Embedding flows into homogeneous potential well, homogeneous NLW, and zero-dimensional NLS

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Abstract

In [Tao16], a solution leading to finite-time blow-up was constructed for a time-averaged Navier-Stokes equation. The solution suggested the possibility of programming an ideal fluid towards blow-up, and in [Tao17], as a proof of concept, it was shown that a Turing machine, formulated by differential geometry, could be embedded into a potential well system. On a related note, embedding into the Euler equation was discussed in [Tao18]. Here we further investigate what kind of flows could be embedded into potential wells, subject to further natural constraints such as homogeneity. We also discuss the case of homogeneous potential functions for nonlinear waves, as well as the universality of zero-dimensional NLS flows.

1 Preliminary definitions

Let $M$ be a smooth manifold and $X \in \mathfrak{X}M$ be a smooth vector field of $M$. Then $(M, X)$ is called a smooth flow (or flow). The flow is called compact if $M$ is compact, and nonsingular if $M$ is nonsingular ($X_p \neq 0 \forall p \in M$).

When $M = T^* \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ and $X(q, p) = (p, -\nabla V(q))$ where $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth potential function, we denote the flow as Well($\mathbb{R}^n, V$). The associated Hamiltonian is $H(q, p) = \frac{|p|^2}{2} + V(q)$, while the Lagrangian is $L(q, p) = \frac{|p|^2}{2} - V(q)$. The symplectic potential is defined in coordinates as $\Theta = p_i dq^i$, while the canonical symplectic form is $\Omega = -d\Theta = dp^i \wedge dq_i$.

Let $T = \mathbb{R}/\mathbb{Z}$ and $H^m_d = C^\infty(T^d \rightarrow \mathbb{R}^m)$. When $M = T^*(H^m_d) = H^m_d \times H^m_d$ (a Frechet manifold/vector space) and $X(q, p) = (p, \Delta_{\mathbb{R}^m} q - \nabla_{\mathbb{R}^m} V(q))$ where $V : \mathbb{R}^m \rightarrow \mathbb{R}$ is a smooth potential function, we denote the flow as NLW$(T^d, \mathbb{R}^m, V)$. We can treat Well($\mathbb{R}^m, V$) as a subset of NLW$(T^d, \mathbb{R}^m, V)$ where $q, p$ are constant functions.

An embedding of the flow $(N, Y)$ into the flow $(M, X)$ is an embedding $\phi : N \rightarrow M$ such that $d\phi \cdot Y = X$. When $M$ is $T^*(H^m_d)$, an embedding means a smooth (in a Gateaux sense) injective immersion.

A 1-form $\theta$ of the flow $(M, X)$ is called (weakly) adapted when $\theta \cdot X \geq 0$ and $L_X(\theta)$ is exact. $\theta$ is called strongly adapted if $\theta \cdot X > 0$.

Then by [Tao17], we know there is an embedding from $(M, X)$ into Well($\mathbb{R}^m, V$) for some $n$ and $V$ if and only if there is a strongly adapted 1-form for $(M, X)$. Under this embedding, $\theta \cdot X$ corresponds to the kinetic energy $|p|^2$ and $L_X(\theta)$ corresponds to $d\mathcal{L}$ where $\mathcal{L}$ is the Lagrangian.

If the flow Well($\mathbb{R}^n, V$) further satisfies that $V$ is homogeneous of order $k$ (away from the origin), i.e. $V(tx) = t^k V(x) \forall t \geq 1, \forall x \in \mathbb{R}^n \setminus B(0, c)$ for some $c \in (0, 1)$, then denote the flow as HomWell$_k(\mathbb{R}^n, V)$. An embedding $\phi : N \rightarrow \mathbb{R}^n \times \mathbb{R}^n, a \mapsto (Q(a), P(a))$ from $(N, Y)$ into HomWell$_k(\mathbb{R}^n, V)$ is understood to satisfy $|Q| > 0$ and $\text{Im} Q$ lies in the region where $V$ is homogeneous. The embedding is also called spherical if $Q(N) \subseteq S^{n-1}$.

Similarly for NLW$(T^d, \mathbb{R}^m, V)$, if $V$ is homogeneous of order $k$, we denote it as HomNLW$_k(T^d, \mathbb{R}^m, V)$. 
Finally we note that a smooth potential $V$ is homogeneous of order $k$ away from the origin $\iff \langle \nabla V(x), x \rangle = kV(x) \forall x \in \mathbb{R}^n \setminus B(0, e)$ for some $e > 0$ (Euler’s homogeneous function theorem).

