Confoliations Transverse to Vector Fields

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Let M be a closed, oriented 3-manifold, and X a nonsingular (= nowhere zero) vector field on M. The main question we address in this paper is: what are the necessary and sufficient conditions on X for there to be a *positive confoliation* ξ transverse to X? Recall a *positive confoliation* ξ is a 2-plane field distribution given locally as the kernel of a 1-form α satisfying $\alpha \wedge d\alpha \geq 0$. Since we are only interested in confoliations transverse to a vector field, we may assume that ξ is orientable, and α is a globally defined 1-form.

One of the principal motifs in the theory of contact 3-manifolds is the study of dynamical properties of vector fields transverse to a contact ξ. For example, Hofer has extensively studied the dynamics of Reeb vector fields X of the contact 1-form α, and in particular, has obtained existence results for closed orbits of Reeb vector fields (c.f. [11]). In this paper we will examine two classes of vector fields, namely the Morse-Smale vector fields and the vector fields whose orbits are the circle fibers of Seifert fibered spaces, and determine necessary and sufficient dynamical conditions on X for the existence of a transverse positive confoliation (or contact structure) ξ.

The theorems will be stated for contact structures, and the full statements for confoliations will be deferred until later. But first let us introduce some terminology. Let $D^2 \subset M$ be an embedded disk with the following properties: (i) ∂D^2 is tangent to X, i.e., ∂D^2 is a closed orbit of X, and (ii) X is transverse to $D^2 - \partial D^2$. Orient D^2 so that X is the oriented normal on $D^2 - \partial D^2$. If the induced boundary orientation on ∂D^2 is the same as the orientation given by restricting X to ∂D^2, then we call D^2 a *right-handed* disk. Otherwise, D^2 is *left-handed.* (See Figure 1.)

A left-handed or right-handed disk D^2 is said to *link* a periodic orbit γ if γ intersects the interior of D^2. A left- or right-handed disk D^2_1 *shadows* another left- or right-handed disk D^2_2 if $int(D^2_2) \subset \phi(\mathbf{R} \times D^2_1)$, where $int(D^2_2)$ is the interior of D^2_2, and $\phi : \mathbf{R} \times D^2_1 \to M$ maps $(t, x) \in \mathbf{R} \times D^2_1$ to the time-t flow of x under X. We now state our result:

Theorem 1 *Let X be a nonsingular Morse-Smale vector field. Then there exists a positive contact structure ξ transverse to X if and only if (i) every 2-sphere transverse to X intersects an attracting or repelling periodic orbit, and (ii) every left-handed disk either links an*

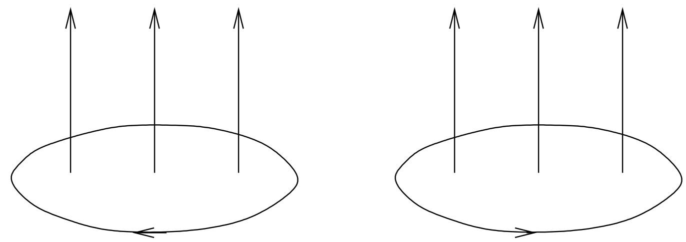


Figure 1: Left-handed vs. right-handed

attracting or repelling periodic orbit or shadows a right-handed disk of an untwisted index 1 (saddle) orbit.

There always exist overtwisted contact structures transverse to Morse-Smale flows which satisfy the conditions of Theorem 1. It is still not understood which Morse-Smale flows admit transverse tight contact structures.

We now state our theorem for contact structures transverse to the circle fibers of Seifert fibered spaces. An n-tuple $(\mu_1, \cdots, \mu_n) \in (0,1)^n$ is *realizable* if there exists a permutation $(\gamma_1, \cdots, \gamma_n)$ of $(a/m, (m-a)/m, 1/m, \cdots, 1/m)$ with $0 < a < m$ and $(a, m) = 1$ such that $\mu_i < \gamma_i$ for all i.

Theorem 2 *Let $\pi : M \to \Sigma$ be a Seifert fibered space with Seifert invariants $(g(\Sigma); b, (\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_n, \beta_n))$, where $b \in \mathbf{Z}$ and $0 < \beta_i/\alpha_i < 1$. Then there exists a positive contact structure ξ transverse to the fibers if and only if one of the following holds:*

1. *$g > 0$ and $b \leq 2g - 2$.*
2. *$g = 0$ and*
   - (a) *$b \leq -2$, or*
   - (b) *$b = -1$, $n \geq 3$, and $(\beta_1/\alpha_1, \cdots, \beta_n/\alpha_n)$ is realizable, or*
   - (c) *$b = -1$, $n = 2$, and $\beta_1/\alpha_1 + \beta_2/\alpha_2 \leq 1$, or*
   - (d) *$b = -1$ and $n = 0$ or $n = 1$.*

The key ingredient in the proof of Theorem 2 is an analysis of commutators of homeomorphisms of $\mathbf{R}$ which descend to $\mathbf{R}/\mathbf{Z} = S^1$. The transverse contact structures are always tight for Seifert fibered spaces.

After this work was completed, we noticed that our motivating question above was asked by Thurston in [20]. This paper is intended to be a first step towards answering this question regarding tight contact structures.

1 Holonomy

This section is devoted to the study of holonomy on flow cylinders and the relationship to overtwisted disks. For related ideas and inspiration, the reader is referred to Section 1.3 of [5]

In what follows, M will be a closed, oriented 3-manifold and X a nonsingular flow. Let $\phi_0 : \Sigma \to M$ be an immersion transverse to X. A *flow cylinder* of ϕ_0 is an immersion $\phi : \mathbf{R} \times \Sigma \to M$, so that $\phi(t, x)$ is the time-t flow of X, starting at $\phi_0(x)$ at $t = 0$. We will slightly abuse notation and refer to the map ϕ and the space $\mathbf{R} \times \Sigma$ both as *flow cylinders*.

Consider the flow cylinder $Z = \mathbf{R} \times D^2$ with cylindrical coordinates (z, r, θ), such that $D^2 = \{r \leq 1\}$. Let us pull X and ξ back to $\mathbf{R} \times D^2$ via ϕ (we will still call them X and ξ). We then have $X = \frac{\partial}{\partial z}$ and ξ is given by a contact 1-form $\alpha = dz - f dr - g d\theta$, for some functions f, g on $\mathbf{R} \times D^2$. On ∂Z there is an induced *characteristic foliation* $\partial Z \bigcap \xi$. It has no singularities (i.e., the characteristic foliation is a genuine foliation), since X is tangent to ∂Z.

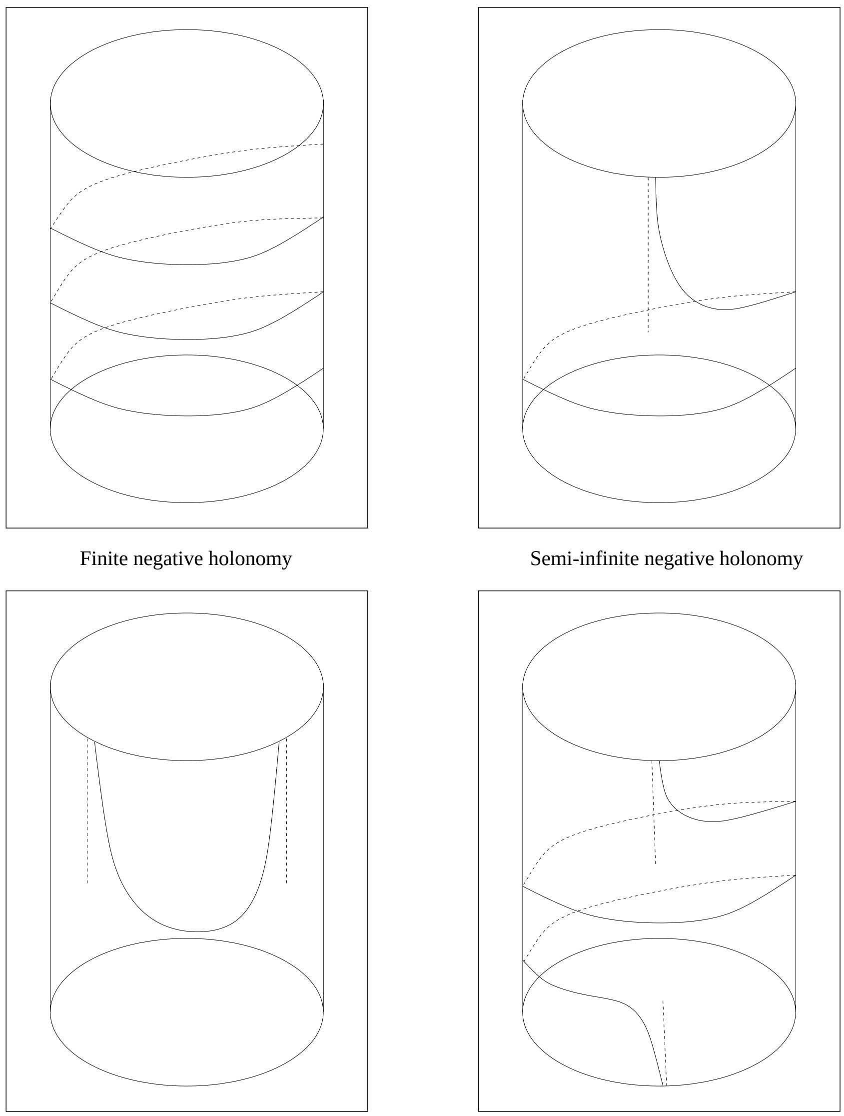
Consider a maximal integral curve $\gamma : \mathbf{R} \to \partial Z$, $\gamma(t) = (z(t), r(t), \theta(t))$, with $\dot\theta(t) > 0$, of the characteristic foliation. γ is then immersed in ∂Z, and either gives an isomorphism to a leaf of ∂Z, or will $\mathbf{Z}$-cover the leaf. Therefore, we will not distinguish between the integral curve γ and the corresponding leaf.

