Sample problems for midterm exam

(1) True or false. Justify your answer. If something is false, you need to give a counterexample. You will get no credit for simply writing “true” or “false”.
(a) If a topological space $X$ is second countable, then every open cover of $X$ has a finite subcover.
(b) A normal topological space is Hausdorff.
(c) Every open subset of $\mathbb{R}$ (with the usual metric topology) is a union of disjoint open intervals (finite, semi-infinite, or infinite).
(d) Every closed subset of $\mathbb{R}$ (with the usual metric topology) is the union of a sequence of points and a (possibly uncountable) union of disjoint closed intervals (finite, semi-infinite, or infinite).
(e) A totally bounded metric space is bounded.
(f) A bounded metric space is totally bounded.
(g) The closure of a subset $S$ of a topological space $X$ is closed. (Here the closure of $S$ is the set of points in $X$ which are adherent to $S$; and a point of $x \in X$ is adherent to $S$ if $S$ meets every neighborhood of $x$.)

(2) Let $(X, d)$ be a metric space and $E$ be a subset of $X$. Show that the boundary $\partial E$ of $E$ is closed in $X$.

(3) Let $(X, d)$ be a metric space and $E$ be a subset of $X$. Show that if $E$ is compact, then it must be closed in $X$.

(4) Let $f : X \to Y$ and $g : X \to Z$ be continuous maps between topological spaces. Show that $h : X \to Y \times Z$ given by $h(x) = (f(x), g(x))$ is continuous where $Y \times Z$ has the product topology.

(5) Show that if $X$ and $Y$ are regular, then so is $X \times Y$.

(6) Let $B = \{[a, b) \subset \mathbb{R} \mid -\infty < a < b < \infty\}$.
(a) Prove that $B$ is a basis for a topology $\mathcal{T}_B$ of $\mathbb{R}$.
(b) Show that $(\mathbb{R}, \mathcal{T}_B)$ is a $T_3$-space. (You actually showed $T_4$ in your HW, which is a bit hard. $T_3$ is much easier and could be a reasonable exam question.)

(7) Let $X$ be a metric space and let $Y \subset X$ be a subset.
(a) Define the closure $\overline{Y}$ of $Y$.
(b) Show that the closure of $\overline{Y}$ is equal to $\overline{Y}$.

(8) Let $B([0, 1])$ be the space of bounded functions $f : [0, 1] \to \mathbb{R}$. Show that $d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$ is a metric on $B([0, 1])$. Is it separable?

(9) Prove that $[0, 1]/(0 \sim 1)$ and the unit circle $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ are homeomorphic.

(10) Let $Y, Z$ be closed subsets of a normal topological space $X$. Let $f : Y \to \mathbb{R}$ and $g : Z \to \mathbb{R}$ be bounded continuous functions. Show that there exists a bounded continuous function $h : X \to \mathbb{R}$ such that $h|_Y = f$ and $h|_Z = g$. 