2 Embedding into HomNLW

It turns out that any compact nonsingular flow that can be embedded into NLW, can also be embedded into HomNLW$_k$ when $k \neq 2$.

**Theorem 1** (Homogeneous NLW embedding). Let $k \notin \{0, 2\}$, $d \in \mathbb{N}$. Let $(N, Y)$ be a compact nonsingular flow with a strongly adapted 1-form $\theta$. Then there is an embedding from $(N, Y)$ into HomNLW$_k(T^d, \mathbb{R}^m, V)$ for some $V$ and some $m$.

We in fact have a stronger conclusion: given any function $R$ smooth on $N \times T^d$ such that $R > 0$, there is $V$ on $\mathbb{R}^m$ homogeneous of order $k$ and an embedding $\phi : N \times T^d \to \mathbb{R}^m \times \mathbb{R}^m$, $(y, x) \mapsto (Q(y, x), P(y, x))$ such that

$$YQ = P, \ YP = \Delta_x Q - \nabla_{\mathbb{R}^n} V(Q), |Q|^2 - R^2 = \text{constant}$$

**Proof.** First, as in [Tao17], since $\theta \cdot Y > 0$, we can construct a Riemannian metric $g$ on $N$ such that $g \cdot Y = \theta$. Let $L$ be smooth on $N$ such that $dL = L_Y(\theta)$.

Any smooth function $f$ on $N$ can be extended to a smooth function $f(y, x) = f(y)$ on $N \times T^d$. Any vector field $Z \in \mathfrak{X}N$ can be extended to $(Z, 0) \in \mathfrak{X}(N \times T^d)$. Similarly for 1-forms. So under these extensions, we can say $Y \in \mathfrak{X}(N \times T^d), \theta \in \Omega^1(N \times T^d)$ and $dL = L_Y(\theta)$. Let $N$ be locally parametrized by functions $(y^i)$, and $T^d$ globally parametrized by $(x^i)$. Since $Y$ is nonvanishing, by straightening, WLOG assume $[Y, \partial_{y^i}] = 0 \forall i$.

We wish to extend $g$ to a Riemannian metric on $N \times T^d$ and satisfy the Euler condition later. The intuition is that after finding $Q$, the Euler condition will be equivalent to:

$$W \left( kL + \left( 1 - \frac{k}{2} \right) (\theta \cdot Y) - YY \left( \frac{|Q|^2}{2} \right) - \left( 1 - \frac{k}{2} \right) |\nabla_x Q|^2 + \Delta_x \left( \frac{|Q|^2}{2} \right) \right) = k \left( \sum_i \partial_{x^i} \langle \partial_{x^i} Q, WQ \rangle - \langle \partial_{x^i} Q, [\partial_{x^i}, W] Q \rangle \right)$$

$$\forall W \in \mathfrak{X}(N \times T^d).$$

Because $k \neq 2$, we can define $G$ on $N \times T^d$ such that

$$kL + \left( 1 - \frac{k}{2} \right) (\theta \cdot Y) - YY \left( \frac{R^2}{2} \right) - \left( 1 - \frac{k}{2} \right) G + \Delta_x \left( \frac{R^2}{2} \right) = \text{constant} \quad (2)$$

where the constant is chosen so that $G > 0$. Then $G$ is constant in $x$ and we define $g(\partial_{y^i}, \partial_{x^i}) = 0, g(\partial_{y^i}, \partial_{x^j}) = \delta_{ij} \frac{G}{2}$. So $\forall W \in \mathfrak{X}(N \times T^d), g(\partial_{y^i}, W) = dx^i(W) \frac{G}{2}$.

We observe that $G = \sum_{i=1}^d g(\partial_{x^i}, \partial_{x^i})$ and $g(\partial_{x^i}, \partial_{x^j}) > 0 \forall i = 1, \ldots, d$. Then $g$ is now a Riemannian metric on $N \times T^d$ and we still have $g \cdot Y = \theta$. We can also check that

$$\partial_{y^i} \left( kL + \left( 1 - \frac{k}{2} \right) (\theta \cdot Y) - YY \left( \frac{R^2}{2} \right) - \left( 1 - \frac{k}{2} \right) G + \Delta_x \left( \frac{R^2}{2} \right) \right) = \sum_{i=1}^d k \partial_{x^i} g(\partial_{x^i}, \partial_{y^i})$$

$$\partial_{x^i} \left( kL + \left( 1 - \frac{k}{2} \right) (\theta \cdot Y) - YY \left( \frac{R^2}{2} \right) - \left( 1 - \frac{k}{2} \right) G + \Delta_x \left( \frac{R^2}{2} \right) \right) = \sum_{i=1}^d k \partial_{x^i} g(\partial_{x^i}, \partial_{x^i})$$
Then we aim to have \( Q(y, x) = R(y, x)S(y, x) \) where \( R^2 \) is sufficiently large and \( S : N \times T^d \to S^{m-1} \) is a map given by Nash embedding.

