Homework 2

(1) Prove that $S^n = \{x_0^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$ is a smooth n-dimensional manifold, by taking stereographic projections.

(2) Define $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{(0, \ldots, 0)\})/\sim$, where $(z_0, \ldots, z_n) \sim (tz_0, \ldots, tz_n)$, $t \in \mathbb{C} - \{0\}$. Prove that $\mathbb{CP}^n$ is a smooth $2n$-dimensional manifold. (Recall that $\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2$, where $z = x + iy \mapsto (x, y)$.)

(3) Prove that $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is a smooth manifold of dimension n.

(4) (30 points) Let $Gr(k, n)$ be the set of all k-dimensional planes in $\mathbb{R}^n$ that pass through the origin. (This is called the *Grassmannian* of k-planes in $\mathbb{R}^n$.) Prove that $Gr(k, n)$ can be given the structure of a smooth manifold of dimension $k(n - k)$.

(5) (20 points) (Existence of bump functions) Consider the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $x \leq 0$ and $f(x) = e^{-1/x}$ for $x > 0$.
   (a) Show that f is smooth and $f \geq 0$.
   (b) Find a smooth function $g : \mathbb{R} \to \mathbb{R}$ such that $g \geq 0$, $g > 0$ on (a, b), and $g = 0$ on $\mathbb{R} - (a, b)$. Here $a < b$.
   (c) Find a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $h \geq 0$, $h = 1$ on $[a, b]$, and $h = 0$ on $\mathbb{R} - (c, d)$. Here $c < a < b < d$.

(6) Prove that $S^n = \{x_0^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$ can be given the structure of an n-dimensional manifold by showing it is a regular value of some map.

(7) Prove that if M, N are manifolds, $f : M \to N$ is a submersion, and $U \subset M$ is open, then $f(U)$ is open in N.