dg is a smooth map $M \to T^*M$. Hint: in local coordinates dg can be written as $x = (x_1, \dots, x_n) \mapsto (x, \frac{\partial g}{\partial x_1}(x), \dots, \frac{\partial g}{\partial x_n}(x))$.

2. Given $\phi : M \to N$, there is no natural map $T^*M \to T^*N$ unless ϕ is a diffeomorphism. However, there exists $\phi^* : \Omega^1(N) \to \Omega^1(M)$, $\theta \mapsto \phi^*\theta$. HW: check that the map is well-defined, i.e., indeed takes a smooth section of T^*N to a smooth section of T^*M. The 1-form $\phi^*\theta$ is called the *pullback* of θ under ϕ.

3. Let $\psi : L \to M$ and $\phi : M \to N$ be smooth maps between smooth manifolds and let θ be a 1-form on N. Then $(\phi \circ \psi)^*\theta = \psi^*(\phi^*\theta)$. HW: check this. Note however that the order of pulling back is reasonable.

4. There exists a commutative diagram:

$$\begin{array}{ccc} \Omega^0(N) & \xrightarrow{\phi^*} & \Omega^0(M) \\ d\downarrow & \circlearrowleft & \downarrow d \\ \Omega^1(N) & \xrightarrow{\phi^*} & \Omega^1(M) \end{array},$$

i.e., $d \circ \phi^* = \phi^* \circ d$. HW: Check this by unwinding the definitions.

5. $d(gh) = gdh + hdg$. HW: Check this.

Example: Let $\theta = f(x,y)dx + g(x,y)dy$ on $\mathbb{R}^2$. If $i : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (a(t), b(t))$, then

$$i^*\theta = (f(a(t), b(t))a'(t) + g(a(t), b(t))b'(t))dt.$$

13. Lie groups

13.1. **Lie groups.**

Definition 13.1. *A* Lie group G *is a smooth manifold together with smooth maps* $\mu : G \times G \to G$ *(multiplication) and* $i : G \to G$ *(inverse) which make* G *into a group.*

Definition 13.2. *A* Lie subgroup $H \subset G$ *is a subgroup of* G *such that the inclusion map* $H \to G$ *is a* 1-1 *immersion.*[4] *A* Lie group homomorphism $\phi : H \to G$ *is a homomorphism which is also a smooth map of the underlying manifolds.*

Examples of Lie groups:

1. $\mathbb{R}^n$ with the usual addition; the quotient $T^n = \mathbb{R}^n/\mathbb{Z}^n$ with the usual addition.
2. The general linear group $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$. We already showed that this is a manifold. The product AB is defined by a formula which is polynomial in the matrix entries of A and B, so μ is smooth. Similarly one can show that i is smooth.
3. More invariantly, given a finite-dimensional $\mathbb{R}$-vector space V, let $GL(V)$ be the group of $\mathbb{R}$-linear automorphisms $V \xrightarrow{\sim} V$.
4. The special linear group $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$.
5. The orthogonal group $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = I\}$.
6. The special orthogonal group $SO(n, \mathbb{R}) = SL(n, \mathbb{R}) \cap O(n)$.

Examples of Lie subgroups:

1. $SL(n, \mathbb{R})$, $O(n)$, and $SO(n, \mathbb{R})$ are Lie subgroups of $GL(n, \mathbb{R})$; $SO(n, \mathbb{R})$ is a Lie subgroup of both $SL(n, \mathbb{R})$ and $O(n)$.
2. Let V be a subspace of $\mathbb{R}^n$. Then the image of V under the quotient map
$$\mathbb{R}^n \to T^n = \mathbb{R}^n/\mathbb{Z}^n$$
is a Lie subgroup. For example, we can take $V = \mathbb{R}\langle(a, b)\rangle \subset \mathbb{R}^2$. If b/a is irrational, then the image of V is dense in T^2. See Section 8.2. This is why we want to define a Lie subgroup as the image of a 1-1 immersion.

Definition 13.3. *A* Lie group representation *is a Lie group homomorphism* $\phi : G \to GL(V)$ *for some finite-dimensional* $\mathbb{R}$*-vector space* V.

13.2. **Extended example:** $O(n)$**.**

1. $AA^T = I$ implies $\det(AA^T) = \det(I) \Rightarrow \det(A) = \pm 1$. Here we are using $\det(AB) = \det(A) \cdot \det(B)$ and $\det(A^T) = \det(A)$.

2. Note that $O(n) \subset GL(n, \mathbb{R})$ but $O(n)$ is not quite a subgroup of $SL(n, \mathbb{R})$. $O(n)$ has two connected components
$$SO(n) = O(n) \cap \{\det(A) = 1\} = O(n) \cap SL(n, \mathbb{R})$$

[4] In some texts a submanifold is the image of a 1-1 immersion; such a definition would be more natural in Lie theory.

and $A_0 \cdot SO(n) = O(n) \cap \{\det(A) = -1\}$, where $A_0 = \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix} \in O(n) \cap \{\det(A) = -1\}$.

3. Show $O(n)$ is a submanifold of $GL(n, \mathbb{R})$. Consider the map $\phi : GL(n, \mathbb{R}) \to \mathrm{Sym}(n)$ given by $A \mapsto AA^T$, where $\mathrm{Sym}(n)$ is the symmetric $n \times n$ matrices with real coefficients. We compute that

$$d\phi(A)(B) = AB^T + BA^T = AB^T + (AB^T)^T.$$

Since for any symmetric matrix C there is a solution to the equation $AB^T = C$ (here A is fixed and we are solving for B), it follows that $d\phi(A)$ is surjective.

4. $\dim O(n) = \dim GL(n, \mathbb{R}) - \dim \mathrm{Sym}(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

5. $SO(n)$ is the group of "rigid rotations" of $S^{n-1} \subset \mathbb{R}^n$. To see this, write $A = (a_1, \dots, a_n)^T$, where a_i are column vectors. Then $AA^T = id$ implies that $a_i^T a_j = \delta_{ij}$, and $\{a_1, \dots, a_n\}$ forms an orthonormal basis for $\mathbb{R}^n$ with the usual inner product. (The extra ingredient of $\det = 1$ is necessary for A to be a "rigid rotation".)

6. Show compactness of $O(n)$. Since $O(n) = \phi^{-1}(I)$, it follows that $O(n)$ is closed. It is also bounded by the previous paragraph.

7. HW: Show that $SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$ and that $SO(2)$ is diffeomorphic to S^1.

8. Consider $SO(3)$. These are the rigid rotations of $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Prove that every element A of $SO(3)$ which is not the identity has a unique axis of rotation in $\mathbb{R}^3$. In other words, there is a unique line through the origin in $\mathbb{R}^3$ which is fixed by A, and A is given by a rotation about this axis.

14. VECTOR BUNDLES

14.1. **Vector bundles.** The tangent bundle $TM \stackrel{\pi}{\to} M$ and the cotangent bundle $T^*M \stackrel{\pi}{\to} M$ are examples of *vector bundles*.

Definition 14.1. *A* real vector bundle of rank k *over a smooth manifold* M *is a pair* $(E, \pi : E \to M)$*, where:*

(1) E *is a smooth manifold and* $\pi : E \to M$ *is a smooth map.*
(2) $\pi^{-1}(p)$ *has a structure of an* $\mathbb{R}$*-vector space of dimension* k*.*
(3) *There exists an open cover* $\{U_\alpha\}$ *of* M *and identifications* $\pi^{-1}(U_\alpha) \stackrel{\sim}{\to} U_\alpha \times \mathbb{R}^k$ *which restrict to vector space isomorphisms* $\pi^{-1}(p) \stackrel{\sim}{\to} \mathbb{R}^k$ *for all* $p \in M$*.*

A rank 1 *vector bundle is often called a* line bundle*.*

The identifications $\pi^{-1}(U_\alpha) \stackrel{\sim}{\to} U_\alpha \times \mathbb{R}^k$ in (3) are called *local trivializations*.

Definition 14.2. *A* section *of a vector bundle* $\pi : E \to M$ *over an open set* $U \subset M$ *is a smooth map* $s : U \to E$ *such that* $\pi \circ s = \mathrm{id}$*. A section over* M *is called a* global section*.*

We write $\Gamma(E, U)$ for the space of sections of E over U; we also write $\Gamma(E)$ if $U = M$. Note that $\Gamma(E, U)$ has an $\mathbb{R}$-vector space structure.

Sections of TM are called *vector fields* and we often write $\mathfrak{X}(M) = \Gamma(TM)$. Sections of T^*M are called *1-forms* and we often write $\Omega^1(M) = \Gamma(T^*M)$.

14.2. **Transition functions, reinterpreted.** Consider $\pi : TM \to M$ and local trivializations $\pi^{-1}(U) \simeq U \times \mathbb{R}^n$, $\pi^{-1}(V) \simeq V \times \mathbb{R}^n$. Let $x = (x_1, \dots, x_n)$ be the coordinates for U and $y = (y_1, \dots, y_n)$ be the coordinates for V. We already computed the transition functions

$$\phi_{UV} : (U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n$$

$$(x, a) \mapsto (y(x), \tfrac{\partial y}{\partial x}(x) a),$$

where the domain is viewed as a subset of $U \times \mathbb{R}^n$, the range is viewed as a subset of $V \times \mathbb{R}^n$, and $a = (a_1, \dots, a_n)^T$. Alternatively, think of ϕ_{UV} as

$$\Phi_{UV} : U \cap V \to GL(n, \mathbb{R}),$$

$$x \mapsto \tfrac{\partial y}{\partial x}(x).$$

(1) For double intersections $U \cap V$, we have $\Phi_{UV}(p) \cdot \Phi_{VU}(p) = \mathrm{id}$.

(2) For triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$\Phi_{U_\alpha U_\gamma}(p) = \Phi_{U_\beta U_\gamma}(p) \cdot \Phi_{U_\alpha U_\beta}(p).$$

This is called the *cocycle condition*. [If $U_\alpha, U_\beta, U_\gamma$ have coordinates x, y, z, then $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$ by the chain rule.]

What's this cocycle condition? This cocycle condition (triple intersection property) is clearly necessary if we want to construct a vector bundle by patching together $\{U_\alpha \times \mathbb{R}^n\}$. It guarantees that the gluings that we prescribe, i.e., $\Phi_{U_\alpha U_\beta}$ from U_α to U_β, etc. are *compatible*.

HW: On the other hand, if we can find a collection $\{\Phi_{U_\alpha U_\beta}\}$ (for all U_α, U_β), which satisfies Conditions (1) and (2), then we can construct a vector bundle by gluing $\{U_\alpha \times \mathbb{R}^n\}$ using this prescription.

Consider $\pi : T^*M \to M$. In a previous lecture we computed that the transition functions $\Phi_{UV} : U \cap V \to GL(n, \mathbb{R})$ are given by $x \mapsto ((\frac{\partial y}{\partial x}(x))^{-1})^T$. The inverse and transpose are both necessary for the cocycle condition to be met.

14.3. **Constructing new vector bundles out of TM.** Let M be a manifold and $\{U_\alpha\}$ an atlas for M. View TM as being constructed out of $\{U_\alpha \times \mathbb{R}^n\}$ by gluing using transition functions $\Phi_{U_\alpha U_\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ that satisfy Conditions (1) and (2).

Consider a representation $\rho : GL(n, \mathbb{R}) \to GL(m, \mathbb{R})$.

Examples of representations:

(1) $\rho : GL(n, \mathbb{R}) \to GL(1, \mathbb{R}) = \mathbb{R}^\times$, $A \mapsto \det(A)$.
(2) $\rho : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$, $A \mapsto BAB^{-1}$.
(3) $\rho : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$, $A \mapsto (A^{-1})^T$.

We can use ρ and glue $U_\alpha \times \mathbb{R}^m$ together using:

$$\rho \circ \Phi_{U_\alpha U_\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}) \to GL(m, \mathbb{R}).$$

Observe that Conditions (1) and (2) are satisfied since ρ is a representation. Therefore we obtain a new vector bundle $TM \times_\rho \mathbb{R}^m$, called TM *twisted by* ρ.

Examples of bundles obtained by twisting:

(1) Gives rise to a line bundle which is usually denoted $\wedge^n TM$.
(3) Gives the cotangent bundle T^*M.

15. MORE ON VECTOR BUNDLES; ORIENTABILITY

15.1. **Orientability.** Let $GL^+(k,\mathbb{R}) \subset GL(k,\mathbb{R})$ be the Lie subgroup of $k \times k$ matrices with positive determinant. Observe that $GL(k,\mathbb{R})$, like $O(k)$, is not connected, and has two connected components $GL^+(k;\mathbb{R})$ and $\mathrm{diag}(-1,1,\dots,1)\cdot GL^+(k;\mathbb{R})$.

Let $E \stackrel{\pi}{\to} M$ be a rank k vector bundle, $\{U_\alpha\}$ be a maximal open cover of M on which $\pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{R}^k$, and

$$\Phi_{U_\alpha U_\beta} : U_\alpha \cap U_\beta \to GL(k,\mathbb{R}),$$

be the transition functions.

Definition 15.1. *$E \to M$ is an* orientable vector bundle *if there exists a subcover such that every $\Phi_{U_\alpha U_\beta}$ factors through $GL^+(k,\mathbb{R})$ (i.e., the image of $\Phi_{U_\alpha U_\beta}$ is in $GL^+(k,\mathbb{R})$).*

Definition 15.2. *A manifold M is* orientable *if $TM \to M$ is an orientable vector bundle.*

It is not easy to prove, directly from the definition, that the following examples are *not* orientable.

Example: The Möbius band B is obtained from $[0,1]\times\mathbb{R}$ by identifying $(0,t) \sim (1,-t)$. By following an oriented basis along the length of the band, we see that the orientation is reversed when we cross $\{1\}\times\mathbb{R}$. Hence B is not orientable.

Example: The Klein bottle K is obtained from $[0,1]\times[0,1]$ by identifying $(0,t) \sim (1,1-t)$ and $(s,0)\sim(s,1)$. It is not orientable.

Example: $\mathbb{RP}^2 = \mathbb{R}^3 - \{0\}/\sim$, where $x \sim tx$, $t \in \mathbb{R}-\{0\}$. This is the set of lines through the origin of $\mathbb{R}^3$. Take the unit sphere S^2. Then $\mathbb{RP}^2$ is the quotient of S^2, obtained by identifying $x \sim -x$. By following an oriented basis from x to $-x$, we find that $\mathbb{RP}^2$ is not orientable. Observe that there is a two-to-one map $\pi : S^2 \to \mathbb{RP}^2$, which is called the *orientation double cover*.

Classification of compact 2-manifolds (surfaces). The oriented ones are: S^2, T^2, surface of genus g. The nonorientable ones are: $\mathbb{RP}^2$, Klein bottle, and one whose orientation double cover is an orientable surface of genus g.

Remark 15.3. *Recall $\wedge^n TM$ from last time. The orientability of M is equivalent to the existence of a global section $s \in \Gamma(\wedge^n TM, M)$ which is nonvanishing (i.e., never zero).*

15.2. **Complex manifolds.** Let $GL(n,\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det A \neq 0\}$.

Claim 15.4. *$GL(n,\mathbb{C})$ is a Lie subgroup of $GL(2n,\mathbb{R})$.*

Proof. We'll explain $\rho : GL(1,\mathbb{C}) \hookrightarrow GL(2,\mathbb{R})$ and leave the general case as an exercise. Consider $z \in GL(1,\mathbb{C})$. If we write $z = x+iy$, then ρ maps

$$z = x+iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

It is easy to verify that ρ is a homomorphism (i.e., a representation). ☐

Example: Recall S^2 as a complex manifold, obtained by gluing together $U = \mathbb{C}$ and $V = \mathbb{C}$ via the map $z \mapsto \frac{1}{z}$. The transition function, written in real coordinates (x, y), was:

$$\Phi_{UV} : U \cap V \to GL(2, \mathbb{R}),$$
$$(x, y) \mapsto \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} y^2 - x^2 & -2xy \\ 2xy & y^2 - x^2 \end{pmatrix}.$$

By Claim 15.4 exists a factorization:

$$\Phi_{UV} : U \cap V \to GL(1, \mathbb{C}) \to GL(2, \mathbb{R}).$$

Also note that the determinant is positive, so S^2 is oriented.

HW: Show that $GL(n, \mathbb{C}) \subset GL^+(2n, \mathbb{R})$. This implies that complex manifolds are always orientable.

16. Integrating 1-forms; tensor products

16.1. **Integrating 1-forms.** Let C be an embedded arc in M, i.e., it is the image of some embedding $\gamma : [c,d] \to M$. In addition, we assume that C is *oriented*. In this case, the direction/orientation on C arises from the usual ordering on the real line. Let ω be a 1-form on M. Then we define the *integral* of ω over C to be:

$$\int_C \omega \stackrel{def}{=} \int_c^d \gamma^*\omega.$$

If t is the coordinate on $[c,d]$, then $\gamma^*\omega$ (in fact any 1-form) will have the form $f(t)dt$.

Lemma 16.1. *The definition does not depend on the particular orientation-preserving parametrization $\gamma : [c,d] \to M$.*

Proof. Take a different parametrization $\gamma_1 : [a,b] \to M$ of C. Then there exists an orientation-preserving diffeomorphism $g : [a,b] \to [c,d]$ such that $\gamma_1 = \gamma \circ g$. (In our case, orientation-preserving means $g(a) = c$ and $g(b) = d$.) Now, $\gamma_1^*\omega = (\gamma \circ g)^*\omega = g^*(\gamma^*\omega)$, and

$$\int_c^d \gamma^*\omega = \int_c^d f(t)dt = \int_a^b f(g(s))dg(s) = \int_a^b \gamma_1^*\omega.$$

□

Now we know how to integrate 1-forms. Over the next few weeks we will define objects that we can integrate on higher-dimensional submanifolds (not just curves), called *k-forms.* For this we need to do quite a bit of preparation.

16.2. **Tensor products.** We first review some notions in linear algebra and then define the tensor product. Let V, W be vector spaces over $\mathbb{R}$. (The vector spaces do not need to be finite-dimensional or over $\mathbb{R}$, but you may suppose they are if you want.)