For the following classification of integral curves on ∂Z, the reader is referred to Figure 2. We say γ has *finite holonomy* if for each $t_0 \in \mathbf{R}$ there exist times $t_1 > t_0$ and $t_{-1} < t_0$ for which $\theta(t_0) = \theta(t_1) = \theta(t_{-1}) \bmod 2\pi$. A finite holonomy curve has *positive holonomy* (resp. *negative holonomy*), if there exists a time $t_1 > t_0$ for which $\theta(t_0) = \theta(t_1) \bmod 2\pi$, and $z(t_0) < z(t_1)$ (resp. $z(t_0) > z(t_1)$). γ is said to have *zero holonomy* if for some time $t_1 > t_0$, $\theta(t_0) = \theta(t_1) \bmod 2\pi$ and $z(t_0) = z(t_1)$.

We also need to classify curves which do not have finite holonomy. γ has *semi-infinite positive holonomy* if either (i) $\lim_{t\to+\infty} z(t) = +\infty$, $\theta(t)$ is bounded above, and $\theta(t)$ is unbounded below, or (ii) $\lim_{t\to-\infty} z(t) = -\infty$, $\theta(t)$ is bounded below, and $\theta(t)$ is unbounded above. γ has *infinite positive holonomy* if $\lim_{t\to+\infty} z(t) = +\infty$, $\lim_{t\to-\infty} z(t) = -\infty$, and $\theta(t)$ is bounded from above and below. We can similarly define *semi-infinite negative holonomy*, and *infinite negative holonomy*. Finally, if $\lim_{t\to\pm} z(t)$ are both $+\infty$ or both $-\infty$, then we say γ is an *infinite curve with zero holonomy*.

If a cylinder ∂Z has a curve γ of positive holonomy (finite, semi-infinite or infinite), we say that ∂Z (or simply Z, in a slight abuse of language) *has positive holonomy* along γ, or that Z or ∂Z has (a region of) positive holonomy.

Example: Consider $\mathbf{R}^3$ with coordinates (x, y, z) and a cylinder $Z_R = \{x^2 + y^2 = R^2\}$. If $\alpha = dz + x dy - y dx$, i.e., α is positive, then for any R, all the integral curves on Z_R have negative holonomy. On the other hand, if $\alpha = dz - x dy + y dx$, i.e., α is negative, then all the curves on Z_R have positive holonomy.



Finite negative holonomy

Semi-infinite negative holonomy

Infinite curve with zero holonomy

Infinite negative holonomy

Figure 2: Integral curves on the flow cylinder

It often is the case that if ξ is a positive contact structure, then all the integral curves on the boundary of flow cylinders have negative holonomy. However, the following example shows that this is not always the case:

Example: Consider a positive overtwisted structure on $\mathbf{R}^3$ with coordinates (r, θ, z), given by $\alpha = \cos r dz + r \sin r d\theta$. Take any disk (usually called an *overtwisted disk*) satisfying the following: (i) its characteristic foliation has one elliptic point and one limit cycle in the interior, and (ii) the disk is transverse to ξ away from the elliptic point. One such disk is $D^2 = \{(r, \theta, f(r, \theta)) | r \leq \pi + \varepsilon, f(r, \theta) = r^2\}$. To obtain a vector field transverse to D^2 and ξ, start with a vector field Y on D^2, nonsingular away from the elliptic point, and transverse to $\xi \bigcap D^2$. Then either $Y + \varepsilon N$ or $-Y + \varepsilon N$, where N is an oriented normal, can be extended to a vector field transverse to D^2 and ξ near the elliptic point as well.

In fact, the existence of finite or semi-infinite curves with positive holonomy on ∂Z forces the contact structure to be overtwisted. This can be seen from the Thurston-Bennequin inequality (c.f. p.49 of [5]).

We will also need the following additivity lemma:

Lemma 1 *(Additivity of holonomy) Let ξ be a positive contact structure on $Z = \mathbf{R} \times D^2$, transverse to $X = \frac{\partial}{\partial z}$.*

1. *Let $D^2 = \bigcup_i \Delta_i$ be a subdivision into 2-simplices Δ_i so that the $\partial(\mathbf{R} \times \Delta_i)$ have negative holonomy (finite, semi-infinite, or infinite) along every integral curve. Then every integral curve on ∂Z has negative holonomy.*

2. *Let $D^2 = \Delta_1 \cup \Delta_2$ be a subdivision so that each integral curve on $\partial(\mathbf{R} \times \Delta_i)$ has negative holonomy or is infinite with zero holonomy. Assume, in addition, that (i) none of the infinite curves with zero holonomy on $\partial(\mathbf{R} \times \Delta_i)$ which intersect $\partial(\mathbf{R} \times D^2)$ have both positive and negative ends lie on $\mathbf{R} \times (\Delta_1 \cap \Delta_2)$, and (ii) two infinite curves γ_i with zero holonomy on $\partial(\mathbf{R} \times \Delta_i)$, $i = 1, 2$, satisfying $\lim_{t \to \pm\infty} z_1(t) = +\infty$ and $\lim_{t \to \pm\infty} z_2(t) = -\infty$ (or vice versa), do not intersect along $\mathbf{R} \times (\Delta_1 \cap \Delta_2)$. Then every integral curve of ∂Z has negative holonomy or is infinite with zero holonomy.*

Notice that Part 2 of Lemma 1 is a slight strengthening of Part 1 to include certain types of infinite curves with zero holonomy. The additivity lemma would not be true if we allowed all possible curves with zero holonomy, and this is the primary difficulty of dealing with infinite cylinders $\mathbf{R} \times D^2$ instead of $S^1 \times D^2$ or infinite cylinders with ends which are standard at $\pm\infty$.

Proof: (1) It suffices to consider the subdivision $D^2 = \Delta_1 \cup \Delta_2$. We sum together integral curves γ_i of $\partial(\mathbf{R} \times \Delta_i)$ which are *adjacent* (i.e., share a common edge along $\mathbf{R} \times pq$, where we write $\Delta_1 \cap \Delta_2 = pq$). Note that there cannot be any vertical asymptotes for curves with semi-infinite or infinite negative holonomy, along $\mathbf{R} \times pq$. This is because a curve γ_1 with semi-infinite or infinite negative holonomy on $\partial(\mathbf{R} \times \Delta_1)$ must be summed with γ_2 on $\partial(\mathbf{R} \times \Delta_2)$ with semi-infinite or infinite *positive* holonomy.

Hence the intersection of γ_1 and γ_2 is a curve with finite holonomy along $\mathbf{R} \times pq$. This ensures that the sum is always negative.

(2) Let γ be an integral curve on $\partial(\mathbf{R} \times D^2)$, which intersects $\mathbf{R} \times pq$. If it does not, it is clearly an infinite curve with negative or zero holonomy, since it sits on one of the $\partial(\mathbf{R} \times \Delta_i)$. Therefore, we only consider γ which are obtained by summing γ_1 on $\partial(\mathbf{R} \times \Delta_1)$ and γ_2 on $\partial(\mathbf{R} \times \Delta_2)$. If γ_1 and γ_2 both have negative holonomy, then we are in the situation of (1).

Suppose now that γ_1 is infinite with zero holonomy. We have the following cases: (i) The positive end of γ_1 has an asymptote on $\mathbf{R} \times pq$, and $\lim_{t \to +\infty} z(t) = -\infty$. Since the negative end of γ_2 has an asymptote on $\mathbf{R} \times pq$ and $z \to -\infty$, the positive end of γ_2 must have an asymptote away from $\mathbf{R} \times pq$ and $z \to -\infty$. Hence, in this case, the sum is infinite with zero holonomy. (ii) The positive end of γ_1 has an asymptote on $\mathbf{R} \times pq$, and $z \to +\infty$. The negative end of γ_2 must then have an asymptote on $\mathbf{R} \times pq$ and $z \to +\infty$. Therefore, γ_2 can be semi-infinite or infinite negative, or infinite with zero holonomy, all of which will give negative holonomy curves or infinite curves with zero holonomy, when summed with γ_1. (iii) The negative end of γ_1 has an asymptote on $\mathbf{R} \times pq$, and $z \to -\infty$. The positive end of γ_2 has $z \to -\infty$ on $\mathbf{R} \times pq$, which means that γ_2 can be semi-infinite or infinite negative, or infinite with zero holonomy. The sum will again be negative or an infinite curve with zero holonomy. (iv) The negative end of γ_1 has an asymptote on $\mathbf{R} \times pq$, and $z \to +\infty$. The positive end of γ_2 satisfies $z \to +\infty$ on $\mathbf{R} \times pq$, so γ_2 is infinite with zero holonomy, as is the sum. (v) Both ends of γ_1 have asymptotes away from $\mathbf{R} \times pq$. If γ_2 is negative, then the sum is negative or infinite with zero holonomy. If γ_2 is infinite with zero holonomy, the sum is infinite with zero holonomy, if both γ_1 and γ_2 have their z coordinates go to $+\infty$ or both go to $-\infty$. Otherwise, the summation will create a pair of infinite curves - one positive and one negative. Our conditions allow us to avoid this possibility. The situation where γ_2 is infinite with zero holonomy is identical. □

2 Criterion for Morse-Smale flows

We will now present a necessary and sufficient condition for a Morse-Smale vector field X to have a transverse confoliation. The following is the confoliation version of Theorem 1 (a description of the type of orbits for a Morse-Smale flow will follow later):

Theorem 3 *Let X be a nonsingular Morse-Smale vector field on a closed, oriented 3-manifold M. Then X is transverse to a positive confoliation ξ if and only if (i) every 2-sphere transverse to X intersects an attracting or repelling periodic orbit, and (ii) every* *left-handed disk either links an attracting or repelling periodic orbit or shadows a right-handed disk of a standard (untwisted) index 1 orbit.*

We can apply a theorem of Eliashberg and Thurston to restate the result in terms of contact structures:

Theorem 4 *(Eliashberg-Thurston) Any smooth confoliation not equal to the foliation by* $\{pt.\} \times S^2$ *of* $S^1 \times S^2$ *can be* C^0*-approximated by a smooth contact structure.*

Theorems 3 and 4 together imply Theorem 1.