We introduce another Riemannian metric on \( N \times T^d \); let \( h_{\alpha \beta} = \frac{g_{\alpha \beta} - (\partial_\alpha R)(\partial_\beta R)}{R^2} \). Then \( h \) is positive definite iff \( (g_{\alpha \beta} - (\partial_\alpha R)(\partial_\beta R))_{\alpha \beta} > 0 \). But \( g_{\alpha \beta} - (\partial_\alpha R)(\partial_\beta R) = g_{\alpha \beta} - \frac{(\partial_\alpha R)^2(\partial_\beta R)}{4R^2} \). As we can add a positive constant to \( R^2 \) and \( g > 0 \), WLOG \( h \) is a Riemannian metric on \( N \times T^d \). Then by Nash embedding, and the fact that any compact regular submanifold of \( \mathbb{R}^d \) (with the usual Euclidean metric) can be embedded into \( S^{2j} \) for some \( c > 0 \), we can rescale \( g, h \) and \( \theta \) to get an isometric embedding \( S : (N \times T^d, h) \to S^{m-1} \) for some \( m \).

Then \( \langle \partial_\alpha S, \partial_\beta S \rangle = h_{\alpha \beta} = \frac{g_{\alpha \beta} - (\partial_\alpha R)(\partial_\beta R)}{R^2} \). Let \( Q = RS \). Then \( \langle \partial_\alpha Q, \partial_\beta Q \rangle = (\partial_\alpha R)(\partial_\beta R) + R^2 \langle \partial_\alpha S, \partial_\beta S \rangle = g_{\alpha \beta} \) and \( Q : (N \times T^d, g) \to \mathbb{R}^m \) is also an isometric embedding. This means for any \( W \in X(N \times T^d) \) we have \( \theta(W) = g(Y, W) = (YQ, WQ) \).

Let \( W \) be \( \partial_y \) or \( \partial_{y_i} \) for some \( j \), then \( W, Y, \partial_{y_i} \) commute and we have:

\[
W \left( kL + \left( 1 - \frac{k}{2} \right) (\theta \cdot Y) - Y \left( \frac{|Q|^2}{2} \right) \right) = k \sum_i \langle \partial_{x_i}Q, WQ \rangle
\]

\[
\Leftrightarrow kY \left( \theta \cdot W \right) + W \left( \left( 1 - \frac{k}{2} \right) |Q|^2 - Y \left( YQ, Q \right) \right) = k \sum_i \langle \partial_{x_i}Q, WQ \rangle
\]

\[
= k \left( \left( 1 - \frac{k}{2} \right) |YQ|^2 - \sum_i \langle \partial_{x_i}Q, WQ \rangle \right)
\]

\[
\Leftrightarrow W \langle -YYQ + \Delta_x Q, WQ \rangle = k \langle -YYQ + \Delta_x Q, WQ \rangle
\]

By \( C^\infty(N \times T^d) \)-linearity, we conclude that equation (1) and \( W \langle -YYQ + \Delta_x Q, WQ \rangle = k \langle -YYQ + \Delta_x Q, WQ \rangle \) hold true \( \forall W \in X(N \times T^d) \). Now we construct the potential.

We define \( v \) on \( N \times T^d \) such that \( kv = \langle -YYQ + \Delta_x Q, WQ \rangle \). Note that this is possible since \( k \neq 0 \).

Then let \( V_0 = v \circ Q^{-1} \) be the restricted potential on \( \text{Im}Q \). Then equation (1) gives \( \langle -YYQ + \Delta_x Q, WQ \rangle = Wv = W(V_0 \circ Q) = \langle \nabla_{\text{Im}Q} V_0(Q), WQ \rangle \).

Let us define \( A = -YYQ + \Delta_x Q \). Then \( \text{proj}_{\text{Im}Q} A \) is the potential on \( \text{Im}Q \). Then \( \langle A, WQ \rangle = kV_0(Q) \). We hope to extend \( V_0 \) to \( V \) such that \( \nabla_{\mathbb{R}^m} V(Q) = A \). We first work on the unit sphere.