1. (Direct sum) $V \oplus W$. As a set, $V \oplus W = V \times W$. The addition is given by $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and the scalar multiplication is given by $c(v,w) = (cv, cw)$. $\dim(V \oplus W) = \dim(V) + \dim(W)$.

2. $\mathrm{Hom}(V,W) = \{\mathbb{R}\text{-linear maps } \phi : V \to W\}$. In particular, we have $V^* = \mathrm{Hom}(V,\mathbb{R})$. $\dim(\mathrm{Hom}(V,W)) = \dim(V) \cdot \dim(W)$.

We now define the *tensor product* $V \otimes W$ of V and W.

Informal definition. Suppose V and W are finite-dimensional and let $\{v_1, \dots, v_m\}$, $\{w_1, \dots, w_n\}$ be bases for V and W, respectively. Then the tensor produce $V \otimes W$ is a vector space which has

$$\{v_i \otimes w_j \mid i = 1, \dots, m; j = 1, \dots, n\}$$

as a basis. Elements of $V \otimes W$ are linear combinations $\sum_{ij} a_{ij} v_i \otimes w_j$.

Definition 16.2. *Let $V_1, \dots, V_k, U$ be vector spaces. A map $\phi : V_1 \times \cdots \times V_k \to U$ is multilinear if ϕ is linear in each V_i separately, i.e., $\phi : \{v_1\} \times \cdots \times V_i \times \cdots \times \{v_k\} \to U$ is linear for each $v_j \in V_j$, $j \neq i$. If $k = 2$, we say ϕ is bilinear.*

Formal definition. The tensor product $V \otimes W$ of V and W is a vector space Z together with a bilinear map $i : V \times W \to Z$, which satisfies the following *universal mapping property*: Given any bilinear map $\phi : V \times W \to U$, there exists a *unique* linear map $\tilde{\phi} : Z \to U$ such that $\phi = \tilde{\phi} \circ i$.

Actual construction. Start with the *free vector space* $F(S)$ generated by a set S. By this we mean $F(S)$ consists of finite linear combinations $\sum_i a_i s_i$, where $s_i \in S$, $a_i \in \mathbb{R}$, and we are treating distinct $s_i \in S$ as linearly independent. (In other words, S is a basis for $F(S)$.) In our case we take $F(V \times W)$.

In $V \otimes W$, we want finite linear combinations of things that look like $v \otimes w$. We also would like the following:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2,$$
$$c(v \otimes w) = (cv) \otimes w = v \otimes (cw).$$

We therefore consider $R(V, W) \subset F(V \times W)$, the vector space generated by the "bilinear relations"

$$(v_1 + v_2, w) - (v_1, w) - (v_2, w),$$
$$(v, w_1 + w_2) - (v, w_1) - (v, w_2),$$
$$(cv, w) - c(v, w),$$
$$(v, cw) - c(v, w).$$

Then the quotient space $F(V \times W)/R(V, W)$ is $V \otimes W$.

Verification of universal mapping property. With $V \otimes W$ defined as above, define

$$i : V \times W \to V \otimes W, \quad (v, w) \mapsto v \otimes w.$$

The bilinearity of i follows from the construction of $V \otimes W$. (For example, $i(v_1 + v_2, w) = (v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w = i(v_1, w) + i(v_2, w)$.) We can define

$$\tilde{\phi} : V \otimes W \to U, \quad \sum_i a_i v_i \otimes w_i \mapsto \sum_i a_i \phi(v_i, w_i).$$

This map is well-defined because all the elements of $R(V, W)$ are mapped to 0. The uniqueness of $\tilde{\phi}$ is clear.

17. Tensor and exterior algebra

17.1. **More on tensor products.** Recall the definition of the tensor product as $V \otimes W = F(V \times W)/R(V,W)$ and the universal property. The universal property is useful for the following reason: If we want to construct a linear map $V \otimes W \to U$, it is equivalent to check the existence of a bilinear map $V \times W \to U$.

Dimension of $V \otimes W$. Suppose V and W are finite-dimensional. Then we claim that

$$\dim(V \otimes W) = \dim V \cdot \dim W.$$

To see this, consider the map

$$\tau : V^* \otimes W \to \mathrm{Hom}(V,W), \quad f \otimes w \mapsto fw,$$

where $fw : v \mapsto f(v)w$. The universal mapping property guarantees the well-definition of τ. In order to verify that τ is an isomorphism, we define its inverse η. Let $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_n\}$ be bases for V and W and let $\{f_1, \dots, f_m\}$ be dual to $\{v_1, \dots, v_m\}$, i.e., $f_i(v_j) = \delta_{ij}$. Then we set $\eta(\sum a_{ij} f_i w_j) = \sum a_{ij} f_i \otimes w_j$. τ and η are clearly inverses. Then $\dim(\mathrm{Hom}(V,W)) = \dim(V) \cdot \dim(W)$, which implies that $\dim(V \otimes W) = \dim V \cdot \dim W$.

Properties of tensor products.

(1) $V \otimes W \simeq W \otimes V$.
(2) $(V \otimes W) \otimes U \simeq V \otimes (W \otimes U)$.

Proof. (1) It is difficult to directly get a well-defined map $V \otimes W \to W \otimes V$, so start with a bilinear map $V \times W \to W \times V \to W \otimes V$, where $(v,w) \mapsto w \otimes v$. The universal property gives a map $V \otimes W \to W \otimes V$ which sends $v \otimes w \mapsto w \otimes v$. The map is an isomorphism since there is an inverse $w \otimes v \mapsto v \otimes w$. (2) is left as an exercise. □

(2) ensures us that we do not need to write parentheses when we take a tensor product of several vector spaces.

Let $A : V \to V$ and $B : W \to W$ be linear maps. Then we have

$$A \oplus B : V \oplus W \to V \oplus W, \qquad A \otimes B : V \otimes W \to V \otimes W.$$

We denote $V^{\otimes k}$ for the k-fold tensor product of V. Then we have a representation

$$\rho : GL(V) \to GL(V^{\otimes k}), \quad A \mapsto A \otimes \cdots \otimes A.$$

17.2. **Tensor and exterior algebra.** The *tensor algebra* $T(V)$ of V is

$$T(V) = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots,$$

where the multiplication is given by

$$(v_1 \otimes \cdots \otimes v_s)(v_{s+1} \otimes \cdots \otimes v_t) = v_1 \otimes \cdots \otimes v_t.$$

Here s or t may be zero, in which case $1 \in \mathbb{R}$ acts as the multiplicative identity.

The *exterior algebra* $\wedge V$ of V is $T(V)/\mathcal{I}$, where $\mathcal{I}$ is the 2-sided ideal generated by elements of the form $v \otimes v$, $v \in V$, i.e., elements of $\mathcal{I}$ are finite sums of terms that look like $\eta_1 \otimes v \otimes v \otimes \eta_2$, where $\eta_1, \eta_2 \in T(V)$ and $v \in V$. Elements of $\wedge V$ are linear combinations of terms of the form $v_1 \wedge \cdots \wedge v_k$, where $k \in \{0, 1, 2 \dots\}$ and $v_i \in V$. By definition, in $\wedge V$ we have

$$v \wedge v = 0. \tag{4}$$

Also note that $(v_1 + v_2) \wedge (v_1 + v_2) = v_1 \wedge v_1 + v_1 \wedge v_2 + v_2 \wedge v_1 + v_2 \wedge v_2$. The first and last terms are zero, so

$$v_1 \wedge v_2 = -v_2 \wedge v_1. \tag{5}$$

$\wedge V$ is clearly an algebra, i.e., there is a multiplication $\omega \wedge \eta$ given elements ω, η in $\wedge V$.

We define $\wedge^k V$ to be the degree k terms of $\wedge V$, i.e., linear combinations of terms of the form $v_1 \wedge \cdots \wedge v_k$ where $v_i \in V$.

Alternating multilinear forms. A multilinear form $\phi : V \times \cdots \times V \to U$ is *alternating* if

$$\phi(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = -1 \cdot \phi(v_1, \dots, v_{i+1}, v_i, \dots v_k).$$

Recall that transpositions generate the full symmetric group S_k on k letters. If $\sigma \in S_k$ maps $(1, \dots, k) \mapsto (\sigma(1), \dots, \sigma(k))$ and $\text{sgn}(\sigma)$ is the number of transpositions mod 2 that is needed to write σ as a product of transpositions, then

$$\phi(v_1, \dots, v_k) = (-1)^{\text{sgn}(\sigma)} \phi(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Universal property. $\wedge^k V$ and $i : V \times \cdots \times V \to \wedge^k V$ satisfy the following: Given an alternating multilinear map $\phi : V \times \cdots \times V \to U$, there is a unique linear map $\tilde{\phi} : \wedge^k V \to U$ such that $\phi = \tilde{\phi} \circ i$.

18. Differential k-forms

18.1. **Basis for $\wedge^k V$.**

Proposition 18.1. *Given a basis $\{e_1, \dots, e_n\}$ for V, a basis for $\wedge^k V$ consists of degree k monomials $e_{i_1} \wedge \dots \wedge e_{i_k}$ with $i_1 < \dots < i_k$. Therefore,* $\dim \wedge^k V = 0$ *for $k > n$ and* $\dim \wedge^k V = \binom{n}{k}$ *for $k \leq n$.*

Example. If $V = \mathbb{R}^3$ with basis $\{e_1, e_2, e_3\}$, then $\wedge^0 V = \mathbb{R}\langle 1 \rangle$, $\wedge^1 V = \mathbb{R}\langle e_1, e_2, e_3 \rangle$, $\wedge^2 V = \mathbb{R}\langle e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3 \rangle$, $\wedge^3 V = \mathbb{R}\langle e_1 \wedge e_2 \wedge e_3 \rangle$, and $\wedge^k V = 0$ for $k > 3$.

Proof. The proof is given in several steps. It is easy to see that $\{e_{i_1} \wedge \dots \wedge e_{i_k} \mid 1 \leq i_1, \dots, i_k \leq n\}$ spans $\wedge^k V$. Moreover, using Equations (4) and (5), we can shrink the spanning set to:

$$\{e_{i_1} \wedge \dots \wedge e_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq n\}.$$

1. If $k > n$, $\wedge^k V = 0$. This is clear since it is impossible to find $1 \leq i_1 < \dots < i_k \leq n$.

2. If $k = n$, $\wedge^k V = \mathbb{R}$, and the basis is given by $e_1 \wedge \dots \wedge e_n$. Since $e_1 \wedge \dots \wedge e_n$ spans $\wedge^k V$, it remains to show that $e_1 \wedge \dots \wedge e_n$ is nonzero! This is done by defining a (nontrivial!) alternating multilinear form $V \times \dots \times V \to \mathbb{R}$, where we are taking n copies of V. Then, by the universal property, $\wedge^n V$ cannot be zero and hence must be $\mathbb{R}$. The details are left as an exercise.

3. If $1 \leq k < n$, then we show that $\{e_{i_1} \wedge \dots \wedge e_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq n\}$ is linearly independent. Indeed, suppose $\omega = \sum_{i_1 < \dots < i_k} a_{i_1,\dots,i_k} e_{i_1} \wedge \dots \wedge e_{i_k} = 0$. For each summand, there is a unique term $e_{j_1} \wedge \dots \wedge e_{j_{n-k}}$ which, when multiplied to ω, kills all the other summands and gives $\pm a_{i_1,\dots,i_k} e_1 \wedge \dots \wedge e_n$. Hence $a_{i_1,\dots,i_k} = 0$, which proves the linear independence. □

Remark 18.2. *For any $v_1, \dots, v_k \in V$, $v_1 \wedge \dots \wedge v_k \neq 0$ in $\wedge^k V$ if and only if $v_1, \dots, v_k$ are linearly independent.*

18.2. **Tensor calculus on manifolds.**

We have now constructed $V^{\otimes k}$ and $\wedge^k V$, given a finite-dimensional vector space V. There exists a natural representation

$$\rho_0 : GL(V) \to GL(V^{\otimes k}), \quad A \mapsto A \otimes \dots \otimes A,$$

and an analogously defined representation

$$\rho_1 : GL(V) \to GL(\wedge^k V), \quad A \mapsto A \wedge \dots \wedge A.$$

Example. $\dim V = 2$. Basis $\{v_1, v_2\}$. $\wedge V$ has basis $\{1, v_1, v_2, v_1 \wedge v_2\}$. If $A : V \to V$ is linear and sends $v_i \mapsto a_{1i} v_1 + a_{2i} v_2$, $i = 1, 2$, then $A(v_1 \wedge v_2) = Av_1 \wedge Av_2 = \det(A) v_1 \wedge v_2$.

Thus we can form $TM \times_{\rho_0} V^{\otimes k} = \otimes_k TM$ and $TM \times_{\rho_1} \wedge^k V = \wedge^k TM$. We can also form $\otimes_k T^* M$ and $\wedge^k T^* M$.

We will focus on $\wedge^k T^*M$ in what follows. Sections of $\wedge^k T^*M$ are called k-forms and can be written in local coordinates $x_1, \dots, x_m$ as:

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1,\dots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Denote by $\Omega^k(M)$ the sections of $\wedge^k T^*M$.

Pullback. Let $\phi : M \to N$ be a smooth map between manifolds, $x_1, \dots, x_m$ and $y_1, \dots, y_n$ local coordinates on M and N, and ω a k-form. Then we can define the pullback $\phi^*\omega$ of $\omega = \sum_{i_1<\cdots<i_k} f_{i_1,\dots,i_k} dy_{i_1} \wedge \cdots \wedge dy_{i_k}$ as follows, using the definition for 1-forms:

$$\phi^*\omega = \sum_{i_1<\cdots<i_k} (f_{i_1,\dots,i_k} \circ \phi) \cdot d(y_{i_1} \circ \phi) \wedge \cdots \wedge d(y_{i_k} \circ \phi).$$

HW: Check that the above definition is independent of choice of coordinates.

18.3. **The exterior derivative.** We can define the extension $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ of $d_0 = d : \Omega^0(M) \to \Omega^1(M)$ as follows (in local coordinates $x_1, \dots, x_n$):

(1) If $f \in \Omega^0(M)$, then $d_0 f = df = \sum_i \frac{\partial f}{\partial x_i} dx_i$.
(2) If $\omega = \sum_I f_I dx_I \in \Omega^k(M)$, then $d_k\omega = \sum_I df_I \wedge dx_I$.

Here $I = (i_1, \dots, i_k)$ is an indexing set and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. We will often suppress the k in d_k.

HW: Check that d_k is independent of the choice of local coordinates.

Example on $\mathbb{R}^3$. Consider $\mathbb{R}^3$ with coordinates (x, y, z). Consider

$$\Omega^0(\mathbb{R}^3) \xrightarrow{d_0} \Omega^1(\mathbb{R}^3) \xrightarrow{d_1} \Omega^2(\mathbb{R}^3) \xrightarrow{d_2} \Omega^3(\mathbb{R}^3).$$

d_0 is the *gradient*

$$d_0 : f \mapsto \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

d_1 is the *curl*

$$d_1 : f dx + g dy + h dz \mapsto \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dy \wedge dz + \dots.$$

d_2 is the *divergence*

$$d_2 : f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy \mapsto \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx \wedge dy \wedge dz.$$

Remark 18.3. *In what follows we will often omit $\wedge$.*

19. De Rham cohomology

This material is nicely presented in Bott & Tu.

Lemma 19.1. *The exterior derivative d satisfies the (skew-commutative) Leibniz rule:*

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta), \tag{6}$$

where $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, and k or l may be zero (in which case we ignore the $\wedge$).

Proof. HW. □

Lemma 19.2. $d^2 = 0$.

Proof. For $d^2 : \Omega^0(M) \to \Omega^2(M)$, we compute:

$$d \circ df = d\left(\sum_i \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \wedge dx_i = 0.$$

When $\alpha = dx_I$, we verify that $d\alpha = d(dx_I) = 0$.

Now if $\alpha = f_I dx_I$, then

$$d\alpha = df_I \wedge dx_I + f_I d(dx_I) = df_I \wedge dx_I,$$
$$d^2\alpha = (d^2 f_I) \wedge dx_I - df_I \wedge d(dx_I) = 0,$$

which proves the lemma. □

Example: If $M = \mathbb{R}^3$, then $d_0 = grad$, $d_1 = curl$, $d_2 = div$. Then $d^2 = 0$ is equivalent to $div(curl) = 0$, $curl(grad) = 0$.

Consider the sequence

$$0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \dots \xrightarrow{d_{n-1}} \Omega^n(M) \to 0, \tag{7}$$

where $n = \dim M$. Since $d_k \circ d_{k-1} = 0$, we have $\operatorname{Im} d_{k-1} \subset \operatorname{Ker} d_k$. This leads to the following definition:

Definition 19.3. *The kth* de Rham cohomology group *of M is given by:*

$$H^k_{dR}(M) \stackrel{def}{=} \operatorname{Ker} d_k / \operatorname{Im} d_{k-1}.$$

Definition 19.4. *Let $\omega \in \Omega^k(M)$. Then ω is closed if $d\omega = 0$ and is* exact *if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$.*

Fact 19.5. *The de Rham cohomology groups are diffeomorphism invariants of the manifold M (this means that if there is a diffeomorphism $\phi : M \xrightarrow{\sim} N$, then the de Rham cohomology groups of M and N are isomorphic), and are finite-dimensional if M is compact.*

Definition 19.6. *A sequence of vector spaces $\cdots \longrightarrow C^k \xrightarrow{d_k} C^{k+1} \xrightarrow{d_{k+1}} C^{k+2} \longrightarrow \dots$ is said to be* exact *if $\operatorname{Im} d_{k-1} = \operatorname{Ker} d_k$ for all k.*

The de Rham cohomology groups measure the failure of Equation (7) to be *exact*.

We will often write $H^k(M)$ instead of $H^k_{dR}(M)$.

Examples:

1. $M = \{pt\}$. Then $\Omega^0(pt) = \mathbb{R}$ and $\Omega^k(pt) = 0$ for $i > 0$. Hence $H^0(pt) = \mathbb{R}$ and $H^k(pt) = 0$ for $k > 0$.