We will say that X satisfies the *linking property for confoliations* if every left-handed disk D^2 links an attracting or repelling periodic orbit. Note that there is a linking property for foliations which is similar - see [9] and [10].

The proof for the existence of a transverse positive confoliation is based on the construction of Sue Goodman [9] on vector fields transverse to foliations, supplemented in places by the extra maneuvering room for the contact part, although the situation for confoliations is more complicated. This will be done in the next section.

In this section we shall prove that if X is transverse to a confoliation ξ, then (i) and (ii) hold in Theorem 3. Unlike Sue Goodman's proof of the necessity of the linking property for foliations, which holds for any nonsingular vector field X, the proof of our version is rather dependent on the following description of the Morse-Smale vector field X:

M has a *round handle decomposition* - a filtration of codimension 0 submanifolds $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_n = M$, where M_1, $M_2 - M_1$, ... , $M_n - M_{n-1}$ are round handles homeomorphic to $S^1 \times D^2$ - with the following properties:

1. X is transverse to ∂M_i, and is an inward normal for M_i on ∂M_i.

2. Each $M_i - M_{i-1}$ is a tubular neighborhood of a periodic orbit γ_i, $i \geq 1$.

3. If Φ_t is the time-t flow of X, then $\gamma_i = \bigcap_t \Phi_t(M_i - M_{i-1})$, $i \geq 1$.

By analogy with Morse theory, we consider the eigenvalues λ_1, λ_2 of the derivative of the Poincaré return map along a periodic orbit γ, and say that the corresponding handle has *index k* if there are k eigenvalues > 1. (See [16] for a careful discussion. Also see [1].) Index 0 handles are attracting, and index 2 handles are repelling, while index 1 handles are saddles. It is important to note that, although there is only one type of index 0 handle or index 2 handle each, there are two kinds of index 1 handles - the standard one and the twisted one, corresponding to the two different splittings of the rank 2 bundle $S^1 \times \mathbf{R}^2 \to S^1$ into rank 1 bundles.

Proof: (Necessity of conditions (i) and (ii)) If ξ is the foliation by $\{pt.\} \times S^2$ of $S^1 \times S^2$, then a transverse X cannot have nullhomotopic periodic orbits, and our linking property is vacuously true. (i) is also immediate. Without loss of generality, we may then assume ξ is contact by Theorem 4.

Let $\widetilde{D}^2$ be a left-handed disk. Choose a slightly smaller subdisk of $D^2 \subset \widetilde{D}^2$, whose boundary is still transverse to ξ. Consider the flow cylinder $\phi : \mathbf{R} \times D^2 \to M$, so that $\mathbf{R} \times D^2$ has cylindrical coordinates (z, r, θ). As before, we pull ξ and X back to $Z = \mathbf{R} \times D^2$, and let $\alpha = dz - f dr - g d\theta$ be a contact 1-form. Notice that $\widetilde{D}^2$ being left-handed implies that $g > 0$ along $z = 0$, $r = 1$. (To see this, observe that $\ker \alpha$ is the span of $\frac{\partial}{\partial r} + f \frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \theta} + g \frac{\partial}{\partial z}$.) This implies that we have positive holonomy along any integral curve γ on $\partial(\mathbf{R} \times D^2)$ and passing through $z = 0$.

Assume that the left-handed disk does not link an attracting or periodic orbit of the Morse-Smale flow. Then the flow cylinder $Z = \mathbf{R} \times D^2 \to M$ is 1-1. The existence of a curve with positive holonomy on ∂Z will give us a contradiction, as long as the left-handed disk does not shadow a right-handed disk of a standard index 1 orbit.

Since X is Morse-Smale, most of D^2 (D^2 minus a 1-dimensional set K, possibly empty) will flow towards attracting periodic orbits, and will have come from repelling periodic orbits. Take a fine subdivision of D^2 into 2-simplices Δ_i. The flow cylinders $\mathbf{R} \times \Delta_i \to M$ for which all of Δ_i flows towards an attracting periodic orbit must have positive ends which are tight, and their boundaries must have finite negative holonomy at the positive end. This is because ξ must be tight on a small enough tubular neighborhood of a periodic orbit. The same holds for flow cylinders which originated from one repelling periodic orbit. Hence, if $\mathbf{R} \times \Delta_i$ does not intersect stable or unstable manifolds of index 1 orbits, then $\partial(\mathbf{R} \times \Delta_i)$ will have integral curves, all of which have finite negative holonomy at the ends. Moreover, by subdividing Δ_i if necessary, we can get $[-R, R] \times \Delta_i$ to be contained inside a standard tight contact structure for $\mathbf{R}^3$, for fixed large R. This implies that all the integral curves of $\partial(\mathbf{R} \times \Delta_i)$ for small enough Δ_i's have finite negative holonomy. Now, by the additivity lemma, any Δ_i whose flow cylinders do not intersect stable or unstable manifolds of periodic 1 orbits will have all integral curves with finite negative holonomy on $\partial(\mathbf{R} \times \Delta_i)$.

We must now proceed with care in dealing with the portion of D^2 which intersect stable manifolds $W_s(\gamma)$ or unstable manifolds $W_u(\gamma)$ of index 1 orbits γ - we shall take small neighborhoods of K, which we will subdivide into small simplices Δ_i. Denote $K_s = D^2 \cap (\bigcup W_s(\gamma))$ and $K_u = D^2 \cap (\bigcup W_u(\gamma))$, where the union is over all the index 1 orbits γ. K_s and K_u will meet transversely because X is Morse-Smale.

Lemma 2 *Let $p \in K$. Then there exists a small enough $\Delta \ni p$, such that $\partial(\mathbf{R} \times \Delta)$ will not have any integral curves with positive holonomy.*

Proof of Lemma: Let us consider the the following simplification first: $\Delta \cap K_s = \delta$, which is a (connected) arc on $\partial\Delta$. (In particular, the intersection has only one component.) Let γ

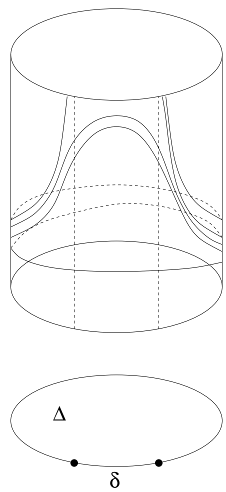


Figure 3: Integral curves on $\mathbf{R} \times \Delta$

be the index 1 orbit corresponding to δ. For the time being, we will concern ourselves with the positive end of the flow cylinder. Note that if α is an arc in D^2 which avoids K, then all the integral curves on $\mathbf{R} \times \alpha$ are *finite*, i.e., do not have asymptotes on $\mathbf{R} \times \alpha$. This implies that any kind of infinite (asymptotic) behavior must occur as the integral curves approach vertical lines on $\mathbf{R} \times \delta$. Moreover, since δ flows into an index 1 orbit γ, we know that all the integral curves on $\mathbf{R} \times \delta$ are finite as well. Hence, the only type of infinite behavior at the positive end would be as in Figure 3.

We can rule out positive holonomy curves by arguing as follows: Shrink Δ sufficiently (still keeping $\Delta \cap K_s \neq \emptyset$), so that $[0, R] \times \Delta$ sits inside a standard contact structure on $\mathbf{R}^3$, for large R. Then there must exist a zero holonomy curve between the positive holonomy curves and the negative holonomy curves. We will modify the flow cylinder for Δ to obtain a time-dependent flow cylinder which has both ends which are tight. Since both ends are tight, there exists a subdivision of Δ for which all the flow cylinders of the 2-simplices have finite negative holonomy, and, by the additivity lemma, the time-dependent flow cylinder of Δ cannot have zero holonomy. In order to obtain a time-dependent flow cylinder, it suffices to define a time-dependent vector field. First define a small perturbation X' of X near γ, which is still Morse-Smale but whose index 1 orbit γ' has been slightly pushed away from γ so that the flow cylinder of Δ (for X) no longer intersects $W_s(X')$. Pick any z_0 so that $\partial([0, z_0] \times \Delta)$ for X contains the zero holonomy curve. Now define the time-dependent vector field X_z by setting $X_z = X$ for $z \leq z_0$, $X_z = X'$ for $z \geq z_0 + 1$, and interpolating inbetween. By construction, this flow cylinder is still transverse to ξ and has positive end which is contained in a neighborhood of an index 0 orbit. Since we may also attach a standard negative end, we have shown that there can be no positive holonomy curves on $\partial(\mathbf{R} \times \Delta) \cap \{z \geq 0\}$, for our original X.

Next consider the situation where $\Delta \cap K_s$ is an arc δ, which subdivides Δ into Δ_1 and Δ_2 of the type treated above. The D_i satisfy the conditions of Part 2 of the additivity lemma, and hence cannot have any integral curves with positive holonomy on $\partial(\mathbf{R} \times D_i)$.