Homogeneity suggests we should define \( V_1 \) on \( \text{Im}S \) as \( V_1(S) = \frac{1}{R^k} V_0(RS) \). Then we have

\[
W(V_1 \circ S) = W \left( \frac{1}{R^k} V_0 \circ Q \right) = -\frac{k}{R^k + T}(WR)(V_0 \circ Q) + \frac{1}{R^k} W(V_0 \circ Q)
\]

\[
= \frac{1}{R^k + T} \left( -\frac{k}{R^k} (WR)(V_0 \circ Q) + \frac{1}{R} \langle A, WQ \rangle \right)
\]

\[
= \frac{1}{R^k - 1} \left( -\frac{(WR)}{R} (A, Q) + \frac{1}{R} \langle A, (WR)S \rangle + \frac{1}{R} \langle A, R(WS) \rangle \right) = \langle A \frac{1}{R^k - 1}, WS \rangle
\]

This means \( \langle \nabla_{\text{Im}S} V_1(S), WS \rangle = \langle A \frac{1}{R^k - 1}, WS \rangle \). Write \( B = \frac{A}{R^k - 1} \). Then \( \text{proj}_{\text{Im}S} B = \nabla_{\text{Im}S} V_1(S) \) and \( \langle B, S \rangle = \langle A \frac{1}{R^k}, WS \rangle = \frac{k}{R^k} V_0(Q) = kV_1(S) \).

We locally parametrize \( \text{Im}S, S^{m-1} \) and \( \mathbb{R}^m \) in a neighborhood \( U \) by \( (a', a', b') \) and \( (a', b', c) \).
respectively since $\text{Im} S \leftrightarrow S^{m-1} \leftrightarrow \mathbb{R}^m$. Let $a = (a^i), b = (b^i)$. WLOG assume $\{b = 0, c = 1\} = U \cap \text{Im} S, \{c = 1\} = U \cap S^{m-1}$.

Then in local coordinates:

$$B(a) = \partial_a V_1(a) \partial_a + F(a) \partial_b + k V_1(a) \partial_c$$

where $F \in C^\infty(\text{Im} S \cap U)$. Then define $V_1$ on $S^{m-1} \cap U$:

$$V_1(a, b) = V_1(a) + (F(a), b)$$

Then $\partial_a V_1(a, 0) = \partial_a V_1(a)$ and $\partial_b V_1(a, 0) = F(a)$ so $\text{proj}_{T(S^{m-1})} B = \nabla_{S^{m-1}} V_1(S)$ on $S^{-1}(U)$. From this local result, by partition of unity, we can extend $V_1$ to all of $S^{m-1}$ to have $V_1$ smooth on $S^{m-1}$ and $\text{proj}_{T(S^{m-1})} B = \nabla_{S^{m-1}} V_1(S)$.

Then by homogeneity we get $V$ on $\mathbb{R}^m$ (except possibly a small neighborhood near the origin). Because $\langle B, S \rangle = kV(S) = \langle \nabla_{\mathbb{R}^m} V, S \rangle$ and $\text{proj}_{T(S^{m-1})} B = \nabla_{S^{m-1}} V_1(S) = \text{proj}_{T(S^{m-1})} \nabla_{\mathbb{R}^m} V(S)$, we conclude

$$B = \nabla_{\mathbb{R}^m} V(S)$$

Homogeneity implies that $\nabla_{\mathbb{R}^m} V(Q) = \nabla_{\mathbb{R}^m} V(RS) = R^{k-1} \nabla_{\mathbb{R}^m} V(S) = R^{k-1} B = A$ and we are done. \hfill $\Box$

**Remark 2.** When $k = 0$, equation 2 yields

$$\theta \cdot Y - YY \left( \frac{R^2}{2} \right) - G + \Delta_x \left( \frac{R^2}{2} \right) = \text{const}$$

However, we also require $0 = kv = \langle -YYQ + \Delta_x Q, Q \rangle = \theta \cdot Y - YY \left( \frac{R^2}{2} \right) - G + \Delta_x \left( \frac{R^2}{2} \right)$ and $G > 0$. If we set $R = v = 1$, these conditions can be satisfied and we can embed $(N, Y)$ into HomNLW$_0(\mathbb{T}^d, \mathbb{R}^m, V)$. So we lose some freedom in controlling $|Q|^2$, but embedding is still possible when $k = 0$.