2. $M = \mathbb{R}$. Then $\Omega^0(M) = C^\infty(\mathbb{R})$ and $\Omega^1(M) \simeq C^\infty(\mathbb{R})$ because every 1-form is of the form $f dx$. Now, $d : f \mapsto \frac{df}{dx} dx$ can be viewed as the map

$$d : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}),$$
$$f \mapsto f'.$$

It is easy to see that $\operatorname{Ker} d = \{\text{constant functions}\}$ and hence $H^0(\mathbb{R}) = \mathbb{R}$. Next, $\operatorname{Im} d$ is all of $C^\infty(\mathbb{R})$, since given any f we can take its antiderivative $\int_0^x f(t)dt$. Therefore, $H^1(\mathbb{R}) = 0$. Also $H^k(\mathbb{R}) = 0$ for $k > 1$ since $\Omega^k(\mathbb{R}) = 0$ for $k > 1$.

3. $M = S^1$. View S^1 as $\mathbb{R}/\mathbb{Z}$ with coordinates x. Then

$$\Omega^0(S^1) = \{\text{periodic functions on } \mathbb{R} \text{ with period } 1\}.$$

$\Omega^1(S^1)$ is also the set of periodic functions on $\mathbb{R}$ by identifying $f(x)dx \mapsto f(x)$. $H^0(S^1) = \mathbb{R}$ as before. Now, $H^1(S^1) = \Omega^1(S^1)/\operatorname{Im} d$ and $\operatorname{Im} d$ is the space of all C^∞-functions $f(x)$ with integral $\int_0^1 f(x)dx = 0$. Hence $H^1(S^1) = \mathbb{R}$. We also have an *exact sequence*:

$$0 \to \mathbb{R} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{\int} \mathbb{R} \to 0.$$

Basic properties.

1. $H^0(M) \simeq \mathbb{R}$ if M is connected. Proof: $df = 0$ if and only if f is locally constant.

2. $H^k(M) = 0$ if $\dim M < k$. Proof: $\Omega^k(M) = 0$.

3. If $M = M_1 \sqcup M_2$, then $H^k(M) \simeq H^k(M_1) \oplus H^k(M_2)$ for all $k \geq 0$. Proof: HW.

20. Mayer-Vietoris sequences; Poincaré lemma

20.1. **Pullback.** Let $\phi : M \to N$ be a smooth map between manifolds.

Lemma 20.1. $d \circ \phi^* = \phi^* \circ d$.

HW: Verify the lemma. This follows easily by computing in local coordinates.

Corollary 20.2. *There is an induced map $\phi^* : H^k(N) \to H^k(M)$ on the level of cohomology.*

Proof. If ω is a closed k-form on N, i.e., $\omega \in \Omega^k(N)$ satisfies $d\omega = 0$, then $\phi^*\omega$ satisfies $d\phi^*\omega = \phi^*(d\omega) = 0$. If ω is exact, i.e., $\omega = d\eta$, then $\phi^*\omega = \phi^* d\eta = d(\phi^*\eta)$ is exact as well. □

20.2. **Mayer-Vietoris sequences.** This is a method for effectively decomposing a manifold and computing its cohomology from its components.

Let $M = U \cup V$, where U and V are open sets. Then we have natural inclusion maps

$$U \cap V \overset{i_U, i_V}{\longrightarrow} U \sqcup V \overset{i}{\to} M. \tag{8}$$

Here i_U and i_V are inclusions of $U \cap V$ into U and into V.

Example: $M = S^1, U = V = \mathbb{R}, U \cap V = \mathbb{R} \sqcup \mathbb{R}$.

Theorem 20.3. *We have the following long exact sequence:*

$$\begin{aligned} 0 \to H^0(M) &\overset{i^*}{\to} H^0(U) \oplus H^0(V) \overset{i_U^* - i_V^*}{\longrightarrow} H^0(U \cap V) \to \\ \to H^1(M) &\overset{i^*}{\to} H^1(U) \oplus H^1(V) \overset{i_U^* - i_V^*}{\longrightarrow} H^1(U \cap V) \to \\ \to \dots. & \end{aligned}$$

Remark 20.4. $0 \to A \to B$ *exact means* $A \to B$ *is injective.* $A \to B \to 0$ *exact means* $A \to B$ *is surjective. Hence* $0 \to A \to B \to 0$ *exact means* $A \to B$ *is an isomorphism.*

The proof of Theorem 20.3 will be given over the next couple of lectures, but for the time being we will apply it:

Example: Compute $H^k(S^1)$ using Mayer-Vietoris using the above decomposition of S^1 as $\mathbb{R} \cup \mathbb{R}$.

20.3. **Poincaré lemma.** The following lemma is an important starting point when using the Mayer-Vietoris sequence to compute cohomology groups.

Lemma 20.5 (Poincaré lemma)**.** *Let $\omega \in \Omega^k(\mathbb{R}^n)$ for $k \geq 1$. Then ω is closed if and only if it is exact.*

In other words, $H^k(\mathbb{R}^n) = 0$ for $k \geq 1$. We will give the proof later, when we discuss homotopy-theoretic properties of de Rham cohomology.

20.4. **Partitions of unity.**

Definition 20.6. *Let* $\{U_\alpha\}$ *be an open cover of* M. *Then a collection of real-valued functions* $\{f_\alpha \geq 0\}$ *on* M *is a* partition of unity subordinate to $\{U_\alpha\}$ *if:*

(1) $\mathrm{Supp}(f_\alpha) \subset U_\alpha$. *Here the* support $\mathrm{Supp}(f_\alpha)$ *of* f_α *is the closure of* $\{x \in M \mid f_\alpha(x) \neq 0\}$.
(2) *At every point* $x \in M$, *there exists a neighborhood* $N(x)$ *of* x *such that the set*
$$\{f_\alpha \mid f_\alpha|_{N(x)} \neq 0\}$$
is finite. If we write $f_1, \dots, f_k$ *for the nonzero functions, then* $\sum_{i=1}^k f_i(x) = 1$.

Proposition 20.7. *For any open cover* $\{U_\alpha\}$ *of* M, *there exists a partition of unity subordinate to* $\{U_\alpha\}$.

Proof. The proof is done in stages.

Step 1. Consider the function $f : \mathbb{R} \to \mathbb{R}$:
$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$
It is easy to show that $f \geq 0$ and f is smooth.

Step 2. Let $a < b$. Then the function
$$g = g_{ab} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x-a) \cdot f(b-x)$$
is a *bump function*, i.e., satisfies $g \geq 0$; $g > 0$ on (a, b); and hence $\mathrm{Supp}(g) = [a, b]$.

Step 3. Construct a bump function on $[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ with coordinates $(x_1, \dots, x_n)$ by letting $\phi(x) = g_{a_1 b_1}(x_1) \dots g_{a_n b_n}(x_n)$. Then ϕ is supported on $[a_1, b_1] \times \cdots \times [a_n, b_n]$ and is positive on the interior.

Step 4. Suppose M is compact. For each $p \in M$, choose an open neighborhood U_p of p of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$ whose closure is contained inside some U_α. For each U_p, construct ϕ_p as in Step 3. Now, since M is compact, there exists a finite collection of $\{p_1, \dots, p_k\}$ where $\{U_{p_i}\}$ cover M. Note that $\sum_{i=1}^k \phi_p > 0$ everywhere on M. If we let $\psi_p = \frac{\phi_p}{\sum \phi_p}$, then $\sum \psi_p = 1$. Finally, we associate to each ψ_p an open set U_α for which $U_p \subset U_\alpha$. Then ψ_α is the sum of all the ψ_p associated to U_α.

Step 5. Not so easy exercise: prove the general case!! $\square$

21. Some homological algebra

21.1. Short exact sequence. Suppose $M = U \cup V$. Then we have

$$U \cap V \stackrel{i_U, i_V}{\longrightarrow} U \sqcup V \stackrel{i}{\to} M,$$

where i_U and i_V are two inclusions, one into U and the other into V.

Lemma 21.1. *The following is a short exact sequence:*

$$0 \to \Omega^k(M) \stackrel{i^*}{\to} \Omega^k(U) \oplus \Omega^k(V) \stackrel{i_U^* - i_V^*}{\longrightarrow} \Omega^k(U \cap V) \to 0. \tag{9}$$

Proof. We prove that $i_U^* - i_V^*$ is surjective. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to $\{U, V\}$; in particular $\rho_U + \rho_V = 1$. Given $\omega \in \Omega^k(U \cap V)$, we have $\rho_V \omega \in \Omega^k(U)$ and $-\rho_U \omega \in \Omega^k(V)$. Then $i_U^*(\rho_V \omega) - i_V^*(-\rho_U \omega) = \omega$. The rest of the exact sequence is left as an exercise. $\square$

21.2. Short exact sequences to long exact sequences. Getting from the short exact sequence to the long exact sequence is a purely algebraic operation.

Define a *cochain complex* $(\mathcal{C}, d)$: $\dots \to C^k \stackrel{d_k}{\longrightarrow} C^{k+1} \stackrel{d_{k+1}}{\longrightarrow} C^{k+1} \to \dots$ to be a sequence of vector spaces and maps with $d_{k+1} \circ d_k = 0$ for all k. $(\mathcal{C}, d)$ gives rise to $H^k(\mathcal{C}) = \operatorname{Ker} d_k / \operatorname{Im} d_{k-1}$, the kth cohomology of the complex.

A *cochain map* $\phi : \mathcal{A} \to \mathcal{B}$ is a collection of maps $\phi_k : A^k \to B^k$ such that the squares of the following diagram commute (i.e., $d_k \circ \phi_k = \phi_{k+1} \circ d_k$):

$$\begin{array}{ccccccccc}
\stackrel{d_{k-2}}{\longrightarrow} & A^{k-1} & \stackrel{d_{k-1}}{\longrightarrow} & A^k & \stackrel{d_k}{\longrightarrow} & A^{k+1} & \stackrel{d_{k+1}}{\longrightarrow} \\
 & \downarrow{\scriptstyle \phi_{k-1}} & & \downarrow{\scriptstyle \phi_k} & & \downarrow{\scriptstyle \phi_{k+1}} & \\
\stackrel{d_{k-2}}{\longrightarrow} & B^{k-1} & \stackrel{d_{k-1}}{\longrightarrow} & B^k & \stackrel{d_k}{\longrightarrow} & B^{k+1} & \stackrel{d_{k+1}}{\longrightarrow}
\end{array}$$

A cochain map $\phi : \mathcal{A} \to \mathcal{B}$ induces a map $\phi : H^k(\mathcal{A}) \to H^k(\mathcal{B})$ on cohomology. The verification is identical to that of the special case of de Rham.

Given an exact sequence $0 \to \mathcal{A} \stackrel{\phi}{\to} \mathcal{B} \stackrel{\psi}{\to} \mathcal{C} \to 0$ of cochain maps, i.e., we have collections of $0 \to A^k \stackrel{\phi_k}{\to} B^k \stackrel{\psi_k}{\to} C^k \to 0$ and commuting squares

$$\begin{array}{ccccccccc}
 & & \uparrow{\scriptstyle d_{k+1}} & & \uparrow{\scriptstyle d_{k+1}} & & \uparrow{\scriptstyle d_{k+1}} & & \\
0 & \longrightarrow & A^{k+1} & \stackrel{\phi_{k+1}}{\longrightarrow} & B^{k+1} & \stackrel{\psi_{k+1}}{\longrightarrow} & C^{k+1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_k} & & \uparrow{\scriptstyle d_k} & & \uparrow{\scriptstyle d_k} & & \\
0 & \longrightarrow & A^k & \stackrel{\phi_k}{\longrightarrow} & B^k & \stackrel{\psi_k}{\longrightarrow} & C^k & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_{k-1}} & & \uparrow{\scriptstyle d_{k-1}} & & \uparrow{\scriptstyle d_{k-1}} & &
\end{array}$$

we obtain a long exact sequence:

$$\begin{aligned} \dots \longrightarrow \; & H^k(\mathcal{A}) \xrightarrow{\phi_k} H^k(\mathcal{B}) \xrightarrow{\psi_k} H^k(\mathcal{C}) \xrightarrow{\delta_k} \\ \longrightarrow \; & H^{k+1}(\mathcal{A}) \xrightarrow{\phi_{k+1}} H^{k+1}(\mathcal{B}) \xrightarrow{\psi_{k+1}} H^{k+1}(\mathcal{C}) \xrightarrow{\delta_{k+1}} \dots . \end{aligned}$$

Verification of $\operatorname{Ker}\psi_k \supset \operatorname{Im}\phi_k$**.** Suppose $[b] \in \operatorname{Im}\phi_k$. Then $b = \phi_k a + db'$, where $a \in A^k$ and $b' \in B^{k-1}$. Now, $\psi_k b = \psi_k(\phi_k a) + \psi_k(db') = d(\psi_{k-1} b')$. Therefore, $[\psi_k b] = 0 \in H^k(\mathcal{C})$.

Verification of $\operatorname{Ker}\psi_k \subset \operatorname{Im}\phi_k$**.** Suppose $[b] \in \operatorname{Ker}\psi_k$. Then $\psi_k b = dc'$, $c' \in C^{k-1}$. Next, use the fact that $B^{k-1} \to C^{k-1} \to 0$ to find $b' \in B^{k-1}$ such that $c' = \psi_{k-1} b'$. Then $\psi_k b = d(\psi_{k-1} b') = \psi_k(db')$. Hence, by the exactness, $b - db' = \phi_k(a)$ for some $a \in A^k$. (Check that $da = 0$.) Thus, $\phi_k[a] = [b]$.

Definition of $\delta_k : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$**.** Let $[c] \in H^k(\mathcal{C})$. Then $dc = 0$. Also we have $b \in B^k$ with $\psi_k b = c$ by the surjectivity of $B^k \to C^k$. Consider db. Since $\psi_{k+1}(db) = d(\psi_k b) = dc = 0$, there exists an $a \in A^{k+1}$ such that $\phi_{k+1} a = db$. Let $[a] = \delta_k[c]$. Here, $da = 0$, since $\phi_{k+2}(da) = d(\phi_{k+1} a) = d(db) = 0$, and $A^{k+2} \to B^{k+2}$ is injective. We need to show that this definition is independent of the choice of c, choice of b, and choice of a. This is left as an exercise.

HW: Verify the rest of the exactness.

22. Integration

Let M be an n-dimensional manifold and $\omega \in \Omega^n(M)$. The goal of this lecture is to motivate and define $\int_M \omega$.

22.1. **Review of integration on $\mathbb{R}^n$.** For more details see Spivak.

Given a function $f : R = [a_1, b_1] \times [a_n, b_n] \to \mathbb{R}$, take a partition $P = (P_1, \dots, P_k)$ of R into small rectangles. Consider the upper and lower bounds

$$U(f, P) := \sum_i \sup(f|_{P_i}) \cdot \mathrm{vol}(P_i), \quad L(f, P) := \sum_i \inf(f|_{P_i}) \cdot \mathrm{vol}(P_i).$$

Definition 22.1. *The function $f : R \to \mathbb{R}$ is* Riemann-integrable *(or simply* integrable*) if for any $\varepsilon > 0$ there exists a partition P such that $U(f, P) - L(f, P) < \varepsilon$. If f is integrable, then we define*

$$\int_R f dx_1 \dots dx_n := \lim_P U(f, P) = \lim_P L(f, P).$$

Similarly we define $\int_U f dx_1 \dots dx_n$, if $U \subset \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}$ is a function with compact support.

Theorem 22.2. *If f is a continuous function with support on a compact set, then f is integrable.*

Change of variables formula. If $g : [a, b] \xrightarrow{\sim} [c, d]$ is a smooth reparametrization and x, y are coordinates on $[a, b], [c, d]$, then

$$\int_{g(a)}^{g(b)} f(y) dy = \int_a^b (f \circ g)(x) \cdot g'(x) dx.$$

This can be rewritten as:

$$\int_{g([a,b])} f(y) dy = \int_{[a,b]} (f \circ g)(x) \cdot |g'(x)| dx.$$

More generally, let U and $V \subset \mathbb{R}^n$ be open sets with coordinates $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, and $\phi : U \xrightarrow{\sim} V$ a diffeomorphism. Then:

$$\int_V f(y) dy_1 \dots dy_n = \int_U f(\phi(x)) \left|\det\left(\frac{\partial \phi}{\partial x}\right)\right| dx_1 \dots dx_n.$$

Remark 22.3. *In light of the change of variables formula, $\int_M \omega$ makes sense only when M is orientable, since the change of variables for an n-form does not have the absolute value. At any rate, n-forms have the wonderful property of having the correct transformation property (modulo sign) under diffeomorphisms.*

22.2. **Orientation.** Recall that M is *orientable* if there exists a subatlas $\{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ such that the Jacobians $J_{\alpha\beta} = d(\phi_\beta \circ \phi_\alpha^{-1})$ have positive determinant. We will refer to such an atlas as an *oriented atlas*.

Proposition 22.4. *M is orientable if and only if there exists a nowhere zero n-form ω on M.*

Proof. Suppose M is orientable. Let $\{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ be an oriented atlas, i.e., a subatlas whose transition functions have Jacobians $J_{\alpha\beta}$ with positive determinant. Take a partition of unity $\{f_\alpha\}$ subordinate to U_α and let $x_1^\alpha, \dots, x_n^\alpha$ be the coordinates on U_α. Construct $\omega_\alpha = f_\alpha dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$, which can be viewed as a smooth n-form on M with support in U_α. Let $\omega = \sum_\alpha \omega_\alpha$. This is nowhere zero, since

$$(\phi_\beta \circ \phi_\alpha^{-1})^* \omega_\beta = f_\beta \circ (\phi_\beta \circ \phi_\alpha^{-1}) \det(J_{\alpha\beta}) dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha.$$

Since $\det(J_{\alpha\beta})$ is positive, $f_\beta \circ (\phi_\alpha \circ \phi_\beta^{-1}) \det(J_{\alpha\beta}) \geq 0$. At any point $p \in M$, at least one f_α is positive and all the terms that are added up contribute nonnegatively with respect to coordinates $dx_1^\alpha, \dots, dx_n^\alpha$, so ω is nowhere zero on M.