Finally, assume $\Delta \cap K_s$ consists of infinitely many parallel arcs. Let δ be a limit arc which corresponds to γ_1, the first index 1 orbit encountered by Δ. Subdivide along δ into Δ_i. As in the simpler case above, we replace the flow cylinder with a time-dependent flow cylinder if there exist integral curves with positive holonomy on $\partial(\mathbf{R} \times \Delta_i)$. This new flow cylinder does not meet γ_1, so we can proceed inductively. Subdivide Δ_i into small enough simplices Δ_{ij} so that Δ_{ij} (i) does not intersect any stable manifolds, (ii) intersects $W_s(X')$ in one arc, or (iii) intersects $W_s(X')$ in infinitely many arcs. If Δ_{ij} are of type (i) or (ii), then their (again modified time-dependent) flow cylinders do not have any positive holonomy curves. Since $\Delta \cap K_s$ consisted of parallel arcs to begin with, the additivity lemma guarantees us that the $\partial(\mathbf{R} \times \Delta_i)$ do not have any positive holonomy curves. If there are type (iii) Δ_{ij}, we must once again subdivide and continue until we exhaust all the index 1 orbits.

The situation at the negative end is identical. This proves the lemma. $\square$

Now, given $p \in K$, there exists $\Delta \ni p$ which has a flow cylinder without positive holonomy. Also notice we may shrink Δ without loss of generality. Since K is compact, we cover K with simplices Δ and shrink them, obtaining a subdivision of a neighborhood of K by Δ_i for which none of the $\mathbf{R} \times \Delta_i$ have positive holonomy. We also throw in Δ_i which do not meet K.

Notice that K_s (resp. K_u) is a lamination on D^2. If it has a closed leaf, then $\widetilde{D}^2$ shadows another left- or right-handed disk. If the former happens, we will take a minimal left-handed disk which does not shadow another left-handed disk. The latter is excluded by our conditions, except when the right-handed disk arises from a twisted index 1 orbit. Hence, we have two possibilities: (1) K_s (resp. K_u) is a union of arcs, or (2) there exist circular components of K_s (resp. K_u), all of which arise from a twisted index 1 orbit.

Case 1: (K_s and K_u are unions of arcs.) In this case we can inductively use the additivity lemma and show that $\mathbf{R} \times D^2$ does not have a curve with positive holonomy. Since X is Morse-Smale, K_s and K_u intersect transversely in D^2. Hence we normalize $\Delta_i = [-1,1] \times [-1,1]$ with coordinates (x, y) (we now think of it as a 2-cell), so that K_s are of the form $x = c$ and K_u are of the form $y = d$, and the constants satisfy $-1 < c, d < 1$. Let us add adjacent Δ_1, Δ_2. Assume without loss of generality that their common edge pq intersects K_s (but not K_u).

Criterion (i) for Part 2 of the additivity lemma is met; otherwise we would have a infinite curve with zero holonomy which begins and ends on $\mathbf{R} \times pq$, wraps around Δ_1, and has $z \to +\infty$ (without loss of generality). Let pq be $x = 1$ and $(1, 1/2), (1, -1/2)$ be the asymptotes for the infinite curve with zero holonomy. If we split Δ_1 along $y = 0$, then the region $\Delta_1 \cap \{y \geq 0\}$ will have flow cylinder with infinite positive holonomy, contradicting our construction from the previous lemma.

Criterion (ii) also follows by noting that all the infinite curves with zero holonomy satisfying $z \to +\infty$ must have asymptotes along pq, whereas all the infinite curves with zero holonomy satisfying $z \to -\infty$ cannot have asymptotes along pq, and hence the infinite curves of the two types can never overlap.

Thus we may add Δ_1 and Δ_2 to get Ω, and still have no integral curves with positive holonomy on $\partial(\mathbf{R} \times \Omega)$. We now inductively attach Δ to Ω - the important point here is that we can attach so that Ω will always be a disk, and that $\Delta \cap \Omega$ consists of at most two (consecutive) edges of Δ, viewed as a 2-cell above. These attachments still satisfy Criteria (i) and (ii) of the additivity lemma (the argument is similar), and thus D^2 cannot have a curve with positive holonomy.

Case 2: (There exists a circular component δ of K_s, coming from a twisted index 1 orbit.) Let $D_2^2 \subset D^2$ be the disk with the circular boundary, and $\Omega_1 \subset D_2^2 \subset \Omega_2$ be slightly smaller and larger disks in D^2, respectively. Assume that the circle is the innermost on D^2. Then, from the above considerations, $\partial(\mathbf{R} \times \Omega_1)$ does not have any curve of positive holonomy.

Claim: $\partial(\mathbf{R} \times \Omega_2)$ for a perturbed vector field X' cannot have a curve of positive holonomy.

Proof of Claim: Essentially, this is because the flow cylinders $\partial(\mathbf{R} \times \Omega_1)$ and $\partial(\mathbf{R} \times \Omega_2)$ are close at the positive end. Notice that a twisted index 1 orbit γ has a 1-component unstable manifold $W_u(\gamma) = \{x \in M \text{ which flow towards } \gamma \text{ as } t \to -\infty\}$, instead of a 2-component one for the regular index 1 orbit. Hence there exist C^∞-close curves $\beta_i \in N(W_u(\gamma)) \cap \partial(\mathbf{R} \times \Omega_i)$, on opposite sides of $W_u(\gamma)$, by picking $\partial\Omega_i$ C^∞-close to δ. Thus, we may perturb the vector field X so that the flow cylinder $\partial(\mathbf{R} \times \Omega_2)$ for X' switches from the flow cylinder $\partial(\mathbf{R} \times \Omega_2)$ for X to $\partial(\mathbf{R} \times \Omega_1)$ for X after some $z_0 > 0$. Since $\partial(\mathbf{R} \times \Omega_1)$ for X did not have any curve of positive holonomy, neither does $\partial(\mathbf{R} \times \Omega_2)$ for X'. □

The lemma allows us to continue our induction. This proves the necessity of condition (ii).

The necessity of condition (i) follows easily from the above discussion. Let S^2 be the transverse sphere which does not link an attracting or repelling orbit. Intersect S^2 with $\cup W_s$ and $\cup W_u$ (union over the index 1 orbits) to get K_s and K_u. Take a closed curve δ in S^2 which cuts through every closed leaf of K_s and K_u - such a curve always exists. Then the closure D^2 of one of the components of $S^2 - \delta$ has flow cylinder with positive or zero holonomy, which is impossible because $K_s \cap D^2$ and $K_u \cap D^2$ are arcs, and because of the discussion above. □

3 Construction of transverse confoliations

3.1 Confoliated I-bundles

Let $I = [0,1]$ and N a compact manifold with boundary. Then $(N \times I, \xi)$ is a *foliated I-bundle* if ξ is a codimension-1 foliation on $N \times I$ transverse to the I-fibers, and $N \times \{0,1\}$ are leaves of ξ. Similarly, $(N \times I, \xi)$ is a *positive confoliated I-bundle* if ξ is a positive confoliation on $N \times I$ transverse to the I-fibers, and $N \times \{0,1\}$ are leaves of ξ.

Suppose that $N = \Sigma$, a Riemann surface with boundary, $\partial\Sigma = S^1$, and $\mathcal{F}$ is a foliated I-bundle on $\partial\Sigma \times I$. If $\mathcal{F}$ is of class C^1, then prescribing $\mathcal{F}$ is equivalent to giving a C^1-diffeomorphism ϕ of I which fixes endpoints. Then we have the following:

Proposition 1 *There exists a positive confoliation ξ on $\Sigma \times I$ with $(\partial\Sigma \times I) \cap \xi = \mathcal{F}$ which makes $(\Sigma \times I, \xi)$ into a confoliated I-bundle, if*

1. *$\Sigma = D^2$, and ϕ is nonincreasing (i.e., $\phi(z) \leq z$ for all $z \in I$), or*
2. *$\Sigma \neq D^2$, and $\phi(z) \leq z$ near $z = 0$ and $z = 1$.*

Proof: (1) Take the coordinates on $D^2 \times I$ to be (r, θ, z), and let ∂D^2 be given by $r = 1$. Then without loss of generality $\mathcal{F}$ on $S^1 \times I$ can be given by $dz - g(\theta, z)d\theta$ where $g(\theta, z) \leq 0$, $g(\theta, 0) = g(\theta, 1) = 0$. Simply extend to the confoliated 1-form

$$\alpha(r, \theta, z) = dz - \tilde{g}(r, \theta, z)d\theta,$$

satisfying $\frac{\partial \tilde{g}}{\partial r} \leq 0$, $\tilde{g}(0, \theta, z) = 0$, and $\tilde{g}(1, \theta, z) = g(\theta, z)$.