**Remark 3.** When $k = 2$, the Euler condition will be equivalent to

$$\forall W \in \mathcal{X}(N \times \mathbb{T}^d) : W \left( 2L - YY \left( \frac{|Q|^2}{2} \right) + \Delta_x \left( \frac{|Q|^2}{2} \right) \right) = 2 \sum_i \partial_{x^i} \langle \partial_{x^i} Q, WQ \rangle - \langle \partial_{x^i} Q, [\partial_{x^i}, W] Q \rangle$$

Note that $Q : N \rightarrow H^m_{\mathbb{d}}$ being an embedding does not imply $Q : N \times \mathbb{T}^d \rightarrow \mathbb{R}^m$ is also an embedding.

Let $W = \partial_{y^j}$. Then by integrating $x$ over $\mathbb{T}^d$ we conclude $\partial_{y^j} \left( 2L - YY \left( \int_{\mathbb{T}^d} \frac{|Q|^2}{2} \right) \right) = 0$. This implies $2L - YY \left( \int_{\mathbb{T}^d} \frac{|Q|^2}{2} \right)$ is constant on $N$. Since it’s understood that $|Q| > 0$, we have the following theorem.

**Theorem 4.** Let $d \in \mathbb{N}$ and $(N, Y)$ be a compact nonsingular flow. There is an embedding from $(N, Y)$ into HomNLW$_2(\mathbb{T}^d, \mathbb{R}^m, V)$ for some $m$ and $V$ iff there is a function $R > 0$ smooth on $N$ and a strongly adapted 1-form $\theta$ such that

$$d \left( YY \left( \frac{R^2}{2} \right) \right) - 4L_Y \theta = 0$$

**Proof.** To prove necessity, follow the proof in [Tao17] to get a weakly adapted 1-form $\theta$. The Euler condition will then, by remark 3, give us the equation above. The averaging process used to upgrade to a strongly adapted 1-form will preserve the equation.

To prove sufficiency, we trivially extend $R$ to $N \times \mathbb{T}^d$ by $R(y, x) := R(y)$, and set $G = 1$. Then
Remark 5. The above condition would be satisfied if we can find a \( Y \)-invariant strongly adapted 1-form \( \hat{\theta} \) (i.e. \( \mathcal{L}_Y \hat{\theta} = 0 \)). The natural question to ask is whether these conditions are superfluous, once given the existence of strongly adapted 1-forms. The answer is no, as the following example will show.

Example 6. Let \( N = T^2 \) and \( Y = f(y) \partial_x \) where \( f \in C^\infty(T) \). For instance: \( f(y) = \sin(2\pi y) + 2 \). Then \( f \) never vanishes and \( \text{sgn} f \) does not change. Because we can replace \( Y \) by \( -Y \) and \( \theta \) by \(-\theta\), WLOG we assume \( f > 0 \).

Then we note that \( \mathcal{L}_Y (dx) = d(i_Y (dx)) = df = \partial_y f \ dy \) and \( \mathcal{L}_Y (dy) = d(i_Y (dy)) = 0 \).

Let \( \theta = \theta_1 dx + \theta_2 dy \). Then \( \mathcal{L}_Y \theta = \int T \theta_1 (x, y) \partial_y f (y) + f(y) \partial_x \theta_2 (x, y) \ dy = 0 \). We note that a one form is exact on \( T^2 \) iff it is closed and vanishes under integration in the \( x \)-direction and \( y \)-direction (the generators of the fundamental group).

So \( \theta \) is strongly adapted if and only if

\[
\begin{align*}
\partial_y (f \partial_x \theta_1) - \partial_x (\theta_1 \partial_y f + f \partial_x \theta_2) &= 0 \\
\int T f(y) \partial_x \theta_1 (x, y) \ dx &= 0 \quad \text{(always true)} \\
\int T \theta_1 (x, y) \partial_y f (y) + f(y) \partial_x \theta_2 (x, y) \ dy &= 0 \\
f \theta_1 > 0
\end{align*}
\]

The last condition just means \( \theta_1 > 0 \).

An easy pick is \( \theta_1 = 1, \theta_2 = \text{const.} \) Or \( \theta_1(x, y) = f(y), \theta_2 = \text{const.} \) Or \( \theta_1 = \frac{1}{f}, \theta_2 = \text{const.} \) So strongly adapted 1-forms exist here and there are many kinds of them.