On the other hand, suppose there exists a nowhere zero n-form ω on M. We choose a subatlas $\{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ whose coordinate functions $x_1^\alpha, \dots, x_n^\alpha$ satisfy the condition that $dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$ is a positive function times ω. Then clearly $J_{\alpha\beta}$ has positive determinant on this subatlas. □

If M is orientable, then there is a nowhere zero n-form ω, which in turn implies that $\wedge^n T^*M$ is isomorphic to $M \times \mathbb{R}$ as a vector bundle, i.e., is a *trivial* vector bundle.

On a connected manifold M, any two nowhere zero n-forms ω and ω' differ by a function, i.e., $\exists$ a positive (or negative) function f such that $\omega = f\omega'$. We define an equivalence relation $\omega \sim \omega'$ if $\omega = f\omega'$ and $f > 0$. Then there exist two equivalence classes of $\sim$ and each equivalence class is called an *orientation* of M.

The standard orientation on $\mathbb{R}^n$ is $dx_1 \wedge \cdots \wedge dx_n$.

Equivalent definition of orientation. Let $Fr(V)$ be the set of ordered bases (or *frames*) of a finite-dimensional $\mathbb{R}$-vector space V of dimension n. It can be given a smooth structure so that it is diffeomorphic to $GL(V)$, albeit not naturally: Fix an ordered basis $(v_1, \dots, v_n)$. Then any other basis $(w_1, \dots, w_n)$ can be written as $(Av_1, \dots, Av_n)$, where $A \in GL(V)$. Hence there is a bijection $Fr(V) \simeq GL(V)$ and we induce a smooth structure on $Fr(V)$ from $GL(V)$ via this identification. The non-naturality comes from the fact that there is a distinguished point $\mathrm{id} \in GL(V)$ but no distinguished basis in $Fr(V)$.

Since $GL(V)$ has two connected components, $Fr(V)$ also has two connected components, and each component is called an *orientation* for V. An *orientation* for M is a choice of orientation for each T_pM which is smooth in $p \in M$.

We can also construct the *frame bundle* $Fr(M) = \sqcup_{p\in M} Fr(T_pM)$ by topologizing as follows: Identify a neighborhood of $p \in M$ with $\mathbb{R}^n$ and identify $\sqcup_{p\in\mathbb{R}^n} Fr(T_p\mathbb{R}^n) = Fr(\mathbb{R}^n) \times \mathbb{R}^n$. The frame bundle is a *fiber bundle* over M whose fibers are diffeomorphic to $GL(V)$.

22.3. **Definition of the integral.** Suppose M is orientable. Choose an oriented atlas $\{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ for M. We then define:

$$\int_M \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(f_\alpha \omega),$$

where $\{f_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$. We will often write $\int_{U_\alpha} f_\alpha \omega$ instead of $\int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(f_\alpha \omega)$.

HW: Check that the definition of $\int_M \omega$ does not depend on the choice of oriented atlas $\{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ as well as on the choice of $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$.

23. Stokes' Theorem

23.1. **Manifolds with boundary.** We now enlarge the class of manifolds by allowing those "with boundary". These are locally modeled on the half-plane $\mathbb{H}^n = \{x_1 \leq 0\} \subset \mathbb{R}^n$ with coordinates $x_1, \dots, x_n$. Let $\partial\mathbb{H}^n = \{x_1 = 0\}$.

Definition 23.1. *An n-*dimensional manifold with boundary *is a pair $(M, \mathcal{A} = \{\phi_\alpha : U_\alpha \to \mathbb{H}^n\})$, where M is a Hausdorff, second countable topological space. ϕ_α is a homeomorphism of an open set $U_\alpha \subset M$ onto an open subset of $\mathbb{H}^n$, and the transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ are smooth. The* boundary ∂M of M *is the set of points of M that are mapped to $\partial\mathbb{H}^n$ under some ϕ_α.*

Example: The n-dimensional unit ball $B^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n | x_1^2 + \dots + x_n^2 \leq 1\}$. $\partial B^n = S^{n-1}$.

Lemma 23.2. *If M is a manifold with boundary, then ∂M is a manifold of dimension* $\dim M - 1$.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M. The key point that you should verify is that if any $p \in M$ is mapped to $\partial\mathbb{H}^n$ under some ϕ_α, then p cannot be mapped to the interior of $\mathbb{H}^n$ under any other coordinate chart ϕ_β. We then take the atlas $\{V_\alpha\}$ for ∂M where $V_\alpha = U_\alpha \cap \phi_\alpha^{-1}(\{x_1 = 0\})$. □

Lemma 23.3. *If M is an orientable manifold with boundary, then ∂M is an orientable manifold.*

Proof. Suppose that $\{(U_\alpha, \phi_\alpha)\}$ is an oriented atlas for M. If $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$ is an oriented basis for TM at each point of U_α, then let $(\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$ be an oriented basis for $T(\partial M)$ at each point of $U_\alpha \cap \partial M$. Since an outward-pointing vector $\frac{\partial}{\partial x_1}$ will go to another outward-pointing vector under a change of coordinates we have consistency. □

Remark 23.4. *The choice of orientation in the above lemma is the* boundary orientation *of ∂M induced from the orientation of M.*

23.2. **Stokes' Theorem.**

Theorem 23.5 (Stokes' Theorem)**.** *Let ω be an $(n-1)$-form on an oriented manifold with boundary M of dimension n. Then $\int_M d\omega = \int_{\partial M} \omega$.*

Zen: The significance of Stokes' Theorem is that a topological operation ∂ is related to an analytic operation d.

Proof. Take an open cover $\{U_\alpha\}$ where U_α is diffeomorphic to (i) $(0,1) \times \dots \times (0,1)$ (i.e., U_α does not intersect ∂M) or (ii) $(0,1] \times (0,1) \times \dots \times (0,1)$ (i.e., $U_\alpha \cap \partial M = \{x_1 = 1\}$). (Note that in the definition of a manifold with boundary, we could have allowed $\mathbb{H}^n$ to be $\{x_1 \leq c\}$, where $c \in \mathbb{R}$ is a constant. Here we are taking $c = 1$.) Let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. By linearity, it suffices to compute $\int_{U_\alpha} d(f_\alpha \omega) = \int_{\partial M \cap U_\alpha} f_\alpha \omega$, i.e., assume ω is supported on one U_α.

We will treat the $n = 2$ case. Let ω be an $(n-1)$-form of type (ii). Then on $[0,1] \times [0,1]$ we can write $\omega = f_1 dx_1 + f_2 dx_2$.

$$\int_{\partial M} \omega = \int_{\partial M} f_1 dx_1 + f_2 dx_2 = \int_0^1 f_2(1, x_2) dx_2.$$

On the other hand,

$$\begin{aligned}
\int_M d\omega &= \int_0^1 \int_0^1 \left(-\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right) dx_1 dx_2 \\
&= \int_0^1 \left(\int_0^1 \frac{\partial f_2}{\partial x_1} dx_1\right) dx_2 + \int_0^1 \left(\int_0^1 -\frac{\partial f_1}{\partial x_2} dx_2\right) dx_1 \\
&= \int_0^1 (f_2(1,x_2) - f_2(0,x_2)) dx_2 + \int_0^1 (f_1(x_1,0) - f_1(x_1,1)) dx_1 \\
&= \int_0^1 f_2(1,x_2) dx_2
\end{aligned}$$

HW: Check that $n > 2$ also works in the same way. □

Example: (Green's Theorem) Let $\Omega \subset \mathbb{R}^2$ be a compact domain with smooth boundary, i.e., Ω is a 2-dimensional manifold with boundary $\partial\Omega = \gamma$. Then

$$\int_\gamma f dx + g dy = \int_\Omega \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx dy.$$

Example: Consider ω defined on $\mathbb{R}^2 - \{0\}$:

$$\omega(x,y) = \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy.$$

Let $C = \{x^2 + y^2 = R^2\}$. Then $x = R\cos\theta$, $y = R\sin\theta$, and we compute

$$\int_C \omega = 2\pi.$$

One can directly verify that ω is closed. We claim that ω is not exact. Indeed, if $\omega = d\eta$, then

$$0 = \int_{\partial C} \eta = \int_C d\eta = 2\pi,$$

which is a contradiction.

24. Applications of Stokes' Theorem

24.1. The Divergence Theorem.

Theorem 24.1. *Let $\Omega \subset \mathbb{R}^3$ be a compact domain with smooth boundary. Let $F = (F_1, F_2, F_3)$ be a vector field on Ω. Then*

$$\int_\Omega \operatorname{div} F \, dxdydz = \int_{\partial\Omega} \langle n, F \rangle dA,$$

where $n = (n_1, n_2, n_3)$ is the unit *outward normal to $\partial\Omega$,*

$$dA = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy,$$

and $\langle \cdot \rangle$ is the standard inner product.

Let $\omega = F_1 dydz + F_2 dzdx + F_3 dxdy$. Then $d\omega = (\operatorname{div} F) dxdydz$. It remains to see why $\int_{\partial\Omega} \omega = \int_{\partial\Omega} \langle n, F \rangle dA$.

Evaluating forms. We explain what it means to take $\omega(v_1, \dots, v_k)$, where ω is a k-form and v_i are tangent vectors. Let V be a finite-dimensional vector space. There exists a map:

$$(\wedge^k V^*) \times (V \times \cdots \times V) \to \mathbb{R}$$

$$(f_1 \wedge \cdots \wedge f_k, (v_1, \dots, v_k)) \mapsto \sum (-1)^\sigma f_1(v_{i_1}) \dots f_k(v_{i_k}),$$

where the sum ranges over all permutations of $(1, \dots, k)$ and σ is the number of transpositions required for the transposition $(1, \dots, k) \mapsto (i_1, \dots, i_k)$. Note that this alternating sum is necessary for the well-definition of the map.

Example: Let $\omega = F_1 dydz + F_2 dzdx + F_3 dxdy$. Then $\omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = -F_2$.

Interior product. We can define the interior product as follows: $i_v : \wedge^k V^* \to \wedge^{k-1} V^*$, $i_v \omega = \omega(v, \cdot, \cdot, \dots, \cdot)$. (Insert v into the first slot to get a $(k-1)$-form.)

Example: On $\mathbb{R}^3$, let $\eta = dxdydz$. Also let n be the unit normal vector to $\partial\Omega$. Then, along $\partial\Omega$ we can define $i_n \eta = n_1 dydz + n_2 dzdx + n_3 dxdy$.

Why is this dA? At any point of $p \in \partial\Omega$, take tangent vectors v_1, v_2 of $\partial\Omega$ so that n, v_1, v_2 is an oriented orthonormal basis. Then the *area form* dA should evaluate to 1 on v_1, v_2. Since $\eta(n, v_1, v_2) = 1$ (since η is just the determinant), we see that $dA = i_n \eta$.

Explanation of $\langle n, F \rangle dA = F_1 dydz + F_2 dzdx + F_3 dxdy$. Also note that $i_F \eta = F_1 dydz + \dots$. But now, $i_F \eta(v_1, v_2) = \eta(F, v_1, v_2) = \eta(\langle n, F \rangle n, v_1, v_2) = \langle n, F \rangle dA$ (by Gram-Schmidt).

24.2. Evaluating cohomology classes.

In this section let M be a compact oriented n-manifold without boundary.

Lemma 24.2. *There exists a well-defined, nonzero map $\int : H^n(M) \to \mathbb{R}$ which sends $[\omega] \to \int_M \omega$.*

Proof. Given $\omega \in \Omega^n(M)$, we map $\omega \mapsto \int_M \omega$. Note that every n-form ω is closed. To show the map descends to a map $\int : H^n(M) \to \mathbb{R}$, let ω be an exact form, i.e., $\omega = d\eta$. Then

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0.$$

Next we prove the nontriviality of $\int$: if ω is an *orientation form* (i.e., ω is nowhere zero), then $\int_M \omega > 0$ or < 0, since on each oriented coordinate chart ω is some positive function times $dx_1 \dots dx_n$. □

The lemma implies that $\dim H^n(M) \geq 1$. In fact, we have the following, which will be proved in a later lecture.

Theorem 24.3. $H^n(M) \simeq \mathbb{R}$.

Example. If $M = S^n$, then $H^i(S^n) = \mathbb{R}$ for $i = 0$ or n and $H^i(S^n) = 0$ for all other i.

25. Evaluating cohomology classes; degree

25.1. **Evaluating cohomology classes.** Let $\phi : M \to N$ be a smooth map between compact oriented manifolds of dimensions m and n, respectively, and $\phi^* : H^k(N) \to H^k(M)$ the induced map on cohomology. If $\omega \in \Omega^m(N)$ is a closed m-form such that $\int_M \phi^*\omega \neq 0$, then $\phi^*\omega$ represents a nonzero element in $H^m(M)$. This implies that $[\omega]$ is a nonzero cohomology class in $H^m(N)$.

Example: On $\mathbb{R}^2 - \{0\}$, consider the closed 1-form

$$\omega(x,y) = \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy.$$

We computed $\int_{S^1} \phi^*\omega = 2\pi$, where $\phi : S^1 \to \mathbb{R}^2 - \{0\}$ mapped $\theta \mapsto (R\cos\theta, R\sin\theta)$. Since $[\phi^*\omega]$ is a nonzero cohomology class in $H^1(S^1)$, so is $[\omega] \in H^1(\mathbb{R}^2 - \{0\})$.

Definition 25.1. *Two maps* $\phi_0, \phi_1 : M \to N$ *are* (smoothly) homotopic *if there exists a map* $\Phi : M \times [0,1] \to N$ *where* $\Phi(x,t) = \phi_t(x)$ *for* $t = 0, 1$.

Lemma 25.2. *If* $\phi_0, \phi_1 : M \to N$ *are homotopic and* $\omega \in \Omega^m(N)$, $m = \dim M$, *is closed, then* $\int_M \phi_0^*\omega = \int_M \phi_1^*\omega$.

Proof.

$$\int_M \phi_1^*\omega - \int_M \phi_0^*\omega = \int_{\partial(M\times[0,1])} \Phi^*\omega = \int_M d(\Phi^*\omega) = \int_M \Phi^*(d\omega) = 0,$$

since ω is closed. □

Example, continued: Since ω is a closed 1-form on $N = \mathbb{R}^2 - \{0\}$, $\int_C \omega = \int_{C'} \omega$ if C and C' are homotopic. That's why the integral did not depend on the radius R of the circle.

25.2. **Definition of degree.** This material can be found in Guillemin & Pollack.

Let $\phi : M \to N$ be a smooth map between compact oriented n-manifolds M and N. Let $y \in N$ be a regular value of ϕ. (Recall $y \in N$ is a *regular value* if, for all $x \in \phi^{-1}(y)$, $df(x)$ is surjective. $y \in N$ which is not a regular value is a *critical value*.)

Claim 25.3. $\phi^{-1}(y)$ *is a finite set* $\{x_1, \dots, x_k\}$.

Proof. Suppose $\phi^{-1}(y)$ is infinite. By the compactness of M, there exists a sequence $x_1, x_2, \dots$ of distinct points in $\phi^{-1}(y)$ and an accumulation point $x = \lim_{i\to\infty} x_i$, which itself must also be in $\phi^{-1}(y)$. However, for every $x \in \phi^{-1}(y)$ there exists an open set U_x which maps diffeomorphically onto an open set around y. Therefore, x could not have been the limit of $x_i \in \phi^{-1}(y)$. □

The claim implies that for a sufficiently small open set V_y containing y, $\phi^{-1}(V_y)$ is a finite disjoint union of open sets $U_{x_1}, \dots, U_{x_k}$, each of which is diffeomorphic to V_y.

Definition 25.4. *The* degree *of a smooth map* $\phi : M \to N$ *between compact oriented* n*-manifolds is the sum of orientation numbers* ± 1 *for each* x_i *in the preimage of a regular value* y*. Here the* *sign is* $+1$ *if the map from a neighborhood of* x_i *to a neighborhood of* y *is orientation-preserving and* -1 *otherwise.*

The following theorem will be explained in a couple of lectures.

Theorem 25.5 (Degree Theorem)**.** *The degree of a mapping* $\phi : M \to N$ *is well-defined.*

26. Sard's Theorem

26.1. **Sard's Theorem and measure zero.**

Theorem 26.1 (Sard). *Let $f : M \to N$ be a smooth map. Then the set of critical values of f (i.e., the set of points $y \in N$ such that $df(x)$ is not surjective for some $x \in f^{-1}(y)$) has measure zero.*

Definition 26.2. *A set $S \subset N$ has* measure zero *if there exists a countable atlas $\{(U_i, \phi_i)\}_{i=1}^{\infty}$ such that, for each $\phi_i(U_i) \subset \mathbb{R}^n$ (here $\dim N = n$) and $\varepsilon > 0$, $\phi_i(U_i \cap S)$ can be covered by a countable union of rectangles $[a_1, b_1] \times \cdots \times [a_n, b_n]$ with total volume $\leq \varepsilon$.*

From now on we will not distinguish between U_i and its image $\phi_i(U_i) \subset \mathbb{R}^n$.

Remark 26.3. *The measure in "measure zero" refers to the Lebesgue measure μ when restricted to $\mathbb{R}^n$. Note that the Lebesgue measure μ is multiplied by a positive smooth function under a change of coordinates. Hence "measure zero" is easier to define than a measure on all of N.*

Remark 26.4. *If $S \subset N$ has measure zero, then S itself can be covered by a countable union of rectangles with total volume $\leq \varepsilon$ (here to each rectangle we need to assign the open set U_i with respect to which we compute the volume): For U_i, take rectangles so that the total volume $\leq \varepsilon \left(\frac{1}{2}\right)^i$. Adding up over all the U_i, we get $\varepsilon \left(\frac{1}{2} + \frac{1}{4} + \dots\right) = \varepsilon$.*

Remark 26.5. *Open subsets of $\mathbb{R}^n$ have nonzero Lebesgue measure. Hence the set of regular values of a smooth map $f : M \to N$ is dense in N.*

26.2. **Proof of Sard's Theorem.**

The proof closely follows that of Milnor, *Topology from the Differentiable Viewpoint*.