(2) This assertion essentially amounts to the following statement, which we prove in the next paragraphs: Let $\mathcal{D} = \mathrm{Homeo}^+(I)$ be the orientation-preserving homeomorphisms of I. Then there exist ϕ_1, $\phi_2 \in \mathcal{D}$ such that

$$\phi(z) \leq [\phi_2^{-1}, \phi_1^{-1}](z) = \phi_2^{-1} \circ \phi_1^{-1} \circ \phi_2 \circ \phi_1(z), \tag{1}$$

for all $z \in I$. Note that if $\Sigma \neq D^2$, then $\Sigma = \Sigma' - D^2$, where Σ' is a closed Riemann surface of genus $g \geq 1$. If we cut along $2g$ suitable simple closed curves γ_i, then $\Sigma - \cup_{i=1}^{2g} \gamma_i$ is an annulus with boundary $\partial\Sigma - (\gamma_{2g}^{-1}\gamma_{2g-1}^{-1} \cdots \gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1)$. Then we can extend $(\partial\Sigma \times I, \xi)$ to a confoliated I-bundle on $\Sigma \times I$ if there exist $\phi_i \in \mathcal{D}$ corresponding to γ_i, $i = 1, \cdots, 2g$, such that

$$\phi(z) \leq [\phi_{2g}^{-1}, \phi_{2g-1}^{-1}] \cdots [\phi_2^{-1}, \phi_1^{-1}](z).$$

For I-bundles it suffices to set $\phi_3 = \cdots \phi_{2g} = id$, and show Equation 1. (See the discussion in Section 4 for a similar discussion in the context of S^1-bundles.) The condition that $\phi(z) \leq z$ near the endpoints is a technical one which allows us to extend the foliated I-bundle $(\partial\Sigma \times I, \xi)$ to a confoliated I-bundle on $\Sigma \times I$ in a C^1-differentiable manner. $\square$

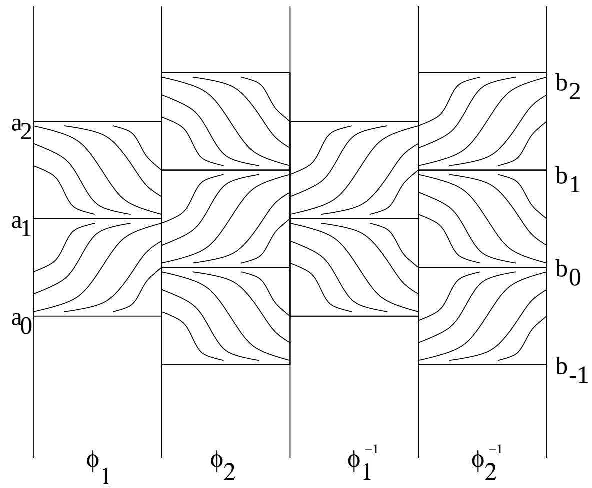


Figure 4: Laddering via commutators of diffeomorphisms of $\mathbf{R}$

Laddering procedure: Consider $\mathbf{a} = (\cdots, a_{-1}, a_0, a_1, a_2, \cdots)$, $\mathbf{b} = (\cdots, b_{-1}, b_0, b_1, \cdots)$ such that $a_i < b_i < a_{i+1}$ for all $i \in \mathbf{Z}$. Denote by $\mathrm{Fix}(\mathbf{n})$, where $\mathbf{n} = (\cdots, n_{-1}, n_0, n_1, \cdots)$ and $n_i < n_{i+1}$, the space of orientation-preserving homeomorphisms $\phi : \mathbf{R} \to \mathbf{R}$ satisfying $\phi(n_i) = n_i$ for all $i \in \mathbf{Z}$, $\phi(z) > z$ for $n_{2k} < z < n_{2k+1}$, $k \in \mathbf{Z}$, and $\phi(z) < z$ for $n_{2k-1} < z < n_{2k}$, $k \in \mathbf{Z}$. We consider commutators $[\phi_2^{-1}, \phi_1^{-1}] = \phi_2^{-1} \circ \phi_1^{-1} \circ \phi_2 \circ \phi_1$, where $\phi_1 \in \mathrm{Fix}(\mathbf{a})$ and $\phi_2 \in \mathrm{Fix}(\mathbf{b})$. Refer to Figure 4.

From the diagram we see that, given $a_0 + \varepsilon > x > a_0$ and $a_2 > y > a_2 - \varepsilon$ (ε as small as we want), there exist $\phi_1 \in \mathrm{Fix}(\mathbf{a})$ and $\phi_2 \in \mathrm{Fix}(\mathbf{b})$ such that $\psi = [\phi_2^{-1}, \phi_1^{-1}]$ satisfies $b_2 - \varepsilon < \psi(x) < b_2$ and $b_2 - \varepsilon < \psi(y) < b_2$. That is, by making the slopes steeper, we can 'ladder up' from very close to a_0 to very close to b_2.

Next, given $\phi \in \mathrm{Homeo}^+(\mathbf{R})$, we explain how to pick $\mathbf{a}$, $\mathbf{b}$, so that ϕ_1 and ϕ_2 satisfy $[\phi_2^{-1}, \phi_1^{-1}](z) \geq \phi(z)$ for all $z \in \mathbf{R}$. The condition to be satisfied is $b_{2k} > \phi(a_{2k})$. Pick in order a_0, $b_0 > \phi(a_0)$, $a_2 > b_0$, $b_2 > \phi(a_2)$, and so on. Also pick $b_{-2} < a_0$, $a_{-2} < \phi^{-1}(b_{-2})$, $b_{-4} < a_{-2}$, and so on. Finally fill in with a_{2k+1} and b_{2k+1}. If we choose ϕ_1, ϕ_2 with 'large slopes', we obtain the desired result.

The same procedure works for $\phi \in \mathrm{Homeo}^+(I)$, the only drawback being that the ϕ_i constructed may not be C^1-differentiable at the endpoints. In order to make the proof work for C^1-functions ϕ, we need to assume $\phi(z) \leq z$ near $z = 0$ and $z = 1$, i.e., allow a 'buffering region' where we can perturb our continuous ϕ_i into C^1-functions. First take germs of C^1-functions ϕ_1, ϕ_2 near $z = 0, 1$, so that $[\phi_2^{-1}, \phi_1^{-1}](z) > \phi(z)$ away from $z = 0, 1$, ϕ_1, ϕ_2 are increasing near $z = 0$, and ϕ_1, ϕ_2 are decreasing near $z = 1$. For example, near $z = 0$, $\phi(z) = 2z$, $\phi(z) = z + z^2$ will do. Pick $0 < a_0 < \varepsilon$ small enough so that $b_0 = \phi_1(a_0)$ and $\phi_2(b_0) < \varepsilon$, and assume ϕ_1 is already defined on $[0, \phi_1^{-1} \circ \phi_2(b_0)] \supset [0, a_0]$, and ϕ_2 is already defined on $[0, b_0]$. Take $\varepsilon > a_1 > \phi_2(b_0)$ and $\varepsilon > b_1 > a_1$. Similarly pick a_2, a_3, b_2, b_3 in an

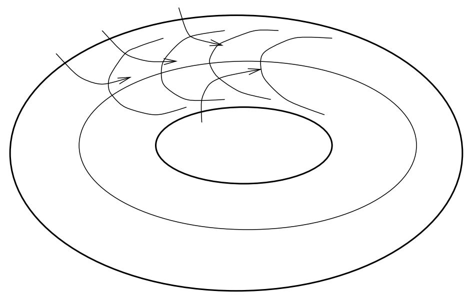


Figure 5: Reeb foliation

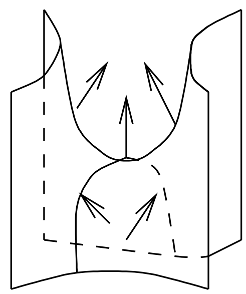


Figure 6: Saddle

ε-neighborhood of $z = 1$. As long as ε is small enough, we can extend ϕ_1 and ϕ_2 to all of I so that they satisfy Equation 1, by making the 'slopes steep enough'.

3.2 Proof of sufficiency of Theorem 3

We will now prove the converse, namely, if conditions (i) and (ii) of Theorem 3 are satisfied, then there exists a confoliation ξ transverse to X.

What follows is essentially Sue Goodman's construction, supplemented by an argument for the contact part. This involves decomposing M into round handles ($=S^1 \times D^2$), constructing a foliation on each handle, and extending the foliation to a confoliation when attaching the round Morse handles.

Step 1: (Index 0) We foliate each $S^1 \times D^2$ by a Reeb component (see Figure 5).

Step 2: (Index 1) We foliate by stacking saddles (see Figure 6).

The resulting characteristic foliation on $\partial(S^1 \times D^2)$ will look like Figure 7, when $S^1 \times D^2$ is viewed as $I \times D^2$ with ends identified ($I = [0, 1]$). Note that D^2 is best thought of as an octagon inside $\mathbf{R}^2$. Four sides a, c, e, g are parallel to the x-axis or the y-axis and X is transverse to the faces $I \times a$, etc. We can turbularize the stacked saddles along these faces,

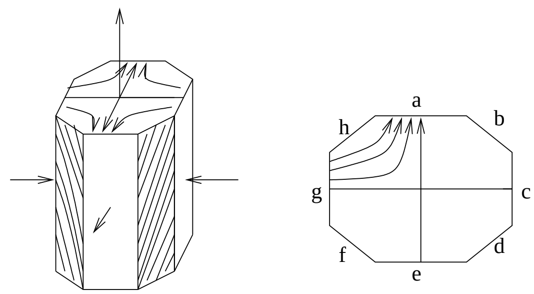


Figure 7: Characteristic foliation

and obtain a foliation on $I \times D^2$ with leaves $I \times a$, $I \times c$, and so on. On the other hand, the other four faces (e.g. $I \times b$) are foliated I-bundles $I \times I$ (where the second $I = b$, and so on) with leaves $I \times \{0\}$ and $I \times \{1\}$.

Now, if $S^1 \times D^2$ is standard, then $I \times D^2$ has two faces on which X is an outward normal. On the other hand, a twisted handle is obtained by gluing via a rotation by π, and there is only one face on which X is an outward normal. These are the faces along which we glue our 1-handles onto the solid tori in Step 1.

Step 3: Attaching index 1 handles. This is where we will require contact structures in certain situations. We will prove this by induction, using the fact that ∂M_{i-1} will always be a union of tori.

Lemma 3 *∂M_i is a union of tori.*

Proof: We use the fact that $\sum_j \chi(N_j) = 0$, if N_j are the components of ∂M_i, and the nonsingular vector field X is transverse to ∂M_i. If not all N_j are tori, then some $N_j = S^2$. Such an S^2 violates condition (i) of Theorem 3. $\square$

Assume the confoliation has already been constructed on M_{i-1}, and the toral components of ∂M_{i-1} are closed leaves of the confoliation. We glue a 1-handle $S^1 \times D^2$ onto ∂M_{i-1} either along one or two annuli A, depending on whether the 1-handle is standard or twisted.