We ask whether there is a strongly adapted \( \theta \) such that \( \mathcal{L}_Y \theta = d(YYr) \) for some \( r \in C^\infty(N) \). Since

\[
d(YYr) = d \left( f^2 \partial_{xx} r \right) = f^2 \partial_{xx} r dx + \left( 2f \partial_y f \partial_{xx} r + f^2 \partial_{xy} r \right) dy
\]

this would mean

\[
\begin{align*}
f \partial_{xx} r - \partial_x \theta_1 &= 0 \\
2f \partial_y f \partial_{xx} r + f^2 \partial_{xy} r - \theta_1 \partial_y f - f \partial_x \theta_2 &= 0
\end{align*}
\]

But by integrating the second equation in \( x \), we get \( \partial_y f (y) \int T \theta_1 (x, y) dx = 0 \ \forall y \). If there is \( y_0 \) such that \( f'(y_0) \neq 0 \), such as when \( f(y) = \sin(2\pi y) + 2 \), then \( \int T \theta_1 (x, y_0) dx = 0 \). But \( \theta_1 > 0 \) so this is a contradiction. Therefore the condition in theorem \( \ref{thm:strongly-adapted-1-forms} \) as well as the existence of a \( Y \)-invariant strongly adapted 1-form, is not superfluous once given the existence of strongly adapted 1-forms.

Effectively, it means there are flows that can embed into \( \text{NLW}(T^d, \mathbb{R}^m, V) \) but not \( \text{HomNLW}_2(T^d, \mathbb{R}^m, V) \), which suggests the case \( k = 2 \) is much more different than other \( k \).

Remark 7. We are sometimes interested in having \( V \geq 0 \). In the case of \( k = 0 \), we already set \( v = 1 > 0 \) so if we take care to work on a small neighborhood when extending from the restricted potential \( V_0 \) to the full potential \( V \), we can get \( V \geq 0 \) for free.

When \( k \neq 0 \), to get \( V \geq 0 \), it is sufficient to have \( v = \frac{1}{k} (-YYQ + \Delta_x Q, Q) > 0 \). This is equivalent to \( k \left( \theta \cdot Y - |\nabla_x Q|^2 + \Delta_x \left( \frac{|Q|^2}{2} \right) - YY \left( \frac{|Q|^2}{2} \right) \right) > 0 \). If \( k < 0 \), this is equivalent to \( \theta \cdot Y + \Delta_x \left( \frac{|Q|^2}{2} \right) - YY \left( \frac{|Q|^2}{2} \right) < |\nabla_x Q|^2 \). Since we can freely add any positive constant to \( G \), we can have \( V \geq 0 \) for free when \( k < 0 \).

However, when \( k > 0 \), the condition becomes \( \theta \cdot Y + \Delta_x \left( \frac{|Q|^2}{2} \right) - YY \left( \frac{|Q|^2}{2} \right) > |\nabla_x Q|^2 \geq 0 \) which is more difficult to satisfy because in equation \( \ref{eq:strongly-adapted-1-forms} \), \( |\nabla_x Q|^2 \) depends on \( |Q|^2 \) and \( \theta \cdot Y \). There is little control over
how \( YY \left( \frac{Q^2}{2} \right) \) will behave so we are tempted to set \(|Q| = 1\). But a particular obstruction to our method then becomes finding a choice of \( L \) (which is defined up to a constant) such that \( \theta \cdot Y > \frac{k}{2} - 1 \). In certain cases, finding \( L \) is possible. For instance, assuming we can find a \( \gamma \)-invariant strongly adapted 1-form \( \theta \) (i.e. \( \mathcal{L}_\gamma \theta = 0 \)), set \( R = |Q| = 1 \), then find \( G \) so that \( \theta \cdot Y > G > 0 \) and \( \left( 1 - \frac{k}{2} \right) (\theta \cdot Y - G) = \text{constant} \).

Alternatively, if we can find a strongly adapted 1-form \( \theta \) such that \( 2k \mathcal{L}_\gamma (\theta) = (k - 2) d(\theta \cdot Y) \), then we can use corollary 13 below to embed into HomWell\(_k(\mathbb{R}^m, V)\), which trivially embeds into HomNLW\(_k(\mathbb{T}^d, \mathbb{R}^m, V)\), and get \( V \geq 0 \).

### 3 Embedding into HomWell

The case of embedding \((N, Y)\) into HomWell\(_k(\mathbb{R}^m, V)\) can be thought of as a special case where \( Q : N \to H_\gamma^0 \) maps points on \( N \) to constant functions, or when \( d = 0 \). As \( |\nabla x Q| \) becomes zero, the Euler condition gives us the following theorem.