It suffices to prove Sard's Theorem in the following local situation.

Theorem 26.6. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map. If we set $C = \{p \mid \operatorname{rank} df(p) < n\}$, then $f(C)$ has measure zero in $\mathbb{R}^n$.*

Proof. We will prove the theorem for $n = 1$, i.e., $f : \mathbb{R}^m \to \mathbb{R}$. The general case is left to the reader.

Define the following subsets of $\mathbb{R}^m$:

$$\begin{aligned} C_1 &= \left\{ p \in \mathbb{R}^m \,\middle|\, \frac{\partial f}{\partial x_i}(p) = 0, \forall i = 1, \dots, m \right\}, \\ C_k &= \{ p \in \mathbb{R}^m \mid \text{all partial derivatives of } f \text{ up to and including order } k \text{ vanish at } p \}. \end{aligned}$$

Then clearly $C = C_1 \supset C_2 \supset C_3 \supset \dots$.

Strategy. We prove the theorem by induction on m. For $m = 0$ the theorem is clear. Assuming the theorem holds for smooth maps $\mathbb{R}^{m-1} \to \mathbb{R}$, we show that for $f : \mathbb{R}^m \to \mathbb{R}$:

1. $f(C_1 - C_2)$ has measure zero.
2. $f(C_k - C_{k+1})$ has measure zero for any $k \geq 1$.
3. For k large enough ($k \geq m$), $f(C_k)$ has measure zero.

Step 1. Let $p \in C_1 - C_2$. We claim there exists a neighborhood V of p for which $f((C_1 - C_2) \cap V)$ has measure zero. (This suffices because if we can cover $C_1 - C_2$ with countably many such V's, the total measure of $f(C_1 - C_2)$ is zero by the argument of Remark 26.4.) If $p \in C_1 - C_2$, then $\frac{\partial f}{\partial x_1}(p) = \cdots = \frac{\partial f}{\partial x_m}(p) = 0$, but some $\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \neq 0$. Without loss of generality assume that $\frac{\partial^2 f}{\partial x_1 \partial x_j}(p) \neq 0$. Then consider the map

$$h : \mathbb{R}^m \to \mathbb{R}^m, \quad (x_1, \dots, x_m) \mapsto (\tfrac{\partial f}{\partial x_j}, x_2, \dots, x_m).$$

Since the Jacobian is invertible at p, by the Inverse Function Theorem the map h restricts to a diffeomorphism $V \overset{\sim}{\to} V'$, where V, V' are open sets containing $p, f(p)$. Let $(\tilde{x}_1, \dots, \tilde{x}_m) = (\frac{\partial f}{\partial x_j}, x_2, \dots, x_m)$ be coordinates on V'. Observe that:

- $\{$critical values of $f : V \to \mathbb{R}\} = \{$critical values of $f \circ h^{-1} : V' \to \mathbb{R}\}$; and
- $\{$critical values of $f \circ h^{-1} : V' \to \mathbb{R}\} \subset \{$critical values of $f \circ h^{-1} : \{\tilde{x}_1 = 0\} \to \mathbb{R}\}$.

We can then apply the inductive assumption to $f \circ h^{-1}$ whose domain is $(m-1)$-dimensional.

Step 2. Similar to Step 1.

Step 3. Let $p \in C_k$, $k \geq m$. In local coordinates we assume $f : [-\frac{1}{2}, \frac{1}{2}]^m \to \mathbb{R}$ and $p = 0$. Taylor's Theorem with remainder gives us

$$f(x+h) = f(x) + R(x;h),$$

where $|R(x;h)| \leq C|h|^{k+1}$, for all $x \in C_k \cap [-\frac{1}{2}, \frac{1}{2}]^m$ and $|h| < \delta$, where $\delta > 0$ is small.

We cover $[-\frac{1}{2}, \frac{1}{2}]^m$ by cubes of length $\delta > 0$ small. For this we need roughly $\frac{1}{\delta^m}$ cubes. Consider one such cube Q which nontrivially intersects C_k. Then its volume is δ^m, whereas its image under f has length on the order of magnitude of δ^{k+1} by Taylor's Theorem. Adding up the total volume of the image, we obtain an upper bound of $C\frac{1}{\delta^m}\delta^{k+1}$, which can be made arbitrarily small by choosing δ small. □

27. Degree

27.1. $H^n(M)$ of a n-dimensional manifold.

Theorem 27.1. *If M is an oriented, compact n-manifold (without boundary), then $\int : H^n(M) \to \mathbb{R}$ is an isomorphism.*

Proof. By Lemma 24.2, $\int : H^n(M) \to \mathbb{R}$ is well-defined and nonzero. It remains to show that if $\int \omega = 0$, then ω is exact. Let $\{U_i\}$ be a cover of M which is finite and has the property that every U_i is diffeomorphic to $\mathbb{R}^n$. Using a partition of unity $\{f_\alpha\}$ subordinate to $\{U_i\}$, we can split ω into the sum $\sum_i \omega_i$, where ω_i is supported inside U_i. Note that $\int_{U_i} \omega_i$ may not be zero.

Lemma 27.2. *If ω is an n-form with compact support and zero integral on $\mathbb{R}^n$, then $\omega = d\eta$, where η has compact support.*

Proof. We will prove this for $n = 2$. Then $\omega = f(x,y)dxdy$. Define $g(x) = \int_{-\infty}^{\infty} f(x,y)dy$. By Fubini's theorem and the hypothesis that $\int_{\mathbb{R}^2} \omega = 0$, we have $\int_{-\infty}^{\infty} g(x)dx = 0$. Define $G(x,y) = \varepsilon(y)g(x)$, where $\varepsilon(y)$ is a bump function with total area 1. Then write:

$$\eta(x,y) = -\left(\int_{-\infty}^{y} [f(x,t) - G(x,t)]dt\right) dx + \left(\int_{-\infty}^{x} G(t,y)dt\right) dy.$$

Clearly, $d\eta = [f(x,y) - G(x,y)]dxdy + G(x,y)dxdy$ and η has compact support. □

By Lemma 27.2 we can replace ω_i by a cohomologically equivalent n-form which is supported on a small neighborhood of a point $x_i \in M$, i.e., we may assume that ω_i is a bump n-form. The total volume of the ω_i is still zero. Now, engulf all the x_i in an open set $U \subset M$ which is diffeomorphic to $\mathbb{R}^n$ so that ω is compactly supported in U and has total area zero. We use the lemma again to complete the proof of Theorem 27.1. □

27.2. **Relationship to the degree.** Recall the definition of degree: Let $\phi : M \to N$ be a smooth map between compact, oriented manifolds without boundary of dimension n. By Sard's Theorem, the set of regular values of ϕ has full measure in N. Let $y \in N$ be a regular value and let $\phi^{-1}(y) = \{x_1, \dots, x_k\}$. Then $\deg(\phi) = \sum_{i=1}^{k} \pm 1$, where the contribution is $+1$ when ϕ is orientation-preserving near x_i and -1 is otherwise.

We will explain why the degree is well-defined. The map $\phi : M \to N$ induces the map $\phi^* : H^n(N) \to H^n(M)$ and we have a commutative diagram:

$$\begin{array}{ccc} H^n(N) & \xrightarrow{\phi^*} & H^n(M) \\ {\scriptstyle\int}\downarrow & & {\scriptstyle\int}\downarrow \\ \mathbb{R} & \xrightarrow{c} & \mathbb{R} \end{array}$$

where the map $\mathbb{R} \to \mathbb{R}$ is multiplication by some real number c. The following proposition calculates this constant c to be $\deg \phi$:

Proposition 27.3. $\deg\phi$ *satisfies*

$$\int_M \phi^*\omega = \deg\phi \int_N \omega. \tag{10}$$

Proof. Once we can prove Equation (10) for a suitable ω of our choice, the proposition follows. Take ω to be supported on a small open disk V_y about y with positive integral. Then $\int_M \phi^*\omega$ will be the sum of $\int_{U_{x_i}} \phi^*\omega$, where U_{x_i} are connected components of the preimage of V_y. Noting that ϕ is a diffeomorphism from U_{x_i} to V_y, we have $\int_{U_{x_i}} \phi^*\omega = \pm\int_{V_y}\omega$, depending on whether the orientations agree or not. This proves Equation (10). □

Corollary 27.4. *The degree of* ϕ *is independent of the choice of regular value* y.

28. Lie derivatives

We define the *interior product* on the linear algebra level as follows:[5] If $v \in V$, then $i_v : \wedge^k V^* \to \wedge^{k-1} V^*$ is given by:

$$f_1 \wedge \cdots \wedge f_k \mapsto \sum_l (-1)^{l+1} f_1 \wedge \ldots f_l(v) \cdots \wedge f_k.$$

Check this is well-defined!

Let X be a vector field on M. Then the we can globalize the above pointwise definition to obtain *interior product* $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ such that $(i_X\omega)(x) = i_{X(x)}\omega(x)$. It satisfies the following properties:

1. For 1-forms ω, $i_X(\omega) = \omega(X)$.
2. In general, we obtain the relation:

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge i_X\beta.$$

Convention: We are taking $\Omega^{-1}(M) = 0$ and $i_X : \Omega^0(M) \to \Omega^{-1}(M)$ to be the zero map.

Given $\omega \in \Omega^k(M)$ and $X_1, \ldots, X_k \in \mathfrak{X}(M)$, we define:

$$\omega(X_1, \ldots, X_k) := i_{X_k} \circ \cdots \circ i_{X_1}\omega.$$

In particular, if $\omega = df_1 \wedge \cdots \wedge df_k$, then

$$\omega(X_1, \ldots, X_k) = \sum_{\sigma \in S_k} (-1)^{\mathrm{sgn}(\sigma)} df_1(X_{\sigma(1)}) \ldots df_k(X_{\sigma(k)}).$$

Definition 28.1. *The Lie derivative with respect to X is the map*

$$\mathcal{L}_X := d \circ i_X + i_X \circ d : \Omega^k(M) \to \Omega^k(M).$$

Lemma 28.2.

1. *If $f \in \Omega^0(M)$, then $\mathcal{L}_X f = d(i_X f) + i_X(df) = df(X) = X(f)$. Hence $\mathcal{L}_X : \Omega^0(M) \to \Omega^0(M)$ satisfies the (unsigned) Leibniz rule.*
2. $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X\omega)$.
3. *$\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ satisfies $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta)$, i.e., the unsigned Leibniz rule.*

Proof. The proof is a simple computation and is left as an exercise. □

By Lemma 28.2, $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ naturally extends the derivation $X : \Omega^0(M) \to \Omega^0(M)$. (We will usually call anything that satisfies the unsigned Leibniz rule a "derivation".)

The following is also a source of derivations $\Omega^k(M) \to \Omega^k(M)$: Let $\phi_t : M \overset{\sim}{\to} M$ be a *1-parameter family of diffeomorphisms*, i.e., there exists a smooth map $\Phi : M \times [-1, 1] \to M$ such

[5] This looks slightly different from the previous definition.

that $\phi_t(x) := \Phi(x,t)$, $t \in [-1,1]$, is a diffeomorphism. Assume in addition that $\phi_0 = \mathrm{id}$. Then

$$\frac{d}{dt}\phi_t^*\omega|_{t=0}$$

is a derivation (verification is easy). If $f \in \Omega^0(M)$, then

$$\frac{d}{dt}\phi_t^* f|_{t=0} = \frac{d}{dt} f(\phi_t)|_{t=0} = df(X) = X(f),$$

where X is the vector field such that $X(x)$, $x \in M$, is viewed as the arc $\phi_t(x)$, $t \in [-1,1]$, that passes through x. We will often write $X = \frac{d\phi_t}{dt}|_{t=0}$.

Proposition 28.3 (Cartan formula)**.** *If $\phi_t : M \stackrel{\sim}{\to} M$ is a 1-parameter family of diffeomorphisms such that $\phi_0 = id$, then $\frac{d}{dt}\phi_t^*|_{t=0} : \Omega^k(M) \to \Omega^k(M)$ is given by $d \circ i_X + i_X \circ d$, where $X = \frac{d\phi_t}{dt}|_{t=0}$.*

Proof. It suffices to check the following:

- $\frac{d}{dt}\phi_t^*|_{t=0}$ and $\mathcal{L}_X$ both satisfy the Leibniz rule. (Already verified!)
- $\frac{d}{dt}\phi_t^*|_{t=0}$ and $\mathcal{L}_X$ agree on $\Omega^0(M)$. (Yes, they are both vector fields.)
- d commutes with $\frac{d}{dt}\phi_t^*|_{t=0}$ and with $\mathcal{L}_X$.

The above three properties allow us to do an induction on the degree. □

Remark 28.4. *We can generalize the Cartan formula slightly as follows: Let $\phi_t : M \stackrel{\sim}{\to} M$, $t \in [a,b]$, be a 1-parameter family of diffeomorphisms and let X_{t_0}, $t_0 \in (a,b)$, be the vector field on M where the arc $\phi_t(x) : [t_0 - \varepsilon, t_0 + \varepsilon] \to M$ is assigned at the point $\phi_{t_0}(x)$ (note NOT at x). Then*

$$\frac{d}{dt}\phi_t^*\omega|_{t=t_0} = \phi_{t_0}^*(d \circ i_{X_{t_0}} + i_{X_{t_0}} \circ d)\omega.$$

29. Homotopy properties

29.1. Homotopy properties of de Rham cohomology.

Proposition 29.1. *Let $\phi_t : M \xrightarrow{\sim} M$, $t \in [0,1]$, be a 1-parameter family of diffeomorphisms. Then ϕ_t^* induce the same map $H^k(M) \to H^k(M)$ for all $t \in [0,1]$.*

Proof. Consider a closed k-form ω on M. In view of Remark 28.4,

$$\frac{d}{dt}\phi_t^*\omega|_{t=t_0} = \phi_{t_0}^*(d \circ i_{X_{t_0}} + i_{X_{t_0}} \circ d)\omega = d(\phi_{t_0}^* i_{X_{t_0}}\omega).$$

Therefore it is exact. Now, $\phi_t^*\omega - \omega = \int_0^t \frac{d}{dt}\phi_t^*\omega|_{t=s} ds$, and the difference is exact as well. (This is evident by thinking of the integral as a limit of Riemann sums.) □

Next we say two maps $\phi_0, \phi_1 : M \to N$ are *(smoothly) homotopic* if there exists a smooth map $\Phi : M \times [0,1] \to N$ with $\phi_t(\cdot) = \Phi(\cdot, t)$, $t = 0, 1$. ϕ_t is said to be the *homotopy* from ϕ_0 to ϕ_1.

Proposition 29.2 (Homotopy invariance)**.** *If $\phi_t : M \to N$, $t \in [0,1]$, is a homotopy from ϕ_0 to ϕ_1, then $\phi_t^* : H^k(N) \to H^k(M)$ is independent of t.*

Proof. Consider $\Phi : M \times \mathbb{R} \to N$. (There exists an extension $\Phi : M \times [0,1] \to N$ to $\Phi : M \times \mathbb{R} \to N$; this is easy for example when M is compact.) We have inclusions $i_t : M \to M \times \mathbb{R}$, $x \mapsto (x,t)$, and clearly $\phi_t = \Phi \circ i_t$. Now take a diffeomorphism $\Psi_t : M \times \mathbb{R} \to M \times \mathbb{R}$, $(x,s) \mapsto (x, s+t)$. Since $i_t = \Psi_t \circ i_0$, $i_t^* = i_0^* \circ \Psi_t^*$. By the previous proposition, Ψ_t^* is independent of t on the level of cohomology. Hence so are i_t^* and ϕ_t^*. □

29.2. Homotopy equivalence.

We say $\phi : M \to N$ is a *homotopy equivalence* if there exists $\psi : N \to M$ such that $\phi \circ \psi : N \to N$ and $\psi \circ \phi : M \to M$ are homotopic to $\mathrm{id} : N \to N$ and $\mathrm{id} : M \to M$. Using Proposition 29.2, it is easy to show:

Proposition 29.3 (Homotopy equivalence)**.** *A homotopy equivalence $\phi : M \to N$ induces an isomorphism $\phi^* : H^k(N) \to H^k(M)$.*

Proof. This is because $\phi^* \circ \psi^* = \mathrm{id}$ and $\psi^* \circ \phi^* = \mathrm{id}$ by homotopy invariance. This proves that ϕ^* and ψ^* are left and right inverses (as linear maps) and are isomorphisms. □

Corollary 29.4 (Poincaré lemma)**.** $H^k_{dR}(\mathbb{R}^n) = 0$ *if* $k > 0$.

Proof. We will show that $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{pt\}$. Consider maps $\phi : \mathbb{R}^n \to \mathbb{R}^0$, $(x_1, \dots, x_n) \mapsto 0$, and $\psi : \mathbb{R}^0 \to \mathbb{R}^n$, $0 \mapsto (0, \dots, 0)$. Clearly, $\phi \circ \psi : \mathbb{R}^0 \to \mathbb{R}^0$, $0 \mapsto 0$, is the identity map. Next, $\psi \circ \phi : \mathbb{R}^n \to \mathbb{R}^n$, $(x_1, \dots, x_n) \mapsto 0$ is homotopic to the identity map. In fact, consider $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$, $((x_1, \dots, x_n), t) \mapsto (tx_1, \dots, tx_n)$. □

Example: Consider a band $S^1 \times (-1,1)$. It is homotopy equivalent to S^1. We have maps $\phi : S^1 \times (-1,1) \to S^1$, $(\theta, t) \mapsto \theta$ and $\psi : S^1 \to S^1 \times (-1,1)$, $\theta \mapsto (\theta, 0)$. $\phi \circ \psi : S^1 \to S^1$

is id. $\psi \circ \phi : S^1 \times (-1,1) \to S^1 \times (-1,1)$ is $(\theta, t) \mapsto (\theta, 0)$ is homotopic to id. In fact, take $F : S^1 \times (-1,1) \times [0,1] \to S^1 \times (-1,1)$, $(\theta, t, s) \mapsto (\theta, ts)$. Therefore, we have:

$$H^k(S^1 \times (-1,1)) \simeq H^k(S^1).$$

Example: Similarly, $H^k(M \times \mathbb{R}^n) \simeq H^k(M)$. More generally, if E is a vector bundle over M, then $H^k(E) \simeq H^k(M)$.