Let us attach along the first annulus. What we have is an extension problem: Find a confoliation on $(T^2 - A) \times I$ so that $(T^2 - A) \times \{0, 1\}$ are leaves of a foliation and $\partial(T^2 - A) \times I$ agrees with $\partial A \times I$, which are the foliated I-bundles coming from the faces b, d, f, h above. Our essential distinction is whether the annulus is *homotopically essential* on the 0-handle or *homotopically trivial.*

If the annulus is homotopically essential, then $T^2 - A$ is an annulus, and it is easy to extend the foliation on the boundary of $(T^2 - A) \times I$ to its interior.

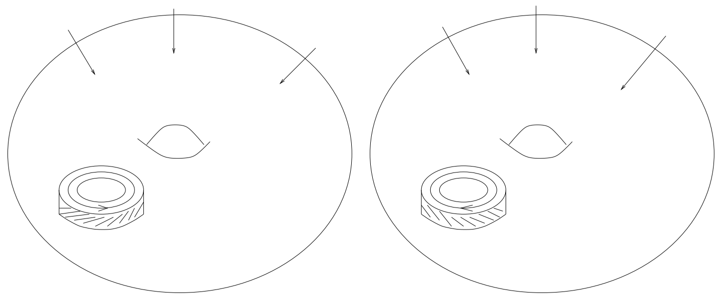


Figure 8: Attaching a 1-handle

On the other hand, if the annulus is not essential, then we have situations as in Figure 8. Due to holonomy, one sees that the component $D^2 \times I$ of $(T^2 - A) \times I$ cannot be foliated, and the contact structures make their first appearance.

The diagram to the right in Figure 8 has a component $D^2 \times I$ with a characteristic foliation of nonpositive holonomy on $\partial D^2 \times I$, which can be extended to a confoliation which is contact on the interior of $D^2 \times I$ and has leaves $D^2 \times \{0,1\}$, by Proposition 1. The other component $(T^2 - D^2) \times I$ can also be made into a positive confoliated I-bundle by Proposition 1.

The diagram to the left, on the other hand, gives rise to a left-handed disk which does not link an attracting or repelling periodic orbit. However, if it shadows a right-handed disk, we can construct a confoliation on M_i by slightly modifying the existing confoliation on M_{i-1} before extending. Since this is a bit involved, we will do this in Step 4.

Hence, we can glue a 1-handle onto T^2 along an annulus A, provided the attachment is not homotopically trivial and left-handed. If the 1-handle was twisted, we would be done. Otherwise, we need to attach the other annulus A onto either the same T^2 or another T^2. Note that the homotopically trivial attachment of *both* annuli is disallowed by Lemma 3.

We will now perform all the possible attachments of index 1 handles and extend the confoliation from M_{i-1} to M_i. If we attach the two annuli onto two distinct T^2, then each can be glued the same way, subject to the restriction that not both are homotopically trivial. If the annuli are glued onto the same T^2, either (1) one annulus is homotopically trivial and the other is homotopically essential, or (2) we have two parallel copies of homotopically nontrivial annuli. In case (1), we first glue along the nontrivial annulus and perform the extension as before. We then glue along the trivial annulus, and want to 'cap off' the two components of the foliated I-bundle $\partial A \times I$. We can extend one of the components via a contact component $D^2 \times I$ as before, where D^2 bounds one of the components of $\partial A \times I$. The other component $S^1 \times I$ looks like Figure 9, which can be turbularized away. In case (2), glue the first annulus (and extend). If the second annulus points in the same direction as the first one, each of the components of the foliated I-bundle $\partial A \times I$ will be an essential annulus on a different toral boundary (c.f. Figure 9), which can be turbularized. If the annuli point

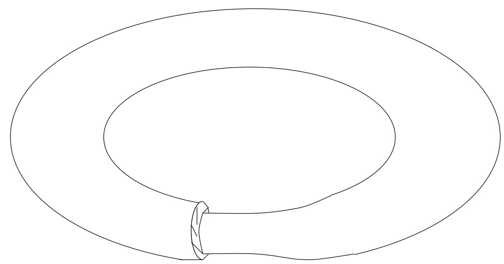


Figure 9: Foliated I-bundle to be turbularized

in opposite directions, then the components of $\partial A \times I$ are parallel essential annuli on the same toral boundary, which we turbularize simultaneously.

Step 4: (Left-handed disks shadowing right-handed disks) We now attach an index 1 handle H_i for the orbit γ_i to ∂M_{i-1}, so that one of the annuli (say A_1) glues onto a component T of ∂M_{i-1} in the fashion depicted on the left-hand side of Figure 8. γ_i then has a left-handed disk which shadows the right-handed disk for γ_j. We have two cases: $j < i$ or $j > i$.

Case 1: ($j < i$) Here, the handle H_j for γ_j has already been attached to ∂M_{j-1}, in one of two ways:

1. The two annuli of attachment were attached to different tori T_1 and T_2.

2. The two annuli were glued to the same torus T_1.

One of the annuli must have been homotopically essential because of Lemma 3. In the first situation, assume that the homotopically essential annulus is glued onto T_1; in the second, recall that the construction above required the essential annulus be glued *first* onto T_1.

$T - A_1$ has two components, $T - D^2$ and D^2. The foliated I-bundle $\partial A_1 \times I$ can be easily extended to $(T - D^2) \times I$, but cannot be extended to $D^2 \times I$. Let $\delta = W_s(\gamma_j) \cap T$, and δ_1 (resp. δ_2) be the inner (resp. outer) component of δ. (This makes sense because they are both null-homotopic.) Let $\Omega \subset D^2$ be a disk containing δ_1 and avoiding δ_2. We can clearly extend our foliation to $(D^2 - \Omega) \times I$. See Figure 10.

We now take an annulus $B \subset \Omega$ with $\partial\Omega \subset \partial B$ and $B \cap \delta = \emptyset$, form its flow cylinder $\mathbf{R} \times B$, and dig out the portion of $\mathbf{R} \times B$ intersecting $M_{i-1} - M_{j-1}$, i.e., the portion that is 'sitting above' T_1. We also replace T_1 by $T_1 \times I$, with leaves $T \times \{0, 1\}$, i.e., we 'blow air through'. See Figure 11.

We now tuck in and turbularize as in Figure 12. Therefore, instead of the foliated I-bundle coming from H_i, we now have a foliated I-bundle coming from H_j, facing the other direction. When attaching H_i along the other (essential) annulus A_2, we turbularize one of the $S^1 \times I$ inward and the other one outward.

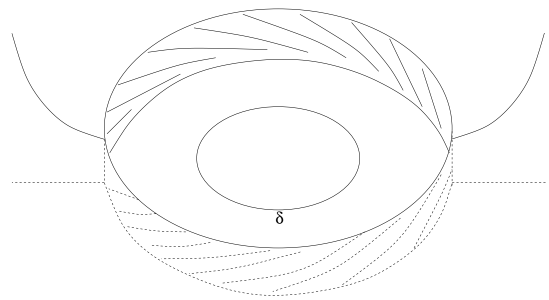


Figure 10: Extension of foliation

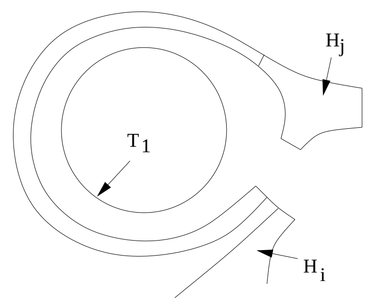


Figure 11: Digging a ditch and blowing air

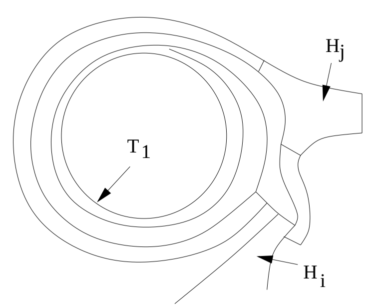


Figure 12: Tucking and turbularizing

Case 2: ($j > i$) This is the only situation where M_i cannot be fully confoliated until we get to M_j because of the left-handed attachment. When we glue the handle H_i onto M_{i-1} along the leaf T, confoliate all of M_i except for $D^2 \times I$, which we leave empty except for leaves $D^2 \times \{0, 1\}$. When it is time to attach H_j onto M_{j-1} along a homotopically trivial annulus A, we take an annulus $B \subset M_{j-1}$ on the outside of A, dig down as before, blow air around T, and insert the foliated I-bundle coming from H_j below that of H_i. The remaining attachments can be easily turbularized.

Step 5: (Index 2) We finally complete the construction by gluing on the 2-handles along tori. This is identical to Step 1. This completes the proof. □

Remark: The above results should be extendable to the case of flows with hyperbolic 1-dimensional chain recurrent set, à la Goodman's subsequent paper [10].

Question: Characterize when the contact structures constructed above are tight.

Note that since all the confoliations in our construction have Reeb components, it is possible to insert a foliated I-bundle $T^2 \times I$ in place of a toral leaf, with the right holonomy which makes the perturbed contact structure overtwisted. This follows from the perturbation on p.62 of [5]. A better question is therefore:

Question: When can we construct a transverse tight structure?

A necessary condition is the nonexistence of left-handed disks. Example 3.3.6 in [5] indicates that transverse tight structures exist in certain situations; however the mechanism is not understood at the moment.

4 Seifert fibered spaces

Let us start with the Milnor-Wood inequality both for foliations and confoliations. The classical result for foliations is due to Milnor [15] and Wood [21], and the version for confoliations was first observed by Giroux and Sato-Tsuboi [19]. The case $g(\Sigma) = 0$ is included in Eliashberg-Thurston [5].