**Theorem 8.** Given \( k \neq 0 \), the compact nonsingular flow \((N, Y)\) can be embedded into HomWell\(_k(\mathbb{R}^m, V)\) for some \( m \) and \( V \) iff there is a strongly adapted 1-form \( \theta \) and a function \( R > 0 \) smooth on \( N \) such that

\[
k \mathcal{L}_\gamma \theta + \left( 1 - \frac{k}{2} \right) d(\theta \cdot Y) - d \left( YY \left( \frac{R^2}{2} \right) \right) = 0
\]

**Remark 9.** By example \( \Box \) when \( k = 2 \), this condition is not superfluous.

When \( k \neq 2 \), it’s possible to set \( r = 1, \theta_1 = \int^k_{1 + 2}, \theta_2 = \text{const} \), which then gives \( 2k \mathcal{L}_\gamma (\theta) = (k - 2) d(\theta \cdot Y) \). So the example doesn’t help, and we do not know whether this condition is superfluous.

**Remark 10.** Once again, the case \( k = 0 \) is a bit special. The Euler condition forces \( 0 = -YY \left( \frac{R^2}{2} \right) + \theta \cdot Y \). So \( Y \left( \frac{R^2}{2} \right) \) is strictly increasing along flow of \( Y \), and the rate of increasing \((\theta \cdot Y)\) has a minimum positive value, so it will blow up, which cannot happen on a compact space. Therefore we have the following theorem:

**Theorem 11.** A compact nonsingular flow \((N, Y)\) can never be embedded into HomWell\(_0(\mathbb{R}^m, V)\) for any \( m \) or \( V \).

Consequently, we have flows that can be embedded into HomNLW\(_0\) but not HomWell\(_0\).

**Remark 12.** Sometimes we want \( V \geq 0 \) (assuming \( k \neq 0 \)). However, for \( k < 0 \), we can not have \( V \geq 0 \), since that would lead to \( -YY \left( \frac{R^2}{2} \right) + \theta \cdot Y \leq 0 \) and blowups just as described above. For \( k > 0 \), whether we can get \( V \geq 0 \) is unknown.

When \( k \neq 0 \), we have the following corollary, which is of some interest.

**Corollary 13** (Homogeneous potential well spherical embedding). Let \( k \neq 0 \) and \((N, Y)\) be a compact nonsingular flow.

1. If there is a spherical embedding from \((N, Y)\) into HomWell\(_k(\mathbb{R}^m, V)\) then \((N, Y)\) has a strongly adapted 1-form \( \theta \) where \( 2k \mathcal{L}_\gamma (\theta) = (k - 2) d(\theta \cdot Y) \).

2. If \((N, Y)\) has a strongly adapted 1-form \( \theta \) and there is \( k \neq 0 \) such that \( 2k \mathcal{L}_\gamma (\theta) = (k - 2) d(\theta \cdot Y) \), there is an embedding from \((N, Y)\) into HomWell\(_k(\mathbb{R}^n, V)\) for some \( n \) and \( V \).

**Remark 14.** We also see that the condition \( \mathcal{L}_\gamma (\theta) = \frac{k-2}{2} d(\theta \cdot Y) \) implies \( Y(\theta \cdot Y) = \mathcal{L}_\gamma (\theta) \cdot Y = \frac{k-2}{2} Y(\theta \cdot Y) \). If \( k \neq -2 \), this means that the kinetic energy \( \theta \cdot Y \) and the Lagrangian \( L \) are constant along the flows of \( Y \).

**Corollary 15.** If \( \mathcal{L}_\gamma (\theta) = 0 \) we can embed \((N, Y)\) into HomWell\(_2(\mathbb{R}^n, V)\) for some \( n \) and \( V \).
Again, by example 6, there are flows which can be embedded into \( \text{Well}(\mathbb{R}^n, V) \), but not \( \text{HomWell}_2(\mathbb{R}^n, V) \).

**Example 16.** Let \((N, Y)\) be a compact non-singular smooth flow.

- When \(N = (S^1)^n \subset \mathbb{C}^n = \mathbb{R}^{2n}\), \(Y(a) = ia\), we can obviously embed \((N, Y)\) into \(\text{HomWell}_2(\mathbb{R}^{2n}, V)\) where \(V(x) = \frac{|x|^2}{2}\) (actually smooth at the origin). With the induced Euclidean metric on \(N\), \(\nabla_Y Y = 0\). Let \(\theta = Y^b\). Then \(\nabla Z \in \mathfrak{X}N\):

  \[
  \mathcal{L}_Y(\theta) \cdot Z = Y(\theta \cdot Z) - \theta \cdot [Y, Z] = Y(\langle Y, Z \rangle) - \langle Y, [Y, Z] \rangle = \langle Y, \nabla_Y Z \rangle - \langle Y, \nabla_Y Z - \nabla_Z Y \rangle = \langle Y, \nabla_Y Z \rangle - \frac{1}{2} Z(|Y|^2) = 0
  \]

so \(\mathcal{L}_Y(\theta) = 0\). Then let \(Q(\alpha) = a, P = YQ = iQ, YP = iP = -Q = -\nabla_{\mathbb{R}^{2n}} V(Q)\). We note that \((N, Y)\) is both isometric and geodesible (defined below).