29.3. **Extended example: surface of genus** g**.** Consider a surface Σ of genus g. If you remove a disk from Σ, you are left with a bouquet of $2g$ bands. You can now use Mayer-Vietoris with U a disk and V a bouquet of $2g$ bands.

29.4. **Euler characteristic.** Let M be an n-dimensional manifold. Then we define the Euler characteristic of M to be:

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H^i_{dR}(M).$$

Examples:

(1) $\chi(\mathbb{R}^n) = 1$.
(2) $\chi(S^2) = 1 + 0 + 1 = 2$.
(3) $\chi(T^2) = 1 - 2 + 1 = 0$.
(4) $\chi(\text{genus } g \text{ surface}) = 2 - 2g$.

Remark 29.5. *For compact surfaces, the Euler characteristic is given by the classical formula* $V - E + F$*, where* V *is the number of vertices,* E *is the number of edges, and* F *is the number of faces of a polyhedron representing the surface.*

HW: Prove that if $0 \to C_1 \to C_2 \to \cdots \to C_k \to 0$ is an exact sequence, then

$$\sum_{i=1}^{k} (-1)^i \dim C_i = 0.$$

Lemma 29.6. *If* $M = U \cup V$*, then* $\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V)$.

Proof. Use the Mayer-Vietoris sequence and add up the dimensions, using the above exercise. $\square$

30. VECTOR FIELDS

Recall a *vector field* on $U \subset M$ is a section of TM defined over U.

30.1. Lie brackets.

Given two vector fields X and Y on M viewed as derivations at each $p \in M$, we can define its Lie bracket $[X, Y] = XY - YX$, i.e., for $f \in C^\infty(M)$,

$$[X, Y](f) = X(Yf) - Y(Xf).$$

Proposition 30.1. *$[X, Y]$ is also a derivation, hence is a vector field.*

Proof. This is a local computation. Take $X = \sum_i a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_j b_j \frac{\partial}{\partial x_j}$. Then:

$$\begin{aligned} [X, Y](f) &= \sum_i a_i \frac{\partial}{\partial x_i} \left(\sum_j b_j \frac{\partial f}{\partial x_j} \right) - \sum_j b_j \frac{\partial}{\partial x_j} \left(\sum_i a_i \frac{\partial f}{\partial x_i} \right) \\ &= \sum_{ij} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} - \sum_{ij} b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} \end{aligned}$$

In other words,

$$\left[\sum_i a_i \frac{\partial}{\partial x_i}, \sum_j b_j \frac{\partial}{\partial x_j} \right] = \sum_{ij} \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

□

Properties of Lie brackets.

(1) (Anticommutativity) $[X, Y] = -[Y, X]$.
(2) (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.
(3) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

These properties are easy to verify, and are left as exercises.

30.2. Fundamental Theorem of Ordinary Differential Equations.

Let $X \in \mathfrak{X}(M)$.

Definition 30.2. *A curve $\gamma : (a, b) \to M$ is an* integral curve of X *if $\frac{d\gamma}{dt}(t) = X(\gamma(t))$. Here $\frac{d\gamma}{dt}(t)$ means $\gamma_*(\frac{\partial}{\partial t}(t))$.*

With respect to local coordinates $x = (x_1, \dots, x_n)$, if we write $X = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}$ and $\gamma(t) = x(t)$, then

$$\frac{dx}{dt}(t) = (a_1(x(t)), \dots, a_n(x(t))).$$

Theorem 30.3 (Fundamental Theorem of ODEs). *Given $X \in \mathfrak{X}(M)$ and $p \in M$, there exist an open set $U \ni p$, $\varepsilon > 0$, and a smooth map $\Phi : U \times (-\varepsilon, \varepsilon) \to M$ such that if we set $\gamma_x(t) = \Phi(x, t)$, $x \in U$, then $\gamma_x(0) = x$ and $\gamma_x(t)$ is an integral curve of X. Moreover, if $\gamma : (-\delta, \delta) \to M$ is another integral curve of X with $\gamma(0) = x$, then $\gamma(t) = \gamma_x(t)$ on $(-\varepsilon, \varepsilon) \cap (-\delta, \delta)$.*

For the Fundamental Theorem of ODEs we need X to be at least C^1. The proof of this theorem will be omitted. We also write $\phi_t(x) := \Phi(x,t)$ and refer to it as the *time-t flow of X*. Note that $\phi_0(x) = \mathrm{id}$.

Remarks.

(1) If M is compact without boundary and X is a vector field on M, then there exists a global flow $\Phi : M \times \mathbb{R} \to M$ with $\phi_0 = \mathrm{id}$ by the uniqueness of the flow of X, where defined. Since M is compact, the finite covering property ensures that we may choose ε to work for all the open sets U. If we know that there is a flow for a short time ε, we can repeat the flow and obtain a flow for an arbitrarily long time.

(2) However, if M is not compact, then there are vector fields X which do not admit global short-time flows $\Phi : M \times (-\varepsilon, \varepsilon) \to M$. (See the example below.)

(3) $\phi_s \circ \phi_t = \phi_{s+t}$ and $\phi_t^{-1} = \phi_{-t}$. In particular, on a compact M, $\{\phi_t\}_{t\in\mathbb{R}}$ forms a 1-parameter *group* of diffeomorphisms isomorphic to the additive group $\mathbb{R}$.

Example. On $M = \mathbb{R} - \{0\}$ consider $X = \frac{\partial}{\partial x}$. The vector field X, considered as a vector field on $\mathbb{R}$, clearly integrates to $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x,t) \mapsto x + t$. However, when $\{0\}$ is removed, no matter how small an ε you take, there is no $\Phi : (\mathbb{R} - \{0\}) \times (-\varepsilon, \varepsilon) \to \mathbb{R} - \{0\}$.

Corollary 30.4. *If $X(p) \neq 0$, then there exist coordinates $(x_1, \dots, x_n)$ near p such that $X = \frac{\partial}{\partial x_1}$.*

Proof. If M is n-dimensional, choose an $(n-1)$-manifold Σ, defined in a neighborhood of p, which is transverse to X. Here Σ is *transverse* to X if $T_q\Sigma$ and $X(q)$ span T_qM at all $q \in \Sigma$. (Why does such a Σ exist?) Now take $\psi : \Sigma \times (-\varepsilon, \varepsilon) \to M$ given by the flow Φ of X restricted to Σ. Since Σ is transverse to X, ψ is a diffeomorphism near p by the inverse function theorem. If $x_2, \dots, x_n$ are coordinates on Σ and x_1 is the coordinate for $(-\varepsilon, \varepsilon)$, then X can be written as $\frac{\partial}{\partial x_1}$.
□

31. Vector fields and Lie derivatives

31.1. **Pullback.**

Lemma 31.1. *Let $f : M \to N$ be a smooth map, $\omega \in \Omega^k(N)$, and $X_1, \dots, X_k \in \mathfrak{X}(M)$. Then we have:*

$$(f^*\omega)(x)(X_1, \dots, X_k) = \omega(f(x))(f_*X_1, \dots, f_*X_k).$$

Proof. It suffices to show the lemma for 1-forms dg. (Why is that?) Then:

$$(f^*dg)(X) = d(g \circ f)(X) = X(g \circ f) = f_*X(g) = dg(f_*X),$$

by definition of the pushforward of X.

Alternate proof. If we write this in local coordinates $x_1, \dots, x_m$ and $y_1, \dots, y_n$ for M and N, then $dg = \sum_i \frac{\partial g}{\partial y_i} dy_i$ and

$$f^*dg = \sum_{i,k} \frac{\partial g}{\partial y_i} \frac{\partial y_i}{\partial x_k} dx_k.$$

Writing $X = \sum_j b_j \frac{\partial}{\partial x_j}$, we obtain

$$f^*dg\left(\sum_j b_j \frac{\partial}{\partial x_j}\right) = \sum_{i,j} \frac{\partial g}{\partial y_i} \frac{\partial y_i}{\partial x_j} b_j,$$

whereas

$$dg\left(f_*\left(\sum_j b_j \frac{\partial}{\partial x_j}\right)\right) = \sum_i \frac{\partial g}{\partial y_i} dy_i \left(\sum_{j,l} b_j \frac{\partial y_l}{\partial x_j} \frac{\partial}{\partial y_l}\right) = \sum_{i,j} \frac{\partial g}{\partial y_i} \frac{\partial y_i}{\partial x_j} b_j.$$

□

31.2. **Lie derivatives.** Let X be a vector field on M. Assume there exists a flow $\Phi : M \times (-\varepsilon, \varepsilon) \to M$, $\phi_t(x) = \Phi(x, t)$, such that $\phi_0(x) = x$. (Since we will be considering derivatives, local flows suffice, but we will assume a global flow for ease of notation.)

The *Lie derivative* $\mathcal{L}_X$ on forms ω is given by:

$$\mathcal{L}_X \omega = \frac{d}{dt} \phi_t^* \omega|_{t=0}.$$

Lie derivatives can be defined on vector fields Y as well:

$$\mathcal{L}_X Y := \frac{d}{dt} (\phi_{-t})_* Y|_{t=0}.$$

Here, vector fields cannot usually be pulled back, but for a diffeomorphism ϕ, there is a suitable substitute, namely $(\phi^{-1})_*$.

More generally, it is easy to see that $\mathcal{L}_X$ can be defined on any section of $\wedge^k T^*M \otimes \wedge^l TM$.

Properties of $\mathcal{L}_X$.

(1) $\mathcal{L}_X f = Xf$.

(2) $\mathcal{L}_X\omega = (d \circ i_X + i_X \circ d)\omega$
(3) $\mathcal{L}_X(\omega(X_1, \dots, X_k)) = (\mathcal{L}_X\omega)(X_1, \dots, X_k) + \sum_i \omega(X_1, \dots, \mathcal{L}_X X_i, \dots, X_k)$.
(4) $\mathcal{L}_X Y = [X, Y]$.

Proof. (1), (2) are already proven. (3) is left for homework. (For example, you can do this in local coordinates; unwinding the definition needs to be done carefully.) We will prove (4), assuming (1), (2), (3). We compute:

$$\begin{aligned} X(Y(f)) &= \mathcal{L}_X(Yf) \\ &= \mathcal{L}_X(df(Y)) \\ &= (\mathcal{L}_X df)(Y) + df(\mathcal{L}_X Y) \\ &= (d \circ \mathcal{L}_X f)Y + df(\mathcal{L}_X Y) \\ &= d(X(f))Y + (\mathcal{L}_X Y)(f) \\ &= Y(X(f)) + (\mathcal{L}_X Y)(f). \end{aligned}$$

Therefore, $(\mathcal{L}_X Y)f = X(Y(f)) - Y(X(f)) = [X, Y](f)$. □

31.3. **Interpretation of $\mathcal{L}_X Y = [X, Y]$.** As before, X, Y may not have global flows, but for simplicity let us assume they do. Let $\phi_s : M \stackrel{\sim}{\to} M$, $s \in \mathbb{R}$, be the 1-parameter group of diffeomorphisms generated by X and $\psi_t : M \stackrel{\sim}{\to} M$, $t \in \mathbb{R}$, be the 1-parameter group of diffeomorphisms generated by Y. Noting that $Y(x) = \lim_{t \to 0} \frac{\psi_t(x) - x}{t}$, we have

$$\begin{aligned} \mathcal{L}_X Y(x) &= \lim_{s \to 0} \frac{((\phi_{-s})_* Y)(x) - Y(x)}{s} \\ &= \lim_{s,t \to 0} \frac{(\phi_{-s} \circ \psi_t \circ \phi_s(x) - x) - (\psi_t(x) - x)}{st} \\ &= \lim_{s,t \to 0} \frac{\phi_{-s} \circ \psi_t \circ \phi_s(x) - \psi_t(x)}{st} \\ &= \lim_{s,t \to 0} \psi_t \left(\frac{\psi_t^{-1} \circ \phi_s^{-1} \circ \psi_t \circ \phi_s(x) - x}{st} \right) \\ &= \lim_{s,t \to 0} \frac{\psi_t^{-1} \circ \phi_s^{-1} \circ \psi_t \circ \phi_s(x) - x}{st}. \end{aligned}$$

Hence, the Lie bracket $[X, Y]$ measures the infinitesimal discrepancy when you flow s units along X, t units along Y, $-s$ units along X and finally $-t$ units along Y.

32. Relationship between d and $[,]$; distributions

32.1. Relationship between d and $[,]$.

Proposition 32.1. *If $\theta \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$, then*

$$d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]).$$

Proof.

$$\begin{aligned} d\theta(X,Y) &= i_Y i_X d\theta = i_Y(\mathcal{L}_X - d \circ i_X)\theta \\ &= i_Y(\mathcal{L}_X\theta - d(\theta(X))) \\ &= (\mathcal{L}_X\theta)Y - Y(\theta(X)) \\ &= X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]). \end{aligned}$$

Here we are using $\mathcal{L}_X(\theta(Y)) = (\mathcal{L}_X\theta)Y + \theta([X,Y])$ to get from the third line to the fourth. □

More generally, for $\omega \in \Omega^k(M)$ and $X_1, \dots, X_{k+1} \in \mathfrak{X}(M)$:

$$\begin{aligned} d\omega(X_1,\dots,X_{k+1}) &= \sum_i (-1)^{i+1} X_i(\omega(X_1,\dots,\widehat{X_i},\dots,X_{k+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j], X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{k+1}). \end{aligned}$$

Here $\widehat{X_i}$ means omit the term with X_i. The proof is an exercise.

32.2. **Distributions.** Recall from Corollary 30.4 that if X is a vector field with $X(p) \neq 0$, then there exist local coordinates $(x_1, \dots, x_n)$ near p such that $X = \frac{\partial}{\partial x_1}$. Can we generalize this? If X, Y are two vector fields such that $X(p), Y(p)$ span a 2-dimensional subspace of T_pM, then the span of $X(x)$ and $Y(x)$ is a 2-plane field for every x in a neighborhood of p.

Let M be an n-dimensional manifold.

Definition 32.2.

1. *A k-dimensional* distribution *$\mathcal{D}$ is a smooth choice of a k-dimensional subspace $\mathcal{D}_p \subset T_pM$ at every point $p \in M$. By a* smooth choice *we mean that for each $p \in M$ there exists a small neighborhood U_p of p and k linearly independent vector fields $X_1, \dots, X_k$ on U_p which span $\mathcal{D}_q$ for each $q \in U_p$. Alternatively, $\mathcal{D}$ is a rank k subbundle of TM.*
2. *An* integral submanifold *N of M is a submanifold where $T_pN \subset \mathcal{D}_p$ at every $p \in N$.* $\dim N$ *is not necessarily equal to* $\dim \mathcal{D}$*, but* $\dim N \leq \dim \mathcal{D}$.
3. *A distribution $\mathcal{D}$ is* integrable *if M is covered by local coordinate charts $(x_1^\alpha, \dots, x_n^\alpha)$ such that $\mathcal{D} = \mathbb{R}\langle \frac{\partial}{\partial x_1^\alpha}, \dots, \frac{\partial}{\partial x_k^\alpha} \rangle$. Equivalently, $\mathcal{D}$ is* integrable *if there locally exist functions $f_1, \dots, f_{n-k}$ such that $\{f_1 = \text{const}, \dots, f_{n-k} = \text{const}\}$ are integral submanifolds of $\mathcal{D}$ and the f_i are independent, i.e., $df_1 \wedge \cdots \wedge df_{n-k} \neq 0$.*

Dually, we can define a k-dimensional distribution on M locally by prescribing $n-k$ linearly independent 1-forms $\omega_1, \dots, \omega_{n-k}$.

Example: On $\mathbb{R}^3$ with coordinates (x, y, z), consider $\omega = dz$. Then $\mathcal{D} = \ker \omega = \mathbb{R}\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$. The *integral surfaces* (i.e., 2-dimensional integral submanifolds) are $z = \mathrm{const}$ and $\mathcal{D}$ is an integrable 2-plane distribution.

Example: On $\mathbb{R}^3$, consider $\omega = dz + (xdy - ydx)$. Then

$$\mathcal{D} = \ker \omega = \mathbb{R}\langle \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \rangle.$$

$\mathcal{D}$ is called a *contact* 2*-plane distribution*. We claim that $\mathcal{D}$ is not integrable. First calculate

$$\omega \wedge d\omega = 2dx \wedge dy \wedge dz \neq 0.$$

If $\mathcal{D} = \mathbb{R}\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \rangle$ for some coordinates (x_1, x_2, x_3), then ω is of the form $f dx_3$. Now, $d\omega = df \wedge dx_3$ and $\omega \wedge d\omega = f dx_3 \wedge df \wedge dx_3 = 0$. This is a contradiction.

33. Frobenius' Theorem

Let M be an n-dimensional manifold. A distribution $\mathcal{D}$ of rank k is a rank k subbundle of TM. Locally, $\mathcal{D}$ is defined as the span of independent vector fields $X_1, \dots, X_k$ or as the kernel of independent 1-forms $\omega_1, \dots, \omega_{n-k}$.

Theorem 33.1 (Frobenius' Theorem). *A distribution $\mathcal{D} \subset TM$ of rank k is integrable if and only if for all $X, Y \in \Gamma(\mathcal{D})$, $[X, Y] \in \Gamma(\mathcal{D})$.*

33.1. **Proof of Frobenius' Theorem.** Suppose $\mathcal{D} \subset TM$ is integrable. Then there exist coordinates $x_1, \dots, x_n$ so that $\mathcal{D} = \mathbb{R}\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \rangle$. Hence $X = \sum_{i=1}^k a_i(x) \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^k b_j(x) \frac{\partial}{\partial x_j}$, and

$$[X, Y] = \sum_{i=1}^k \sum_{j=1}^k \left(a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \frac{\partial}{\partial x_i} \right) \in \Gamma(\mathcal{D}).$$

Suppose for all $X, Y \in \Gamma(\mathcal{D})$, $[X, Y] \in \Gamma(\mathcal{D})$. We will find coordinates $x_1, \dots, x_n$ so that $\mathcal{D} = \mathbb{R}\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \rangle$. Note that all our computations are local, so we restrict to $M = \mathbb{R}^n$. We will first do a slightly easier situation.