Theorem 5 *Consider a circle bundle $\pi : M \to \Sigma$ with Euler class e over a closed Riemann surface Σ of genus $g(\Sigma)$. Then,*

1. *for $g(\Sigma) > 0$, there exists a foliation ξ transverse to the circle fibers if and only if $|e| \leq 2g(\Sigma) - 2$.*

2. *for $g(\Sigma) > 0$, there exists a positive confoliation ξ (hence a positive contact structure ξ) transverse to the circle fibers if and only if $e \leq 2g(\Sigma) - 2$.*

3. *for $g = 0$, there exists a transverse foliation ξ if and only if $e = 0$.*

4. *for $g = 0$, there exists a transverse positive confoliation ξ if and only if $e \leq 0$.*

5. *for $g = 0$, there exists a transverse positive contact structure ξ if and only if $e < 0$.*

In this section we will prove a generalization of this result to Seifert fibered spaces $\pi : M \to \Sigma$. The theorems are proved in much the same way, namely by reducing to considerations of circle homeomorphisms or diffeomorphisms. The case $g(\Sigma) > 0$ follows easily from the analogous result of Eisenbud, Hirsch, and Neumann [2] for foliations, whereas the case $g(\Sigma) = 0$ relies on subsequent papers by Jankins and Neumann [12], [13], and Naimi [17].

Let us first introduce notation. Let $\pi : M \to \Sigma$ be an oriented Seifert fibered space over a closed Riemann surface Σ with Seifert invariants $(g(\Sigma); b, (\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_n, \beta_n))$, where $\alpha_i, \beta_i \in \mathbf{Z}^+$, α_i, β_i are relatively prime, $0 < \beta_i/\alpha_i < 1$ (all for $i = 1, ..., n$), and $b \in \mathbf{Z}$. b, the *background charge*, is the Euler number when $M \to \Sigma$ is a circle bundle. Notice that some authors use the same letter b to stand for minus the background charge. For convenience we will write $(\alpha_0, \beta_0) = (1, b)$. We have branch points $p_i \in \Sigma$, $i = 1, ..., n$ sitting below the singular fibers $\pi^{-1}(p_i)$ of M, as well as $p_0 \in \Sigma$, near which we concentrate the background charge. Choosing disks $D_i \subset \Sigma$, $i = 0, ..., n$, containing p_i, we have a trivial S^1-bundle $\pi^{-1}(\Sigma - \bigcup_{i=0}^n D_i) \simeq (\Sigma - \bigcup D_i) \times S^1$, as well as an identification $\pi^{-1}(D_i) \simeq D^2 \times S^1$, where the $\{pt\} \times S^1$ are not fibers unless $i = 0$.

For each D_i, $i = 0, ..., n$, we have an identification

$$\partial(D^2 \times S^1) = T^2 \to \partial((\Sigma - \bigcup D_i) \times S^1)_{i,opp} = T^2$$

given by

$$A = \begin{pmatrix} \alpha & \gamma \\ -\beta & \delta \end{pmatrix} \in SL(2, \mathbf{Z}),$$

where $\partial()_{i,opp}$ refers to the component corresponding to D_i, with the opposite orientation, and γ, δ are chosen so that $\det A = 1$.

Now, if we have a foliation by disks $D^2 \times \{pt\}$ on $D^2 \times S^1$, then the characteristic foliation on $\partial(D^2 \times S^1)$ becomes the foliation by lines of slope $-\beta_i/\alpha_i$ on $\partial((\Sigma - \bigcup D_i) \times S^1)_{i,opp}$, and hence the foliation by lines of slope β_i/α_i on $\partial((\Sigma - \bigcup D_i) \times S^1)_i$.

The following theorem includes the cases considered in Theorem 2. Recall that an n-tuple $(\mu_1, \cdots, \mu_n) \in (0,1)^n$ is *realizable* if there exists a permutation $(\gamma_1, \cdots, \gamma_n)$ of $(a/m, (m - a)/m, 1/m, \cdots, 1/m)$ with $0 < a < m$ and $(a, m) = 1$ such that $\mu_i < \gamma_i$ for all i.

Theorem 6 *Let $\pi : M \to \Sigma$ be a Seifert fibered space with Seifert invariants $(g(\Sigma); b, (\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_n, \beta_n))$.*

1. *Assume $g > 0$. Then there exists a foliation ξ transverse to the fibers if and only if $b \leq 2g-2$ and $2-2g \leq b+n$.*

2. *Assume $g > 0$. Then there exists a positive confoliation (and hence a positive contact structure) ξ transverse to the fibers if and only if $b \leq 2g-2$.*

3. *Assume $g = 0$. Then there exists a transverse foliation if and only if one of the following holds:*

   (a) *$b \leq -2$ and $2 \leq b+n$.*

   (b) *$b \leq -2$, $1 = b+n$, and $(1-\beta_1/\alpha_1, \cdots, 1-\beta_n/\alpha_n)$ is realizable.*

   (c) *$b = -1$, $n \geq 3$, and $(\beta_1/\alpha_1, \cdots, \beta_n/\alpha_n)$ is realizable.*

   (d) *$b = -1$, $n = 2$, and $\beta_1/\alpha_1 + \beta_2/\alpha_2 = 1$.*

   (e) *$b = 0$ and $n = 0$.*

4. *Assume $g = 0$. Then there exists a transverse positive confoliation if and only if one of the following holds:*

   (a) *$b \leq -2$.*

   (b) *$b = -1$, $n \geq 3$, and $(\beta_1/\alpha_1, \cdots, \beta_n/\alpha_n)$ is realizable.*

   (c) *$b = -1$, $n = 2$, and $\beta_1/\alpha_1 + \beta_2/\alpha_2 \leq 1$.*

   (d) *$b = -1$ and $n = 0$ or $n = 1$.*

   (e) *$b = 0$ and $n = 0$ (this is the only one which cannot be deformed to a positive contact structure.)*

The proof can be reduced to considerations of circle homeomorphisms (or diffeomorphisms), as we will now explain. Let $\mathcal{D} = Homeo^+(S^1)$ be the orientation-preserving homeomorphisms of S^1 and $\widetilde{\mathcal{D}}$ be its universal cover, given by orientation-preserving homeomorphisms of $\mathbf{R}$ which descend to $S^1 = \mathbf{R}/\mathbf{Z}$. For the most part we can think of $\mathcal{D}$ as the orientation-preserving diffeomorphisms - the difference is mostly technical and very slight. Denote by $sh(\gamma) : \mathbf{R} \to \mathbf{R}$ the translation $x \mapsto x + \gamma$. Finding a transverse foliation is equivalent to finding a_i, $b_i \in \mathcal{D}$, $i = 1, ..., g$, and $e_i \in \widetilde{Homeo}(S^1)$ so that

$$[a_1, b_1] \cdots [a_g, b_g] = sh\left(\frac{\beta_0}{\alpha_0}\right)^{e_0} \cdots sh\left(\frac{\beta_n}{\alpha_n}\right)^{e_n}, \tag{2}$$

where we denote $\phi^e = e \circ \phi \circ e^{-1}$. To see why, refer to Figure 13. The ∂D_i contribute $sh(\beta_i/\alpha_i)$,

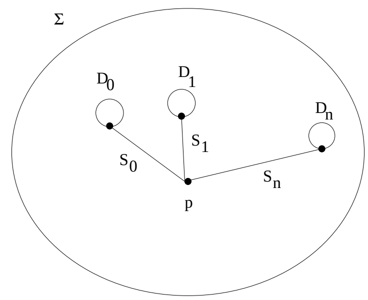


Figure 13: Base space Σ

the slits S_i to the fixed point p contribute the conjugations e_i, and $\partial(\Sigma - \bigcup D_i - \bigcup S_i)$ is now a bouquet of $2g$ circles, contributing to the product of commutators on the left-hand side of the equation. We can also see that finding a transverse positive confoliation is equivalent to solving

$$[a_1, b_1] \cdots [a_g, b_g] \geq sh\left(\frac{\beta_0}{\alpha_0}\right)^{e_0} \cdots sh\left(\frac{\beta_n}{\alpha_n}\right)^{e_n}, \tag{3}$$

If $\phi \in \widetilde{\mathcal{D}}$, define $\underline{m}\phi = \min(\phi(x) - x)$ and $\overline{m}\phi = \max(\phi(x) - x)$. Also, let $\lfloor\gamma\rfloor$ be the greatest integer $\leq \gamma$ and $\lceil\gamma\rceil$ be the least integer $\geq \gamma$. We need two lemmas from [2]. We refer the reader to the proofs there.

Lemma 4 *$\phi \in \widetilde{\mathcal{D}}$ is a g-fold product of commutators in $\widetilde{\mathcal{D}}$ if and only if $\underline{m}\phi < 2g - 1$ and $\overline{m}\phi > 1 - 2g$.*

Lemma 5 *Let ϕ_1, ... $\phi_n \in \widetilde{\mathcal{D}}$ and $r \in \mathbf{R}$ satisfying $\sum\lfloor\underline{m}\phi_i\rfloor < r < \sum\lceil\overline{m}\phi_i\rceil$. Then there exist $e_i \in \widetilde{Homeo}(S^1)$ such that $\underline{m}(\prod \phi_i^{e_i}) \leq r \leq \overline{m}(\prod \phi_i^{e_i})$.*

Proof of Theorem 6: (1) is presented in [2], and (3) is the combination of the efforts of [12], [13], and [17]. The proofs of (2) and (4) are similar - we will provide a brief explanation.

g > 0: Let $g > 0$. We will prove (2). Suppose that there is a transverse positive confoliation. Then taking $\lfloor\underline{m}\rfloor$ of both sides of Equation 3, we obtain

$$\left\lfloor \underline{m}\prod_{i=1}^{g}[a_i, b_i]\right\rfloor \geq \left\lfloor \underline{m}\left(\prod_{i=0}^{n} sh\left(\frac{\beta_i}{\alpha_i}\right)^{e_i}\right)\right\rfloor \geq \sum_{i=0}^{n}\left\lfloor \underline{m}\left(sh\left(\frac{\beta_i}{\alpha_i}\right)^{e_i}\right)\right\rfloor = \sum_{i=0}^{n}\left\lfloor\frac{\beta_i}{\alpha_i}\right\rfloor = b.$$

By Lemma 4, $\underline{m}\prod[a_i, b_i] < 2g - 1$, so $2g - 2 \geq b$.