- When \((N, Y)\) is geodesible, i.e. there is a Riemannian metric \(g\) such that \(\mathcal{L}_Y g = 0\): By letting \(\theta = g \cdot Y\), we have \(\mathcal{L}_Y(\theta) = \mathcal{L}_Y(g) \cdot Y + g \cdot \mathcal{L}_Y(y) = 0\) and corollary \([15]\) applies.

- When \((N, Y)\) is geodesible, i.e. there is a Riemannian metric \(g\) such that \(\nabla_Y Y = 0\): As \(Y\) is nonsingular, \(\tilde{g} = \frac{g}{|Y|^2}\) is also a Riemannian metric. WLOG assume \(g = \tilde{g}\), \(|Y| = 1\). Then let \(\theta = g \cdot Y\) and we have

  \[
  \mathcal{L}_Y(\theta) \cdot Z = Yg(Y, Z) - g(Y, [Y, Z]) = g(Y, \nabla_Y Z) - g(Y, \nabla_Y Z - \nabla_Z Y) = g(Y, \nabla_Y Z) = Z(|Y|^2) = 0.
  \]

So corollary \([15]\) applies.

- When \(N = M \times [0, 1]/ ((y, 1) \sim (\phi(y), 0)) \) where \(\phi : M \to M\) is a diffeomorphism on a compact smooth manifold \(M\), and \(Y = (X, \partial_t)\) where \(X \in \mathfrak{X}M\) such that \(X = X \circ \phi\). Let \(\theta = (0, dt)\), then \(\mathcal{L}_Y(\theta) = 0\). Note that when \(\phi = \text{Id}_M, N = M \times (\mathbb{R}/\mathbb{Z})\) and \(X\) can be any vector field on \(M\).

## 4 Universality of NLS

A perhaps surprising fact is that the class of zero-dimensional NLS flows is universal.

**Definition 17.** When \(M = \mathbb{C}^n = \mathbb{R}^{2n}, X(a) = i\nabla V(a)\) where \(V : \mathbb{C}^n \to \mathbb{R}\) smooth, we denote the flow \((M, X)\) as \(\text{NLS}^0(\mathbb{C}^n, V)\).

**Theorem 18 (Universality of zero-dimensional NLS).** Any compact nonsingular flow \((N, Y)\) can be embedded into \(\text{NLS}^0(\mathbb{C}^n, V)\) for some \(n\) and \(V\).

**Proof.** Let \(Q\) be any embedding of \(N\) into \((S^1)^n\) for some \(n\) (we can use Whitney theorem, for example, then embed the compact image into \((S^1)^n\) as above).

\[
\begin{array}{ccc}
N & \overset{Q}{\longrightarrow} & \mathbb{C}^n = \mathbb{R}^{2n} \\
\downarrow Y & & \downarrow i\nabla V \\
TN & \overset{TQ}{\longrightarrow} & T\mathbb{R}^{2n}
\end{array}
\]

We wish to define \(V\) so that \(YQ = i\nabla V\) or \(-iYQ = \nabla V\). First define \(v = \alpha \in \mathbb{R}\) on \(N\) (constant function) and \(V_0 = v \circ Q\). As before, we want to check that

\[
\forall Z \in \mathfrak{X}N : \langle -iYQ, ZQ \rangle = \left\langle \nabla_{Q(N)} V_0(Q), ZQ \right\rangle = 0
\]
This is true because $Q = (Q_1, \ldots, Q_n)$ where $|Q_i| = 1$, so $Z Q_j \parallel Y Q_j \perp Q_j \forall j = 1, n$. So $-i Y Q = (-i Y Q_1, \ldots, -i Y Q_n) \perp Z Q$. So the projection of $-i Y Q$ onto $T Q(N)$ is $\nabla_{Q(N)} V_0(Q)$.

Then just as in [Tao17], we can extend $V_0$ to $V$ on $R^{2n}$ so that $-i Y Q = \nabla_{R^{2n}} V(Q)$.

If we need $V \geq 0$, just set $\alpha > 0$.

Remark 19. The task becomes more difficult if we require $V(e^{it} a) = V(a) \forall t \in R, \forall a \in C^n$ or $V$ to be homogeneous. That would require $-i Y Q$ to be constant in our torus embedding above, which is unreasonable to expect, so we must find another method of embedding.

References