Proposition 33.2. *Let $X_1, \dots, X_k$ be independent vector fields with $\mathcal{D} = \mathbb{R}\langle X_1, \dots, X_k \rangle$. If $[X_i, X_j] = 0$ for all i, j, then $\mathcal{D}$ is integrable.*

Proof. We will deal with the case where $\dim \mathcal{D} = 2$ and $M = \mathbb{R}^3$. Suppose $[X, Y] = 0$. Using the fundamental theorem of ODE's, we can write $X = \frac{\partial}{\partial x_1}$. Then $Y = \sum_{i=1}^3 b_i \frac{\partial}{\partial x_i}$, and $[X, Y] = 0$ implies that $\frac{\partial b_i}{\partial x_1} = 0$, i.e., $b_i = b_i(x_2, x_3)$ (there is no dependence on x_1). Now take $Y' = Y - b_1 X = b_2(x_2, x_3) \frac{\partial}{\partial x_2} + b_3(x_2, x_3) \frac{\partial}{\partial x_3}$. If we project to $\mathbb{R}^2$ with coordinates x_2, x_3, then Y' can be integrated to $\frac{\partial}{\partial x_2'}$, after a possible change of coordinates. Therefore, $\mathcal{D} = \mathbb{R}\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2'} \rangle$. $\square$

Still assuming $\dim \mathcal{D} = 2$ and $M = \mathbb{R}^3$, suppose $[X, Y] = AX + BY$. Without loss of generality, $X = \frac{\partial}{\partial x_1}$ and $Y = b_2 \frac{\partial}{\partial x_2} + b_3 \frac{\partial}{\partial x_3}$. Then,

$$[X, Y] = \frac{\partial b_2}{\partial x_1} \frac{\partial}{\partial x_2} + \frac{\partial b_3}{\partial x_1} \frac{\partial}{\partial x_3} = A \frac{\partial}{\partial x_1} + B b_2 \frac{\partial}{\partial x_2} + B b_3 \frac{\partial}{\partial x_3}.$$

This implies: $A = 0$, $\frac{\partial b_2}{\partial x_1} = B b_2$, $\frac{\partial b_3}{\partial x_1} = B b_3$. Hence,

$$b_2 = f(x_2, x_3) e^{\int_{t=0}^{t=x_1} B(t, x_2, x_3) dt}, b_3 = g(x_2, x_3) e^{\int_{t=0}^{t=x_1} B(t, x_2, x_3) dt}.$$

Therefore, $Y = e^{\int B} (f(x_2, x_3) \frac{\partial}{\partial x_2} + g(x_2, x_3) \frac{\partial}{\partial x_3})$, and by rescaling Y we get $Y' = f(x_2, x_3) \frac{\partial}{\partial x_2} + g(x_2, x_3) \frac{\partial}{\partial x_3}$. As before, now Y' can be integrated to give $\frac{\partial}{\partial x_2'}$.

HW: Write out a general proof.

33.2. **Restatement in terms of forms.** If $\mathcal{D}$ has rank k on M of dimension n, then locally there exist 1-forms $\omega_1, \dots, \omega_{n-k}$ such that $\mathcal{D} = \{\omega_1 = \dots = \omega_{n-k} = 0\}$.

Proposition 33.3. *$\mathcal{D}$ is integrable if and only if $d\omega_i = \sum_{j=1}^{n-k} \theta_{ij} \wedge \omega_j$, where θ_{ij} are 1-forms.*

Proof. We use the identity

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]). \tag{11}$$

Let $X, Y \in \Gamma(\mathcal{D})$. Then the identity simplifies to:

$$d\omega(X,Y) = -\omega([X,Y]). \tag{12}$$

Suppose $d\omega_i = \sum_{j=1}^{n-k} \theta_{ij} \wedge \omega_j$. Then $d\omega_i(X,Y) = 0$. Hence $\omega_i([X,Y]) = 0$ for all i by Equation (12), which implies that $[X,Y] \in \Gamma(\mathcal{D})$.

Suppose $\mathcal{D}$ is integrable. Complete $\omega_1, \dots, \omega_{n-k}$ to a (pointwise) basis by adding $\eta_1, \dots, \eta_k$. Then

$$d\omega_i = \sum_{i<j} a_{ij}\omega_i \wedge \omega_j + \sum_{i=1}^{k} \sum_{j=1}^{n-k} b_{ij}\eta_i \wedge \omega_j + \sum_{i<j} c_{ij}\eta_i \wedge \eta_j.$$

By Equation (12), $d\omega_i(X,Y) = 0$ for $X, Y \in \Gamma(\mathcal{D})$. Taking $X_1, \dots, X_k \in \Gamma(\mathcal{D})$ dual to $\eta_1, \dots, \eta_k$, we find that $d\omega_i(X_r, X_s) = c_{rs}$ (or $-c_{sr}$). This proves that all the c_{ij} are zero. $\square$

34. Connections

34.1. **Definition.** Let E be a rank k vector bundle over M and let s be a section of E. s may be local (i.e., in $\Gamma(E,U)$) or global (i.e., in $\Gamma(E,M)$). Also let X be a vector field. We want to differentiate s at $p \in M$ in the direction of $X(p) \in T_pM$.

Definition 34.1. *A* connection *or* covariant derivative ∇ *assigns to every vector field* $X \in \mathfrak{X}(M)$ *a differential operator* $\nabla_X : \Gamma(E) \to \Gamma(E)$ *which satisfies:*

(1) $\nabla_X s$ *is* $\mathbb{R}$*-linear in* s*, i.e.,* $\nabla_X(c_1 s_1 + c_2 s_2) = c_1 \nabla_X s_1 + c_2 \nabla_X s_2$ *if* $c_1, c_2 \in \mathbb{R}$.
(2) $\nabla_X s$ *is* $C^\infty(M)$*-linear in* X*, i.e.,* $\nabla_{fX+gY} s = f\nabla_X s + g\nabla_Y s$.
(3) *(Leibniz rule)* $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

Note: The definition of connection is tensorial in X (condition (2)), so $(\nabla_X s)(p)$ depends on s *near* p but only on X *at* p.

34.2. **Flat connections.** We will now present the first example of a connection.

A vector bundle E of rank k is said to be *trivial* or *parallelizable* if there exist sections $s_1, \dots, s_k \in \Gamma(E,M)$ which span E_p at every $p \in M$. Although not every vector bundle is parallelizable, locally every vector bundle is trivial since $E|_U \simeq U \times \mathbb{R}^k$. We will now construct connections on trivial bundles.

Write any section s as $s = \sum_i f_i s_i$, where $f_i \in C^\infty(M)$. Then define

$$\nabla_X s = \sum_i (Xf_i)s_i = (Xf_1)s_1 + \cdots + (Xf_k)s_k \in \Gamma(E).$$

This connection is usually called a *flat connection*.

HW: Check that this satisfies the axioms of a connection.

Note that $\nabla_X s_i = 0$ for all $X \in \mathfrak{X}(M)$. Sections s satisfying such a property are said to be *covariant constant*.

Important remark: We can define a connection ∇ for each trivialization $E|_U = U \times \mathbb{R}^k$, and there is nothing canonical about the connection ∇ above. (It depends on the choice of trivialization.) The space of connections is a large space (to be made more precise later).

Proposition 34.2. *Any two covariant constant frames* $s_1, \dots, s_k$ *and* $\overline{s}_1, \dots, \overline{s}_k$ *differ by an element of* $GL(k, \mathbb{R})$.

Proof. Let $\overline{s}_1, \dots, \overline{s}_k$ be another covariant constant frame, i.e., $\nabla_X \overline{s}_i = 0$. Since we can write

$$\overline{s}_i = \sum_j f_{ij} s_j,$$

with $f_{ij} \in C^\infty(M)$, we have:

$$\begin{aligned} 0 = \nabla_X \overline{s}_i &= \sum_j \nabla_X (f_{ij} s_j) \\ &= \sum_j [(X f_{ij}) s_j + f_{ij} \nabla_X s_j] \\ &= \sum_j (X f_{ij}) s_j. \end{aligned}$$

This proves that $X f_{ij} = 0$ for all X and hence $f_{ij} = \text{const}$. $\square$

Therefore, a flat connection determines a covariant constant frame $\{s_1, \dots, s_k\}$ up to an element of $GL(k, \mathbb{R})$.

35. More on connections

35.1. **Preliminaries on vector bundles.** Let E be a vector bundle over M and $\phi : N \to M$ be a smooth map. Then we can define the *pullback bundle* $\phi^{-1}E$ over N as follows:

(1) The fiber $(\phi^{-1}E)_n$ over $n \in N$ is the fiber $E_{\phi(n)}$ over $\phi(n) \in M$.
(2) There exist sufficiently small open sets $V \subset N$, so that $\phi(V) \subset U$ and $\varphi_U : E|_U \xrightarrow{\sim} U \times \mathbb{R}^k$. The trivialization $\phi^{-1}E|_V \simeq V \times \mathbb{R}^k$ is induced from this.

Next, if E and F are vector bundles over M, then we can define $E \oplus F$ as follows:

(1) The fiber $(E \oplus F)_m$ over $m \in M$ is $E_m \oplus F_m$.
(2) Take $U \subset M$ small enough so that $E|_U \xrightarrow{\sim} U \times \mathbb{R}^k$ and $F|_U \xrightarrow{\sim} U \times \mathbb{R}^l$. Then we get $(E \oplus F)|_U \simeq U \times (\mathbb{R}^k \oplus \mathbb{R}^l)$.

$E \otimes F$ is defined similarly.

35.2. **Existence.** Let M be an n-dimensional manifold and E be a rank k vector bundle over M. Recall a connection ∇ is a way of differentiating sections of E in the direction of a vector field X.

$$\nabla_X : \Gamma(E) \to \Gamma(E),$$
$$\nabla_X(fs) = (Xf)s + f\nabla_X s.$$

Definition 35.1. *A connection ∇ on E is* flat *if there exists an open cover $\{U_\alpha\}$ of M such that $E|_{U_\alpha}$ admits a covariant constant frame $s_1, \dots, s_k$.*

Proposition 35.2. *Connections exist on any vector bundle E over M.*

Note that if E is parallelizable we have already defined connections globally on E. The key point is to pass from local to global when E is not globally trivial.

Let ∇' and ∇'' be two connections on $E|_U$. Let us see whether $\nabla' + \nabla''$ is a connection.

$$\begin{aligned} (\nabla'_X + \nabla''_X)(fs) &= \nabla'_X(fs) + \nabla''_X(fs) \\ &= (Xf)s + f\nabla'_X s + (Xf)s + f\nabla''_X s \\ &= 2(Xf)s + f(\nabla'_X + \nabla''_X)s. \end{aligned}$$

This is not quite a connection, since $2(Xf)$ should be Xf instead. However, a simple modification presents itself:

Lemma 35.3. *Suppose $\lambda_1, \lambda_2 \in C^\infty(U)$ satisfies $\lambda_1 + \lambda_2 = 1$. Then $\lambda_1\nabla' + \lambda_2\nabla''$ is a connection on $E|_U$.*

Proof. HW. □

Proof of Proposition 35.2. Let $\{U_\alpha\}$ be an open cover such that $E|_{U_\alpha}$ is trivial. Let ∇^α be a flat connection on $E|_{U_\alpha}$ associated to some trivialization. Next let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then form $\sum_\alpha f_\alpha \nabla^\alpha$. By the previous lemma, the Leibniz rule is satisfied. □

Remark: Although each of the pieces ∇^α is flat before patching, the patching destroys flatness. There is no guarantee that (even locally) there exist sections $s_1, \dots, s_k$ which are covariant constant. In fact, for a generic connection, there is not even a single covariant constant section. One way of measuring the failure of the existence of covariant constant sections is the *curvature*.

35.3. **The space of connections.** Given two connections ∇ and ∇', we compute their difference:

$$(\nabla_X - \nabla'_X)(fs) = f(\nabla_X - \nabla'_X)s.$$

Therefore, the difference of two connections is *tensorial* in s.

Locally, take sections $s_1, \dots, s_k$ (not necessarily covariant constant). Then $(\nabla_X - \nabla'_X)s_i = \sum_j a_{ij}s_j$, where (a_{ij}) is a $k \times k$ matrix of functions. In other words, $\nabla - \nabla'$ can be represented by a matrix $A = (A_{ij})$ of 1-forms A_{ij}. Here $a_{ij} = A_{ij}(X)$. Hence, locally it makes sense to write:

$$\nabla = d + A.$$

Here $s = \sum f_i s_i$ corresponds to $(f_1, \dots, f_k)^T$ and more precisely

$$\nabla(f_1, \dots, f_k)^T = d(f_1, \dots, f_k)^T + A(f_1, \dots, f_k)^T.$$

Globally, $\nabla - \nabla'$ is a section of $T^*M \otimes End(E)$. Here $End(E) = Hom(E, E)$. The space of such sections is often written as $\Omega^1(End(E))$ and a section is called a "1-form with values in $End(E)$". This proves:

Proposition 35.4. *The space of connections on E is an affine space $\Omega^1(End(E))$.*

Remark: We view $\Omega^1(End(E))$ not as a vector space (which has a preferred zero element) but rather as an affine space, which is the same thing except for the lack of a preferred zero element.

36. CURVATURE

Let $E \to M$ be a rank r vector bundle and ∇ be a connection on E.

Definition 36.1. *The* curvature R_∇ *(or simply R) of a connection ∇ is given by:*

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]},$$

or

$$R(X,Y)s = [\nabla_X, \nabla_Y]s - \nabla_{[X,Y]}s.$$

Proposition 36.2.

(1) *$R(X,Y)s$ is tensorial, i.e., $C^\infty(M)$-linear, in each of X, Y, and s.*
(2) *$R(X,Y) = -R(Y,X)$.*

Proof. (2) is easy. For (1), we will prove that $R(X,Y)$ is tensorial in s and leave the verification for X and Y as an exercise.

$$\begin{aligned} R(X,Y)(fs) &= (\nabla_X\nabla_Y - \nabla_Y\nabla_X)(fs) - \nabla_{[X,Y]}(fs) \\ &= \nabla_X((Yf)s + f\nabla_Y s) - \nabla_Y((Xf)s + f\nabla_X s) - (([X,Y]f)s + f\nabla_{[X,Y]}s) \\ &= fR(X,Y)s \end{aligned}$$

□

Proposition 36.3. *The flat connection $\nabla_X s = \sum (Xf_k)s_k$ has $R = 0$. (Here $s_1, \dots, s_r$ trivializes $E|_U$ and $s = \sum f_k s_k$.)*

Proof. We use $X = \frac{\partial}{\partial x_i}$, $Y = \frac{\partial}{\partial x_j}$. Since $R(X,Y)$ is tensorial, it suffices to compute it for our choices.

$$\begin{aligned} R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)s &= \sum_k \left(\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}} - \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\right) f_k s_k \\ &= \sum_l \left[\nabla_{\frac{\partial}{\partial x_i}}\left(\frac{\partial f_k}{\partial x_j}s_k\right) - \nabla_{\frac{\partial}{\partial x_j}}\left(\frac{\partial f_k}{\partial x_i}s_k\right)\right] \\ &= \sum \left(\frac{\partial^2 f_k}{\partial x_i \partial x_j} - \frac{\partial^2 f_k}{\partial x_j \partial x_i}\right)s_k = 0. \end{aligned}$$

□

36.1. **Interpretations of curvature.** Think of ∇ as $d + A$ in local coordinates if necessary. We have a sequence:

$$\Omega^0(E) \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\nabla} \Omega^2(E) \to \dots.$$

The first map is covariant differentiation (interpreted slightly differently). It turns out that this sequence is not a chain complex, i.e., $\nabla \circ \nabla \neq 0$ usually. In fact the obstruction to this being a chain complex is the curvature. Let us locally write:

$$\nabla \circ \nabla s = (d + A)(d + A)s = (d^2 + Ad + dA + A \wedge A)s = (dA + A \wedge A)s.$$

Proposition 36.4. *$R = dA + A \wedge A$, i.e., $R(X, Y)s = (dA + A \wedge A)(X, Y)s$.*

Proof. It suffices to prove the proposition for $X = \frac{\partial}{\partial x_i}$, $Y = \frac{\partial}{\partial x_j}$, and $s = s_k$, where $s_1, \dots, s_r$ is a local frame for $E|_U$. A is an $r \times r$ matrix of 1-forms $(A^t_{ij} dx_t)$. (We will use the Einstein summation convention – if the same index appears twice we assume it is summed over this index.) Then we compute:

$$\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} s_k = \nabla_{\frac{\partial}{\partial x_i}} (s_m A^j_{mk}) = s_m \frac{\partial A^j_{mk}}{\partial x_i} + s_n A^i_{nm} A^j_{mk} \tag{13}$$

The computation of the rest is left as an exercise. □

37. Riemannian metrics, Levi-Civita

37.1. **Leftovers from last time.** Last time we defined the curvature R_∇ of a connection ∇. Locally, if ∇ is given by $d + A$, then $R = dA + A \wedge A$.

Theorem 37.1. *∇ is a flat connection if and only if $R_\nabla \equiv 0$.*

We have already shown the easy direction: If ∇ is flat, then $R_\nabla \equiv 0$. The other direction will be omitted for now (probably will be given next semester), since a "good proof" will take us a bit far afield. Our only comment is that $R = dA + A \wedge A = 0$ or $dA = A \wedge A$ looks a lot like the Frobenius integrability condition given in terms of forms....

Corollary 37.2. *Let E be a rank r vector bundle over $\mathbb{R}$ and ∇ be a connection on E. Then ∇ is flat.*

Proof. This is because all 2-forms on $\mathbb{R}$ are zero. □

Remark: There are lots of connections which are not flat, since it is easy to find A so that $dA + A \wedge A \neq 0$.