Conversely, assume $b \leq 2g - 2$. Consider first the case $n = 0$, i.e., M is a circle bundle. If $b \geq 2 - 2g$, then by Lemma 4, we can write $sh(b)$ as a g-fold product of commutators. If

$b < 2-2g$, then $sh(b)$ will be smaller than any g-fold product of commutators. At any rate, there exist a_i, b_i such that $\prod_{i=1}^{g}[a_i, b_i] \geq sh(b)$.

Now assume $n > 0$. Then $\sum\lfloor\beta_i/\alpha_i\rfloor < \sum\lceil\beta_i/\alpha_i\rceil$, which implies $\sum\lfloor\underline{m}(sh(\beta_i/\alpha_i))\rfloor < \sum\lceil\overline{m}(sh(\beta_i/\alpha_i))\rceil$. Hence, by Lemma 5, there exist e_i such that $\underline{m}\prod sh(\beta_i/\alpha_i)^{e_i} \leq b+\varepsilon < 2g-1$, and which, by Lemma 4, implies the existence of a_i and b_i satisfying Equation 3.

g = 0: For $g = 0$, Equation 3 reduces to

$$id \geq \prod_{i=0}^{n} sh\left(\frac{\beta_i}{\alpha_i}\right)^{e_i} \tag{4}$$

We will now prove (4) of Theorem 6. If Equation 4 holds, then $0 \geq \lfloor\underline{m}\prod_{i=0}^{n} sh(\beta_i/\alpha_i)^{e_i}\rfloor \geq b$. If $b = 0$, then $n = 0$. If $b = -2$, then we can use Lemma 5 and $\sum_{i=1}^{n}\lfloor\underline{m}(sh(\beta_i/\alpha_i))\rfloor < -2+\varepsilon$ to get $e_i \in \widetilde{\mathcal{D}}$ satisfying $\underline{m}(\prod_{i=1}^{n} sh(\beta_i/\alpha_i)^{e_i}) \leq -2+\varepsilon$, which implies that $\overline{m}(\prod_{i=1}^{n} sh(\beta_i/\alpha_i)^{e_i}) \leq -1+\varepsilon$, proving Equation 4.

g = 0, b = -1: We now have remaining the case $g = 0$ and $b = -1$ (the hard part). $n = 0$, $n = 1$ are automatic. For $n = 2$, we need $\beta_1/\alpha_1 + \beta_2/\alpha_2 \leq 1$: Taking the *rotation number* $\mathrm{rot}(\phi) = \lim_{n\to\infty}\frac{1}{n}(\phi^n(x) - x)$, we find $\mathrm{rot}(sh(1-\beta_1/\alpha_1)^{e_1}) \geq \mathrm{rot}(sh(\beta_2/\alpha_2)^{e_2})$, which evaluates to $1 - \beta_1/\alpha_1 \geq \beta_2/\alpha_2$.

Assume from now on that $n \geq 3$.

Definition: Let $n \in \mathbf{Z}^+$, $J \subset \{1, ..., n\}$, $b \in \mathbf{Z}$, and $(\mu_1, \cdots, \mu_n) \in (0,1)^n$. We say that $(J; b; \mu_1, ..., \mu_n)$ is *F-realizable* if there exist $\phi_i \in \widetilde{\mathcal{D}}$ such that $\mathrm{rot}(\phi_i) = \mu_i$, ϕ_i is conjugate to $sh(\mu_i)$ if $i \in J$, and $\phi_n \circ \cdots \circ \phi_1 = sh(-b)$. $(J; b; \mu_1, \cdots, \mu_n)$ is *C-realizable* if $\phi_n \circ \cdots \circ \phi_1 \leq sh(-b)$. If we suppress the J, we will assume $J = \{1, \cdots, n\}$ and if we suppress the b, we will assume $b = -1$. For $b = -1$ we will extend our previous definition of *realizable* and say that $(J; \mu_1, \cdots, \mu_n)$ is *realizable* if there exists a permutation $(\gamma_1, \cdots, \gamma_n)$ of $(a/m, (m-a)/m, 1/m, \cdots, 1/m)$ with $0 < a < m$ and $(a, m) = 1$ such that $\mu_i < \gamma_i$ for all $i \in J$ and $\mu_i \leq \gamma_i$ for all $i \notin J$.

Let $S_n(J)$ be the set of realizable $(J; \mu_1, \cdots, \mu_n)$ for a fixed index set J. Define $T_3(J) = S_3(J)$, and $T_n(J)$ inductively by $T_n(J) = \{(\mu_1, \cdots, \mu_n) \in (0,1)^n |$ there exists an x for which $(\mu_1, \cdots, \mu_{n-2}, x) \in T_{n-1}(J_1), (1-x, \mu_{n-1}, \mu_n) \in T_3(J_2)\}$, where $J_1 = J \cap \{1, \cdots, n-2\}$ and $J_2 = \emptyset$ or $\{2\}$ or $\{3\}$ or $\{2,3\}$ depending on whether $J \cap \{n-1, n\} = \emptyset$ or $\{n-1\}$ or $\{n\}$ or $\{n-1, n\}$.

Although we are primarily interested in $J = \{1, \cdots, n\}$, we must deal with any $J \subset \{1, 2, 3\}$ due to the following reduction lemma, proved in [2]:

Lemma 6 *$S_n(J) = T_n(J)$.*

Assume for the moment that we can prove that $T_3(J) = S_3(J)$ coincides with the set of C-realizable triples (as well as the F-realizable triples), for all J. Then the Lemma implies that $S_n(J)$ is the set of C-realizable n-tuples (or F-realizable n-tuples) as follows: it is clear that the set of C-realizable n-tuples is a subset of $T_n(J)$. On the other hand, it is shown in [12] and [13] that:

Lemma 7 *$S_n(J) \subset \{$C-realizable n-tuples $(\mu_1, \cdots, \mu_n)\}$.*

We are then left to prove:

Proposition 2 *$S_3(J)$ is the set of C-realizable triples (and is also the set of F-realizable triples).*

Proof: The F-realizable case is proven by Naimi [17]. The C-realizable case is similar, and depends on the following lemmas.

In [12] and [13] it is shown that:

Lemma 8 *There exist vertices $(\gamma_1, \gamma_2, \gamma_3) = (\frac{p-q'}{p}, \frac{q-p'}{q}, \frac{1}{p+q})$ satisfying*

$$pp' + qq' = 1 + pq, \text{ for } p, q > 1, \tag{5}$$

with the property that given $(\mu_1, \mu_2, \mu_3) \in (0,1)^3 - S_3(J)$, there exists some $(\gamma_1, \gamma_2, \gamma_3)$ with $\mu_i \geq \gamma_i$ for all $i \in J$ and $\mu_i > \gamma_i$ for all $i \notin J$.

In [17] it is shown that:

Lemma 9 *Assume $(\gamma_1, \gamma_2, \gamma_3) = (\frac{p-q'}{p}, \frac{q-p'}{q}, \frac{1}{p+q})$ satisfies Equation 5. Then (μ_1, μ_2, μ_3) is not C-realizable if $\mu_1 = \gamma_1$, $\mu_2 = \gamma_2$, and $\mu_3 \geq \gamma_3$. (In this case $J = \{1, 2, 3\}$.)*

Lemma 10 *Fix $(\gamma_1, \gamma_2, \gamma_3) \in (0,1)^3$. If $(\gamma_1, \gamma_2, \gamma)$ is not C-realizable for every $\gamma \geq \gamma_3$, then $(J; \mu_1, \mu_2, \mu_3)$ is not C-realizable for every $J \subset \{1, 2, 3\}$, and $\mu_i \geq \gamma_i$ for $i \in J$ and $\mu_i > \gamma_i$ for $i \notin J$.*

The proofs of the last two lemmas are virtually identical to the proof given by Naimi, with the modification of $f_1 \circ f_2 \circ f_3 = sh(1)$ to $f_1 \circ f_2 \circ f_3 \leq sh(1)$. It turns out that all the inequalities in Naimi's paper are still valid. Since we are not able to add to his exposition, we will leave it to the reader to check this claim. $\square$

We will conclude this paper with a few remarks and questions.

Remark: Let $M \to \Sigma$ be a circle bundle over a compact Riemann surface Σ, possibly with boundary. Then any ξ transverse to the fibers is tight. (This follows from symplectic filling on the unit disk bundle of the complex line bundle over Σ, with Euler class equal to the Euler class of the circle bundle.) The same is true for Seifert fibered spaces - we can proceed either by proving a filling result for symplectic orbifolds (which M bounds), or by showing directly that an overtwisted disk for M (if it exists) can be moved off the singular fibers.

Problem: Classify all tight contact structures on Seifert fibered spaces (or circle bundles). Classify all transverse tight contact structures.

Remark: Unlike taut foliations or essential laminations, tight contact structures cannot always be homotoped so that either all the 2-planes are tangent or all the 2-planes are transverse to the fibers. This is due to the following theorem of Gompf [8]:

Theorem 7 *Let $M \to \Sigma$ be a circle bundle with Euler number e over a closed, connected, oriented surface Σ of genus g. Then M admits at least $\lfloor g - \frac{e}{2} \rfloor$ holomorphically fillable contact structures whose homotopy classes remain distinct after allowing orientation-preserving self-diffeomorphisms of M.*

Problem: Extend the transversality results to all flows in general, or, in particular, to flows such as Anosov flows (Anosov flows already have tangential positive and negative contact structures) or volume-preserving flows.

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