37.2. **Riemannian metrics.**

Definition 37.3. *A* Riemannian metric *$\langle , \rangle$ or g on M is a positive definite symmetric bilinear form (or inner product) $g(x) : T_xM \times T_xM \to \mathbb{R}$ which is smooth in $x \in M$.*

Recall: A bilinear form $\langle , \rangle : V \times V \to \mathbb{R}$ is *positive definite* if $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$. $\langle , \rangle$ is *symmetric* if $\langle v, w\rangle = \langle w, v\rangle$.

Example: On $\mathbb{R}^n$ take the *standard Euclidean metric* $g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{ij}$. This is usually written as $g = \sum_i dx_i \otimes dx_i$. Any other Riemannian metric on $\mathbb{R}^n$ can be written as $g(x) = \sum_{ij} g_{ij}(x) dx_i \otimes dx_j$, where $g_{ij}(x) = g_{ji}(x)$.

Proposition 37.4. *Every manifold M admits a Riemannian metric.*

Proof. Let $\{U_\alpha\}$ be an open cover so that $U_\alpha \simeq \mathbb{R}^n$. On each U_α, we take the standard Euclidean metric g_α. Now let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then $\sum_\alpha f_\alpha g_\alpha$ is the desired metric. □

The pair (M, g) of a manifold M together with a Riemannian metric g on M is called a *Riemannian manifold.*

Let $i : N \to (M, g)$ be an embedding or immersion. Then the *induced Riemannian metric* i^*g on N is defined as follows:

$$i^*g(x)(v, w) = g(i(x))(i_*v, i_*w),$$

where $v, w \in T_xN$. The injectivity of i_* is required for the positive definiteness.

37.3. **Levi-Civita connections.** Connections on $TM \to M$ have extra structure because X and Y are the same type of object in the expression $\nabla_X Y$. In fact, we can define the *torsion*:

$$\mathcal{T}_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Proposition 37.5. *$\mathcal{T}_\nabla(X,Y)$ is tensorial in X and Y.*

This is an easy exercise and is left as an exercise. (Note that the notion of torsion does not depend at all on the Riemannian metric.) We say ∇ is *torsion-free* if $\mathcal{T}_\nabla = 0$.

Definition 37.6. *∇ is* compatible *with g if $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$. Here $X,Y,Z \in \mathfrak{X}(M)$.*

Theorem 37.7. *Let (M,g) be a Riemannian manifold. Then there exists a unique torsion-free connection which is compatible with g.*

Proof. For any vector fields X,Y,Z, we have:

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{14}$$

$$Y\langle X,Z\rangle = \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle, \tag{15}$$

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \tag{16}$$

Taking (14) + (15) − (16), we get:

$$2\langle \nabla_X Y, Z\rangle + \langle [Y,X],Z\rangle + \langle [X,Z],Y\rangle + \langle [Y,Z],X\rangle = X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle, \tag{17}$$

and solving for $\langle \nabla_X Y, Z\rangle$, we get:

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(\langle [X,Y],Z\rangle + \langle [Z,X],Y\rangle + \langle [Z,Y],X\rangle + X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle). \tag{18}$$

It is clear that the values of Equation 18 determine ∇. It remains to show that Equation 18 indeed defines a connection which satisfies the torsion-free and compatibility conditions. It is clear that $\langle \nabla_X Y, Z\rangle = \langle \nabla_Y X, Z\rangle$ when X,Y,Z are of the form $\frac{\partial}{\partial x_i}$, and that Equation 14 can be recovered from Equation 18. The details are left as an exercise. □

The unique torsion-free, compatible connection is called the *Levi-Civita connection* for (M,g).

38. Shape operator

Let Σ be a surface embedded in the standard Euclidean $(\mathbb{R}^3, g)$, and let $\overline{g}$ be the induced metric on Σ. We will denote the Levi-Civita connection on $(\mathbb{R}^3, g)$ by ∇ and the Levi-Civita connection on $(\Sigma, \overline{g})$ by $\overline{\nabla}$.

Claim: ∇ satisfies $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$.

The verification is easy. The claim implies that $\nabla_X Y$ is simply $\frac{d}{dt} Y(\gamma(t))|_{t=0}$, where $\gamma(t)$ is the arc representing X at a given point.

What we will do today is valid for hypersurfaces ($(n-1)$-dimensional submanifolds) M inside (N, g) of dimension n, but we will restrict our attention to $N = \mathbb{R}^3$ for simplicity.

Definition 38.1. *Let X, Y be vector fields of $\mathbb{R}^3$ which are tangent to Σ, and let N be the* unit normal vector field *to Σ inside $\mathbb{R}^3$. The* shape operator *is $S(X, Y) = \langle \nabla_X Y, N \rangle$. In other words, $S(X, Y)$ is the projection in the N-direction of $\nabla_X Y$.*

Proposition 38.2. *$S(X, Y)$ is tensorial in X, Y and is symmetric.*

Proof. $S(X, Y) = S(Y, X)$ follows from the torsion-free condition and the fact that $[X, Y]$ is tangent to Σ. Now,

$$S(fX, Y) = \langle \nabla_{fX} Y, N \rangle = \langle f \nabla_X Y, N \rangle = f S(X, Y).$$

Tensoriality in Y is immediate from the symmetric condition. □

Remark: The shape operator is usually called the *second fundamental form* in classical differential geometry and measures how curved a surface is. (In case you are curious what the *first fundamental form* is, it's simply the induced Riemannian metric.)

Also observe that $S(X, Y) = \langle \nabla_X Y, N \rangle = \langle \nabla_X N, Y \rangle$, by using the fact that $\langle Y, N \rangle = 0$ (since N is a normal vector and Y is tangent to Σ).

38.1. Induced connection vs. Levi-Civita.

If $X, Y \in \mathfrak{X}(M)$, we can write:

$$\nabla_X Y = \nabla^h_X Y + S(X, Y) N,$$

where $\nabla^h_X Y$ denotes the projection of $\nabla_X Y$ onto $T\Sigma$.

Proposition 38.3. *$\nabla^h = \overline{\nabla}$, i.e., ∇^h is the Levi-Civita connection of $(\Sigma, \overline{g})$.*

Proof. We have defined $\nabla^h_X Y = \nabla_X Y - S(X, Y) N$. It is easy to verify that ∇^h satisfies the properties of a connection on Σ.

∇^h is torsion-free:

$$\begin{aligned} \nabla^h_X Y - \nabla^h_Y X &= (\nabla_X Y - S(X, Y) N) - (\nabla_Y X - S(Y, X) N) \\ &= \nabla_X Y - \nabla_Y X \\ &= [X, Y] \end{aligned}$$

∇^h is compatible with $\overline{g}$:

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ &= \langle \nabla^h_X Y, Z\rangle + \langle Y, \nabla^h_X Z\rangle, \end{aligned}$$

since $\langle N, X\rangle = 0$ for any vector field N on Σ. □

It seems miraculous that somehow the induced connection is a Levi-Civita connection. Classically, the induced covariant derivative came first, and Levi-Civita came as an abstraction of the covariant derivative.

39. Gauss' Theorema Egregium

Let $(\Sigma, \overline{g})$ be a 2-dimensional Riemannian submanifold of the standard Euclidean $(\mathbb{R}^3, g)$. The shape operator is a symmetric bilinear form:

$$S : T_x\Sigma \times T_x\Sigma \to \mathbb{R},$$
$$S(X,Y) = \langle \nabla_X Y, N\rangle$$

where N is a unit normal to Σ, X, Y are vectors in $T_x\Sigma$ which are extended to an arbitrary vector field tangent to Σ, and ∇ is the Levi-Civita connection for $(\mathbb{R}^3, g)$. We can represent $S(x)$, $x \in \Sigma$, as a matrix by taking an orthonormal basis $\{e_1, e_2\}$ at $T_x\Sigma$ and taking the entries $S(e_i, e_j)$. The trace of this matrix is called the *mean curvature* and the determinant is called the *scalar curvature* or the *Gaußian curvature*.

Denote by ∇ the Levi-Civita connection for g and $\overline{\nabla}$ the Levi-Civita connection for $\overline{g}$. Also write $R = R_\nabla$ and $\overline{R}$ for $R_{\overline{\nabla}}$.

Theorem 39.1 (Gauß' Theorema Egregium)**.** *If X, Y are vector fields on Σ, then*

$$\langle \overline{R}(X,Y)Y, X\rangle = S(X,X)S(Y,Y) - S(X,Y)^2.$$

What this says is that the right-hand side, an extrinsic quantity (depends on the embedding into 3-space) is equal to the left-hand side, an intrinsic quantity (only depends on the Riemannian metric $\overline{g}$ and not on the particular embedding into $\mathbb{R}^3$). Therefore, the scalar curvature is expressed purely in terms of the curvature of the induced metric.

Proof. Let N be the unit normal vector to Σ.

$$\begin{aligned}
\langle \overline{\nabla}_X \overline{\nabla}_Y Y, X\rangle &= X\langle \overline{\nabla}_Y Y, X\rangle - \langle \overline{\nabla}_Y Y, \overline{\nabla}_X X\rangle \\
&= X\langle \nabla_Y Y - S(Y,Y)N, X\rangle - \langle \nabla_Y Y - S(Y,Y)N, \nabla_X X - S(X,X)N\rangle \\
&= X\langle \nabla_Y Y, X\rangle - \langle \nabla_Y Y, \nabla_X X\rangle + \langle S(X,X)N, \nabla_Y Y\rangle \\
&= \langle \nabla_X \nabla_Y Y, X\rangle + S(X,X)S(Y,Y).
\end{aligned}$$

Similarly,

$$\langle \overline{\nabla}_Y \overline{\nabla}_X Y, X\rangle = \langle \nabla_Y \nabla_X Y, X\rangle - S(X,Y)^2,$$
$$\langle \overline{\nabla}_{[X,Y]} Y, X\rangle = \langle \nabla_{[X,Y]} Y, X\rangle.$$

Finally,

$$\begin{aligned}
\langle \overline{R}(X,Y)Y, X\rangle &= \langle R(X,Y)Y, X\rangle + S(X,X)S(Y,Y) - S(X,Y)^2 \\
&= S(X,X)S(Y,Y) - S(X,Y)^2.
\end{aligned}$$

□

40. EULER CLASS

40.1. **Compatible connections.** Let E be a rank k vector bundle over a manifold M. A *fiber metric* is a family of positive definite inner products $\langle , \rangle_x : E_x \times E_x \to \mathbb{R}$ which varies smoothly with respect to $x \in M$. A connection ∇ is *compatible* with $\langle , \rangle$ if $X\langle s_1, s_2\rangle = \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle$, for all vector fields X and $s_1, s_2 \in \Gamma(E)$.

Remark: We can view the Riemannian metric g on M as a fiber metric of $TM \to M$. When we think of TM as a vector bundle over M, we forget the fact that TM was derived from M.

Let $U \subset M$ be an open set over which E is trivializable, and let $\{s_1, s_2, \dots, s_k\}$ be an orthonormal frame of E over U. An orthonormal frame can be obtained by starting from some frame of E over U and applying the Gram-Schmidt orthogonalization process.

With respect to $\{s_1, \dots, s_k\}$ we can write $\nabla = d + A$, where A is a $k \times k$ matrix with entries which are 1-forms on U.

Lemma 40.1. *A is a skew-symmetric matrix, i.e., $A^T = -A$.*

Proof. If we write $A = (A^k_{ij} dx_k)$, then we have $\nabla_{\frac{\partial}{\partial x_k}} s_j = s_i A^k_{ij}$.

$$\frac{\partial}{\partial x_k}\langle s_i, s_j\rangle = \langle \nabla_{\frac{\partial}{\partial x_k}} s_i, s_j\rangle + \langle s_i, \nabla_{\frac{\partial}{\partial x_k}} s_j\rangle,$$

so we have

$$A^k_{ij} = -A^k_{ij}.$$

□

Lemma 40.2. *Let $\{s'_1, \dots, s'_k\}$ be another orthonormal frame for E over U. If $g : U \to SO(k)$ is the transformation sending coordinates with respect to s_i to coordinates with respect to s'_i (by left multiplication), then the connection matrix transforms as: $A \mapsto g^{-1}dg + g^{-1}Ag$.*

Proof.

$$\begin{aligned} g^{-1}(d + A)g &= g^{-1}dg + g^{-1}gd + g^{-1}Ag \\ &= d + (g^{-1}dg + g^{-1}Ag). \end{aligned}$$

You may want to check that if A is skew-symmetric and g is orthogonal, then $g^{-1}dg + g^{-1}Ag$ is also skew-symmetric. □

40.2. **Rank 2 case.** Suppose from now on that E has rank 2 over M of arbitrary dimension. Then A_U (the connection matrix over U with respect to some trivialization) is given by

$$A_U = \begin{pmatrix} 0 & A_{21} \\ -A_{21} & 0 \end{pmatrix}.$$

Then the curvature matrix R_U is

$$R_U = dA_U + A_U \wedge A_U = \begin{pmatrix} 0 & \omega_U \\ -\omega_U & 0 \end{pmatrix},$$

where ω_U is the 2-form dA_{21}.

Theorem 40.3. *There is a global closed 2-form ω which coincides with ω_U on each open set U. Hence a connection ∇ on E gives rise to an element $[\omega] \in H^2_{dR}(M)$. This cohomology class is independent of the choice of connection ∇ compatible with $\langle,\rangle$, and hence is an invariant of the vector bundle E. It is called the* Euler class *of E and is denoted $e(E)$.*

Proof. We need to show that on overlaps $U \cap V$, $\omega_U = \omega_V$. If $g : U \cap V \to SO(2)$ is the orthogonal transformation taking from U to V, then we compute R with respect to the connection 1-form $g^{-1}dg + g^{-1}A_U g$. It is not hard to see that we still get

$$R = \begin{pmatrix} 0 & \omega_U \\ -\omega_U & 0 \end{pmatrix}.$$

Now, two different connections ∇ and ∇' have difference in $\Omega^1(End(E))$. (Moreover, they have values in 2×2 skew-symmetric matrices.) It is not hard to see that if we pick out the upper right hand corner of the matrix on each local coordinate chart U, then they coincide and yield a global 1-form α, and the difference between R_∇ and $R_{\nabla'}$ will be the exact form $d\alpha$. □

Example: For the Levi-Civita connection $\overline{\nabla}$ on a surface $(\Sigma, \overline{g}) \hookrightarrow (\mathbb{R}^3, g)$, we have, locally,

$$R_U = \begin{pmatrix} 0 & \kappa\theta_1 \wedge \theta_2 \\ -\kappa\theta_1 \wedge \theta_2 & 0 \end{pmatrix},$$

where κ is the scalar curvature, $\{e_1, e_2\}$ is an orthonormal frame, and $\{\theta_1, \theta_2\}$ is dual to the frame (called the *dual coframe*). (The fact that κ is the scalar curvature is the content of the Theorema Egregium!)

40.3. **The Gauß-Bonnet Theorem.** Let (M, g) be an oriented Riemannian manifold of dimension n. Then there exists a naturally defined volume form ω which has the following property: At $x \in M$, let $e_1, \dots, e_n$ be an oriented orthonormal basis for T_xM. Then $\omega(x)(e_1, \dots, e_n) = 1$. If we change the choice of orthonormal basis by multiplying by $A \in SO(n)$, then we have a change of $det(A)$, which is still 1. Therefore, ω is well-defined.

For surfaces (Σ, g), we have an area form dA.

Theorem 40.4 (Gauß-Bonnet)**.** *Let Σ be a compact submanifold of Euclidean space $(\mathbb{R}^3, g)$. Then, for one of the orientations of Σ,*

$$\int_\Sigma \kappa dA = 2\pi\chi(\Sigma).$$

Here κ is the scalar curvature, dA is the area form for $\overline{g}$ induced from $(\mathbb{R}^3, g)$, and $\chi(\Sigma)$ is the Euler characteristic of Σ.

The *Euler characteristic* of a compact manifold M of dimension n is:

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H^i_{dR}(M).$$

Note that a compact surface Σ (without boundary) of genus g has $\chi(\Sigma) = 2 - 2g$.

Proof. Notice that κdA is simply $\kappa\theta_1 \wedge \theta_2$ above in the Example, and hence the Euler class is $e(TM) = [\kappa dA]$. In order to evaluate $\int_\Sigma \kappa dA$, we therefore need to compute $\int_\Sigma \omega$ for the connection of our choice on $T\Sigma$ compatible with g, by using Theorem 40.3.

In what follows we will frequently identify $SO(2)$ with the unit circle $S^1 = \{e^{i\theta} | \theta \in [0, 2\pi]\}$ in $\mathbb{C}$ via

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \leftrightarrow e^{i\theta}.$$

We will do a sample computation in the case of the sphere S^2. Let S^2 be the union of two regions $U = \{|z| \leq 1\}$ and $V = \{|w| \leq 1\}$ identified via $z = 1/w$ along their boundaries. Here z, w are complex coordinates. (Note that U and V are not open sets, but it doesn't really matter....) If we trivialize $T\Sigma$ on U and V using the natural trivialization from $T\mathbb{C}$, then the gluing map $g : U \cap V \to SO(2)$ is given by $\theta \mapsto e^{2i\theta}$. If we set A_V to be identically zero, then $A_U = g^{-1}dg + g^{-1}A_V g = g^{-1}dg = \begin{pmatrix} 0 & 2d\theta \\ -2d\theta & 0 \end{pmatrix}$ along ∂U (after transforming via g). No matter how we extend A_U to the interior of U, we have the following by Stokes' Theorem:

$$\int_U \omega_U = \int_{\partial U} 2d\theta = 4\pi = 2\pi\chi(S^2).$$

Now let Σ be a compact surface of genus g (without boundary). Then we can remove g annuli $S^1 \times [0, 1]$ from Σ so that Σ becomes a disk Σ' with $2g - 1$ holes. We make A flat on the annuli, and see what this induces on Σ'. A computation similar to the one above gives the desired formula. (Check this!!) □